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Sharp corner points and isometric extension problem in Banach spaces

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ABSTRACT

In this article, we begin using some geometric methods to study the isometric extension problem in general real Banach spaces. For any Banach space Y, we define a collection of "sharp corner points" of the unit ball $B_1(Y^*)$, which is empty if Y is strictly convex and $\dim Y \geq 2$. Then we prove that any surjective isometry between two unit spheres of Banach spaces X and Y has a linear isometric extension on the whole space if Y is a Gâteaux differentiability space (in particular, separable spaces or reflexive spaces) and the intersection of "sharp corner points" and weak*-exposed points of $B(Y^*)$ is weak*-dense in the latter. Moreover, we study the "sharp corner points" in many classical Banach spaces and solve isometric extension problem affirmatively in the case that Y is (ℓ^1), $c_0(\Gamma)$, $c(\Gamma)$, $\ell^\infty(\Gamma)$ or some $C(\Omega)$.

1. Introduction and preliminaries

The famous Mazur–Ulam theorem in [32] stated that any surjective isometry V between two real normed spaces with $V(\theta) = \theta$ (zero element) must be linear. In [31], P. Mankiewicz proved that any surjective isometry between the convex bodies (i.e. open connected subsets) of two normed spaces can be extended to a surjective affine isometry on the whole space.

In 1987, D. Tingley proposed the following problem in [37].

Problem 1.1. Let X and Y be real normed spaces with unit spheres $S_1(X)$ and $S_1(Y)$, respectively. Suppose that $V_0 : S_1(X) \to S_1(Y)$ is a surjective isometry. Is V_0 necessarily the restriction of a linear or affine isometry on X?

We only consider the isometric extension problem in real normed spaces, since it is clearly negative in the complex case. This problem is interesting and easy to understand. Moreover, it is very important. If this problem has a positive answer, then the local geometric property of a mapping on the unit sphere will determine the property of the mapping on the whole space.

However, it is very difficult to solve. As Professor E. Odell said (in a recommendation letter for the first author to apply for a Science Award in 2012): "this is a very difficult problem that remains unsolved after 25 years". In [37], D. Tingley only proved that any isometry V_0 between the unit spheres $S_1(X_{(n)})$ and $S_1(Y_{(m)})$ necessarily maps the antipodal points to antipodal points, that is $V_0(-x) = -V_0(x)$ for any $x \in S_1(X_{(n)})$ (both $X_{(n)}$ and $Y_{(m)}$ are real finite-dimensional normed spaces).

For quite a while (about 15 years), there has been no progress at all on this problem, until it was solved in Hilbert space and $\ell^p(\Gamma)$ space $(1 \leq p \leq \infty)$ (see [10,8,9,6,7]). In the past decade, the isometric extension problem was considered in

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various classical Banach spaces and many good results were obtained, through studying the specific form of norm and a lot of special skills (see [12]).

By now, the isometric extension problem has been solved affirmatively if X is any classical Banach space and Y is a general Banach space (see [5,10–15,19,21,22,24,27–30,34,36,38–42]). However, little progress has been obtained if X and Y are both general Banach spaces, even in the two-dimensional case. Recently, the isometric extension problem was considered in somewhere-flat Banach spaces and polyhedral Banach spaces and some impressive results were obtained (see [2,25]). Moreover, this problem was also considered in the F-spaces (see [1,20,35,43]).

In this article, we attempt to study the isometric extension problem in general Banach spaces through some geometric properties of the Banach spaces including weak*-exposed points, Gâteaux differentiability, and so on.

In Section 2, we mainly prove some important lemmas and give the definition of "sharp corner points" of the unit ball $B_1(Y^*)$, where Y^* is the dual space of a Banach space Y, denoted by $\mathcal{SC}(Y^*)$. We show that there exists no such kind of points in $B_1(Y^*)$ if Y is strictly convex and dim$Y \geq 2$. Furthermore, for any Banach space Y, we prove that any smooth point in the unit sphere $S_1(Y^*)$ is not a sharp corner point.

In Section 3, we give some basic definitions and well-known results concerning Gâteaux differentiability space, weak-Asplund space, Asplund generated space, and so on. These well-known results can be found in [3,4,16,26,33] and take an important role in many corollaries of this article.

In Section 4, we prove the main result (Theorem 4.2) of this article.

Theorem 1.2. *Let X be a Banach space and Y be a Gâteaux differentiability space. If $\mathcal{P}(Y^*)$ is the set of* weak**-exposed points in $B_1(Y^*)$ and $\mathcal{P}(Y^*) \cap \mathcal{SC}(Y^*)$ is* weak**-dense in $\mathcal{P}(Y^*)$, then any surjective isometry between two unit spheres $S_1(X)$ and $S_1(Y)$ can be extended to a linear isometry on the whole space.*

From this theorem, we deduce a result (Theorem 4.11) concerning the isometric extension of isometry between unit spheres $S_1(X)$ and $S_1(Y)$, where X is a general Banach space and Y is an Asplund generated space.

Theorem 1.3. *Let X be a Banach space and Y be an Asplund generated space. Suppose that V_0 is an isometric mapping from the unit sphere $S_1(X)$ into $S_1(Y)$, which satisfies the following condition:*

$(*)$ *For any* $x_1, x_2 \in S_1(X)$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\|\lambda_1 V_0 x_1 + \lambda_2 V_0 x_2\| = 1 \Longrightarrow \lambda_1 V_0 x_1 + \lambda_2 V_0 x_2 \in V_0[S(X)].$$

Let $Z = \overline{span}\{V_0 x : x \in S_1(X)\}$. Suppose that $\mathcal{P}(Z^*) \cap \mathcal{SC}(Z^*)$ is weak**-dense in $\mathcal{P}(Z^*)$. Then V_0 can be extended to a linear isometry on the whole space.*

Consequently, we obtain that if $Y = (\ell^1)$, $c_0(\Gamma)$, $c(\Gamma)$, $\ell^\infty(\Gamma)$ or some $C(\Omega)$ (for example, the set of "G_δ-points" is dense in Ω), then the answer for the isometric extension problem is also affirmative.

It is worthwhile to say more about the geometric methods used in this article. In our former papers on this problem, we had to use different kinds of methods in various classical Banach spaces, because we mainly use the analytic methods which concerns the specific form of the norm. However, we can discuss two kinds of spaces together in this article because we use the geometric methods here.

Before we start, let us first set some notations and recall some definitions. In this article, all normed spaces are over $\mathbb{R}$ and Y^* denote the dual space of a normed space Y. $S_1(Y)(B_1(Y))$ denotes the unit sphere (unit ball) of a normed space Y.

