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Decay estimates of the coupled chemotaxis–fluid equations in R^3

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ABSTRACT

In this paper, we are concerned with a Chemotaxis–Navier–Stokes model, arising from biology, which is a coupled system of the chemotaxis equations and the viscous incompressible fluid equations with transport and external force. The optimal convergence rates of classical solutions to the Chemotaxis–Navier–Stokes system for small initial perturbation around constant states are obtained by pure energy method under the assumption the initial data belong to $\dot{H}^{-s} \cap H^N$, $N \geqslant 3$ $(0 \leqslant s < 3/2)$. The $\dot{H}^{-s}$ $(0 \leqslant s < 3/2)$ negative Sobolev norms are shown to be preserved along time evolution. Compared to the result in [5], we obtain the optimal decay rates of the higher-order spatial derivatives of the solutions.

1. Introduction

Chemotaxis is a biological process in which cells (e.g., bacteria) move towards a chemically more favorable environment. For example, bacteria often swim towards higher concentration of oxygen to survive. Generally, the motion of the fluid is determined by the well-known incompressible Navier–Stokes equations or Stokes equations. Thus, this kind of cell–fluid interaction becomes more complicated since it not only consists of chemotaxis and diffusion, but also includes transport and viscous fluid dynamics. In particular, it is interesting and important in biology to study some phenomenon of sedimentation on the basis of the coupled cell–fluid model. In [11], the authors observed large-scale convection patterns in a water drop sitting on a glass surface containing oxygen-sensitive bacteria, oxygen diffusing into the drop through the fluid-air interface and they proposed this model:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \delta \Delta n - \nabla \cdot \big(\chi(c) n \nabla c\big), \\ \partial_t c + u \cdot \nabla c = \mu \Delta c - k(c) n, \\ \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u - n \nabla \phi, \\ \nabla \cdot u = 0, \quad t > 0,\ x \in R^3. \end{cases} \tag{1}$$

Here, the unknowns are $n = n(t,x)$, $c = c(t,x)$, $u = u(t,x)$, $P = P(t,x)$ denoting the cell density, chemical concentration, velocity field and pressure of the fluid, respectively. Ω is a domain where the cells and the fluid move and interact. Positive constants δ, μ and ν are the corresponding diffusion coefficients for the cells, chemical and fluid. $\chi(c)$ is the chemotactic

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sensitivity and $k(c)$ is the consumption rate of the chemical by the cells. $\phi(x)$ is a given potential function accounting the effects of external forces such as gravity. The system (1) is supplied with initial conditions

$$(n,c,u)|_{t=0}=\big(n_0(x),c_0(x),u_0(x)\big),\quad x\in\Omega,$$

and some proper boundary conditions. The experimental set-up corresponds to mixed-type boundary conditions [11]. However, we work in full space R^3.

In this paper, we consider the decay rates of (1) by using an energy method under the assumption that the initial datum (n,c,u) is a small smooth perturbation of the constant state $(\hat{n},0,0)$ with $\hat{n}>0$. In [5] Duan et al. not only obtain the global solution for small smooth perturbation but also get the optimal rates of (1) by L^p estimates method under suitable assumption. In [5], under the assumption $(n_0-\hat{n},c_0)\in L^1(R^3)$ $u_0\in L^q(R^3)$ and $\phi\in L^\infty(R^+;L^{\frac{2q}{2-q}}(R^3))$, $q\in(1,\frac{6}{5})$ they get the decay rate

$$\begin{cases}\|n-\hat{n}\|_{L^p}\leqslant C\|n_0-\hat{n}\|_{L^1\cap L^p}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, & 1\leqslant p<\infty,\\ \|c\|_{L^p}\leqslant C\|c_0\|_{L^1\cap L^p}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, & 1\leqslant p<\infty,\\ \|u\|_{L^2}\leqslant C\big(\|u_0\|_{L^1\cap H^3}+\|(n_0-\hat{n},c_0)\|_{L^1\cap H^3}+\|n_0-\hat{n}\|_{L^1\cap L^2}\|c_0\|_{L^1\cap L^2}\big)(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})}. \end{cases}\tag{2}$$

For $2\leqslant p<\infty$, the decay rates of cell density n and chemical concentration c were obtained by L^p energy method in [7]. The decay estimates of cell density n and chemical concentration c for $1\leqslant p\leqslant 2$ are obtained because they can directly get $(n,c)(t)\in L^1$ under the initial data $(n,c)(0)\in L^1$. By interpolation method, they get the decay estimates of cell density n and chemical concentration c for all $1\leqslant p<\infty$.

By using the pure energy method similar to [12] and supplying with initial data in different $H^3\cap\dot{H}^{-s}$ $(0\leqslant s<\frac{3}{2})$ spaces, we find the (1) enjoy the same decay rates for L^p norm. But we extend the velocity with the same decay rate for all $2\leqslant p<\infty$ and the nonlinear convective term $u\cdot\nabla u$ in (1) is considered too. The decay rate of higher derivative is obtained too. This method contain two parts: $\langle 1\rangle$ closing the energy estimates at each k-th level (referring to the order of the spatial derivatives of the solution), which will be obtained in Section 4; $\langle 2\rangle$ deriving a novel negative Sobolev estimates for nonlinear system which requires $s<3/2$, which will be obtained in Section 5.

Next, let us mention some work concerned to this work. For the system (1) and related systems there is a local existence result in [9]. In [5], the authors proved global existence for (1) with the simpler Stokes equations in R^2 and small perturbation Cauchy solution in R^3. In [6], existence issues and asymptotic behaviour are investigated in R^2 or R^3. In [8], the authors obtain the global existence of weak solutions for the chemotaxis-Stokes system with nonlinear diffusion for the cell density.

The Keller–Segel system is the best-studied model for chemotaxis. In the Keller–Segel system, the chemical is produced and not consumed as in our case. For the elliptic–parabolic Keller–Segel model, in [2], the authors summarises the results, i.e. there is a critical mass M, below M they obtained the global existence and above M they got finite-time blow-up. For the parabolic–parabolic Keller–Segel model recent progress has been achieved in [3]. For more references on the general Keller–Segel system, the interested reader can refer to recent work [1,3]. Kinetic models for chemotaxis can be found in [4].

Notation. In this paper, ∇^k with an integer $k\geqslant 0$ stands for the usual any spatial derivatives of order k. $\|f\|_{L^p}$ denotes the usual norm in L^p spaces. We also use $\langle,\rangle$ denote the inner product in L^2 spaces. In other word, $\langle f,g\rangle=\int_{R^3}f\times g\,dx$. C denotes a constant independent of time t, and C_0 denotes a constant only dependent on initial data. $(\bar{n},0,0)$ is the steady state and the $\rho(t):=n(t,x)-\bar{n}$ denotes the density perturbation around the steady state.

$$\Lambda^s f(x)=\int_{\mathbb{R}^3}|\xi|^s\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi,\tag{3}$$

where $\hat{f}$ is the Fourier transform of f. We define the homogeneous Sobolev space $\dot{H}^s$ of all f for which $\|f\|_{\dot{H}^s}$ is finite, where

$$\|f\|_{\dot{H}^s}:=\big\|\Lambda^s f\big\|_{L^2}=\big\||\xi|^s\hat{f}\big\|_{L^2}.\tag{4}$$

Throughout this paper, we also assume the following conditions:

$$\begin{cases}\text{(i)}, & \delta>0,\ \mu>0,\ \nu>0,\\ \text{(ii)}, & n_0(x)\geqslant 0,\ c_0(x)\geqslant 0,\ \nabla\cdot u_0(x)=0 \text{ for all } x\in R^3,\\ \text{(iii)}, & \chi(\cdot),\ k(\cdot) \text{ and } \phi(\cdot,\cdot) \text{ are smooth with } \chi(0)=k(0)=0,\ \text{and } k'(c)\geqslant 0 \text{ for all } x\in R,\\ \text{(iv)}, & \sup_{t\geqslant 0}\big\|\phi(t,x)\big\|_{L^3}<\infty.\end{cases}\tag{A}$$

Our main results are stated in the following theorem.

