J. Math. Anal. Appl. 415 (2014) 686–712

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Liouville type results and a maximum principle for non-linear differential operators on the Heisenberg group

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ARTICLE INFO

Article history:
Received 7 January 2014
Available online 5 February 2014
Submitted by C. Gutierrez

Keywords:
Liouville theorem
Keller–Osserman
Heisenberg group
Non-linear differential inequalities

ABSTRACT

We prove Liouville type results for non-negative solutions of the differential inequality $\Delta_\varphi u \geqslant f(u)\ell(|\nabla_0 u|)$ on the Heisenberg group under a generalized Keller–Osserman condition. The operator $\Delta_\varphi u$ is the φ-Laplacian defined by $\operatorname{div}_0(|\nabla_0 u|^{-1}\varphi(|\nabla_0 u|)\nabla_0 u)$ and φ, f and ℓ satisfy mild structural conditions. In particular, ℓ is allowed to vanish at the origin. A key tool that can be of independent interest is a strong maximum principle for solutions of such differential inequality.

1. Introduction

Liouville type theorems for non-linear differential operators on Carnot groups have been considered by many authors in the last twenty years. Most of the results are related to the so-called non-coercive case such as, for example, the Yamabe equation. See e.g. [14,1,5,3,7,8] and the references therein. The coercive case has been considered only recently by [7,8,17,4].

In this paper we consider the non-linear differential operator on the Heisenberg group

$$\Delta_\varphi u = \operatorname{div}_0\left(\frac{\varphi(|\nabla_0 u|)}{|\nabla_0 u|}\nabla_0 u\right)$$

where div_0 and ∇_0 denote the horizontal divergence and the horizontal gradient respectively (see later for the relevant definitions). Under suitable assumptions we will show that non-negative entire solutions of the differential inequality

$$\Delta_\varphi u \geqslant f(u)\ell\big(|\nabla_0 u|\big) \tag{1}$$

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http://dx.doi.org/10.1016/j.jmaa.2014.01.087
0022-247X/

are constant. Similar results have been proved also in [17] and [4] under the quite restrictive assumption that the gradient term in the RHS be bounded away from zero. The method used in [17] and [4] and also in this paper has its origins in the techniques introduced by Keller in [16] and Osserman in [19] for the standard Laplacian in $\mathbb{R}^n$ and it is essentially based on the "explicit" construction of a radial supersolution that blows up in finite time and on a comparison argument. When one tries to adapt this argument to Carnot groups, further difficulties arise. In particular, a very technical and tricky point in the comparison argument requires that the gradient of a radial supersolution and the gradient of the solution do not vanish at the same point. In the Euclidean case this is automatic since the gradient of the Euclidean distance never vanishes. In the setting of Carnot groups, however, this is no longer true since the horizontal gradient of the distance may vanish. This difficulty was overcome in [4] assuming $\ell(0) > 0$ whereas in [17] Theorem 1.1 is not correctly stated. In this paper we perform a deep analysis of the set where the horizontal gradient of the solution may vanish in order to avoid such a strong assumption. Among the tools we use is a maximum principle for non-negative solutions of (1), valid under mild assumptions on φ, f, ℓ that can be of independent interest. See Section 4 for a comparison with previous similar results.

For a detailed analysis of existence and non-existence of positive solutions of equations of the kind (1) under various assumptions on the gradient term ℓ in the Euclidean setting we refer the reader to [11].

Let $\mathbb{H}^m$ be the Heisenberg group, i.e. the Lie group with underlying manifold $\mathbb{C}^m \times \mathbb{R} \simeq \mathbb{R}^{2m+1}$ and group structure defined by

$$(z,t) \cdot (z',t') = \left(z + z', t + t' + 2\operatorname{Im}\left(z \cdot \overline{z'}\right)\right)$$

and set

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

where $z_j = x_j + iy_j$, for $j = 1, \dots, m$. The vector fields $\{X_j, Y_j\}$ are left-invariant and their linear span is called the horizontal bundle of $\mathbb{H}^m$. Since

$$[X_j, Y_k] = -4\delta_{jk} T,$$

Hörmander's condition is satisfied and the canonical sub-Laplacian, defined by

$$\Delta_{\mathbb{H}^m} = \sum_{j=1}^{m} \left(X_j^2 + Y_j^2\right),$$

is hypoelliptic by Hörmander's theorem (see [15]). We refer the interested reader to [21] for a detailed introduction to the Heisenberg group. See also the book [2] for an in-depth treatment of general Carnot groups and sub-Laplacians.

