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The variable exponent Sobolev capacity and quasi-fine properties of Sobolev functions in the case $p^- = 1$

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ABSTRACT

In this article we extend the known results concerning the subadditivity of capacity and the Lebesgue points of functions of the variable exponent Sobolev spaces to cover also the case $p^- = 1$. We show that the variable exponent Sobolev capacity is subadditive for variable exponents satisfying $1 \leqslant p < \infty$. Furthermore, we show that if the exponent is log-Hölder continuous, then the functions of the variable exponent Sobolev spaces have Lebesgue points quasieverywhere and they have quasicontinuous representatives also in the case $p^- = 1$. To gain these results we develop methods that are not reliant on reflexivity or maximal function arguments.

1. Introduction

The last two decades have offered great progress in the field of variable exponent function spaces. The current theory has its origins in the study of function spaces and certain variational problems. Nowadays, by virtue of grown understanding of the basic theory, the research has spread out to several fields of analysis, variational problems and PDEs. As an introduction and a history coverage to the subject of variable exponent problems, we advice the reader to see the original article [23], the monograph [3], and the survey articles [4,13,28,29].

In this article, we study Lebesgue points and quasicontinuity of Sobolev functions in the variable exponent setting. The variable exponent Sobolev spaces play an important role in the study of problems with non-standard growth conditions and thus understanding the pointwise behaviour of the functions is essential. The main contribution of this paper is to extend the results obtained in [9] to cover also the case $p^- = 1$.

In [9, Section 4] the authors proved that every Sobolev function has Lebesgue points outside of a set of $p(\cdot)$-capacity zero and that the precise pointwise representative of a Sobolev function is $p(\cdot)$-quasicontinuous. These are extensions of the classical results to the variable exponent setting. However, in [9] the authors had the additional assumption that the variable exponent satisfies $p^- > 1$.

In order to relax the condition to $p^- \geqslant 1$, we first generalize some of the results from [11] to cover the limiting case. This we do by establishing the subadditivity of $p(\cdot)$-capacity also in the case $p^- = 1$. As a consequence of the arguments, we are able to prove the subadditivity of the Sobolev capacity for exponents $p : \mathbb{R}^n \to [1, \infty)$ without the assumption $p^+ < \infty$. The proofs for the corresponding results in [11] and [9] rely on the reflexivity of the Sobolev space and boundedness of the maximal operator respectively. However, since neither of these properties is available in our case, we need to develop different arguments.

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Instead of maximal operators, we use an operator defined as a limit of integral averages in our approach and we obtain a capacitary weak-type estimate for it. This is by virtue of suitable modular versions of Poincaré inequality and extension theorem for Sobolev functions that we establish. These results come with additional correction terms, which cause some technical awkwardnesses to the arguments. However, we are able to overcome these difficulties by using the a priori information that the sets we are studying have small measure. In the end we obtain that Sobolev functions have Lebesgue points quasieverywhere and that each Sobolev function has a quasicontinuous representative.

2. Preliminaries

In this section we recall briefly some basic concepts and definitions from the analysis of variable exponent spaces. We cite the main tools and results when we use them throughout the article, as they are quite multitudinous to list into a single preliminary chapter. See [3] for a good overall reference and a comprehensive introduction to this field.

Let $\Omega \subset \mathbb{R}^n$ be an open set. A measurable function $p : \Omega \to [1,\infty)$ is called a *variable exponent*. Note that we may later on impose additional restrictions on the variable exponent. We denote

$$p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x), \qquad p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x),$$

and for $E \subset \mathbb{R}^n$,

$$p_E^+ := \operatorname*{ess\,sup}_{x\in E} p(x), \qquad p_E^- = \operatorname*{ess\,inf}_{x\in E} p(x).$$

The variable exponent Lebesgue and Sobolev spaces are obtained in the usual way. Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define a *modular* by setting

$$\varrho_{p(\cdot)}(u) = \int_\Omega \left|u(x)\right|^{p(x)}\,dx.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u/\lambda)$ is finite for some $\lambda > 0$. We define a norm on this space as a Luxemburg norm:

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\bigl\{\lambda > 0\colon\ \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\bigr\}.$$

It is known that $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of a *Musielak–Orlicz space*, but here we only consider the Lebesgue and Sobolev type spaces. For constant function p the variable exponent Lebesgue space coincides with the standard Lebesgue space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u has modulus in $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

We also define

$$\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(\nabla u).$$

The log-Hölder continuity of the variable exponent has become a standard assumption in the theory of variable exponent spaces, since it makes several techniques and methods from real and harmonic analysis available. We will also impose this additional condition for the exponent in the latter part of the article.

Definition 2.1. Function $p : \Omega \to \mathbb{R}$ is locally log-Hölder continuous on Ω if there exists $c_1 > 0$ such that

$$\left|p(x) - p(y)\right| \leqslant \frac{c_1}{\log(e + \frac{1}{|x-y|})}$$

for all $x, y \in \Omega$. We say that p is globally log-Hölder continuous on Ω if it is locally log-Hölder continuous on Ω and there exists $p_\infty \geqslant 1$ and a constant $c_2 > 0$ such that

$$\left|p(x) - p_\infty\right| \leqslant \frac{c_2}{\log(e + |x|)}$$

for all $x \in \Omega$. The constant $\max\{c_1, c_2\} =: c$ is called the log-Hölder constant of p.

Remark. We usually replace the constants c_1, c_2 by the maximum c. This is due to the fact that we may extend a log-Hölder continuous function to a larger domain, but in such procedure one of the constants may become larger. However, the maximum c remains in extension.