Theorem 1.1. *Under the assumption* (A) *and* $(n_0(x)-\bar{n}, c_0(x), u_0(x)) \in H^N$, $N \geqslant 3$ *and then there exists a constant* ϵ *such that if*

$$\|n_0-\bar{n}\|_{H^3}+\|c_0\|_{H^3}+\|u_0\|_{H^3} \leqslant \epsilon \tag{5}$$

then the problem (1) *admits a unique global solution* (ρ, c, u) *satisfying that for all* $t \geqslant 0$,

$$\|\rho(t)\|_{H^N}^2+\|u(t)\|_{H^N}^2+\|c(t)\|_{H^N}^2+\int_0^t \|\nabla\rho(\tau)\|_{H^N}^2+\|\nabla u(\tau)\|_{H^N}^2+\|\nabla c(\tau)\|_{H^N}^2\, d\tau \tag{6}$$

$$\leqslant C\left(\|\rho_0\|_{H^N}^2+\|u_0\|_{H^N}^2+\|c_0(x)\|_{H^N}^2\right). \tag{7}$$

If further, $\rho_0(x), u_0(x), c_0(x) \in \dot{H}^{-s}$ *for some* $s \in [0, 3/2)$, *then for all* $t \geqslant 0$,

$$\|\rho(t)\|_{\dot{H}^{-s}}^2+\|u(t)\|_{\dot{H}^{-s}}^2+\|c(t)\|_{\dot{H}^{-s}}^2 \leqslant C_0, \tag{8}$$

and for $\ell = 0, 1, \ldots, N$, *the following decay results hold*:

$$\|\nabla^k\rho(t)\|_{H^{N-\ell}}+\|\nabla^\ell u(t)\|_{H^{N-\ell}}+\|\nabla^\ell c(t)\|_{H^{N-\ell}} \leqslant C_0(1+t)^{-\frac{\ell+s}{2}}. \tag{9}$$

Corollary 1.2. *Under the assumptions of Theorem* 1.1 *except that we replace the* $\dot{H}^{-s}$ *assumption by that* $\rho_0, u_0, c_0 \in L^p$ *for some* $p \in (1, 2]$, *then the following decay results hold*:

$$\|\nabla^\ell\rho(t)\|_{H^{N-\ell}}+\|\nabla^\ell u(t)\|_{H^{N-\ell}}+\|c(t)\|_{H^{N-\ell}} \leqslant C_0(1+t)^{-\sigma_{p,\ell}} \quad \text{for } \ell = 0, 1, \ldots, N. \tag{10}$$

Here the number $\sigma_{p,\ell}$ *is defined by*

$$\sigma_{p,\ell} := \frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{\ell}{2}. \tag{11}$$

The paper is structured as follows. In Section 2 we state the decay rates of linear parabolic equation. In Section 3 we prove the L^2 estimate of nonlinear equations and get the boundness of $\Lambda^{-s}f$. Finally, in Section 4 we can prove the our main result easily.

2. Preliminaries

In this section, we list the inequality we will use frequently in the later.

During this section, we will employ the following Gagliardo–Nirenberg interpolation inequality frequently. $\|u\|_{L^r(\mathbb{R}^3)}$ denotes the usual Sobolev norms.

Lemma 2.1. *For* $m = |\alpha|$, *and* $j = 0, 1, \ldots, m-1$, $1 \leqslant p$, $1 \leqslant q$,

$$\|D^j u\|_{L^r(\mathbb{R}^3)} \leqslant C\left\{\|D^m u\|_{L^p(\mathbb{R}^3)}\right\}^a \left\{\|u\|_{L^q(\mathbb{R}^3)}\right\}^{1-a},$$

here

$$\frac{j}{n}-\frac{1}{r} = a\left(\frac{m}{n}-\frac{1}{p}\right)+(1-a)\left(0-\frac{1}{q}\right),$$

and

$$\begin{cases} a \in \left[\dfrac{j}{m}, 1\right], \text{ unless } 1<p<\infty,\ m-j-\dfrac{n}{p} \in \{0\} \cup N,\ \text{then},\ a \in \left[\dfrac{j}{m}, 1\right), \\ \text{if } j=0,\ pm<n,\ q=\infty,\ \text{then, we need } u \to 0\left(|x| \to \infty\right),\ \text{or } \|u\|_{L^w(\mathbb{R}^3)}<\infty,\ w>0. \end{cases} \tag{12}$$

Lemma 2.2. *Assume that* $\|(\rho, u)\|_{H^3} \leqslant c_0 \leqslant 1$. *Let* $f(\rho)$ *be a smooth function of* ρ, *then for any integer* $k \geqslant 1$ *we have*

$$\|\nabla^l f(\rho)\cdot\nabla^{k-l}u\|_{L^2} \lesssim c_0\|\nabla^k(\rho, u)\|_{L^2}. \tag{13}$$

Proof. For $1 \leqslant k$, $1 \leqslant l \leqslant k$, using the Leibniz formula, for $\sum_{i=1}^n \gamma_i = l$, we have

$$\nabla^l f(n)\cdot\nabla^{k-l}u = \text{ a sum of products } f^{\gamma_1,\ldots,\gamma_n}(\rho)\nabla^{\gamma_1}\rho\cdots\nabla^{\gamma_n}\rho\nabla^{k-l}u, \quad \text{with } \gamma_i \geqslant 1. \tag{14}$$

By the Sobolev interpolation inequality, we have $\|\rho\|_{L^\infty} \leqslant \|\rho\|_{H^3} \leqslant c_0 \leqslant 1$. So, we have

$$\left|f^{\gamma_1,\dots,\gamma_n}(\rho)\right| \leqslant C, \quad C \text{ depends only on function } f.$$

Using the Hölder inequality and the Gagliardo–Nirenberg interpolation inequality, we have

$$\begin{aligned}
\left\|\nabla^l f(n)\nabla^{k-l}u\right\|_{L^2} &\lesssim \left\|\nabla^{\gamma_1}\rho\right\|_{\frac{2k}{\gamma_1}}\left\|\nabla^{\gamma_2}\rho\right\|_{\frac{2k}{\gamma_2}}\cdot\left\|\nabla^{\gamma_n}\rho\right\|_{\frac{2k}{\gamma_n}}\left\|\nabla^{k-l}u\right\|_{\frac{2k}{k-l}} \\
&\lesssim \|\nabla\rho\|_{L^3}^{1-\frac{\gamma_1}{k}}\left\|\nabla^k\rho\right\|_{L^2}^{\frac{\gamma_1}{k}}\cdots\|\nabla\rho\|_{L^3}^{1-\frac{\gamma_n}{k}}\left\|\nabla^k\rho\right\|_{L^2}^{\frac{\gamma_n}{k}}\|\nabla u\|_{L^3}^{1-\frac{k-l}{k}}\left\|\nabla^k u\right\|_{L^2}^{\frac{k-l}{k}} \\
&\lesssim \left\|\nabla(\rho,u)\right\|_{L^3}^{n-1}\left\|\nabla^k(\rho,u)\right\|_{L^2} \\
&\lesssim c_0\left\|\nabla^k(\rho,u)\right\|_{L^2}. \qquad \square
\end{aligned} \tag{15}$$

3. The decay rates of linear parabolic equation

Let $U(x) = (u_1(x),\dots,u_n(x))$, $U(x) \in R^n$. The linear parabolic systems are

$$\partial_t U + \sum_{i=1}^n A_i\partial_i U - \Delta U = 0 \tag{16}$$

supplied with some suitable initial data.