Notation 1.4. *Let Y be a normed space and $y_0^* \in S_1(Y^*)$:*

$A(y_0^*) := \{y \in S_1(Y) : y_0^*(y) = 1\}$;

$\mathcal{A}(Y^*) := \{y^* \in S_1(Y^*) : A(y^*) \neq \emptyset\}$;

$P(y_0^*) := \{y \in S_1(Y) : y_0^*(y) = 1, y^*(y) < 1$ *for any* $y^* \in S_1(Y^*)$ *with* $y^* \neq y_0^*\}$;

$\mathcal{P}(Y^*) := \{y^* \in S_1(Y^*) : P(y^*) \neq \emptyset\}$.

Remark 1.5. Let Y be a normed space and $y_0^* \in S_1(Y^*)$. $A(y_0^*)$ is the set of "norm-attaining points" of y_0^*. $\mathcal{A}(Y^*)$ is the subset of $S_1(Y^*)$ in which any y^* norm-attains at some point in $S_1(Y)$. $P(y_0^*)$ is the set of "peak-functions" $J(y) \in Y^{**}$, which have (only) a peak at y_0^* (where J is the canonical mapping from Y to Y^{**}). $y_0^* \in \mathcal{P}(Y^*)$ is called the weak*-exposed point of unit ball $B_1(Y^*)$. It is evident that any $y_0 \in P(y_0^*)$ is a smooth point of $S_1(Y)$. Conversely, if y_0 is a smooth point of $S_1(Y)$, there exists a unique $y_0^* \in \mathcal{P}(Y^*)$ with $y_0^*(y_0) = 1$ (see [4,23]).

2. Some lemmas

We first introduce Lemma 2.1 in [41] as follows.

Lemma 2.1. *Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. Then we have*

$$\|x_1 + x_2\| = 2 \Longleftrightarrow \|V_0 x_1 + V_0 x_2\| = 2, \quad \forall x_1, x_2 \in S_1(X).$$

Proof. We only need to prove the "$\Longrightarrow$" part, because V_0^{-1} is also a surjective isometry from $S_1(Y)$ onto $S_1(X)$. Suppose that $\|x_1+x_2\|=2$. By the Hahn–Banach theorem, there exists $x_0^* \in S_1(X^*)$ such that $x_0^*(x_1+x_2)=\|x_1+x_2\|=2$. Hence

$$2=\|x_1+x_2\|=|x_0^*(x_1+x_2)|\le |x_0^*(x_1)|+|x_0^*(x_2)|\le 2,$$

and we have

$$x_0^*(x_1)=x_0^*(x_2)=1. \tag{2.1}$$

Let $\bar{x}_n=\left(1-\frac{1}{n}\right)x_1+\frac{1}{n}x_2$ ($\forall\, n\in\mathbb{N}$). By Eq. (2.1), we get a sequence $\{\bar{x}_n\}\subseteq S_1(X)$. For each $n\in\mathbb{N}$ and $x\in S_1(X)$, suppose that

$$\|\bar{x}_n+x\|=2. \tag{2.2}$$

By the Hahn–Banach theorem and the similar method, there exists $x^*_{(n,x)}\in S_1(X^*)$ such that $x^*_{(n,x)}(\bar{x}_n+x)=2$, which implies that

$$x^*_{(n,x)}(x_1)=x^*_{(n,x)}(x_2)=x^*_{(n,x)}(x)=1.$$

Therefore, we obtain

$$\|x_2+x\|=2, \tag{2.3}$$

since

$$2=x^*_{(n,x)}(x_2+x)\le\|x_2+x\|\le 2.$$

Note that

$$\|\bar{x}_n-V_0^{-1}(-V_0\bar{x}_n)\|=\|V_0\bar{x}_n+V_0\bar{x}_n\|=\|2V_0\bar{x}_n\|=2,\quad \forall\, n\in\mathbb{N}. \tag{2.4}$$

By the similar methods we used to deduce (2.3) from (2.2), we have that

$$\|x_2-V_0^{-1}(-V_0\bar{x}_n)\|=2,\quad \forall\, n\in\mathbb{N} \tag{2.5}$$

by (2.4). Note that V_0 is isometric and (2.5). We can obtain

$$\|V_0x_2+V_0\bar{x}_n\|=2,\quad \forall\, n\in\mathbb{N}.$$

Let $n\to\infty$. We get $\|V_0x_2+V_0x_1\|=2$ and complete the proof. $\square$

To get Lemma 2.3, we need to prove the following lemma.

Lemma 2.2. *Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $y_0^*\in\mathcal{P}(Y^*)$, then $V_0^{-1}[A(y_0^*)]\subseteq S_1(X)$ is convex.*

Proof. Since $y_0^*\in\mathcal{P}(Y^*)$, there exists $y_0\in P(y_0^*)(\subseteq A(y_0^*))$. Therefore, for any $x_1,x_2\in V_0^{-1}[A(y_0^*)]$ and $\lambda\in[0,1]$, we have

$$2=y_0^*(y_0+V_0x_1)\le\|y_0+V_0x_1\|\le 2,$$

that is $\|y_0+V_0x_1\|=2$. By Lemma 2.1, we have that $\|V_0^{-1}y_0+x_1\|=2$, and there exists $x_1^*\in S_1(X^*)$ such that

$$x_1^*(V_0^{-1}y_0+x_1)=2,$$

by the Hahn–Banach theorem. Note that $|x_1^*(V_0^{-1}y_0)|\le 1$ and $|x_1^*(x_1)|\le 1$. We get that

$$x_1^*(V_0^{-1}y_0)=x_1^*(x_1)=1,$$

and thus

$$2=x_1^*\left(V_0^{-1}y_0+\frac{V_0^{-1}y_0+x_1}{2}\right)\le\left\|V_0^{-1}y_0+\frac{V_0^{-1}y_0+x_1}{2}\right\|\le 2,$$

that is

$$\left\|V_0^{-1}y_0+\frac{V_0^{-1}y_0+x_1}{2}\right\|=2.$$

By Lemma 2.1, we obtain

$$\left\|y_0+V_0\left(\frac{V_0^{-1}y_0+x_1}{2}\right)\right\|=2.$$

Therefore, there exists $y_1^* \in S_1(Y^*)$ such that

$$y_1^*(y_0) + y_1^*\left[V_0\left(\frac{V_0^{-1}y_0 + x_1}{2}\right)\right] = 2,$$

by the Hahn–Banach theorem. From the similar arguments as above, we get that

$$y_1^*(y_0) = y_1^*\left[V_0\left(\frac{V_0^{-1}y_0 + x_1}{2}\right)\right] = 1. \tag{2.6}$$