One typical assumption concerning the variable exponent is also the strict upper bound $p^+ < \infty$. This implies e.g. that the notion of *convergence in modular* is equivalent to the convergence in norm. In this work, we do not explicitly make this assumption for the whole paper. Some of our main results are true without it and some require a strict upper bound. However, in our case, the assumption is not too restrictive. For instance, the difficulties related to fine continuity properties of Sobolev functions occur for the small values of variable exponent.

We now introduce the basic tool that we need in our study.

Definition 2.2. Let $p : \mathbb{R}^n \to [1, \infty)$ be a variable exponent. The $p(\cdot)$-capacity of set $E \subset \mathbb{R}^n$ is defined as

$$C_{p(\cdot)}(E) = \inf \int\limits_{\mathbb{R}^n} |u|^{p(x)} + |\nabla u|^{p(x)}\, dx,$$

where the infimum is taken over admissible functions $u \in S_{p(\cdot)}(E)$ where

$$S_{p(\cdot)}(E) = \left\{ u \in W^{1,p(\cdot)}\left(\mathbb{R}^n\right):\ u \geqslant 1 \text{ in an open set containing } E \right\}.$$

It is easy to see that if we restrict these admissible functions $S_{p(\cdot)}$ to the case $0 \leqslant u \leqslant 1$, we get the same capacity. In this case it is obviously possible to also drop the absolute value from $|u|^{p(x)}$.

The notion of $p(\cdot)$-capacity plays the expected role in the potential theory and in the study of Sobolev functions in the variable exponent setting, see [1,9–12,14,15,25]. In general, the $p(\cdot)$-capacity is used to measure fine properties of functions and sets. The $p(\cdot)$-capacity enjoys the usual desired properties of capacity when $1 < p^- \leqslant p^+ < \infty$, see [11,3], but some of the properties remain still open for the case $p^- = 1$. In [8] the authors introduced a mixed "BV–Sobolev"-type capacity to study the case $p^- = 1$. This approach produces a capacity with good set function properties, i.e. it is an outer measure and has a nice behaviour with respect to monotonous limits of sets. Moreover, the mixed-type capacity has the same zero sets as $p(\cdot)$-capacity. In this article, we are able to establish the subadditivity for the Sobolev capacity in the case $p^- = 1$. Furthermore, we will explicitly apply this result when studying the fine continuity properties of Sobolev functions. Recall the following terminology.

Definition 2.3. We say that a claim holds $p(\cdot)$-*quasieverywhere* if it holds everywhere except in a set of $p(\cdot)$-capacity zero. A function $u : \Omega \to \mathbb{R}$ is said to be $p(\cdot)$-*quasicontinuous* if for every $\varepsilon > 0$ there exists an open set U with $C_{p(\cdot)}(U) < \varepsilon$ such that u restricted to $\Omega \setminus U$ is continuous.

We now begin with the actual theoretical results of this paper. First we extend the result concerning the subadditivity of variable exponent Sobolev capacity. We then proceed to study the Lebesgue points and fine continuity properties of Sobolev functions.

3. Subadditivity of the capacity and immediate corollaries

In this section we extend the result of subadditivity of the variable exponent Sobolev capacity to cover also the case $1 \leqslant p(x) < \infty$. Note that we allow $p^- = 1$ and $p^+ = \infty$, but not $p(x) = \infty$ pointwise essentially. To this end, we need an argument that does not rely on reflexivity of the function space.

The following lemma is a variable exponent modification of [6, Chapter 4.7.1, Lemma 2(iii)], see also [7]. It holds for arbitrary measurable variable exponents $p : \mathbb{R}^n \to [1, \infty)$.

Lemma 3.1. *Let $u_i \in W^{1,p(\cdot)}(\mathbb{R}^n)$ for $i = 1, 2, \ldots$ and define*

$$u(x) := \sup_i u_i(x),$$
$$g(x) := \sup_i \left|\nabla u_i(x)\right|.$$

If $u, g \in L^{p(\cdot)}(\mathbb{R}^n)$, then $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and $|\nabla u| \leqslant g$ almost everywhere.

Proof. The variable exponent Hölder inequality [3, Lemma 3.2.20] implies that u and g are locally integrable functions. Let us now recall the proof of [6, Chapter 4.7.1, Lemma 2(iii)]. Roughly speaking the strategy is to show that the distributional gradient ∇u is a vector measure that is absolutely continuous with respect to the Lebesgue measure. The main tools are the Riesz representation of a functional [6, Chapter 1.8, Theorem 1] and the Radon–Nikodym theorem [18, Theorem (19.27) and the amendments following it]. The methods of applying the Riesz representation and the Radon–Nikodym theorem remain valid also in our case, even though we work in slightly different function spaces. Finally, we have that u has the weak gradient ∇u dominated by g. This also completes the statement about u belonging to the desired Sobolev space. □

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Remark. See also [2, Lemma 1.28] for a very short proof of a similar result in the metric measure space setting. In the proof, the supremum of the upper gradients corresponds to the function g in our lemma.

Remark. It is not conventional to use the assumption $p^+ = \infty$, so let us spend a few words on this issue. When analyzing the proof of Lemma 3.1 working from [6, Chapter 4.7.1, Lemma 2(iii)], we note that the part where the relaxation $p^+ = \infty$ might come into play is as follows. We need to be able to conclude that the integral

$$\int_E g\,dx$$

vanishes if the measure of E vanishes. Note that a straightforward application of Hölder or Young inequalities does not work in this case due to the possibility of $p^+ = \infty$. We instead use the split [3, Theorem 3.3.11]:

$$\int_E g\,dx = \int_E g_1\,dx + \int_E g_\infty\,dx,$$

where $g_1 \in L^1(\mathbb{R}^n)$, $g_\infty \in L^\infty(\mathbb{R}^n)$. Let now $|E| \to 0$. The L^1 integral vanishes due to the absolute continuity of the Lebesgue integral. The L^∞ integral vanishes due to essential boundedness. These facts are not affected by the case $p^+ = \infty$.