Applying $\nabla^k(16) \times \nabla^k U$ and integrating by part over R^3, we obtain

$$\frac{d}{2dt}\langle\nabla^k U, \nabla^k U\rangle + \langle\nabla^{k+1}U, \nabla^{k+1}U\rangle = 0. \tag{17}$$

Similarly, applying $\Lambda^{-s}(16) \times \Lambda^{-s}U$, and integrating by part over R^3, we have

$$\frac{d}{2dt}\langle\Lambda^{-s}U, \Lambda^{-s}U\rangle + \langle\Lambda^{-s}\nabla U, \Lambda^{-s}\nabla U\rangle = 0. \tag{18}$$

If the initial data $U_0(x) \in \dot{H}^N \cap H^{-s}$, from (18), we have

$$\left\|\Lambda^{-s}U\right\|_{L^2} \leqslant C\left(\left\|\Lambda^{-s}U_0\right\|_{L^2}\right). \tag{19}$$

Applying the Parseval equality and Hölder inequality

$$\left\|\nabla^{k+1}f\right\|_{L^2} \geqslant \left(\left\|\nabla^l f\right\|_{L^2}\right)^{1+\frac{1}{l+s}}\|f\|_{H^{-s}}^{-\frac{1}{l+s}} \tag{20}$$

combining (19).

$$\frac{d}{2dt}\langle\nabla^k U, \nabla^k U\rangle + C_0\left(\langle\nabla^k U, \nabla^k U\rangle\right)^{1+\frac{1}{k+s}} \leqslant 0. \tag{21}$$

So, we have

$$\left\|\nabla^k U\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+s}{2}}, \quad k = 0, 1, \dots, N. \tag{22}$$

For the small perturbation of nonlinear parabolic systems, if we can have the similar inequality such as (17) and (19), the parabolic system's decay rates are obtained.

The linear part of (1) read as:

$$\begin{cases}
\partial_t n - \delta\Delta n = 0, \\
\partial_t c - \mu\Delta c = 0, \\
\partial_t u - \nu\Delta u + n\nabla\phi = 0, \\
\nabla\cdot u = 0, \quad t > 0,\ x \in R^3.
\end{cases} \tag{23}$$

Notice there are some different for the linear part of (1), that is to say $n\nabla\phi$. So, if we want to get the same result applying ∇^k to $(23)_3$, multiplying with $\nabla^k u$, integrating by part over R^3, together with $\nabla\cdot u = 0$ and assumption (A),

$$\frac{d}{2dt}\left\|\nabla^k u\right\|_{L^2}^2 + \nu\left\|\nabla^{k+1}u\right\|_{L^2}^2 = \langle\nabla^{k+1}n\cdot\phi(x), \nabla^k u\rangle \leqslant \left\|\nabla^{k+1}n\right\|_{L^2}\left\|\nabla^k u\right\|_L^6\|\phi\|_L^3 \leqslant C\left\|\nabla^{k+1}u\right\|_{L^2}\left\|\nabla^{k+1}n\right\|_{L^2}. \tag{24}$$

Applying ∇^k to $(23)_1$, and multiplying with $\nabla^k n$, integrating by part over R^3,

$$\frac{d}{2dt}\left\|\nabla^k n\right\|_{L^2}^2 + \delta\left\|\nabla^{k+1}n\right\|_{L^2}^2 = 0. \tag{25}$$

From, (24), using the Cauchy inequality, there exist a positive constant C_1, C_2, satisfying

$$\frac{d}{2dt}\left\|\nabla^k u\right\|_{L^2}^2 + C_1\left\|\nabla^{k+1} u\right\|_{L^2}^2 \leqslant C_2\left\|\nabla^{k+1} n\right\|_{L^2}^2. \tag{26}$$

From, (25) and (26), there exists a positive constant C_3 satisfying

$$\frac{d}{2dt}\left\|\nabla^k (n,u)\right\|_{L^2}^2 + C_3\left\|\nabla^{k+1} (n,u)\right\|_{L^2}^2 \leqslant 0. \tag{27}$$

From all mentioned in this section, we have for the linear part of (1),

$$\frac{d}{2dt}\left\|\nabla^k (n,u,c)\right\|_{L^2}^2 + C_4\left\|\nabla^{k+1} (n,u,c)\right\|_{L^2}^2 \leqslant 0, \tag{28}$$

where C_4 is a positive constant. This means that the linear part of (1) has the similar estimates to the linear parabolic equation.

If we can get the uniform upper bounds of $(\Lambda^{-s}n, \Lambda^{-s}c, \Lambda^{-s}u)$, we can get the same decay rates as (22).

4. The L^2 estimate of nonlinear equations

Let $\rho = n - \bar{n}$, then the (1) can be rewritten to be

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho = \delta\Delta\rho - \nabla\cdot\big(\chi(c)(\rho+\bar{n})\nabla c\big), \\ \partial_t c + u\cdot\nabla c = \mu\Delta c - k(c)(\rho+\bar{n}), \\ \partial_t u + u\cdot\nabla u + \nabla P = \nu\Delta u - (\rho+\bar{n})\nabla\phi, \\ \nabla\cdot u = 0, \quad t>0,\ x\in R^3. \end{cases} \tag{29}$$

Lemma 4.1. *Under the assumption (A) and $\|n_0 - \hat{n}\|_{H^3} + \|c\|_{H^3} + \|u\|_{H^3} \leqslant \epsilon$, where ϵ is small enough, for $k = 0, 1, \ldots, N$*

$$\frac{d}{2dt}\big(\langle\nabla^k\rho,\nabla^k\rho\rangle + \langle\nabla^k c,\nabla^k c\rangle + \langle\nabla^k u,\nabla^k u\rangle\big) + c\big(\langle\nabla^{k+1}\rho,\nabla^{k+1}\rho\rangle + \langle\nabla^{k+1} c,\nabla^{k+1} c\rangle + \langle\nabla^{k+1} u,\nabla^{k+1} u\rangle\big) \leqslant 0. \tag{30}$$

Proof. Applying ∇^k to (29) and multiplying $\nabla^k\rho$, $\nabla^k c$, $\beta\nabla^k u$ respectively, we obtain

$$\begin{aligned} &\frac{d}{2dt}\big(\langle\nabla^k\rho,\nabla^k\rho\rangle + \langle\nabla^k c,\nabla^k c\rangle + \langle\nabla^k u,\nabla^k u\rangle\big) + \delta\langle\nabla^{k+1}\rho,\nabla^{k+1}\rho\rangle + \mu\langle\nabla^{k+1} c,\nabla^{k+1} c\rangle + \nu\langle\nabla^{k+1} u,\nabla^{k+1} u\rangle \\ &\quad = -\langle\nabla^k(u\nabla n),\nabla^k\rho\rangle - \langle\nabla^k\big(\nabla\big(\chi(c)\nabla c\big)\big),\nabla^k\rho\rangle - \langle\nabla^k(u\nabla c),\nabla^k c\rangle - \langle\nabla^k\big(k(c)n\big),\nabla^k c\rangle \\ &\qquad - \langle\nabla^k(u\nabla n),\nabla^k u\rangle - \langle\nabla^k(n\nabla\phi),\nabla^k u\rangle \\ &\quad = I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