Note Eq. (2.6) and $y_0 \in P(y_0^*)$. We have $y_1^* = y_0^*$ and

$$y_0^*\left[V_0\left(\frac{V_0^{-1}y_0 + x_1}{2}\right)\right] = 1. \tag{2.7}$$

Since $x_2 \in V_0^{-1}[A(y_0^*)]$, we get that $y_0^*\left[V_0x_2 + V_0\left(\frac{V_0^{-1}y_0+x_1}{2}\right)\right] = 2$, which implies that $\left\|V_0x_2 + V_0\left(\frac{V_0^{-1}y_0+x_1}{2}\right)\right\| = 2$. By Lemma 2.1, we get that

$$\left\|x_2 + \frac{V_0^{-1}y_0 + x_1}{2}\right\| = 2,$$

and there exists $x_2^* \in S_1(X^*)$ such that

$$x_2^*\left(x_2 + \frac{V_0^{-1}y_0 + x_1}{2}\right) = 2,$$

by the Hahn–Banach theorem. Note that $|x_2^*(x_2)|, |x_2^*(V_0^{-1}y_0)|, |x_2^*(x_1)| \le 1$. We have

$$x_2^*(V_0^{-1}y_0) = x_2^*(x_1) = x_2^*(x_2) = 1,$$

and

$$x_2^*[V_0^{-1}y_0 + (\lambda x_1 + (1-\lambda)x_2)] = 2.$$

Therefore, we get that $\|V_0^{-1}y_0 + (\lambda x_1 + (1-\lambda)x_2)\| = 2$, which implies that

$$\|y_0 + V_0(\lambda x_1 + (1-\lambda)x_2)\| = 2, \tag{2.8}$$

by Lemma 2.1. Then, from (2.8) and the similar argument we used to deduce (2.7), we can also obtain

$$y_0^*[V_0(\lambda x_1 + (1-\lambda)x_2)] = y_0^*(y_0) = 1,$$

that is $\lambda x_1 + (1-\lambda)x_2 \in V_0^{-1}[A(y_0^*)]$. Thus $V_0^{-1}[A(y_0^*)]$ is convex and the proof is completed. □

Lemma 2.3. *Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $y_0^* \in \mathcal{P}(Y^*)$, there exists $x_0^* \in S_1(X^*)$ such that*

$$y_0^*(y) = \pm 1 \Longrightarrow x_0^*(V_0^{-1}y) = y_0^*(y),$$

for any $y \in S_1(Y)$.

Proof. If $y \in S_1(Y)$ and $y_0^*(y) = 1$, then $y \in A(y_0^*)$. By Lemma 2.2, $V_0^{-1}[A(y_0^*)] \subseteq S_1(X)$ is convex and does not meet with the interior of $B_1(X)$. (It is evident that the interior of $B_1(X)$ is not empty.) Therefore, by the Eidelheit Separation theorem, there exists $x_0^* \in S_1(X^*)$ such that

$$\sup\{x_0^*(\bar{x}) : \bar{x} \in B_1(X)\} \le \inf\{x_0^*(x) : x \in V_0^{-1}[A(y_0^*)]\},$$

which implies that

$$1 \le \inf\{x_0^*(x) : x \in V_0^{-1}[A(y_0^*)]\} \le \inf\{\|x_0^*\| \cdot \|x\| : x \in V_0^{-1}[A(y_0^*)]\} = 1,$$

that is $x_0^*(x) = 1$ for any $x \in V_0^{-1}[A(y_0^*)]$.

Furthermore, if $\widetilde{y} \in S_1(Y)$ and $y_0^*(\widetilde{y}) = -1$, then $-\widetilde{y} \in A(y_0^*)$. Since $y_0^* \in \mathcal{P}(Y^*)$, there exists $y_0 \in P(y_0^*)$ ($\subseteq A(y_0^*)$), and we have that

$$2 \ge \|V_0^{-1}\widetilde{y} - V_0^{-1}y_0\| = \|\widetilde{y} - y_0\| \ge |y_0^*(\widetilde{y} - y_0)| = 2,$$

that is $\|V_0^{-1}y_0 + (-V_0^{-1}\widetilde{y})\| = 2$. By Lemma 2.1, we have $\|y_0 + V_0(-V_0^{-1}\widetilde{y})\| = 2$. Therefore, there exists $y_1^* \in S_1(Y^*)$ such that

$$y_1^*(y_0 + V_0(-V_0^{-1}\widetilde{y})) = 2,$$

by the Hahn–Banach theorem. Then we have

$$y_1^*(y_0) = y_1^*(V_0(-V_0^{-1}\widetilde{y})) = 1. \tag{2.9}$$

Note that Eq. (2.9) and $y_0 \in P(y_0^*)$. We have that $y_1^* = y_0^*$ and thus $y_0^*[V_0(-V_0^{-1}\widetilde{y})] = 1$. By the conclusion in the previous part of this proof, we obtain immediately that $x_0^*(-V_0^{-1}\widetilde{y}) = 1$, that is $x_0^*(V_0^{-1}\widetilde{y}) = -1$. Thus the proof is completed. $\square$

We will give the definition of "sharp corner points". These points play an important role in our result concerning the isometric extension problem in Gâteaux differentiability space (in particular, separable spaces or reflexive spaces).

Definition 2.4. Let Y be normed space. Then $y_0^* \in S_1(Y^*)$ is called a sharp corner point of $B_1(Y^*)$, if it satisfies the following conditions:

(i) For any $y \in S_1(Y)$ with $|y_0^*(y)| < 1$ and $\varepsilon > 0$, there exists $\widetilde{y}_\varepsilon \in S_1(Y)$ such that

$$y_0^*(\widetilde{y}_\varepsilon) = 1 \quad \text{and} \quad \|\widetilde{y}_\varepsilon \pm y\| \le 1 + |y_0^*(y)| + \varepsilon.$$

(ii) For any $y \in S_1(Y)$ with $0 < |y_0^*(y)| < 1$ and $\varepsilon > 0$, there exists $\overline{y}_\varepsilon \in S_1(Y)$ such that

$$y_0^*(\overline{y}_\varepsilon) = \frac{y_0^*(y)}{|y_0^*(y)|} \quad \text{and} \quad \|\overline{y}_\varepsilon - y\| \le 1 - |y_0^*(y)| + \varepsilon.$$

These sharp corner points of $B_1(Y^*)$ are denoted by $\mathcal{SC}(Y^*)$. Then we will give an important lemma as follows.

Lemma 2.5. *Let X and Y be normed spaces. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $y_0^* \in \mathcal{P}(Y^*) \cap \mathcal{SC}(Y^*)$, then we have*

$$x_0^*(V_0^{-1}y) = y_0^*(y) \quad \forall\, y \in S_1(Y),$$

where $x_0^* \in S_1(X^*)$ is the functional obtained in Lemma 2.3.

Proof. We take two steps to complete the proof:

(a) $|y_0^*(y)| = |x_0^*(V_0^{-1}y)|$ for any $y \in S_1(Y)$.