A distinguished homogeneous norm can be defined, through the fundamental solution Γ of the sub-Laplacian, by

$$\|q\| = \left\|(z,t)\right\| = \Gamma(z,t)^{\frac{1}{2m}} = \left(|z|^4 + t^2\right)^{\frac{1}{4}}.$$

This norm is homogeneous of degree 1 with respect to the natural dilations $\delta_\lambda : (z,t) \mapsto (\lambda z, \lambda^2 t)$, $\lambda > 0$ and gives rise to the Korányi distance, defined by

$$d(q,q') = \left\|q^{-1} \cdot q'\right\|, \quad q, q' \in \mathbb{H}^m.$$

In addition to the Korányi distance, we shall equip the Heisenberg group with two more distance functions: the Euclidean distance and the metric d_∞ induced by the homogeneous norm

$$\left|(z,t)\right|_\infty = \max\left\{|z|, |t|^{\frac{1}{2}}\right\}.$$

The latter metric is trivially homogeneous of degree 1 with respect to the dilations δ_λ and, as we shall see, it turns out to be a somewhat natural choice for what concerns the Hausdorff measure and dimension of hypersurfaces in the Heisenberg group. By virtue of this fact, all the Hausdorff measures and dimensions on $\mathbb{H}^m$ will be meant with respect to the above metric unless otherwise specified. When both metrics are taken into consideration, we denote by $\dim_E$ (resp. $\dim_\infty$) the Hausdorff dimension with respect to the Euclidean metric (resp. the metric d_∞).

If $\Omega \subseteq \mathbb{H}^m$ is open and $u : \Omega \to \mathbb{R}$, we define the horizontal gradient of u as the horizontal vector field

$$\nabla_0 u = \sum_{j=1}^m (X_j u)X_j + (Y_j u)Y_j$$

and we say that $u \in C^1_H(\Omega)$ if its horizontal gradient is defined and continuous in Ω (see [12, Section 5] for further details). For $k \in \mathbb{N}$ we likewise define the classes $C^k_H(\Omega)$.

For horizontal vector fields $W = \sum(w_j X_j + \tilde{w}_j Y_j)$ and $Z = \sum(z_j X_j + \tilde{z}_j Y_j)$ we define

$$W \cdot Z = \sum_{j=1}^m w_j z_j + \tilde{w}_j \tilde{z}_j$$

and

$$\operatorname{div}_0 W = \sum_{j=1}^m X_j(w_j) + Y_j(\tilde{w}_j),$$

so that, by definition,

$$|\nabla_0 u|^2 = \nabla_0 u \cdot \nabla_0 u$$

and

$$\Delta_{\mathbb{H}^m} u = \operatorname{div}_0 \nabla_0 u.$$

In what follows, we will use radial functions on the Heisenberg group defined by means of the Korányi distance d. As we have already observed, an unpleasant fact about d is that its horizontal gradient vanishes over the line $z = 0$. Indeed,

$$\left|\nabla_0 d(z,t)\right| = \frac{|z|}{d(z,t)}.$$

2. Definitions and assumptions

In this paper we will consider the inequality

$$\Delta_\varphi u \geqslant f(u)\ell\left(|\nabla_0 u|\right) \tag{2}$$

where Δ_φ is the φ-Laplace operator, defined by

$$\Delta_\varphi u = \operatorname{div}_0\left(\frac{\varphi(|\nabla_0 u|)}{|\nabla_0 u|}\nabla_0 u\right),$$

with φ satisfying the following structural conditions:

$$\begin{cases} \varphi \in C^0\big([0,+\infty)\big) \cap C^1\big((0,+\infty)\big), \quad \varphi(0) = 0, \\ \varphi' > 0 \quad \text{on } (0,+\infty). \end{cases} \tag{3}$$

The φ-Laplace operator is a non-linear operator that generalizes the well-known p-Laplace operator, defined, for $p > 1$, as

$$\Delta_p u = \operatorname{div}_0\big(|\nabla_0 u|^{p-2}\nabla_0 u\big).$$

Both these operators are straightforward generalizations of their equivalents in the Euclidean setting and have recently been studied in the context of the Heisenberg group and, more in general, of Carnot groups (see e.g. [4] and the references therein).