As for I_1

$$|I_1| = \left|\langle\nabla^k(u\cdot\nabla u),\nabla^k\rho\rangle\right| \leqslant \left\|\nabla^k(u\cdot\nabla u)\right\|_{L^{\frac{6}{5}}}\left\|\nabla^k\rho\right\|_{L^6} \leqslant \left\|\nabla^k(u\cdot\nabla u)\right\|_{L^{\frac{6}{5}}}\left\|\nabla^{k+1}\rho\right\|_{L^2}. \tag{31}$$

The next is to estimate the $\|\nabla^k(u\nabla u)\|_{L^{\frac{6}{5}}}$ item. To control it by the left item of (29), the following inequality is needed.

$$\begin{aligned} \left\|\nabla^k(u\nabla u)\right\|_{L^{\frac{6}{5}}} &= C_l\sum_{l=0}^{l=k+1}\left\|\nabla^l u\nabla^{k+1-l}u\right\|_{L^{\frac{6}{5}}} = C_l\sum_{l=0}^{l=[\frac{k+1}{2}]}\left\|\nabla^l u\nabla^{k+1-l}u\right\|_{L^{\frac{6}{5}}} + C_l\sum_{l=[\frac{k+1}{2}]+1}^{l=k+1}\left\|\nabla^l u\nabla^{k+1-l}u\right\|_{L^{\frac{6}{5}}} \\ &= W_1 + W_2. \end{aligned}$$

For $l \leqslant [\frac{k+1}{2}]$,

$$\begin{aligned} \|W_1\|_{L^{\frac{6}{5}}} &\leqslant C\sum_{l=0}^{l=[\frac{k+1}{2}]}\left\|\nabla^l u\nabla^{k+1-l}u\right\|_{L^{\frac{6}{5}}} \leqslant \sum_{l=0}^{l=[\frac{k+1}{2}]}\left\|\nabla^l u\right\|_{L^3}\left\|\nabla^{k+1-l}u\right\|_{L^2} \\ &\leqslant C\sum_{l=0}^{l=[\frac{k+1}{2}]}\left\|\nabla^\alpha u\right\|_{L^2}^{1-\theta}\left\|\nabla^{k+1}u\right\|_{L^2}^{1-\theta}\|u\|_{L^2}^{1-\theta}\left\|\nabla^{k+1}u\right\|_{L^2}^{1-\theta} \\ &\leqslant \varepsilon\left\|\nabla^{k+1}u\right\|_{L^2}\left\|\nabla^{k+1}u\right\|_{L^2}, \end{aligned} \tag{32}$$

here, α, θ satisfy:

$$\begin{cases} \dfrac{l}{3} - \dfrac{1}{3} = \left(\dfrac{\alpha}{3} - \dfrac{1}{2}\right)\theta + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)(1-\theta), \\ \dfrac{k+1-l}{3} - \dfrac{1}{2} = \left(\dfrac{0}{3} - \dfrac{1}{2}\right)(1-\theta) + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)\theta. \end{cases} \tag{33}$$

From (33),

$$\theta = \frac{k+1-l}{k+1}, \qquad \alpha = \frac{k+1}{2(k+1-l)},$$

so, $0 < \theta < 1$, $\alpha \in [\frac{1}{2}, 1)$.

When $l \geqslant [\frac{k+1}{2}] + 1$,

$$\begin{aligned} \|W_2\|_{L^{\frac{6}{5}}} &\leqslant C \sum_{l=[\frac{k+1}{2}]+1}^{l=k} \left\|\nabla^l u \nabla^{k+1-l} u\right\|_{L^{\frac{6}{5}}} \leqslant \sum_{l=[\frac{k+1}{2}]+1}^{l=k} \left\|\nabla^l u\right\|_{L^2} \left\|\nabla^{k+1-l} u\right\|_{L^3} \\ &\leqslant C \sum_{j=1}^{l=[\frac{k+1}{2}]} \left\|\nabla^j u\right\|_{L^3} \left\|\nabla^{k+1-j} u\right\|_{L^2} \\ &\leqslant C \sum_{j=1}^{l=[\frac{k+1}{2}]} \left\|\nabla_1^{\alpha} u\right\|_{L^2}^{1-\theta_1} \left\|\nabla^{k+1} u\right\|_{L^2}^{1-\theta_1} \|u\|_{L^2}^{1-\theta_1} \left\|\nabla^{k+1} u\right\|_{L^2}^{1-\theta_1} \\ &\leqslant \varepsilon \left\|\nabla^{k+1} u\right\|_{L^2} \left\|\nabla^{k+1} u\right\|_{L^2}, \end{aligned} \tag{34}$$

here, α_1, θ_1 satisfy:

$$\begin{cases} \dfrac{j}{3} - \dfrac{1}{3} = \left(\dfrac{\alpha_1}{3} - \dfrac{1}{2}\right)\theta_1 + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)(1-\theta_1), \\ \dfrac{k+1-j}{3} - \dfrac{1}{2} = \left(\dfrac{0}{3} - \dfrac{1}{2}\right)(1-\theta_1) + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)\theta_1. \end{cases} \tag{35}$$

From (35),

$$\theta_1 = \frac{k+1-j}{k+1}, \qquad \alpha_1 = \frac{k+1}{2(k+1-j)},$$

so, $0 < \theta < 1$, $\alpha \in [\frac{1}{2}, 1)$

$$\|W_2\|_{L^{\frac{6}{5}}} \leqslant \varepsilon \left\|\nabla^{k+1} u\right\|_{L^2} \left\|\nabla^{k+1} c\right\|_{L^2}. \tag{36}$$

From (31), (32) and (36), we have

$$|I_1| \leqslant \varepsilon\left(\left\langle\nabla^{k+1} u, \nabla^{k+1} u\right\rangle + \left\langle\nabla^{k+1} c, \nabla^{k+1} c\right\rangle + \left\langle\nabla^{k+1} \rho, \nabla^{k+1} \rho\right\rangle\right). \tag{37}$$

Similarly, the estimates for I_3, I_5 are obtained.