Indeed, for any $y \in S_1(Y)$, we can assume that $|y_0^*(y)| < 1$ (otherwise we can immediately get (a) by Lemma 2.3). Note $y_0^* \in \mathcal{SC}(Y^*)$ and Lemma 2.3. For any $\varepsilon > 0$, there exists $\widetilde{y}_\varepsilon \in S_1(Y)$ such that

$$x_0^*(V_0^{-1}\widetilde{y}_\varepsilon) = y_0^*(\widetilde{y}_\varepsilon) = 1,$$

and

$$\begin{aligned} 1 \pm x_0^*(V_0^{-1}y) &= |\pm 1 - x_0^*(V_0^{-1}y)| = |x_0^*(V_0^{-1}(\pm\widetilde{y}_\varepsilon)) - x_0^*(V_0^{-1}y)| \\ &\le \|V_0^{-1}(\pm\widetilde{y}_\varepsilon) - V_0^{-1}y\| = \|\widetilde{y}_\varepsilon \pm y\| \le 1 + |y_0^*(y)| + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we obtain that

$$|x_0^*(V_0^{-1}y)| \le |y_0^*(y)|, \quad \forall\, y \in S_1(Y).$$

If $|y_0^*(y)| = 0$, it is clear that $|x_0^*(V_0^{-1}y)| = 0$. Otherwise, note that $y_0^* \in \mathcal{SC}(Y^*)$ and Lemma 2.3. For any $\varepsilon > 0$, there exists $\overline{y}_\varepsilon \in S_1(Y)$ such that

$$|x_0^*(V_0^{-1}\overline{y}_\varepsilon)| = |y_0^*(\overline{y}_\varepsilon)| = 1,$$

and

$$\begin{aligned} 1 - |x_0^*(V_0^{-1}y)| &= |x_0^*(V_0^{-1}\overline{y}_\varepsilon)| - |x_0^*(V_0^{-1}y)| \\ &\le |x_0^*(V_0^{-1}\overline{y}_\varepsilon) - x_0^*(V_0^{-1}y)| \\ &\le \|V_0^{-1}\overline{y}_\varepsilon - V_0^{-1}y\| \\ &= \|\overline{y}_\varepsilon - y\| \le 1 - |y_0^*(y)| + \varepsilon. \end{aligned}$$

Therefore, we get that

$$|y_0^*(y)| \le |x_0^*(V_0^{-1}y)|, \quad \forall\, y \in S_1(Y)$$

and complete the first step.

(b) $y_0^*(y) = x_0^*(V_0^{-1}y)$ for any $y \in S_1(Y)$.

Indeed, if $y_0^*(y) = 0$, then we have $x_0^*(V_0^{-1}y) = 0$ because of (a). Otherwise, note that $y_0^* \in \mathcal{SC}(Y^*)$ and Lemma 2.3. For any $\varepsilon > 0$, there exists $\overline{y}_\varepsilon \in S_1(Y)$ such that

$$x_0^*(V_0^{-1}\overline{y}_\varepsilon) = y_0^*(\overline{y}_\varepsilon) = \frac{y_0^*(y)}{|y_0^*(y)|},$$

and

$$\begin{aligned} 1 &= |y_0^*(\overline{y}_\varepsilon)| = |x_0^*(V_0^{-1}\overline{y}_\varepsilon)| \le |x_0^*(V_0^{-1}y)| + |x_0^*(V_0^{-1}\overline{y}_\varepsilon) - x_0^*(V_0^{-1}y)| \\ &\le |y_0^*(y)| + |x_0^*(V_0^{-1}\overline{y}_\varepsilon - V_0^{-1}y)| \le |y_0^*(y)| + \|V_0^{-1}\overline{y}_\varepsilon - V_0^{-1}y\| \\ &= |y_0^*(y)| + \|\overline{y}_\varepsilon - y\| \le 1 + \varepsilon. \end{aligned}$$

We can get

$$0 \le |x_0^*(V_0^{-1}\overline{y}_\varepsilon) - x_0^*(V_0^{-1}y)| - (\,|x_0^*(V_0^{-1}\overline{y}_\varepsilon)| - |x_0^*(V_0^{-1}y)|\,) \le \varepsilon,$$

that is

$$0 \le \left|\frac{y_0^*(y)}{|y_0^*(y)|} - x_0^*(V_0^{-1}y)\right| - \left(\left|\frac{y_0^*(y)}{|y_0^*(y)|} - |x_0^*(V_0^{-1}y)|\right|\right) \le \varepsilon.$$

Since ε is arbitrary, we have that $x_0^*(V_0^{-1}y)$ and $y_0^*(y)$ have the same sign because $y_0^*(\overline{y}_\varepsilon) = \frac{y_0^*(y)}{|y_0^*(y)|}$. The proof is completed. $\square$

Remark 2.6. In Section 4, we will show that if Y is one of the spaces (ℓ^1), (c_0) or $C(K)$ (K is a compact metric space), then we have that $\mathcal{P}(Y^*) \subseteq \mathcal{SC}(Y^*)$. If Y is one of the spaces $c_0(\Gamma)$, $c(\Gamma)$, $\ell^\infty(\Gamma)$ (Γ is an infinite index set) and some $C(\Omega)$ (Ω is a compact Hausdorff space), then we have that $\mathcal{P}(Y^*) \cap \mathcal{SC}(Y^*)$ is weak*-dense in $\mathcal{P}(Y^*)$. Therefore, we will apply our main result Theorem 4.2 to these spaces and get a positive answer to the isometric extension problem. Here, the corresponding results for spaces $c_0(\Gamma)$, $c(\Gamma)$ and $C(\Omega)$ are new.

Proposition 2.7. *Let Y be a strictly convex Banach space and* $\dim Y \ge 2$. *Then we have that $\mathcal{SC}(Y^*) = \varnothing$.*

Proof. It is clear that if $y_0^* \in S_1(Y^*)$, there exists at most one element $y_0 \in S_1(Y)$ such that $y_0^*(y_0) = 1$. Otherwise, if there exists $y_1 \in S_1(Y)$ such that $y_0 \ne y_1$ and $y_0^*(y_1) = 1$, then for any $\lambda \in (0, 1)$, we have that

$$1 = y_0^*(\lambda y_0 + (1-\lambda)y_1) \le \|y_0^*\| \cdot \|\lambda y_0 + (1-\lambda)y_1\| < 1,$$

which is impossible. Assume that $\mathcal{SC}(Y^*) \ne \varnothing$ and $y_0^* \in \mathcal{SC}(Y^*)$. Note that $\ker y_0^* \ne \{\theta\}$ since $\dim Y \ge 2$. For any $y \in S_1(Y) \cap \ker y_0^*$, $y \ne \theta$ and $\varepsilon > 0$, there exists unique $\tilde{y}$ such that

$$y_0^*(y) = 1 \quad \text{and} \quad \|y_0 \pm y\| \le 1 + |y_0^*(y)| + \varepsilon = 1 + \varepsilon.$$

Since ε is arbitrary, we get that $\|y_0 \pm y\| \le 1$ and

$$2 = \|y_0 + y + y_0 - y\| \le \|y_0 + y\| + \|y_0 - y\| \le 2,$$

that is

$$\|y_0 + y + y_0 - y\| = \|y_0 + y\| + \|y_0 - y\|.$$

Since Y is strictly convex, we get that $y_0 + y = y_0 - y$, which is impossible. $\square$

Proposition 2.8. *Let Y be a real Banach space. Then any smooth point of $S_1(Y^*)$ is not a sharp corner point.*

Proof. Suppose that f_0 is a smooth point of $S_1(Y^*)$. There is a unique $y_0^{**} \in S_1(Y^{**})$ such that $y_0^{**}(f_0) = 1$. If there does not exist $y_0 \in S_1(Y)$ such that $g(y_0) = y_0^{**}(g)$ for any $g \in Y^*$, that is, $A(f) = \varnothing$, f_0 is clearly not a sharp corner point.