The previous lemma allows us to conclude that the $p(\cdot)$-capacity is subadditive, which implies that it is an outer measure.

Theorem 3.2. *Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function. If $E_i \subset \mathbb{R}^n$ for $i = 1, 2, \dots,$ then*

$$C_{p(\cdot)}\left(\bigcup_{i=1}^\infty E_i\right) \leqslant \sum_{i=1}^\infty C_{p(\cdot)}(E_i).$$

Proof. Let $\varepsilon > 0$. We may assume that

$$\sum_{i=1}^\infty C_{p(\cdot)}(E_i) < \infty,$$

or otherwise there is nothing to prove. For each E_i, we choose $u_i \in S_{p(\cdot)}(E_i)$ with $u_i \geqslant 0$ and

$$\int_{\mathbb{R}^n} u_i^{p(x)} + |\nabla u_i|^{p(x)}\,dx < C_{p(\cdot)}(E_i) + 2^{-i}\varepsilon.$$

Define

$$u(x) := \sup_i u_i(x),$$
$$g(x) := \sup_i \left|\nabla u_i(x)\right|$$

for $x \in \mathbb{R}^n$. Now

$$\int_{\mathbb{R}^n} u^{p(x)} + g^{p(x)}\,dx \leqslant \sum_{i=1}^\infty \int_{\mathbb{R}^n} u_i^{p(x)} + |\nabla u_i|^{p(x)}\,dx \leqslant \sum_{i=1}^\infty C_{p(\cdot)}(E_i) + \varepsilon.$$

Note that now

$$\int_{\mathbb{R}^n} u^{p(x)} + g^{p(x)}\,dx < \infty.$$

Lemma 3.1 now implies that u has a weak gradient in $L^{p(\cdot)}(\mathbb{R}^n)$ with $|\nabla u| \leqslant g$ almost everywhere. It is now clear that u is an admissible function for the union:

$$u \in S_{p(\cdot)}\left(\bigcup_{i=1}^\infty E_i\right).$$

The proof is now settled as we let $\varepsilon \to 0$:

$$\begin{aligned} C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right) &\leqslant \int_{\mathbb{R}^n} u^{p(x)} + |\nabla u|^{p(x)}\,dx \\ &\leqslant \int_{\mathbb{R}^n} u^{p(x)} + g^{p(x)}\,dx \\ &\leqslant \sum_{i=1}^{\infty} C_{p(\cdot)}(E_i) + \varepsilon. \qquad \square \end{aligned}$$

We may now repeat the claims in [11, Section 5] in fuller generality. In [11], the authors note that the results are modified from [22]. These results are also reproduced in [3, Section 11.1].

Lemma 3.3. *(See Lemma 5.1 in [11].) Let $p^+ < \infty$. For each Cauchy sequence of functions in $C(\mathbb{R}^n) \cap W^{1,p(\cdot)}(\mathbb{R}^n)$ there is a subsequence which converges pointwise $p(\cdot)$-quasieverywhere in $\mathbb{R}^n$. Moreover, the convergence is uniform outside a set of arbitrarily small $p(\cdot)$-capacity.*

Remark. Confer also to [3, Lemma 11.1.2].

Proof. For the possible notation in this proof, please consult the proof in [11]. In the proof presented in that article, the only part in which the authors use the property $p^- > 1$ is the argument using the subadditivity of the capacity, that is

$$C_{p(\cdot)}(F_j) \leqslant \sum_{i=j}^{\infty}(E_i) \leqslant \sum_{i=j}^{\infty} 2^{-i} \leqslant 2^{1-j}.$$

We now know subadditivity also for the case $p^- = 1$ per Theorem 3.2, so the whole proof may be reconstructed also in this case. $\square$

Recall that the variable exponent p is said to satisfy the *density condition* if smooth Sobolev functions are dense in $W^{1,p(\cdot)}$ with respect to the Sobolev norm. The density condition is an abstract condition in the sense that exponents p with or without the density property have not been fully characterized. Two important cases are known and also some partial other results.

We know that we have the density property in the case of a log-Hölder continuous exponent [27] and in the case of suitably cone-monotonous exponent [5], see also the remark in [16, Section 2.2]. Let us also mention that e.g. Zhikov has given an example of a 'worse' exponent which still gives the density condition [30] and in [17, Chapter 4] we see a completely different kind of condition for the denseness of continuous functions in the Sobolev space.

In the following, we simply assume the density condition without regard to what explicit conditions for p eventually guarantee it.

Theorem 3.4. *(See Theorem 5.2 in [11].) Let p be a bounded variable exponent with the density condition. For each Sobolev function $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ there is a $p(\cdot)$-quasicontinuous function $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u = v$ a.e. in $\mathbb{R}^n$.*

Remark. Confer also to [3, Theorem 11.1.3].

Proof. The proof in that article only uses properties of the variable exponent function space through Lemma 5.1 in that article. We may simply replace this by our Lemma 3.3 and the rest of the proof remains the same. $\square$

The remaining two statements do not depend on $p^- = 1$ or $p^- > 1$ in any way in the original [11]. We repeat them for completeness. They are originally stemming from ideas in [24, Chapter 2].