The functions f and ℓ on the RHS of (2) are required to satisfy the following structural conditions:

$$\begin{cases} f \in C^0\big([0,+\infty)\big), \quad f > 0 \text{ on } (0,+\infty); \\ f \text{ is increasing on } [0,+\infty); \end{cases} \tag{4}$$

$$\begin{cases} \ell \in C^0\big([0,+\infty)\big), \quad \ell > 0 \text{ on } (0,+\infty); \\ \ell \text{ is } B\text{-monotone non-decreasing on } [0,+\infty). \end{cases} \tag{5}$$

We recall that ℓ is said to be *B-monotone non-decreasing* on $[0,+\infty)$ if, for some $B \geqslant 1$,

$$\sup_{s\in[0,t]} \ell(s) \leqslant B\ell(t), \quad \forall t \in [0,+\infty).$$

Clearly, if ℓ is monotone non-decreasing on $[0,+\infty)$, then it is 1-monotone non-decreasing on the same set. We stress that conditions (5) do allow ℓ to vanish at the origin, while this is not the case in [17] and [4].

We will prove that, under the above structural conditions and a few more assumptions on φ and ℓ, inequality (2) has no non-negative, non-constant, entire solutions if and only if φ, f and ℓ satisfy a certain request of integrability at infinity known as the Keller–Osserman condition.

First of all, let us establish that, by an entire solution to inequality (2), we mean a function $u : \mathbb{H}^m \to \mathbb{R}$ such that $u \in C^1_H(\mathbb{H}^m)$ and u satisfies the inequality in the weak sense, that is, for every $\zeta \in C_0^\infty(\mathbb{H}^m)$, $\zeta \geqslant 0$,

$$-\int\limits_{\mathbb{R}^{2m+1}} |\nabla_0 u|^{-1}\varphi\big(|\nabla_0 u|\big)\nabla_0 u \cdot \nabla_0 \zeta \geqslant \int\limits_{\mathbb{R}^{2m+1}} f(u)l\big(|\nabla_0 u|\big)\zeta.$$

In order to express the generalized Keller–Osserman condition, from now on we shall assume that

$$\frac{t\varphi'(t)}{\ell(t)} \in L^1\big(0^+\big) \setminus L^1(+\infty) \tag{6}$$

and

$$\lim_{t\to 0^+} \frac{\varphi(t)}{\ell(t)} = 0. \tag{7}$$

Note that often (e.g. in the case of the p-Laplacian) the latter condition directly assures integrability at 0^+ in the former. We define

$$K(t)=\int_0^t \frac{s\varphi'(s)}{\ell(s)}\,ds; \tag{8}$$

observe that $K:[0,+\infty)\to[0,+\infty)$ is a C^1-diffeomorphism with

$$K'(t)=\frac{t\varphi'(t)}{\ell(t)}>0 \quad \text{for } t>0,$$

thus the existence of the increasing inverse $K^{-1}:[0,+\infty)\to[0,+\infty)$. Finally we set

$$F(t)=\int_0^t f(s)\,ds.$$

Definition 2.1. The ***generalized Keller–Osserman condition*** for inequality

$$\Delta_\varphi u \geqslant f(u)\ell\big(|\nabla_0 u|\big)$$

is the request:

$$\frac{1}{K^{-1}(F(t))}\in L^1(+\infty). \tag{9}$$

In order to deal with the fact that $|\nabla_0 d|$ is not constant and vanishes on a line, we need to assume one further request binding φ and ℓ together, i.e. there exists $C\geqslant 1$ such that

$$\frac{s^2\varphi'(st)}{\ell(st)}\leqslant C\frac{\varphi'(t)}{\ell(t)}, \quad \forall s\in[0,1],\ t\in[0,+\infty). \tag{10}$$

We stress that this is a mild requirement and, since it replaces the more demanding set of homogeneity assumptions present in both [17] and [4], it allows to improve the Liouville-type result of [17], which can now be stated for a wider class of differential operators. Nonetheless, it should be mentioned that the pairing of (10) and the B-monotonicity of ℓ does affect the class of admissible functions φ, indeed the two conditions combined together imply that

$$s^2\varphi'(st)\leqslant BC\varphi'(t)$$

for every $s\in[0,1]$ and for every $t\in[0,+\infty)$, which in turn implies that φ' must be bounded below by t^{-2} for large t.