If $y_0 \in S_1(Y)$ given above exists, we assume that f_0 is also a sharp corner point. For any $y \in S_1(Y)$ with $0 < f_0(y) < 1$ and $\varepsilon > 0$, we see that $\|y - y_0\| \le 1 - f_0(y) + \varepsilon$, that is,

$$\|y - y_0\| \le 1 - f_0(y) = f_0(y_0) - f_0(y).$$

Note that $f_0(y_0) - f_0(y) \le \|y - y_0\|$. We have that

$$\|y - y_0\| = f_0(y_0) - f_0(y) = f_0(y_0 - y),$$

which implies that

$$f_0\left(\frac{y_0 - y}{\|y - y_0\|}\right) = 1.$$

However, it is impossible since $f_0 \in S_1(Y^*)$ is a smooth point. $\square$

Remark 2.9. Note that (ℓ^2) is strictly convex and $(\ell^2)^* = (\ell^2)$. By Proposition 2.7, there exist no sharp corner points in the unit ball of $(\ell^2)^*$ and so does (ℓ^2). More generally, for any self-conjugate and uniformly convex Banach space Y, we can prove that $\mathcal{SC}(Y^*) = \varnothing$ by Proposition 2.7 since any uniformly convex Banach space is strictly convex. Moreover, we can also get that $\mathcal{SC}(Y^*) = \varnothing$ by Proposition 2.8 since Y^* is uniformly smooth.

3. Some well-known results for Gâteaux differentiability spaces

In this section, let us recall some results for Gâteaux differentiability space, separable space, Asplund generated space, and so on (see [3,4,16,26,33]).

Definition 3.1. A Banach space E is said to be a Gâteaux differentiability space (weak-Asplund space) if for any continuous convex function f on it, there exists a dense (dense G_δ) subset $E_0 \subseteq E$ such that f is Gâteaux differentiable at any $x_0 \in E_0$.

Remark 3.2. Since the Gâteaux differentiability condition for a norm is homogeneous, a norm is differentiable at x if it is differentiable at λx for some scalar λ. Consequently, if E is a Gâteaux differentiability space, there exists a dense subset of the unit sphere $S_1(E)$ where $\|x\|$ is Gâteaux differentiable.

Proposition 3.3. *A Banach space E is a Gâteaux differentiability space if and only if any* weak* *compact convex subset of E^* is the* weak* *closed convex hull of its* weak**-exposed points (see [26]).*

Proposition 3.4. *Let E and E_1 be Banach spaces. Suppose that $T : E \to E_1$ is linear and continuous. If E is a Gâteaux differentiability space and $T(E)$ is dense in E_1, then E_1 is also a Gâteaux differentiability space. In particular, if a Banach space F is the image of a Gâteaux differentiability space by a linear continuous mapping, then F is also a Gâteaux differentiability space.*

Definition 3.5. A Banach space E is called Asplund generated if there exists an Asplund space X and a linear continuous operator $T : X \to E$ such that $T(X)$ is dense in E.

Remark 3.6. Recall that a Banach space E is called an Asplund space if for any continuous convex function f on it, there exists a dense G_δ subset $E_0 \subseteq E$ such that f is Fréchet differentiable at any $x_0 \in E_0$. Moreover, we have the following important facts:

(i) A Banach space E is an Asplund space if and only if E^* has the Radon–Nikodym property.
(ii) All the reflexive spaces and $c_0(\Gamma)$ space (for any index set Γ) are Asplund spaces.

Proposition 3.7. *Any weakly compactly generated space is an Asplund generated space. Any subspace of an Asplund generated space is a weak-Asplund space.*

Proposition 3.8. *Any separable Banach space is a weak-Asplund space. Moreover, if a Banach space E whose dual space E^* admits a strictly convex norm, then E is also a weak-Asplund space (see [3]).*

Definition 3.9. Let Ω be a compact space. Then $t_0 \in \Omega$ is called a G_δ-point if there exists a countable collection of open subsets $\{G_n \subseteq \Omega : n \in \mathbb{N}\}$ such that $\{t_0\} = \bigcap_{n=1}^{\infty} G_n$. Ω is said to be *scattered* if any subset of Ω has an isolated point.

Proposition 3.10. *Let Ω be a compact space. Then $C(\Omega)$ is Asplund if and only if Ω is scattered (see [16]).*

Remark 3.11. It is still an open question: What additional properties should Ω have such that $C(\Omega)$ is a Gâteaux differentiability space (a weak-Asplund space)? We only know the following result: if $C(\Omega)$ is a Gâteaux differentiability space (a weak-Asplund space), then Ω is sequentially compact, and for any closed subset $F \subseteq \Omega$, the set F_0 of G_δ-points of F is dense in F (F_0 contains a dense G_δ completely metrizable subset of F) (see [16, Theorem 2.2.3]). Therefore, we have to point out that a Banach space E may not be a Gâteaux differentiability space if the set of Gâteaux differentiable points of $S_1(E)$ is a dense subset.

4. Main theorems

In this section, we first introduce a result which can be seen in [17,43] (in [43], the proof was simplified).

Theorem 4.1. *Let X and Y be normed spaces. Suppose that V_0 is an isometry from $S_1(X)$ into $S_1(Y)$ and*

$$\|V_0x - |\lambda|V_0y\| \le \|x - |\lambda|y\|, \quad \forall\, x, y \in S_1(X),\ \lambda \in \mathbb{R}.$$

Then V_0 can be extended to an isometry on the whole space. Moreover, if V_0 is surjective, then V_0 can be linearly extended too.

Sketch of proof. For integrating this paper, we write the main idea of the proof as follows: Let

$$V\,x = \begin{cases} \|x\| V_0\left(\dfrac{x}{\|x\|}\right), & x \neq \theta; \\ \theta, & x = \theta. \end{cases}$$

Then we have that $\|Vx - Vy\| \le \|x - y\|$ for any $x, y \in S_1(Y)$ and $\|Vx - Vy\| = \|x - y\|$ if $\|x\| = \|y\|$, $x = \theta$ or $y = \theta$. Indeed, V is an isometry. Otherwise, there exist $x_0, y_0 \in X$ with $\|y_0\| > \|x_0\| > 0$ such that $\|Vx_0 - Vy_0\| < \|x_0 - y_0\|$. We can take $z_0 \in X$ such that $\|z_0\| = \|y_0\|$ and $z_0 \in \overrightarrow{y_0x_0}$ (the semi-line with the starting point y_0 and crossing x_0). Then we get the following inequality:

$$\begin{aligned} \|z_0 - y_0\| &= \|z_0 - x_0\| + \|x_0 - y_0\| > \|Vz_0 - Vx_0\| + \|Vx_0 - Vy_0\| \\ &\ge \|Vz_0 - Vy_0\|, \end{aligned}$$

which is impossible. If V_0 is surjective, we can also get a linear isometric extension by the Mazur–Ulam theorem. □

We can now show our main result as follows.