Lemma 3.5. *(See Lemma 5.3 in [11].) Let p be a bounded variable exponent, and let $E \subset \mathbb{R}^n$. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be $p(\cdot)$-quasicontinuous, non-negative, and $u \geqslant 1$ in E. Then for each $\varepsilon > 0$ there is a function $v \in S_{p(\cdot)}(E)$ with $\varrho_{1,p(\cdot)}(u - v) < \varepsilon$.*

Remark. See also [3, Lemma 11.1.7] which has a slightly relaxed condition.

Proof. The proof is exactly the same as in the original. The fact that $p^- = 1$ plays no role. $\square$

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Theorem 3.6. *(See Theorem 5.4 in [11].) Let p be a bounded variable exponent. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be $p(\cdot)$-quasicontinuous and take $\lambda > 0$. We have*

$$C_{p(\cdot)}\big(\big\{x \in \mathbb{R}^n \;\big|\; |u(x)| > \lambda\big\}\big) \leqslant \int\limits_{\mathbb{R}^n} \left|\frac{u(x)}{\lambda}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx.$$

Remark. Confer [3, Corollary 11.1.9].

Proof. The proof is exactly the same as in the original. The fact that $p^- = 1$ plays no role. Note that the original proof uses Theorem 2.2 and Lemmata 2.6, 5.3 in [11]; $p^- = 1$ plays no role in any of these. □

In the next section of this note we complete an already known result concerning Lebesgue points of Sobolev functions. It has been shown for $p^- > 1$ in [9] and that proof is itself based on [21]. We incorporate some of the proofs from [9] and ideas from [6] to complete the result to the case $p^- = 1$.

4. Lebesgue points in Sobolev spaces and quasi-fine properties of functions

In [9, Section 4] the authors proved that every Sobolev function has Lebesgue points outside of a set of $p(\cdot)$-capacity zero and that the precise pointwise representative of a Sobolev function is $p(\cdot)$-quasicontinuous. These considerations extend the classical results from the fixed exponent case to the variable exponent setting.

However, in [9] the authors had the additional assumption that the variable exponent $p(\cdot)$ satisfies the condition $p^- > 1$. In this section we complete the theory by extending these results also to cover the case $p^- = 1$. To this end, we need to develop different arguments.

We begin by adapting some auxiliary results from [3]. These are either rephrases from that monograph or adaptations which follow with elementary modifications of the original proofs. Note that the results might not be stated in their full generality here compared to the monograph. From now on, we will make the following assumption:

For the rest of the paper, we will assume that $p^+ < \infty$ and that the variable exponent p is globally log-Hölder continuous.

Observe that for exponents satisfying $p^+ < \infty$, the global log-Hölder continuity is equivalent to $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, see [3, Remark 4.1.5].

Lemma 4.1 *(Estimate for the integral average). (See Corollary 4.2.5 in [3].) Let Ξ be a ball or a cube in $\mathbb{R}^n$, let $u \in L^{p(\cdot)}(\mathbb{R}^n)$, and $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant 1$. Then*

$$\int\limits_{\Xi} \left| \fint\limits_{\Xi} |u|\,dy \right|^{p(x)} dx \leqslant C \int\limits_{\Xi} |u|^{p(y)}\,dy + C|\Xi|.$$

The constant C depends only on p.

Proof. This statement is almost the same as [3, Corollary 4.2.5]. We estimate the β in that statement up to C on the right hand side. We may fix any $m > 0$ appearing in that statement and we always have

$$\int\limits_{\Xi} (e + |y|)^{-m}\,dy \leqslant |\Xi|.$$

Note that the β only depends on the choice of m, which we may consider fixed, and on the properties of p. Our statement now follows. □

The following is a standard trick in the study of variable exponent modular estimates, but we choose to explicitly make this short comment. In Lemma 4.1 the assumption $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant 1$ is not an actual restriction. We can handle the case $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} > 1$ simply by scaling u suitably either by its norm or its modular. In this case the constant C will depend also on either the norm or the modular. The same is true throughout Lemmata 4.2–4.4, 4.6.

Often the estimates in the case of norms greater than one are not very useful. Note that we are going to use the lemmata only in a setting where we can control the norms in question and make them as small as we wish. This is, namely, the proof of Theorem 4.9.

Lemma 4.2 *(Estimates for the averaging operator). (See Theorems 4.4.8, 4.4.15 in [3].) Let $\mathcal{Q}$ be a family of balls or cubes that have suitable finite overlap, $u \in L^{p(\cdot)}(\mathbb{R}^n)$, and $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant 1$. Then*

$$\int\limits_{\mathbb{R}^n} |T_{\mathcal{Q}} u|^{p(x)}\,dx \leqslant C \sum_{\Xi \in \mathcal{Q}} \biggl(\int\limits_{\Xi} |u|^{p(y)}\,dy + |\Xi| \biggr).$$

Here we have denoted the averaging operator

$$T_{\mathcal{Q}}u(x) := \sum_{\Xi\in\mathcal{Q}} \fint_{\Xi} |u|\,dy\,\chi_{\Xi}(x).$$

The constant C depends only on p and the overlapping properties of the family $\mathcal{Q}$.

Let further $\lambda \geqslant 1$, assume that if $\Xi\in\mathcal{Q}$, then $\Xi\subset\lambda\Xi^*$ for some ball or cube Ξ^*, and assume that the sets Ξ^* have a suitable finite overlap. Then

$$\int_{\mathbb{R}^n}\left|\sum_{\Xi\in\mathcal{Q}}\left|\fint_{\Xi^*}|u|\,dy\right|\chi_{\Xi}(x)\right|^{p(x)}dx \leqslant C\int_{\mathbb{R}^n}|u|^{p(y)}\,dy + C\sum_{\Xi\in\mathcal{Q}}|\Xi|.$$

The constant C depends only on n, λ, properties of p and the overlapping properties of the families $\mathcal{Q}$ and $\{\Xi^*\}_{\Xi\in\mathcal{Q}}$.