We also observe that (10) does not automatically imply the integrability at 0^+ in (6). For instance if $\varphi(t)=t^{p-1}$ and $\ell(t)=t^p$, then (10) is satisfied, but $\frac{t\varphi'(t)}{\ell(t)}\notin L^1(0^+)$.

Under the previous assumptions, we shall prove that, if the Keller–Osserman condition is satisfied, inequality (2) has no non-constant, non-negative entire solutions that are C^1_H on the whole $\mathbb{H}^m$ and C^2_H on $\mathbb{H}^m$ except for a vertical line (see Theorem 5.1). Moreover, the result is essentially sharp in the following sense: under the structural conditions (3), (4), (5), (7), (10) and a slightly stronger version of (6), if the Keller–Osserman condition (9) is not satisfied, then (2) admits non-negative, non-constant entire solutions of class C^1_H on the whole $\mathbb{H}^m$ and C^2_H on $\mathbb{H}^m$ save for a vertical line, as shown in Theorem 6.1.

In order to clarify the meaning of the assumptions stated above, we shall briefly focus on the special case when the functions involved are powers, and see what these assumptions translate into. To this end, let $p > 1$ and $\theta \geqslant 0$ and set $\varphi(t) = t^{p-1}$ and $\ell(t) = t^\theta$, i.e. the differential inequality (2) becomes

$$\Delta_p u \geqslant f(u)|\nabla_0 u|^\theta.$$

It is immediate to see that assumptions (3) and (5) are always satisfied for these choices of φ and ℓ, while (6) and (7) become the request $\theta < p - 1$. The Keller–Osserman condition (9) translates into the request that

$$\big[F(t)\big]^{-\frac{1}{p-\theta}} \in L^1(+\infty),$$

which generalizes the usual Keller–Osserman condition for the p-Laplacian (see, for instance, [18]). Finally, assumption (10) becomes $\theta \leqslant p$, so that ultimately, in the very special case of powers, the whole set of assumptions (save for the Keller–Osserman condition) is satisfied for $0 \leqslant \theta < p - 1$.

It is worth mentioning that, in [9], the authors consider Liouville-type results for coercive elliptic equations and inequalities in the Euclidean space and, in doing so, they find a condition for the non-existence of solutions that resembles (and actually slightly improves) the one obtained above for the special case of the p-Laplacian.

We end the section remarking that there are many examples of differential operators other than the p-Laplacian that can be considered in this setting. Such operators include for instance generalizations of the mean curvature operator of the kind

$$Lu = \operatorname{div}_0\left(\frac{|\nabla_0 u|^{p-1}}{(1 + |\nabla_0 u|^2)^{q/2}} \nabla_0 u\right)$$

with $p > 1$ and $q < p$ (see e.g. [7] or [18]), operators of the kind

$$Lu = \operatorname{div}_0\big(\big(|\nabla_0 u|^{p-2} + |\nabla_0 u|^{q-2}\big)\nabla_0 u\big)$$

with $p, q > 1$ (see e.g. [20]) or

$$Lu = \operatorname{div}_0\left(\frac{\sinh(|\nabla_0 u|)}{|\nabla_0 u|} \nabla_0 u\right)$$

that for suitable choices of the gradient term ℓ satisfy the structural conditions (3), (5), (6), (7) and (10).

3. Radial supersolutions

A key tool in the proof of the Liouville-type theorem is the construction of an appropriate radial supersolution, i.e. a function $v = \alpha \circ d$ satisfying

$$\Delta_\varphi v \leqslant f(v)\ell\big(|\nabla_0 v|\big)$$

where $d(p) = d_q(p) = d(p, q)$ denotes the distance from a fixed point q. Keeping in mind the definition of the φ-Laplacian and the properties of the horizontal divergence, such as the following

$$\operatorname{div}_0(fW) = f \operatorname{div}_0 W + \nabla_0 f \cdot W,$$

together with the fact that

$$\Delta_{\mathbb{H}^m} d = |\nabla_0 d|^2 \frac{2m+1}{d},$$

some computation yields the following expression for the φ-Laplacian of a radial function v:

$$\Delta_\varphi v = \varphi'\big(\alpha'|\nabla_0 d|\big)\alpha''|\nabla_0 d|^2 + \varphi\big(\alpha'|\nabla_0 d|\big)|\nabla_0 d|\frac{2m+1}{d} \tag{11}$$

where, for ease of notation, we have assumed α increasing. As we shall see, this is not restrictive for our purposes.