Theorem 4.2. *Let X be a Banach space and Y be a Gâteaux differentiability space. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $\mathcal{P}(Y^*) \cap \mathcal{SC}(Y^*)$ is weak*-dense in $\mathcal{P}(Y^*)$, then V_0 can be extended to a linear isometry on the whole space.*

Proof. For any $x_1, x_2 \in S_1(X)$ and $\lambda \in \mathbb{R}$, we have that

$$\|V_0x_1 - |\lambda| V_0x_2\| = \sup_{y^* \in S_1(Y^*)} |y^*(V_0x_1 - |\lambda| V_0x_2)|.$$

By Proposition 3.3, we get that

$$\begin{aligned} \|V_0x_1 - |\lambda| V_0x_2\| &= \sup_{y_0^* \in \mathcal{P}(Y^*)} |y_0^*(V_0x_1 - |\lambda| V_0x_2)| \\ &= \sup_{y_0^* \in \mathcal{P}(Y^*) \cap \mathcal{SC}(Y^*)} |y_0^*(V_0x_1 - |\lambda| V_0x_2)|. \end{aligned} \tag{4.1}$$

By Lemma 2.5, for any $y_0^* \in \mathcal{P}_0(Y^*)$, there exists $x_0^* \in S_1(X^*)$ (x_0^* is obtained in Lemma 2.3) such that

$$\begin{aligned} |y_0^*(V_0x_1 - |\lambda| V_0x_2)| &= |y_0^*(V_0x_1) - y_0^*(|\lambda| V_0x_2)| = |x_0^*(x_1) - x_0^*(|\lambda| x_2)| \\ &\le \|x_1 - |\lambda| x_2\|. \end{aligned} \tag{4.2}$$

Note Eqs. (4.1) and (4.2). We get immediately that

$$\|V_0x_1 - |\lambda| V_0x_2\| \le \|x_1 - |\lambda| x_2\|, \quad \forall x_1, x_2 \in S_1(X),\ \lambda \in \mathbb{R},$$

and complete the proof because of Theorem 4.1. □

Corollary 4.3. *Let X be a Banach space and Y be a separable Banach space (more generally, Y^* admits a strictly convex norm). Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. If $\mathcal{P}(Y^*) \cap \mathcal{SC}(Y^*)$ is weak*-dense in $\mathcal{P}(Y^*)$, then V_0 can be extended to a linear isometry on the whole space.*

Proof. Note that any weak-Asplund space is a Gâteaux differentiability space. We get the conclusion immediately by Proposition 3.8. □

Corollary 4.4. *Let X be a Banach space and $Y = (\ell^1)$. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. Then V_0 can be extended to a linear isometry on the whole space.*

Proof. Note that Y is separable and Corollary 4.3. We only need to check that $\mathcal{P}(Y^*) \subseteq \mathcal{SC}(Y^*)$. It is easy to see that

$$\mathcal{P}(Y^*) = \{\, \{\theta_n\} : \{\theta_n\} \in (\ell^\infty), \theta_n = \pm 1, n \in \mathbb{N}\}.$$

Let $y_0^* \in \mathcal{P}(Y^*)$ and $y \in S_1(Y)$ with $|y_0^*(y)| < 1$. If $y_0^* = \{\theta_n^0\}$ and $y = \{y(n)\}$, we can take $\widetilde{y} = \{\widetilde{y}(n)\}$ such that

$$\widetilde{y}(n) = \theta_n^0 |y(n)|, \quad \forall\, n \in \mathbb{N}.$$

Then we have that $\{\widetilde{y}(n)\} \in S_1(Y)$, $y_0^*(\widetilde{y}) = 1$ and

$$\begin{aligned} \|\widetilde{y} \pm y\| &= \sum_{n=1}^{\infty} |\widetilde{y}(n) \pm y(n)| = \sum_{n=1}^{\infty} |\, \theta_n^0 |y(n)| \pm y(n)\,| \\ &= \sum_{n=1}^{\infty} |\, |y(n)| \pm \theta_n^0 y(n)| = \sum_{n=1}^{\infty} |y(n)| \pm \sum_{n=1}^{\infty} \theta_n^0 y(n) \\ &= 1 \pm y_0^*(y) \le 1 + |y_0^*(y)|. \end{aligned}$$

Moreover, if $y_0^*(y) \neq 0$, we can also take $\bar{y} = \frac{y_0^*(y)}{|y_0^*(y)|} \cdot \widetilde{y}$ and have that

$$\begin{aligned}\|\bar{y} - y\| &= \sum_{n=1}^{\infty} \left| \frac{y_0^*(y)}{|y_0^*(y)|} \cdot \theta_n^0 |y(n)| - y(n) \right| = \sum_{n=1}^{\infty} \left| |y(n)| - \frac{y_0^*(y)}{|y_0^*(y)|} \cdot \theta_n^0 y(n) \right| \\ &= \sum_{n=1}^{\infty} |y(n)| - \frac{y_0^*(y)}{|y_0^*(y)|} \sum_{n=1}^{\infty} \theta_n^0 y(n) = 1 - \frac{y_0^*(y)}{|y_0^*(y)|} y_0^*(y) = 1 - |y_0^*(y)|.\end{aligned}$$

Then we complete the proof. □

Corollary 4.5. *Let X be a Banach space and $Y = (c_0)$. Suppose that V_0 is a surjective isometry between $S_1(X)$ and $S_1(Y)$. Then V_0 can be extended to a linear isometry on the whole space.*

Proof. Note that Y is separable and Corollary 4.3. We only need to check that $\mathcal{P}(Y^*) \subseteq \mathcal{SC}(Y^*)$. It is easy to see that

$$\mathcal{P}(Y^*) = \{\pm e_n^* : n \in \mathbb{N}\},$$

where $e_n^* = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in (\ell^1)$ for any $n \in \mathbb{N}$. Let $e_{n_0}^* \in \mathcal{P}(Y^*)$ and $y \in S_1(Y)$ with $|e_{n_0}^*(y)| < 1$. We can take $\widetilde{y} = e_{n_0} \in S_1(Y)$. Then we have that

$$\begin{aligned}\|\widetilde{y} \pm y\| &= \|\{e_{n_0}(n) \pm y(n)\}\| = \sup_{n\in\mathbb{N}} |e_{n_0}(n) \pm y(n)| \\ &\leq 1 + |y(n_0)| = 1 + |e_{n_0}^*(y)|.\end{aligned}$$