As for I_3

$$|I_3| = \left|\left\langle\nabla^k (u \cdot \nabla u), \nabla^k \rho\right\rangle\right| \leqslant \left\|\nabla^k (u \cdot \nabla u)\right\|_{L^{\frac{6}{5}}} \left\|\nabla^k \rho\right\|_{L^6} \leqslant \left\|\nabla^k (u \cdot \nabla u)\right\|_{L^{\frac{6}{5}}} \left\|\nabla^{k+1} \rho\right\|_{L^2}. \tag{38}$$

Similarly,

$$\left\|\nabla^k (u \nabla c)\right\|_{L^{\frac{6}{5}}} = C_l \sum_{l=0}^{l=k+1} \left\|\nabla^l u \nabla^{k+1-l} c\right\|_{L^{\frac{6}{5}}} = C_l \sum_{l=0}^{l=[\frac{k+1}{2}]} \left\|\nabla^l u \nabla^{k+1-l} c\right\|_{L^{\frac{6}{5}}} + C_l \sum_{l=[\frac{k+1}{2}]+1}^{l=k+1} \left\|\nabla^l u \nabla^{k+1-l} c\right\|_{L^{\frac{6}{5}}} = D_1 + D_2$$

for $l \leqslant [\frac{k+1}{2}]$,

$$
\begin{aligned}
\|D_1\|_{L^{\frac{6}{5}}} &\leqslant C \sum_{l=0}^{l=[\frac{k+1}{2}]} \big\|\nabla^l u \nabla^{k+1-l} c\big\|_{L^{\frac{6}{5}}} \leqslant \sum_{l=0}^{l=[\frac{k+1}{2}]} \big\|\nabla^l u\big\|_{L^3} \big\|\nabla^{k+1-l} c\big\|_{L^2} \\
&\leqslant C \sum_{l=0}^{l=[\frac{k+1}{2}]} \big\|\nabla^{\alpha} u\big\|_{L^2}^{1-\theta} \big\|\nabla^{k+1} u\big\|_{L^2}^{1-\theta} \|c\|_{L^2}^{1-\theta} \big\|\nabla^{k+1} c\big\|_{L^2}^{1-\theta} \\
&\leqslant \varepsilon \big\|\nabla^{k+1} u\big\|_{L^2} \big\|\nabla^{k+1} c\big\|_{L^2}. \qquad (39)
\end{aligned}
$$

Here, α, θ satisfy:

$$
\begin{cases}
\dfrac{l}{3} - \dfrac{1}{3} = \left(\dfrac{\alpha}{3} - \dfrac{1}{2}\right)\theta + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)(1-\theta), \\
\dfrac{k+1-l}{3} - \dfrac{1}{2} = \left(\dfrac{0}{3} - \dfrac{1}{2}\right)(1-\theta) + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)\theta
\end{cases} \qquad (40)
$$

from (40),

$$
\theta = \frac{k+1-l}{k+1}, \qquad \alpha = \frac{k+1}{2(k+1-l)}.
$$

So, $0 < \theta < 1$, $\alpha \in [\frac{1}{2}, 1)$.

When $l \geqslant [\frac{k+1}{2}] + 1$,

$$
\begin{aligned}
\|D_2\|_{L^{\frac{6}{5}}} &\leqslant C \sum_{l=[\frac{k+1}{2}]+1}^{l=k} \big\|\nabla^l u \nabla^{k+1-l} c\big\|_{L^{\frac{6}{5}}} \leqslant \sum_{l=[\frac{k+1}{2}]+1}^{l=k} \big\|\nabla^l u\big\|_{L^2} \big\|\nabla^{k+1-l} c\big\|_{L^3} \\
&\leqslant C \sum_{j=1}^{l=[\frac{k+1}{2}]} \big\|\nabla^j c\big\|_{L^3} \big\|\nabla^{k+1-j} u\big\|_{L^2} \\
&\leqslant C \sum_{j=1}^{l=[\frac{k+1}{2}]} \big\|\nabla_1^{\alpha} c\big\|_{L^2}^{1-\theta_1} \big\|\nabla^{k+1} u\big\|_{L^2}^{1-\theta_1} \|u\|_{L^2}^{1-\theta_1} \big\|\nabla^{k+1} u\big\|_{L^2}^{1-\theta_1} \\
&\leqslant \varepsilon \big\|\nabla^{k+1} c\big\|_{L^2} \big\|\nabla^{k+1} u\big\|_{L^2}. \qquad (41)
\end{aligned}
$$

Here, α_1, θ_1 satisfy:

$$
\begin{cases}
\dfrac{j}{3} - \dfrac{1}{3} = \left(\dfrac{\alpha_1}{3} - \dfrac{1}{2}\right)\theta_1 + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)(1-\theta_1), \\
\dfrac{k+1-j}{3} - \dfrac{1}{2} = \left(\dfrac{0}{3} - \dfrac{1}{2}\right)(1-\theta_1) + \left(\dfrac{k+1}{3} - \dfrac{1}{2}\right)\theta_1
\end{cases} \qquad (42)
$$

from (42),

$$
\theta_1 = \frac{k+1-j}{k+1}, \qquad \alpha_1 = \frac{k+1}{2(k+1-j)}.
$$

So, $0 < \theta_1 < 1$, $\alpha_1 \in [\frac{1}{2}, 1)$

$$
\|D_2\|_{L^{\frac{6}{5}}} \leqslant \varepsilon \big\|\nabla^{k+1} u\big\|_{L^2} \big\|\nabla^{k+1} c\big\|_{L^2} \qquad (43)
$$

from (38), (39) and (43), we have

$$
\begin{aligned}
|I_3| &\leqslant \varepsilon \big(\big\langle \nabla^{k+1} u, \nabla^{k+1} u \big\rangle + \big\langle \nabla^{k+1} c, \nabla^{k+1} c \big\rangle \big), \qquad (44) \\
|I_5| &\leqslant \varepsilon \big(\big\langle \nabla^{k+1} u, \nabla^{k+1} u \big\rangle \big). \qquad (45)
\end{aligned}
$$

As for I_2

$$\begin{aligned}
|I_2| &\leqslant \langle \nabla^k(\chi(c)(\rho+\hat{n})\nabla c), \nabla^{k+1}\rho\rangle \\
&\leqslant \langle \nabla^k((\rho+\hat{n})\nabla h(c)), \nabla^{k+1}\rho\rangle \\
&\leqslant C\langle(\rho+\hat{n})\nabla^{k+1}h(c), \nabla^{k+1}\rho\rangle + \sum_{l=1}^{l=k} C_l\nabla^l\rho\nabla^{k+1-l}h(c), \nabla^{k+1}\rho\rangle \\
&\leqslant W_{21} + W_{22},
\end{aligned} \tag{46}$$

$$\begin{aligned}
|W_{21}| &\leqslant C\varepsilon(\langle\nabla^{k+1}c, \nabla^{k+1}c\rangle + \langle\nabla^{k+1}\rho, \nabla^{k+1}\rho\rangle) + \sum_{l=1}^{l=k} C_l\langle\nabla^l\chi(c)\nabla^{k+1-l}c, \nabla^{k+1}\rho\rangle \\
&\leqslant C\varepsilon(\langle\nabla^{k+1}c, \nabla^{k+1}c\rangle + \langle\nabla^{k+1}\rho, \nabla^{k+1}\rho\rangle) + W_{211}.
\end{aligned} \tag{47}$$

As for W_{22}, with the help of Lemma 2.2,

$$\begin{aligned}
|W_{22}| &\leqslant C\sum_{l=1}^{l=k}|\langle\nabla^l\rho\nabla^{k+1-l}h(c), \nabla^{k+1}\rho\rangle| \\
&\leqslant C\sum_{l+k_1+k_2+\cdots+k_m=k+1,\, k_i\geqslant 1}\|\nabla^l\rho\nabla_1^k c\nabla_2^k c\cdots\nabla_m^k c\|_{L^2}\|\nabla^{k+1}\rho\|_{L^2} \\
&\leqslant C\|\nabla^l\rho\|_{L^{\frac{2(k+1)}{l}}}\cdots\|\nabla^{k_m}c\|_{L^{\frac{2(k+1)}{k_m}}}\|\nabla^{k+1}\rho\|_{L^2} \\
&\leqslant C\varepsilon(\langle\nabla^{k+1}\rho, \nabla^{k+1}\rho\rangle + \langle\nabla^{k+1}c, \nabla^{k+1}c\rangle),
\end{aligned} \tag{48}$$

$$|W_{211}| \leqslant C\varepsilon\langle\nabla^{k+1}c, \nabla^{k+1}c\rangle. \tag{49}$$