Proof. The calculations in the proofs in the monograph are done on modulars and passed to norms only at the end. Thus these estimates may be read inside the proofs of [3, Theorem 4.4.8] and [3, Theorem 4.4.15]. The proofs utilize results related to our Lemma 4.1 from the book Chapter 4.2. □

Lemma 4.3 *(Modular Poincaré inequality). (See Proposition 8.2.8(b) in [3].) Let B be a ball in $\mathbb{R}^n$, $u\in W^{1,p(\cdot)}(B)$, and $\|u\|_{W^{1,p(\cdot)}(B)}\leqslant 1$. Then*

$$\int_B\left|\frac{u-u_B}{\operatorname{diam} B}\right|^{p(x)}dx \leqslant C\int_B |\nabla u|^{p(x)}\,dx + C|B|.$$

The constant C depends only on n and the properties of p; note that the constant does not depend on the size or geometry of B in this case.

Proof. This result uses [3, Lemma 8.2.1(b)]. In the case of a ball we'll have $B=\Omega$ with the notation of that result and the multiplicative constant c does not depend on the geometry of Ω, i.e. of B. Finally the proof uses our Lemma 4.1. The constant will now depend only on n and p. □

Lemma 4.4 *(Analogue of the extension theorem). (See Theorem 8.5.12 in [3].) Let B be a ball in $\mathbb{R}^n$, $\operatorname{diam} B<1$, $u\in W^{1,p(\cdot)}(B)$, and $\|u\|_{W^{1,p(\cdot)}(B)}\leqslant 1$. Then there is an extension v of u with*

$$\int_{\mathbb{R}^n}|v|^{p(x)}+|\nabla v|^{p(x)}\,dx \leqslant C\int_B\left|\frac{u}{\operatorname{diam} B}\right|^{p(x)}+|\nabla u|^{p(x)}\,dx + C|B|.$$

The constant C depends on n, p, and the geometry of a ball; note that the constant does not depend on size of B, and note also that since we state the result for a ball, the geometric properties may be considered fixed.

Proof. The proof of [3, Theorem 8.5.12] depends on the other results in [3, Chapter 8.5] and also on earlier results which we have presented above as Lemmata 4.1–4.2. The proof is almost the same, except that we now work with modulars instead of norms. Note that the choice of modular versus norm *has not yet been made* in the results of the book Chapter 8.5 prior to [3, Theorem 8.5.12]. Note also that the results [3, 8.5.8–8.5.11] concern *pointwise behaviour* and they do not feature $p(\cdot)$-norms or modulars.

The extension is defined as follows. We use u itself in B. We use a "slightly spread-out" averaged Taylor polynomial in small Whitney cubes in the part of $\mathbb{R}^n\setminus B$ which is close to B. We use "almost zero" in larger Whitney cubes of $\mathbb{R}^n\setminus B$ which are further away from B. By almost zero we mean that the "spreading out" that occurs in the family of small Whitney cubes may cause the extension to attain non-zero values also outside the family of small cubes. However, this only happens for a relatively short distance, and further outside the extension is zero. We then apply [3, Corollary 8.5.9] pointwise in the family of small Whitney cubes and [3, Corollary 8.5.11] pointwise in the family of large Whitney cubes. This we do for formulas featuring modulars instead of norms, but confer above.

When applying [3, Corollary 8.5.11] and noting that the cubes Q are of scale relative to the scale of B, a multiplicative $(\operatorname{diam} B)^{-1}$ will appear in front of a term including u, but not ∇u. When applying [3, Corollary 8.5.9] we note that neither u nor ∇u will have multiplicatives related to the size of B. We then estimate

$$|u| \leqslant \left|\frac{u}{\operatorname{diam} B}\right|$$

pointwise by our assumption $\operatorname{diam} B<1$.

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One more note to present is that when we apply our Lemmata 4.1–4.2, the averaging operator applied multiple times may increase the norm above one. This is obviously bypassed by first dealing with $\frac{u}{c}$ where the constant c depends only on n, on the geometry of a ball, and on the properties of p. More precisely, the dependence on geometry appears as the John domain ε of a ball in $\mathbb{R}^n$. The idea is that averaging operators applied multiple times now increase the norm only to at most one.

Finally the modular version follows and an additional correction term in the form of $C|B|$ appears. The multiplicative constants on the right hand side will also depend on the size of c above, but since c depends only on n, p, and the geometric properties of a ball, this is not an issue. □

For the remainder of this section, let us denote the limes superior operator

$$\mathcal{A}u(x) := \limsup_{r\to 0} \fint_{B(x,r)} |u|\,dy,$$

where $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The operator is well defined in the whole $\mathbb{R}^n$.

The following elementary auxiliary estimate is a direct consequence of the Lebesgue differentiation theorem. This deals with the measure of an exceptional set. After this we are about to deal with the capacity of an exceptional set.

Lemma 4.5. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\lambda > 0$. Let $V \subset \mathbb{R}^n$ be a measurable set and let*

$$A := \{x \in V \mid \mathcal{A}u(x) > \lambda\}.$$

Then

$$|A| \leqslant \frac{1}{\lambda} \int_V |u|\,dy.$$

The weak-type estimate or Chebyshev inequality of Lemma 4.5 will be sufficient for our study of capacity. We are now ready to present a capacity estimate derived from [6]. Note that the appearance of the correction term $|A|$ is how we handle the variable exponent situation. It seems to us that the result cannot be proved without a correction term in the variable exponent case. We will use the following and Lemma 4.5 to control the capacity of an exceptional set. Note that the appearance of the measure term $|A|$ is especially the reason for the necessity of the Chebyshev inequality.