Lemma 3.1. *Let $\sigma \in (0,1]$; then the generalized Keller–Osserman condition (9) implies*

$$\frac{1}{K^{-1}(\sigma F(t))} \in L^1(+\infty). \tag{12}$$

The proof of this lemma is achieved through a change of variable. For the details, we refer the reader to [17].

We pass now to the construction of radial supersolutions of (2). The next proposition will be used to prove our Liouville-type result in $\mathbb{H}^1$, while the subsequent proposition will be needed in $\mathbb{H}^m$ for $m > 1$.

Proposition 3.2. *Assume the validity of (3), (4), (5), (6), (7) and (10). Fix $0 < t_0 < t_1$, $0 < \varepsilon < \eta < A$, where A may possibly be equal to $+\infty$ if (9) holds. Let $h_1, h_2 : [t_0, +\infty) \to \mathbb{R}$ be C^β functions and let E be a subset of $\mathbb{R}$ such that $\dim(E) < \beta$, for some $\beta \leqslant 1$. Then there exist $T > t_1$ and a strictly increasing and convex function $\alpha \in C^2([t_0, T))$ such that for every $q \in \mathbb{H}^1$ the radial function $v = \alpha \circ d_q$ satisfies*

$$\begin{cases} \Delta_\varphi v \leqslant f(v)\ell\big(|\nabla_0 v|\big) & \text{on } B_T(q) \setminus B_{t_0}(q), \\ v = \varepsilon & \text{on } \partial B_{t_0}(q), \\ v = A & \text{on } \partial B_T(q), \\ \varepsilon \leqslant v \leqslant \eta & \text{on } B_{t_1}(q) \setminus B_{t_0}(q). \end{cases}$$

Moreover, for every $t \in E \cap [t_0, T]$,

$$\alpha'\big(t^{\frac{1}{2}}\big) \neq h_1(t) \quad \text{and} \quad \alpha'\big(t^{\frac{1}{2}}\big) \neq h_2(t). \tag{13}$$

Proposition 3.3. *Let $m > 1$. Assume the validity of (3), (4), (5), (6), (7) and (10). Fix $0 < t_0 < t_1$, $0 < \varepsilon < \eta < A$, where A may possibly be equal to $+\infty$ if (9) holds. Let $h_1, h_2 : [t_0, +\infty) \to \mathbb{R}$ and let E be an at most countable subset of $\mathbb{R}$. Then there exist $T > t_1$ and a strictly increasing and convex function $\alpha \in C^2([t_0, T))$ such that for every $q \in \mathbb{H}^m$ the same conclusions of Proposition 3.2 hold.*

Proof of Propositions 3.2 and 3.3. Consider $\sigma \in (0,1]$ to be determined later and let $T_\sigma > t_0$ be such that

$$T_\sigma - t_0 = \int_\varepsilon^A \frac{ds}{K^{-1}(\sigma F(s))}.$$

Note that, when $A = +\infty$ and (9) holds, the RHS is well defined by Lemma 3.1. Moreover, since the RHS diverges as $\sigma \to 0^+$, up to choosing σ sufficiently small we can shift T_σ in such a way that $T_\sigma > t_1$. We implicitly define the C^2 function $\alpha_\sigma(t)$ by requiring

$$T_\sigma - t = \int_{\alpha_\sigma(t)}^A \frac{ds}{K^{-1}(\sigma F(s))} \quad \text{on } [t_0, T_\sigma).$$

We observe that, by construction, $\alpha_\sigma(t_0) = \varepsilon$ and, since $K^{-1} > 0$, $\alpha_\sigma(t) \uparrow A$ as $t \to T_\sigma$. A first differentiation yields

$$\frac{\alpha'_\sigma}{K^{-1}(\sigma F(\alpha_\sigma))} = 1,$$

hence α_σ is monotone increasing and $\sigma F(\alpha_\sigma) = K(\alpha'_\sigma)$. Differentiating once more we deduce

$$\sigma f(\alpha_\sigma)\alpha'_\sigma = K'(\alpha'_\sigma)\alpha''_\sigma = \frac{\alpha'_\sigma \varphi'(\alpha'_\sigma)}{\ell(\alpha'_\sigma)}\alpha''_\sigma.$$