So, from (46), (47), (48) and (49)

$$|I_2| \leqslant C\varepsilon(\langle\nabla^{k+1}c, \nabla^{k+1}c\rangle + \langle\nabla^{k+1}\rho, \nabla^{k+1}\rho\rangle + \langle\nabla^{k+1}u, \nabla^{k+1}u\rangle). \tag{50}$$

The estimates of I_4 are more complicate.

$$\begin{aligned}
I_4 &= \langle\nabla^k(k(c)n), \nabla^k c\rangle \\
&= -\langle\nabla^k((k(c)-k'(0)c)n), \nabla^k c\rangle - \langle\nabla^k(k'(0)c(\rho+\hat{n})), \nabla^k c\rangle \\
&= -\langle\nabla^k((k(c)-k'(0)c)n), \nabla^k c\rangle - k'(0)\langle(\rho+\hat{n})\nabla^k c, \nabla^k c\rangle - \sum_{l=1}^{k}\langle k'(0)\nabla^l\rho\nabla^{k-l}c, \nabla^k c\rangle \\
&= T_1 - k'(0)\langle(\rho+\hat{n})\nabla^k c, \nabla^k c\rangle - T_2,
\end{aligned} \tag{51}$$

$$|T_1| \leqslant C\langle\nabla^k f(c)n, \nabla^k c\rangle \leqslant C\sum_{k_1+k_2=k}\|\nabla_1^k h(c)\nabla_2^k n\|_{L^2}\|\nabla^k c\|_{L^2}. \tag{52}$$

For $k_1 \leqslant [\frac{k}{2}]$,

$$\begin{aligned}
\|\nabla_1^k h(c)\nabla_2^k n\|_{L^2} &\leqslant C\|\nabla_1^k c\|_{L_\infty}\|\nabla_2^k n\|_{L^2} \\
&\leqslant C\|\nabla^\alpha c\|_{L^2}^{\frac{k_2}{k+1}}\|\nabla^{k+1}u\|_{L^2}^{\frac{k_1+1}{k+1}}\|n\|_{L^2}^{\frac{k_1+1}{k+1}}\|\nabla^{k+1}n\|_{L^2}^{\frac{k_2}{k+1}} \\
&\leqslant C\varepsilon\|\nabla^{k+1}u\|_{L^2}\|\nabla^{k+1}n\|_{L^2}.
\end{aligned} \tag{53}$$

Where $\alpha = \frac{k+1}{2k_2} \leqslant 3$.

For $k_2 \geqslant [\frac{k}{2}] + 1$, by using the same method of treat (53), we have

$$\begin{aligned}
\|\nabla_1^k h(c)\nabla_2^k n\|_{L^2} &\leqslant C\|\nabla_2^k n\|_{L_\infty}\|\nabla_1^k h(c)\|_{L^2} \\
&\leqslant C\|\nabla_2^k n\|_{L_\infty}\|\nabla^{k_1-1}(h'(c)\nabla c)\nabla_2^k n\|_{L^2} \\
&\leqslant C\|\nabla_2^k n\|_{L_\infty}\left(\|h'(c)\nabla_1^k c\nabla_2^k n\|_{L^2} + \sum_{m=1}^{k_1-1}\|\nabla^m h'(c)\nabla^{k_1-m}c\nabla_2^k n\|_{L^2}\right) \\
&\leqslant C\varepsilon\|\nabla^{k+1}u\|_{L^2}\|\nabla^{k+1}n\|_{L^2}.
\end{aligned} \tag{54}$$

From (52), (53) and (54),

$$|T_1| \leqslant C\varepsilon\big(\langle\nabla^{k+1}\rho, \nabla^{k+1}\rho\rangle + \langle\nabla^{k+1}c, \nabla^{k+1}c\rangle\big). \tag{55}$$

Similarly, the estimate of T_2

$$|T_2| \leqslant C\varepsilon\big(\langle\nabla^{k+1}\rho, \nabla^{k+1}\rho\rangle + \langle\nabla^{k+1}c, \nabla^{k+1}c\rangle\big). \tag{56}$$

From (51), (55) and (56), we have

$$|I_4| \leqslant C\varepsilon\big(\langle\nabla^{k+1}c, \nabla^{k+1}c\rangle + \langle\nabla^{k+1}n, \nabla^{k+1}n\rangle\big) - C\langle\nabla^k c, \nabla^k c\rangle \leqslant C\varepsilon\big(\langle\nabla^{k+1}c, \nabla^{k+1}c\rangle + \langle\nabla^{k+1}n, \nabla^{k+1}n\rangle\big). \tag{57}$$

Choose sufficely small β, and from (37), (44), (45), (50) and (57), we conclude the lemma. □

5. The boundness of Λ^{-s}

Denote $\mathrm{F}(t) = \langle\Lambda^{-s}\rho, \Lambda^{-s}\rho\rangle + \langle\Lambda^{-s}c, \Lambda^{-s}c\rangle + \langle\Lambda^{-s}u, \Lambda^{-s}u\rangle$. The goal is to verify $\mathrm{F}(t) \leqslant C_0$ for all $t \geqslant 0$. The following inequality is necessary when we prove Lemma 4.1. This inequality can be founded in [10] on p. 119.

$$\|\Lambda^{-s}f\|_{L^q} \leqslant C\|f\|_{L^p}; \quad \text{where, } s \in (0,3),\ 1 < p < q < \infty,\ \frac{1}{q} + \frac{s}{3} = \frac{1}{p}. \tag{58}$$

Lemma 5.1. *Under the assumption (A) and $\|\rho(t)\|^2_{\dot H^{-s}} + \|u(t)\|^2_{\dot H^{-s}} + \|c(t)\|^2_{\dot H^{-s}} \leqslant C_0$, then for all $t > 0$,*

$$\mathrm{F}(t) = \langle\Lambda^{-s}\rho, \Lambda^{-s}\rho\rangle + \langle\Lambda^{-s}c, \Lambda^{-s}c\rangle + \langle\Lambda^{-s}u, \Lambda^{-s}u\rangle \leqslant C. \tag{59}$$

Proof. Applying Λ^{-s} to (29) and multiplying them by $\Lambda^{-s}\rho$, $\Lambda^{-s}c$, $\beta\Lambda^{-s}u$ respectively, and integrating over R^3

$$\begin{aligned}
&\frac{d}{2dt}\big(\langle\Lambda^{-s}\rho, \Lambda^{-s}\rho\rangle + \langle\Lambda^{-s}c, \Lambda^{-s}c\rangle + \beta\langle\Lambda^{-s}u, \Lambda^{-s}u\rangle\big) + \delta\langle\Lambda^{-s}\nabla\rho, \Lambda^{-s}\nabla\rho\rangle + \mu\langle\Lambda^{-s}\nabla c, \Lambda^{-s}\nabla c\rangle \\
&\quad + \beta\nu\langle\Lambda^{-s}\nabla u, \Lambda^{-s}\nabla u\rangle \\
&\quad = -\langle\Lambda^{-s}(u\nabla\rho), \Lambda^{-s}\rho\rangle - \langle\Lambda^{-s}(u\nabla c), \nabla^{-s}c\rangle - \beta\langle\Lambda^{-s}(u\nabla u), \Lambda^{-s}u\rangle \\
&\qquad - \langle\nabla\cdot\big(\chi(c)(\rho+\hat n)\nabla c\big), \nabla^{-s}\rho\rangle - \langle\nabla^{-s}\big(k(c)(\rho+\hat n)\big), \nabla^{-s}c\rangle - \beta\langle\big((\rho+\hat n)\nabla\phi\big), \nabla^{-s}u\rangle \\
&\quad = M_1 + M_2 + M_3 + M_4 + M_5 + M_6.
\end{aligned} \tag{60}$$