Lemma 4.6. *Let $0 < \lambda \leqslant 1$, $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $u \geqslant 0$, $\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} \leqslant \lambda$, and let $V \subset \mathbb{R}^n$ be a bounded measurable set. Denote*

$$A := \{x \in V \mid \mathcal{A}u(x) > \lambda\}.$$

Then

$$C_{p(\cdot)}(A) \leqslant \frac{C}{\lambda^{p_+}} \int_{\mathbb{R}^n} |u|^{p(x)} + |\nabla u|^{p(x)}\,dx + C|A|.$$

The constant C depends only on n and the properties of p.

Proof. We adapt the proof from [6, Chapter 4.8, Lemma 1], but we introduce some modifications in order to handle the variable exponent. Let $\varepsilon > 0$. We use the definition of the Lebesgue measure and the relation between measures of cubes and balls to find a family of open balls with

$$A \subset \bigcup_{\nu=1}^{\infty} B'_\nu \quad \text{and} \quad \sum_{\nu=1}^{\infty} |B'_\nu| \leqslant C|A| + \varepsilon.$$

For each $x \in A$ we may find a ball with

$$\fint_{B(x,r_x)} \frac{u(y)}{\lambda}\,dy > 1.$$

Note that the function u was assumed to be non-negative. Also, due to properties of the limes superior, we may choose r_x smaller than any upper bound we wish. Now, due to this smallness and the choice of the family $\{B'_\nu\}$, we may have $B(x, r_x) \subset B'_\nu$ for some index ν. We also choose r_x so small that

$$\operatorname{diam} B(x, r_x) < \min\left\{1, \frac{\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{c\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\}$$

where c is the constant from the variable exponent Poincaré inequality for norms [3, Theorem 8.2.4(b)]. Note that c depends only on n and properties of p. We now use the Besicovitch covering theorem for open balls upon $\{B(x,r_x)\}$ to find the following cover:

$$A \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{\infty} B_{ij}, \quad N < \infty,\ \{B_{ij}\}_{j=1}^{\infty} \text{ are disjoint.}$$

This version of Besicovitch covering theorem can be proved by modifying the proof of [26, Theorem 2.7], see [19, Theorem 2.1] and also [20, pp. 482–486]. Observe that by the earlier choice of r_x we have

$$\bigcup_{j=1}^{\infty} B_{ij} \subset \bigcup_{\nu=1}^{\infty} B'_{\nu}.$$

By the disjointness we now deduce

$$\sum_{j=1}^{\infty} |B_{ij}| \leqslant C|A| + \varepsilon,$$

and by the finiteness we then deduce

$$\sum_{i=1}^{N} \sum_{j=1}^{\infty} |B_{ij}| \leqslant CN|A| + N\varepsilon.$$

Note that $(\frac{u_{B_{ij}}-u}{\lambda})_+$ is a Sobolev function in B_{ij}. We see by the Poincaré inequality and the properties of weak differentiation that

$$\begin{cases} \left\| \left(\dfrac{u_{B_{ij}} - u}{\lambda} \right)_+ \right\|_{L^{p(\cdot)}(B_{ij})} \leqslant \dfrac{c(\operatorname{diam} B_{ij})}{\lambda} \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \\ \left\| \nabla \left(\dfrac{u_{B_{ij}} - u}{\lambda} \right)_+ \right\|_{L^{p(\cdot)}(B_{ij})} \leqslant \dfrac{1}{\lambda} \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{cases}$$

By the smallness assumption of the radii and the assumption $||u\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} \leqslant \lambda$ we now see that

$$\left\| \left(\frac{u_{B_{ij}} - u}{\lambda} \right)_+ \right\|_{W^{1,p(\cdot)}(B_{ij})} \leqslant 1$$

so we use Lemma 4.4 to extend each of them to

$$\begin{cases} v_{ij} \in W^{1,p(\cdot)}(\mathbb{R}^n), \\ \displaystyle\int_{\mathbb{R}^n} |v_{ij}|^{p(x)} + |\nabla v_{ij}|^{p(x)}\, dx \leqslant C \int_{B_{ij}} \left| \frac{u_{B_{ij}} - u}{\lambda \operatorname{diam} B_{ij}} \right|^{p(x)} + \left| \frac{\nabla u}{\lambda} \right|^{p(x)} dx + C|B_{ij}|. \end{cases}$$

Set the function

$$v := \sup_{i,j} v_{ij},$$

whence $\frac{u}{\lambda} + v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and $\frac{u}{\lambda} + v \geqslant 1$ in an open neighbourhood of A; e.g., the union of the balls B_{ij} is a suitable open set. Note that the Sobolev'ness of v is deduced from Lemma 3.1, where the required condition $v, g \in L^{p(\cdot)}(\mathbb{R}^n)$ is obtained by similar arguments as in the estimate we are going to do next.