Moreover, if $e_{n_0}^*(y) \neq 0$, we can take

$$\bar{y} = y + \left(\frac{e_{n_0}^*(y)}{|e_{n_0}^*(y)|} - e_{n_0}^*(y) \right) e_{n_0} \in S_1(Y),$$

that is, $\bar{y} = \{\bar{y}(n)\}$ with

$$\bar{y}(n) = \begin{cases} \dfrac{y(n_0)}{|y(n_0)|}, & \text{if } n = n_0; \\ y(n), & \text{if } n \neq n_0. \end{cases}$$

We can get that

$$\begin{aligned}\|\bar{y} - y\| &= \sup_n |\bar{y}(n) - y(n)| = \left| \frac{y(n_0)}{|y(n_0)|} - y(n_0) \right| \\ &= 1 - |y(n_0)| = 1 - |e_{n_0}^*(y)|.\end{aligned}$$

Then we complete the proof. □

Corollary 4.6. *Let X be a Banach space and $Y = C(K)$ (K is a compact metric space). Suppose that $Z \subseteq Y$ is a linear closed subspace, and there exists a dense subset $T \subseteq K$ such that all the "peak functions" whose peak is $t \in T$ are in Z. If V_0 is an isometric mapping from $S_1(X)$ onto $S_1(Z)$, then V_0 can be extended to a linear isometry on the whole space.*

Proof. Note that $C(K)$ is a separable Banach space and

$$\mathcal{P}(Y^*) = \{\pm\delta_k^* : \ k \in K\} \quad (\delta_{k_0}^*(y) = y(k_0) \text{ for every } y = y(k) \in Y).$$

It is easy to see that

$$\{\pm\delta_t^* : \ t \in T\} \subseteq \mathcal{P}(Z^*)$$

and $\{\pm\delta_t^* : \ t \in T\}$ is weak*-dense in $\mathcal{P}(Z^*)$. By Corollary 4.3, we only need to prove that $\delta_{t_0}^* \in \mathcal{SC}(Z^*)$ for any $t_0 \in T$ (because it is similar to prove that $-\delta_{t_0}^* \in \mathcal{SC}(Z^*)$ for any $t_0 \in T$).

For any $\delta_{t_0}^* \in \mathcal{P}(Y^*)$, $z \in S_1(Z)$ with $|\delta_{t_0}^*(z)| = |z(t_0)| \leq 1$, and $\varepsilon > 0$ (if $z(t_0) \neq 0$, we also assume that $\varepsilon < \frac{|z(t_0)|}{2}$), there exists an open neighborhood $G(t_0)$ of t_0 in K such that

$$|z(k) - z(t_0)| < \varepsilon, \quad \forall\, k \in G(t_0). \tag{4.3}$$

By Urysohn's Lemma we can get $y(k) \in C(K)$ such that

$$y(t_0) = 1, \qquad y(k) \equiv 0 \quad (\forall\, k \in K \setminus G(t_0))$$

and

$$0 \le y(k) \le 1, \quad \forall\, k \in K.$$

Then we can make a "peak function" $p_{t_0}(k) \in C(K)$ (whose peak is t_0 and $p_{t_0}(t_0) = 1$), which is equal to 0 on $K \setminus G(t_0)$ and takes non-negative value on K. Let

$$\widetilde{z}_\varepsilon(k) = \min(y(k), p_{t_0}(k)).$$

It is easy to see that $\widetilde{z}_\varepsilon(k)$ is also a "peak function" on K whose peak is t_0 and $0 \le \widetilde{z}_\varepsilon(k) \le 1$, and thus $\widetilde{z}_\varepsilon \in S_1(Z)$ by the hypotheses of Z. By (4.3), we have that $\widetilde{z}_\varepsilon \pm z \in Z$ and

$$\begin{aligned}
\|\widetilde{z}_\varepsilon \pm z\| &= \max\left(\max_{k\in G(t_0)} |\widetilde{z}_\varepsilon(k) \pm z(k)|, \max_{k\in K\setminus G(t_0)} |z(k)| \right) \\
&\le \max\left(\max_{k\in G(t_0)} |\widetilde{z}_\varepsilon(k)| + \max_{k\in G(t_0)} |z(k)|, \max_{k\in K\setminus G(t_0)} |z(k)| \right) \\
&\le 1 + (|z(t_0)| + \varepsilon) = 1 + \delta^*_{t_0}(z) + \varepsilon.
\end{aligned}$$

Moreover, if $\delta^*_{t_0}(z) = z(t_0) \neq 0$, we first change above "peak function" $p_{t_0}(k)$ into $\overline{p}_{t_0}(k)$ which may be very sharp in above neighborhood $G(t_0)$, and let it satisfy the following condition:

$$\overline{p}_{t_0}(k) \le 1 - \frac{|z(k)| - |z(t_0)|}{1 - |z(t_0)|}, \quad \forall\, k \in G(t_0). \tag{4.4}$$

When we take

$$\overline{z}_\varepsilon = z + \left(\frac{\delta^*_{t_0}(z)}{|\delta^*_{t_0}(z)|} - \delta^*_{t_0}(z) \right) \overline{p}_{t_0},$$

by the hypotheses of Z, we have that $\overline{z}_\varepsilon \in Z$ and

$$\overline{z}_\varepsilon(k) = \begin{cases} \dfrac{z(t_0)}{|z(t_0)|}, & \text{if } k = t_0; \\ z(k) + (1 - |z(t_0)|)\dfrac{z(t_0)}{|z(t_0)|}\overline{p}_{t_0}(k), & \text{if } k \in G(t_0) \setminus \{t_0\}; \\ z(k), & \text{if } k \in K \setminus G(t_0). \end{cases}$$

Note that both $z(k)$ and $(1 - |z(t_0)|)\frac{z(t_0)}{|z(t_0)|}\overline{p}_{t_0}(k)$ have the same sign because of (4.3). By (4.4), we obtain that

$$\left| z(k) + (1 - |z(t_0)|)\frac{z(t_0)}{|z(t_0)|}\overline{p}_{t_0}(k) \right| = |z(k)| + (1 - |z(t_0)|)\overline{p}_{t_0}(k) \le 1.$$

Then we have that $\overline{z}_\varepsilon \in S_1(Z)$, $\overline{z}_\varepsilon - z \in Z$ and

$$\|\overline{z}_\varepsilon - z\| = \left\| \left(\frac{\delta^*_{t_0}(z)}{|\delta^*_{t_0}(z)|} - \delta^*_{t_0}(z) \right) \overline{p}_{t_0} \right\| = 1 - |\delta^*_{t_0}(z)|.$$

Then we complete the proof by Corollary 4.3. □

We write Corollary 4.6 in such a form as above because it will be used in Theorem 4.7. As we stated in Remark 3.11, $C(\Omega)$ (Ω is a compact Hausdorff space) may not be a Gâteaux differentiability space even if the set of G_δ-points of Ω is dense in Ω. However, we can also get the conclusion of Corollary 4.6 by the similar methods.