For M_1,

$$|M_1| \leqslant \big|\langle\nabla^{-s}(u\cdot\nabla\rho), \nabla^{-s}\rho\rangle\big| \leqslant C\|u\nabla\rho\|_{L^{\frac{1}{\frac12+\frac s3}}}\|\nabla^{-s}\rho\|_{L^2} \leqslant C\|u\|_{L^{\frac3s}}\|\nabla\rho\|_{L^2}\|\nabla^{-s}\rho\|_{L^2}. \tag{61}$$

Similarly,

$$|M_2| \leqslant C\|u\|_{L^{\frac3s}}\|\nabla c\|_{L^2}\|\nabla^{-s}\rho\|_{L^2}, \tag{62}$$

$$|M_3| \leqslant C\|u\|_{L^{\frac3s}}\|\nabla u\|_{L^2}\|\nabla^{-s}u\|_{L^2}. \tag{63}$$

For M_4,

$$\begin{aligned}
|M_4| &\leqslant \big|\langle\nabla^{-s}\big(\nabla\chi(c)(\rho+\hat n)\nabla c\big), \nabla^{-s}c\rangle\big| \\
&\leqslant C\big(\big|\langle\nabla^{-s}\big(\nabla\rho\cdot\nabla h(c)\big), \nabla^{-s}c\rangle\big| + \big|\langle\nabla^{-s}\big[\chi(c)(\rho+\hat n)\big], \nabla^{-s}c\rangle\big|\big) \\
&\leqslant C\big(\|\nabla\rho\|_{L^{\frac3s}}\|\nabla c\|_{L^2} + \|\nabla c\|_{\frac3s}\big)\|\nabla^{-s}c\|_{L^2}. \qquad \square
\end{aligned} \tag{64}$$

For M_6,

$$\begin{aligned}
|M_6| &\leqslant \beta C\langle\Lambda^{-s}\big((\rho+\hat n)\nabla\phi\big), \Lambda^{-s}u\rangle \\
&\leqslant \beta C\langle\Lambda^{-s}(\nabla\rho\phi), \Lambda^{-s}u\rangle \\
&\leqslant \beta C\|\nabla\rho\|_{L^{\frac3s}}\|\phi\|_{L^{\frac65}}\|\Lambda^{-s}u\|_{L^6} \\
&\leqslant \varepsilon\langle\Lambda^{-s}\nabla u, \Lambda^{-s}\nabla u\rangle + C_\varepsilon\|\nabla\rho\|^2_{H^2}.
\end{aligned} \tag{65}$$

For M_5,

$$\begin{aligned}
|M_5| &= -\langle \Lambda^{-s}\big(k(c)(\rho+\hat{n})\big), \Lambda^{-s}c\rangle \\
&= -\hat{n}\langle \Lambda^{-s}k(c), \Lambda^{-s}c\rangle - \langle \Lambda^{-s}\big(k(c)\rho\big), \Lambda^{-s}c\rangle \\
&\leqslant -C_1\langle \Lambda^{-s}c, \Lambda^{-s}c\rangle + \|\nabla^{-s}c\|_{L^2}\|\Lambda^{-s}(c\rho)\|_{L^2} \\
&\leqslant -C_1\langle \Lambda^{-s}c, \Lambda^{-s}c\rangle + \varepsilon\langle \Lambda^{-s}c, \Lambda^{-s}c\rangle + \|\nabla\rho\|_{H^2}^2 \\
&\leqslant C\|\nabla\rho\|_{H^2}^2.
\end{aligned} \tag{66}$$

So from, (61), (62), (63), (64), (65), (66)

$$\begin{aligned}
&\frac{d}{2dt}\big(\langle \Lambda^{-s}\rho, \Lambda^{-s}\rho\rangle + \langle \Lambda^{-s}c, \Lambda^{-s}c\rangle + \beta\langle \Lambda^{-s}u, \Lambda^{-s}u\rangle\big) + \delta\langle \Lambda^{-s}\nabla\rho, \Lambda^{-s}\nabla\rho\rangle + \mu\langle \Lambda^{-s}\nabla c, \Lambda^{-s}\nabla c\rangle \\
&\quad + \beta\nu\langle \Lambda^{-s}\nabla u, \Lambda^{-s}\nabla u\rangle \\
&\leqslant C\big\{\|\nabla\rho, \nabla c, \nabla u\|_{H^2}^2 + \|u\|_{L^{\frac{3}{s}}}\|\nabla\rho\|_{L^2}\|\Lambda^{-s}\rho\|_{L^2} + \|u\|_{L^{\frac{3}{s}}}\|\nabla c\|_{L^2}\|\Lambda^{-s}c\|_{L^2}\big\} \\
&\quad + \big\{\|u\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\|\Lambda^{-s}u\|_{L^2} + \|c\|_{L^{\frac{3}{s}}}\|\nabla^2 c\|_{L^2}\|\Lambda^{-s}c\|_{L^2}\big\}.
\end{aligned} \tag{67}$$

The next, we turn to estimate $\|(\rho, \nabla\rho, c, u)\|_{L^{\frac{3}{s}}}$.

Case 1. $s \in (0, \frac{1}{2}]$.

Noticing $\frac{3}{s} \geqslant 6$, to estimate $\|(\rho, \nabla\rho, c, u)\|_{L^{\frac{3}{s}}}$, the higher-order of (ρ, c, u) is needed.

$$\|u\|_{\frac{3}{s}} \leqslant C\|\nabla u\|_{L^2}^{\frac{1}{2}-s}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}+\frac{s}{2}} \leqslant C\big(\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\big). \tag{68}$$

Similarly,

$$\|c\|_{\frac{3}{s}} \leqslant C\|\nabla c\|_{L^2}^{\frac{1}{2}-s}\|\nabla^2 c\|_{L^2}^{\frac{1}{2}+\frac{s}{2}} \leqslant C\big(\|\nabla c\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2\big), \tag{69}$$

and

$$\|\nabla\rho\|_{\frac{3}{s}} \leqslant C\|\nabla\nabla\rho\|_{L^2}^{\frac{1}{2}-s}\|\nabla^2\nabla\rho\|_{L^2}^{\frac{1}{2}+\frac{s}{2}} \leqslant C\big(\|\nabla^2\rho\|_{L^2}^2 + \|\nabla^3\rho\|_{L^2}^2\big). \tag{70}$$

So, the estimate of $\|(u, \nabla\rho, c, \rho)\|_{L^2}$

$$\|(u, \nabla\rho, c, \rho)\|_{L^{\frac{3}{s}}}C \leqslant \|(\nabla\rho, \nabla c, \nabla u)\|_{H^2}^2. \tag{71}$$