We use the above definitions and estimates. We also use the modular Poincaré inequality, i.e. Lemma 4.3, and calculate

$$\int_{\mathbb{R}^n} \left| \frac{u}{\lambda} + v \right|^{p(x)} + \left| \nabla \left(\frac{u}{\lambda} + v \right) \right|^{p(x)} dx$$
$$\leqslant C \int_{\mathbb{R}^n} \left| \frac{u}{\lambda} \right|^{p(x)} + \left| \frac{\nabla u}{\lambda} \right|^{p(x)} dx + C \sum_{i=1}^{N} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |v_{ij}|^{p(x)} + |\nabla v_{ij}|^{p(x)}\, dx$$

$$\leqslant C\int_{\mathbb{R}^n}\left|\frac{u}{\lambda}\right|^{p(x)}+\left|\frac{\nabla u}{\lambda}\right|^{p(x)}dx+C\sum_{i=1}^{N}\sum_{j=1}^{\infty}\left(\int_{B_{ij}}\left|\frac{u-u_{B_{ij}}}{\lambda\operatorname{diam}B_{ij}}\right|^{p(x)}+\left|\frac{\nabla u}{\lambda}\right|^{p(x)}dx+|B_{ij}|\right)$$

$$\leqslant C\int_{\mathbb{R}^n}\left|\frac{u}{\lambda}\right|^{p(x)}+\left|\frac{\nabla u}{\lambda}\right|^{p(x)}dx+C\sum_{i=1}^{N}\sum_{j=1}^{\infty}\left(\int_{B_{ij}}\left|\frac{\nabla u}{\lambda}\right|^{p(x)}dx+|B_{ij}|\right).$$

We then use the disjointness with respect to j and finiteness with respect to i to see

$$\int_{\mathbb{R}^n}\left|\frac{u}{\lambda}+v\right|^{p(x)}+\left|\nabla\left(\frac{u}{\lambda}+v\right)\right|^{p(x)}dx\leqslant C(N+1)\int_{\mathbb{R}^n}\left|\frac{u}{\lambda}\right|^{p(x)}+\left|\frac{\nabla u}{\lambda}\right|^{p(x)}dx+C\sum_{i=1}^{N}\sum_{j=1}^{\infty}|B_{ij}|.$$

Recall that

$$\sum_{i=1}^{N}\sum_{j=1}^{\infty}|B_{ij}|\leqslant CN|A|+N\varepsilon.$$

Also, we have deduced that $\frac{u}{\lambda}+v$ is a suitable test function for the capacity of A. Then we let $\varepsilon\to 0$ and conclude the proof. Note that the N from the Besikovitch theorem only depends on the dimension n. □

We now repeat some results from [9, Chapter 4].

Lemma 4.7. *Let $u\in W^{1,p(\cdot)}(\mathbb{R}^n)$. Then*

$$C_{p(\cdot)}\left(\left\{x\in\mathbb{R}^n\;\middle|\;\limsup_{r\to 0} r^{p(x)}\fint_{B(x,r)}|\nabla u|^{p(y)}\,dy>0\right\}\right)=0.$$

Proof. This is [9, Lemma 4.9]. The proof is the same for $p^-=1$. The proof only relies on results from [9, Chapter 4] which are true in the case $p^-=1$. The proof also relies on the subadditivity of the capacity. We have shown that the capacity is subadditive in the case $p^-=1$ in Theorem 3.2. □

Lemma 4.8. *Let $1\leqslant p^-\leqslant p^+<n$, $u\in W^{1,p(\cdot)}(\mathbb{R}^n)$, and $x\in\mathbb{R}^n$ be a point where*

$$\limsup_{r\to 0} r^{p(x)}\fint_{B(x,r)}|\nabla u|^{p(y)}\,dy=0.$$

Then

$$\limsup_{r\to 0}\fint_{B(x,r)}|u-u_{B(x,r)}|^{p^*(y)}\,dy=0.$$

Proof. This is proven as a part of the proof of [9, Theorem 4.12], see also [3, Theorem 11.4.10]. Note that the whole statement in that paper requires the boundedness of the maximal function, which is not true in the case $p^-=1$, but the part that covers our statement does not depend on the maximal function. The proof relies on the Sobolev–Poincaré inequality for norms, but both the Sobolev and the Poincaré inequalities are now known to hold in the case $p^-=1$. In this case the inequalities can be established by the averaging operator instead of the maximal operator. See [3, Theorem 8.3.1(b) among others] for details. Our statement now follows from the proof of [9, Theorem 4.12]. □

The following is the main result of this section. We adapt the proof from [6, Chapter 4.8, Theorem 1].

Theorem 4.9. *Let $1\leqslant p^-\leqslant p^+<n$ and $u\in W^{1,p(\cdot)}(\mathbb{R}^n)$. Then the limit*

$$\lim_{r\to 0}\fint_{B(x,r)}u(y)\,dy=:u^*(x)$$

exists quasieverywhere in $\mathbb{R}^n$ and

$$\lim_{r\to 0}\fint_{B(x,r)}\left|u(y)-u^*(x)\right|^{p^*(x)}dy=0$$

quasieverywhere in $\mathbb{R}^n$. Let us call u^* the precise representative *of u. The precise representative is quasicontinuous.*

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Proof. The variable exponent forces, it seems, us to arrange this proof locally. Let $R > 0$; later on R shall be as large as we might require. Denote, in the spirit of Lemmata 4.5–4.6, $V := B(0, R)$. Denote the set

$$D := \left\{x \in V \;\middle|\; \limsup_{r\to 0} r^{p(x)} \fint_{B(x,r)} |\nabla u|^{p(y)}\,dy > 0\right\}.$$

Lemma 4.7 is true also locally, i.e. with respect to V instead of $\mathbb{R}^n$, so the set D is of zero capacity. By Lemma 4.8 we see that

$$\limsup_{r\to 0} \fint_{B(x,r)} |u - u_{B(x,r)}|^{p^*(y)}\,dy = 0$$

for all $x \in V \setminus D$. By the density of smooth functions we choose an approximating sequence $u_i \in W^{1,p(\cdot)}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$:

$$\begin{cases} \|u - u_i\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} \leqslant 2^{-p^+ i - i - 1}, \\ \|u - u_i\|_{W^{1,1}(V)} \leqslant 2^{-2i-1}. \end{cases}$$