Canceling α'_σ throughout, we obtain

$$\varphi'(\alpha'_\sigma)\alpha''_\sigma = \sigma f(\alpha_\sigma)\ell(\alpha'_\sigma). \tag{14}$$

Now we set $v = \alpha_\sigma \circ d_q$ and observe that v is a C^2_H radial function on $B_{T_\sigma} \setminus B_{t_0}$ whose φ-Laplacian has the expression (11). We will show that we can find a value of σ (independent of q) such that v is a supersolution, using the properties of α_σ that we discussed above and (10). In the following computation we omit the subscript σ in α_σ and rename $|\nabla_0 d| = s$ for ease of notation. By (11)

$$\frac{\Delta_\varphi v}{f(v)\ell(|\nabla_0 v|)} = \frac{s^2\varphi'(s\alpha')\alpha''}{f(\alpha)\ell(s\alpha')} + \frac{s\varphi(s\alpha')}{f(\alpha)\ell(s\alpha')}\frac{2m+1}{d}. \tag{15}$$

We can estimate the first term on the RHS of (15) using (10) and (14):

$$\frac{s^2\varphi'(s\alpha')\alpha''}{f(\alpha)\ell(s\alpha')} \leqslant C\frac{\varphi'(\alpha')\alpha''}{f(\alpha)\ell(\alpha')} = C\sigma.$$

As for the second term, we first observe that

$$\varphi(s\alpha') = \varphi(s\alpha'(t_0)) + \int_{t_0}^{d} \big[\varphi(s\alpha'(u))\big]'\,du = \varphi(s\alpha'(t_0)) + \int_{t_0}^{d} s\varphi'(s\alpha'(u))\alpha''(u)\,du,$$

so that

$$\frac{s\varphi(s\alpha')}{f(\alpha)\ell(s\alpha')}\frac{2m+1}{d} = \frac{2m+1}{d}\left(\frac{s\varphi(s\alpha'(t_0))}{f(\alpha)\ell(s\alpha')} + \frac{\int_{t_0}^{d} s^2\varphi'(s\alpha'(u))\alpha''(u)\,du}{f(\alpha)\ell(s\alpha')}\right).$$

Now the first term of this can be estimated using (4) and (5) by

$$\frac{2m+1}{d}\frac{s\varphi(s\alpha'(t_0))}{f(\alpha)\ell(s\alpha')} \leqslant \frac{2m+1}{t_0}\frac{B\varphi(s\alpha'(t_0))}{f(\alpha(t_0))\ell(s\alpha'(t_0))}$$

and since $K(0) = 0$, $\alpha_\sigma(t_0) = \varepsilon$ and $\alpha'_\sigma(t_0) = K^{-1}(\sigma F(\varepsilon)) \to 0$ as $\sigma \to 0$, we can use (7) to deduce that this term goes to 0 as $\sigma \to 0$. As for the last term, using (10) we have that

$$\begin{aligned}\frac{2m+1}{d}\frac{\int_{t_0}^{d} s^2\varphi'(s\alpha'(u))\alpha''(u)\,du}{f(\alpha)\ell(s\alpha')} &\leqslant \frac{2m+1}{f(\alpha)\ell(s\alpha')d}\int_{t_0}^{d} C\varphi'(\alpha'(u))\alpha''(u)\frac{\ell(s\alpha'(u))}{\ell(\alpha'(u))}\,du \\ &= \frac{2m+1}{f(\alpha)\ell(s\alpha')d}\int_{t_0}^{d} C\sigma f(\alpha(u))\ell(s\alpha'(u))\,du\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{2m+1}{f(\alpha)\ell(s\alpha')d}(d-t_0)C\sigma f(\alpha)B\ell\big(s\alpha'\big)\\&\leqslant \sigma(2m+1)CB.\end{aligned}$$

Putting these estimates together, we find that

$$\Delta_\varphi v \leqslant \tilde{C}\left\{\sigma + \frac{2m+1}{t_0}\frac{B\varphi(\alpha'_\sigma(t_0)|\nabla_0 d|)}{f(\alpha_\sigma(t_0))\ell(\alpha'_\sigma(t_0)|\nabla_0 d|)}\right\} f(v)\ell\big(|\nabla_0 v|\big)$$

and the whole bracket tends to 0 as $\sigma \to 0$ whence, for every σ in a neighborhood of 0, the radial function v is indeed a supersolution.