Combining (67) and (71),

$$\begin{aligned}
&\frac{d}{2dt}\big(\langle \Lambda^{-s}\rho, \Lambda^{-s}\rho\rangle + \langle \Lambda^{-s}c, \Lambda^{-s}c\rangle + \beta\langle \Lambda^{-s}u, \Lambda^{-s}u\rangle\big) + \delta\langle \Lambda^{-s}\nabla\rho, \Lambda^{-s}\nabla\rho\rangle + \mu\langle \Lambda^{-s}\nabla c, \Lambda^{-s}\nabla c\rangle \\
&\quad + \beta\nu\langle \Lambda^{-s}\nabla u, \Lambda^{-s}\nabla u\rangle \\
&\leqslant \varepsilon C\big(\|(\nabla u, \nabla c, \nabla\rho)\|_{H^2}^2\big)\|(\Lambda^{-s}\rho, \Lambda^{-s}u, \Lambda^{-s}c)\|_{L^2} + C\|(\nabla u, \nabla c, \nabla\rho)\|_{H^2}^2.
\end{aligned} \tag{72}$$

From (72),

$$\begin{aligned}
&\frac{d}{2dt}\mathrm{F}(t) + \delta\langle \Lambda^{-s}\nabla\rho, \Lambda^{-s}\nabla\rho\rangle + \mu\langle \Lambda^{-s}\nabla c, \Lambda^{-s}\nabla c\rangle + \beta\nu\langle \Lambda^{-s}\nabla u, \Lambda^{-s}\nabla u\rangle \\
&\quad \leqslant C\big(\|(\nabla u, \nabla c, \nabla\rho)\|_{H^2}\big)^2 \sup_{0\leqslant s\leqslant t}\big(\mathrm{F}(s)\big)^{\frac{1}{2}} + C\|(\nabla u, \nabla c, \nabla\rho)\|_{H^2}^2.
\end{aligned} \tag{73}$$

Integrating (73) from 0 up to t,

$$\frac{1}{2}\mathrm{F}(t) + \int_0^t \mu\langle \Lambda^{-s}\nabla c, \Lambda^{-s}\nabla c\rangle + \beta\nu\langle \Lambda^{-s}\nabla u, \Lambda^{-s}\nabla u\rangle \leqslant C\Big(\Big(\sup_{0\leqslant s\leqslant t}\mathrm{F}(s)\Big)^{\frac{1}{2}} + 1\Big). \tag{74}$$

From (74), we have

$$\mathrm{F}(t) \leqslant C_0, \quad \forall t \geqslant 0. \tag{75}$$

From (19), (21), (22), (30) and (74), we have

$$\|\nabla^k U\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+s}{2}}, \quad k=0,1,\dots,N,\ s \in \left(0, \frac{1}{2}\right]. \tag{76}$$

Case 2. $s \in [\frac{1}{2}, \frac{3}{2})$.

This case is more complicate, as for $s \in [\frac{1}{2}, \frac{3}{2})$, $2 \leqslant \frac{3}{s} \leqslant 6$. So, we can estimate $\|u\|_{L^{\frac{3}{s}}}$ by interpolating it between $\|u\|_{L^2}$ and $\|u\|_L^6$

$$\|u\|_{L^{\frac{3}{s}}} \leqslant \left(\|u\|_L^2\right)^{s-\frac{1}{2}}\left(\|u\|_{L^6}\right)^{\frac{3}{2}-s}, \tag{77}$$

$$\|c\|_{L^{\frac{3}{s}}} \leqslant \left(\|c\|_L^2\right)^{s-\frac{1}{2}}\left(\|c\|_{L^6}\right)^{\frac{3}{2}-s}, \tag{78}$$

$$\|\rho\|_{L^{\frac{3}{s}}} \leqslant \left(\|\rho\|_L^2\right)^{s-\frac{1}{2}}\left(\|\rho\|_{L^6}\right)^{\frac{3}{2}-s}. \tag{79}$$

So from (67), (77), (78), (79) and integrating from $(0,t)$,

$$\begin{aligned}
&\frac{1}{2}\mathrm{F}(t) + \int_0^t \left[\delta\langle \Lambda^{-s}\nabla\rho, \Lambda^{-s}\nabla\rho\rangle + \mu\langle \Lambda^{-s}\nabla c, \Lambda^{-s}\nabla c\rangle + \beta\nu\langle \Lambda^{-s}\nabla u, \Lambda^{-s}\nabla u\rangle\right] d\tau \\
&\quad \leqslant C\left(\int_0^t \|(\rho,c,u)\|_{L^{\frac{3}{s}}} \|(\nabla\rho,\nabla c,\nabla u)\|_{H^1}\left(\mathrm{F}(t)\right)^{\frac{1}{2}} d\tau + 1\right) \\
&\quad \leqslant C\left[\left(\sup_{0\leqslant\tau\leqslant t}\mathrm{F}(\tau)\right)^{\frac{1}{2}} \int_0^t \left\{\|(\rho,c,u)\|_{L^2}\right\}^{s-\frac{1}{2}}\left\{\|(\nabla\rho,\nabla c,\nabla u)\|_{H^1}\right\}^{\frac{5}{2}-s} d\tau + 1\right] \\
&\quad = C\left[\left(\sup_{0\leqslant\tau\leqslant t}\mathrm{F}(\tau)\right)^{\frac{1}{2}}\mathrm{Q}(t) + 1\right] \\
&\quad \leqslant C\left(\left(\sup_{0\leqslant\tau\leqslant t}\mathrm{F}(\tau)\right)^{\frac{1}{2}} + 1\right).
\end{aligned} \tag{80}$$

Where we have to prove $Q(t) \leqslant C$. Notice that we have $n_0, u_0, c_0 \in \dot{H}^{-1/2}$ since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0,s]$, we then deduce from what we have proved for Theorem 1.1 with $s = 1/2$ that the following decay result holds: By (76) for $s = \frac{1}{2}$. We have

$$\|\nabla^k \rho\|_{L^2} \leqslant C_0(1+t)^{-\frac{1}{4}}, \tag{81}$$

$$\|\nabla^k c\|_{L^2} \leqslant C_0(1+t)^{-\frac{1}{4}}, \tag{82}$$

$$\|\nabla^k u\|_{L^2} \leqslant C_0(1+t)^{-\frac{1}{4}}. \tag{83}$$

So,

$$\left\{\|(\rho,c,u)\|_{L^{\frac{3}{s}}}\right\}^{s-\frac{1}{2}}\left\{\|(\nabla\rho,\nabla c,\nabla u)\|_{H^1}\right\}^{\frac{5}{2}-s} \leqslant C(1+t)^{-\frac{7}{4}+\frac{s}{2}}. \tag{84}$$

From (84),

$$\mathrm{Q}(t) \leqslant C\int_0^t (1+\tau)^{-\frac{7}{4}+\frac{s}{2}} d\tau \leqslant C. \tag{85}$$

From (80), we have

$$\mathrm{F}(t) \leqslant C_0, \quad \forall t \geqslant 0. \tag{86}$$

From (75) and (86), we conclude Lemma 4.1.

6. The proof of Theorem 1.1

According to Section 3, we have got all we need in Section 4 and 5. Integrating from 0 to t, we obtain (6). The result of Theorem 1.1 are obvious by Section 3. So we conclude Theorem 1.1.

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