Theorem 4.7. *Let X be a Banach space and $Y = C(\Omega)$ (Ω is a compact Hausdorff space). Suppose that there exists a dense subset $T \subseteq \Omega$ such that T contains all the G_δ-points of Ω. If a linear closed subspace $Z \subseteq Y$ contains all such "peak functions" whose peak is $t \in T$ and V_0 is an isometric mapping from $S_1(X)$ onto $S_1(Z)$, then V_0 can be extended to a linear isometry on the whole space.*

Proof. It is the case that $\{\pm\delta^*_t : t \in T\} \subseteq \mathcal{P}(Y^*)$ and $\delta^*_t \in \mathcal{SC}(Z^*)$ for any $t \in T$ by the similar arguments of Corollary 4.6. There exists $x^*_t \in S_1(X^*)$ such that

$$\delta^*_t(z) = x^*_t(V_0^{-1}z), \quad \forall\, z \in S_1(Z),$$

by Lemma 2.5. Note that $\overline{T} = \Omega$. We have

$$\begin{aligned}\|V_0x_1 - |\lambda|V_0x_2\| &= \sup_{\omega\in\Omega} |(V_0x_1)(\omega) - |\lambda|(V_0x_2)(\omega)| \\ &= \sup_{t\in T} |(V_0x_1)(t) - |\lambda|(V_0x_2)(t)| \\ &= \sup_{t\in T} |\delta_t^*(V_0x_1) - |\lambda|\delta_t^*(V_0x_2)| \\ &= \sup_{t\in T} |x_t^*(x_1) - |\lambda|x_t^*(x_2)| \\ &\le \|x_1 - |\lambda|x_2\|, \quad \forall x_1, x_2 \in S_1(X).\end{aligned}$$

Then we complete the proof by Theorem 4.1. □

Remark 4.8. Both Corollary 4.6 and Theorem 4.7 generalize the corresponding results in [18].

In Remark 3.11, we stated that (ℓ^∞) is not a Gâteaux differentiability space. However, we can get the conclusion of Corollary 4.5 in (ℓ^∞) by the similar methods.

Theorem 4.9. *Let X be a Banach space and $Y = c_0(\Gamma), c(\Gamma)$ or $\ell^\infty(\Gamma)$ (Γ is an infinite index set). Suppose that $Z \subseteq Y$ is a linear closed subspace and $\{e_\gamma : \gamma \in \Gamma\} \subseteq Z$. If V_0 is a surjective isometry between $S_1(X)$ and $S_1(Z)$, then V_0 can be extended to a linear isometry on the whole space.*

Proof. Note that $\{\pm e_\gamma^* : \gamma \in \Gamma\} \subseteq \mathcal{P}(Y^*)$ where

$$e_{\gamma_0}^*(e_\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_0; \\ 0, & \text{if } \gamma \ne \gamma_0; \end{cases}$$

for any $\gamma \in \Gamma$. By the similar arguments of Corollary 4.6, we have that $e_\gamma^* \in \mathcal{SC}^*(Z^*)$ for any $\gamma \in \Gamma$. Therefore there exists $x_\gamma^* \in S_1(X^*)$ such that

$$e_\gamma^*(z) = x_\gamma^*(V_0^{-1}z), \quad \forall z \in S_1(Z),$$

by Lemma 2.5. We can get that

$$\begin{aligned}\|V_0x_1 - |\lambda|V_0x_2\| &= \sup_{\gamma\in\Gamma} |(V_0x_1)(\gamma) - |\lambda|(V_0x_2)(\gamma)| \\ &= \sup_{\gamma\in\Gamma} |e_\gamma^*(V_0x_1) - |\lambda|e_\gamma^*(V_0x_2)| \\ &= \sup_{\gamma\in\Gamma} |x_\gamma^*(x_1) - |\lambda|x_\gamma^*(x_2)| \\ &\le \|x_1 - |\lambda|x_2\|, \quad \forall x_1, x_2 \in S_1(X).\end{aligned}$$

Then we complete the proof by Theorem 4.1. □

Remark 4.10. By Remark 3.6, we see that $c_0(\Gamma)$ is an Asplund space and thus a Gâteaux differentiability space. Therefore, we can also get the conclusion of Theorem 4.9 for $c_0(\Gamma)$ by Theorem 4.2.

Theorem 4.11. *Let X be a Banach space and Y be an Asplund generated space. Suppose that V_0 is an isometric mapping from the unit sphere $S_1(X)$ into $S_1(Y)$ which satisfies the following condition:*

$$(*) \text{ For any } x_1, x_2 \in S_1(X) \quad \text{and} \quad \lambda_1, \lambda_2 \in \mathbb{R},$$
$$\|\lambda_1V_0x_1 + \lambda_2V_0x_2\| = 1 \Longrightarrow \lambda_1V_0x_1 + \lambda_2V_0x_2 \in V_0[S(X)].$$

Let $Z = \overline{span}\{V_0x : x \in S_1(X)\}$. Suppose that $\mathcal{P}(Z^*) \cap \mathcal{SC}(Z^*)$ is weak**-dense in $\mathcal{P}(Z^*)$. Then V_0 can be extended to a linear isometry on the whole space.*

Proof. We first prove that $S_1(Z) = V_0[S_1(X)]$. Note the condition $(*)$ and the equality

$$\sum_{k=1}^{n} \lambda_kV_0x_k = \left\|\sum_{k=1}^{n-1} \lambda_kV_0x_k\right\| \sum_{k=1}^{n-1} \frac{\lambda_k}{\left\|\sum_{k=1}^{n-1} \lambda_kV_0x_k\right\|} V_0x_k + \lambda_nV_0x_n.$$

By induction, we get that

$$\left\|\sum_{k=1}^{n} \lambda_kV_0x_k\right\| = 1 \Longrightarrow \sum_{k=1}^{n} \lambda_kV_0x_k \in V_0[S_1(X)]; \quad \forall x_k \in S_1(X),\ \lambda_k \in \mathbb{R}\ (1 \le k \le n),\ n \in \mathbb{N}.$$

Therefore, we have that

$$S_1(Z) = V_0[S_1(X)].$$

Note Proposition 3.7 and that Z is a closed subspace of Y. The conclusion is clear by Theorem 4.2. □

Corollary 4.12. *Suppose that Y in Theorem 4.11 has dual space Y^* with RNP (in particular, Y is either reflexive or $c_0(\Gamma)$-space). Then the conclusion is also valid.*

Proof. It is the direct consequence of Remark 3.6 and Theorem 4.11. □

Corollary 4.13. *Suppose that Y in Theorem 4.11 is weakly compact generated. Then the conclusion is also valid.*

Proof. It is the direct consequence of Proposition 3.7 and Theorem 4.11. □

Corollary 4.14. *Suppose that $Y = C(\Omega)$ (Ω is a scattered compact space). Then the conclusion is also valid.*

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