We still need to show that, possibly with a further reduction of σ, $\alpha_\sigma(t_1) \leqslant \eta$. From the trivial identity

$$\int_{\alpha_\sigma(t_1)}^{A} \frac{ds}{K^{-1}(\sigma F(s))} = T_\sigma - t_1 = (T_\sigma - t_0) + (t_0 - t_1) = \int_{\varepsilon}^{A} \frac{ds}{K^{-1}(\sigma F(s))} + (t_0 - t_1)$$

we deduce

$$\int_{\varepsilon}^{\alpha_\sigma(t_1)} \frac{ds}{K^{-1}(\sigma F(s))} = t_1 - t_0.$$

It suffices to choose σ such that $\int_\varepsilon^\eta \frac{ds}{K^{-1}(\sigma F(s))} > t_1 - t_0$; then obviously $\alpha_\sigma(t_1) < \eta$.

Now, to prove Proposition 3.2, the only thing left to show is that we can choose a value of σ such that the function α_σ satisfies all of the above requirements and the condition

$$\alpha'_\sigma\big(t^{\frac{1}{2}}\big) \neq h_1(t) \quad \text{and} \quad \alpha'_\sigma\big(t^{\frac{1}{2}}\big) \neq h_2(t)$$

for every $t \in E \cap [t_0, T_\sigma]$. Since the other requirements are satisfied for σ in a sufficiently small neighborhood of 0, it suffices to show that there exists at least a sequence of values of σ converging to 0 for which the latter condition holds for every $t \in E$. To this end, we observe that if $\sigma_1 < \sigma_2$, then

$$\frac{1}{K^{-1}(\sigma_1 F(s))} > \frac{1}{K^{-1}(\sigma_2 F(s))}$$

and, since

$$\int_{\varepsilon}^{\alpha_{\sigma_1}(t)} \frac{ds}{K^{-1}(\sigma_1 F(s))} = \int_{\varepsilon}^{\alpha_{\sigma_2}(t)} \frac{ds}{K^{-1}(\sigma_2 F(s))} = t - t_0,$$

this yields $\alpha_{\sigma_1} < \alpha_{\sigma_2}$. Since moreover

$$\alpha'_\sigma(t) = K^{-1}\big(\sigma F\big(\alpha_\sigma(t)\big)\big),$$

we have that

$$\alpha'_{\sigma_1}(t) = K^{-1}\big(\sigma_1 F\big(\alpha_{\sigma_1}(t)\big)\big) < K^{-1}\big(\sigma_2 F\big(\alpha_{\sigma_2}(t)\big)\big) = \alpha'_{\sigma_2}(t),$$

that is, α'_σ is strictly increasing with respect to σ.

Because of this, we can deduce that, for every fixed t, the equation

$$\alpha'_\sigma\bigl(t^{\frac{1}{2}}\bigr) = h_i(t), \tag{16}$$

where we have generically denoted by h_i the functions h_1 or h_2, can be satisfied for at most one value of σ. We define a map Φ_i that associates, to every $t \in E$, the only value of σ (when it exists) such that (16) holds. We shall prove that the map Φ_i is of class C^β; in this way, since the image of an s-dimensional set under a C^β map has dimension at most $\frac{s}{\beta}$ (see Proposition A.1 in Appendix A for the details), we can deduce that the set

$$\Sigma_i = \bigl\{\sigma\colon\ \exists t \in E \text{ such that } \alpha'_\sigma\bigl(t^{\frac{1}{2}}\bigr) = h_i(t)\bigr\}$$

has Hausdorff dimension strictly less than 1 and therefore has Lebesgue measure 0. The same is obviously also true for

$$\Sigma_1 \cup \Sigma_2 = \bigl\{\sigma\colon\ \exists t \in E \text{ such that } \alpha'_\sigma\bigl(t^{\frac{1}{2}}\bigr) = h_1(t) \text{ or } \alpha'_\sigma\bigl(t^{\frac{1}{2}}\bigr) = h_2(t)\bigr\},$$

which thus cannot contain any open interval.