Note that the inequalities are also true for the corresponding modulars, since the bounds are less than one. Please recall that $W^{1,p(\cdot)}(\mathbb{R}^n) \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Denote the sets

$$A_i := \left\{x \in V \;\middle|\; \mathcal{A}|u - u_i| > 2^{-i}\right\}.$$

By Lemmata 4.5, 4.6, and the choice of u_i,

$$C_{p(\cdot)}(A_i) \leqslant 2^{-i} C.$$

Let $x \in V \setminus A_i$. We note the following properties:

$$\begin{cases} \mathcal{A}|u - u_i|(x) \leqslant 2^{-i}, & \text{by the definition of } A_i, \\ \mathcal{A}\big|u_i - u_i(x)\big|(x) = 0, & \text{by the continuity of } u_i. \end{cases}$$

Thus, by the triangle inequality, we see

$$\limsup_{r\to 0} \big|u_{B(x,r)} - u_i(x)\big| \leqslant 2^{-i}.$$

Denote the sets

$$E_k := D \cup \bigcup_{i=k}^{\infty} A_i \quad \text{and} \quad E := \bigcap_{k=1}^{\infty} E_k.$$

We have subadditivity of the capacity per Theorem 3.2 and as per earlier in this proof we have

$$C_{p(\cdot)}(D) = 0 \quad \text{and} \quad C_{p(\cdot)}(A_i) \leqslant 2^{-i} C,$$

so we deduce

$$\begin{cases} C_{p(\cdot)}(E_k) \leqslant \displaystyle\sum_{i=k}^{\infty} 2^{-i} C \to 0 \quad \text{as } k \to \infty, \\ \displaystyle\lim_{k\to\infty} C_{p(\cdot)}(E_k) = 0, \\ C_{p(\cdot)}(E) \leqslant \displaystyle\lim_{k\to\infty} C_{p(\cdot)}(E_k) = 0. \end{cases}$$

Also, if we take $x \in V \setminus E_k$ and $i, j \geqslant k$, we may see

$$\big|u_i(x) - u_j(x)\big| \leqslant 2^{-i} + 2^{-j}.$$

Thus the sequence of smooth functions u_i converges uniformly to a continuous function v in $V \setminus E_k$. We see that

$$v = \lim_{r\to 0} \fint_{B(x,r)} u\,dy \quad \text{in } V \setminus E_k.$$

We deduce that u^* exists everywhere in $V \setminus E$.

We pass from local to global by the subadditivity of the capacity, see Theorem 3.2. We know that $\mathbb{R}^n$ is the union of countably many balls $B(0, R)$. Above we have proved the local version which is valid for each $B(0, R)$. The global exceptional set is the union of countably many sets of zero capacity. The global version now follows.

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Let us next establish the claim of

$$\lim_{r\to 0} \fint_{B(x,r)} \left|u(y) - u^*(x)\right|^{p^*(x)} dy = 0$$

quasieverywhere in $\mathbb{R}^n$. We cover the whole space with countably many balls $B(0,R)$ and from now on, we denote by E the union of the local exceptional sets corresponding to the balls $B(0,R)$. Thus E is a global exceptional set where u^* possibly does not exist. Let $x \in \mathbb{R}^n \setminus E$. Obviously $x \in B(0,R)$ for some $R > 0$. Thus $x \notin D$ for the D corresponding to the ball $B(0,R)$ and hence

$$\lim_{r\to 0} \fint_{B(x,r)} \left|u(y) - u_{B(x,r)}\right|^{p^*(y)} dy = 0.$$

Note that due to the convergence $u_{B(x,r)} \to u^*(x)$ and $p^*(y) \geqslant 1$, for all r small enough we have

$$\left|u_{B(x,r)} - u^*(x)\right|^{p^*(\cdot)} \leqslant \left|u_{B(x,r)} - u^*(x)\right|.$$

Now we see

$$\begin{aligned}\lim_{r\to 0} \fint_{B(x,r)} \left|u_{B(x,r)} - u^*(x)\right|^{p^*(y)} dy &\leqslant \lim_{r\to 0} \fint_{B(x,r)} \left|u_{B(x,r)} - u^*(x)\right| dy \\ &= \lim_{r\to 0} \left|u_{B(x,r)} - u^*(x)\right| \\ &= 0.\end{aligned}$$

The claim follows by observing

$$\lim_{r\to 0} \fint_{B(x,r)} \left|u(y) - u^*(x)\right|^{p^*(y)} dy \leqslant C \lim_{r\to 0} \fint_{B(x,r)} \left|u(y) - u_{B(x,r)}\right|^{p^*(y)} dy + C \lim_{r\to 0} \fint_{B(x,r)} \left|u_{B(x,r)} - u^*(x)\right|^{p^*(y)} dy.$$

Let us finally establish the quasicontinuity of u^*. Choose any $R > 0$. We follow the local results in the first part of the proof. Let $\varepsilon > 0$. By convergence to zero we choose k with

$$C_{p(\cdot)}(E_k) < \frac{\varepsilon}{2}.$$

The Sobolev capacity is an outer capacity, so there exists an open set U with

$$E_k \subset U \quad \text{and} \quad C_{p(\cdot)}(U) < \varepsilon.$$

We know that the approximant sequence u_i converges uniformly to u^* outside E_k with respect to $B(0,R)$, so u^* is continuous in $B(0,R) \setminus U$. Quasicontinuity is a local property, so the claim follows for the whole $\mathbb{R}^n$. $\square$

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