Now, to show that Φ_i is Hölder continuous, define

$$G(\sigma, u, t) = \int_u^A \frac{ds}{K^{-1}(\sigma F(s))} - (T_\sigma - t)$$

and observe that

$$\frac{\partial G}{\partial u} = \frac{-1}{K^{-1}(\sigma F(u))} \neq 0,$$

so that G defines an implicit function $u = \alpha_\sigma(t)$ which is at least of class C^2 with respect to σ and t. In particular, α'_σ is of class C^1 and, as we have seen above, it is strictly increasing with respect to σ. By virtue of this, we can also apply the implicit function theorem to the equality

$$\alpha'_\sigma\bigl(t^{\frac{1}{2}}\bigr) = u$$

and solve the latter equality for σ:

$$\sigma = g(u, t)$$

for some function g which is of class C^1 with respect to u and even t, provided t stays away from 0.

Therefore, the function Φ_i, which is defined by the equality

$$\alpha'_\sigma\bigl(t^{\frac{1}{2}}\bigr) = h_i(t),$$

can now be made explicit as follows

$$\Phi_i(t) = g\bigl(h_i(t), t\bigr),$$

which is trivially of class C^β because of the fact that h_i is C^β by assumption and provided that t stays away from 0, which is not restrictive in our case.

Therefore, fixing a value for σ which satisfies all the above requirements and renaming T and α the corresponding T_σ and α_σ, we have proved the claim.

To complete the proof of Proposition 3.3, define again the map Φ_i as the map that associates to a point $t \in E$ the only value of σ (if it exists) such that

$$\alpha'_\sigma\big(t^{\frac{1}{2}}\big) = h_i(t)$$

and observe that the set of values of σ that we want to avoid is just the union of the images of Φ_1 and of Φ_2, which is at most countable. Therefore, in every neighborhood of 0 we can always find uncountably many values of σ that satisfy the required condition. Picking one of these values sufficiently close to 0 gives rise to the required value of T and function α, and the claim is proved. $\square$

In the next propositions we shall construct a supersolution to (2) which will be needed in the proof of the maximum principle (Theorem 4.3), in the case of $\mathbb{H}^1$ and $\mathbb{H}^m$ for $m > 1$ respectively.

Proposition 3.4. *Assume the validity of (3), (4), (5), (7) and fix $0 < t_0 < t_1$, $0 < k_0 < k_1$. Let E be a subset of $[t_0, t_1]$ such that $\dim(E) < \beta$ and let $h_1, h_2 : [t_0, t_1] \to \mathbb{R}$ be C^β functions for some $\beta \leqslant 1$. Then there exists $\sigma_0 > 0$ such that for a.e. value of $0 < \sigma \leqslant \sigma_0$ and for every $q \in \mathbb{H}^1$, the function $v = \alpha \circ d_q$, with $\alpha(t) = \sigma(t - t_1) + k_1$ satisfies*

$$\begin{cases} \Delta_\varphi v \leqslant f(v)\ell\big(|\nabla_0 v|\big) & \text{in } \overline{B_{t_1}(q)} \setminus B_{t_0}(q), \\ v \geqslant k_0 & \text{on } \partial B_{t_0}(q), \\ v = k_1 & \text{on } \partial B_{t_1}(q) \end{cases}$$

and, for every $t \in E$,

$$\alpha'\big(t^{\frac{1}{2}}\big) \neq h_1(t) \quad \text{and} \quad \alpha'\big(t^{\frac{1}{2}}\big) \neq h_2(t). \tag{17}$$

Proposition 3.5. *Let $m > 1$. Assume the validity of (3), (4), (5), (7) and fix $0 < t_0 < t_1$, $0 < k_0 < k_1$. Let E be an at most countable subset of $[t_0, t_1]$ and let $h_1, h_2 : [t_0, t_1] \to \mathbb{R}$. Then there exists $\sigma_0 > 0$ such that for every $0 < \sigma \leqslant \sigma_0$, save for at most a countable subset and for every $q \in \mathbb{H}^m$ the same conclusions of Proposition 3.4 hold.*

Proof of Propositions 3.4 and 3.5. Consider $\sigma \in (0, 1]$ to be determined later and set

$$\alpha_\sigma(t) = \sigma(t - t_1) + k_1.$$

Then obviously $\alpha_\sigma(t_1) = k_1$ for every σ, while $\alpha_\sigma(t_0) \geqslant k_0$ for $\sigma \leqslant \frac{k_1 - k_0}{t_1 - t_0}$.

Since $\alpha'_\sigma = \sigma$ and $\alpha''_\sigma = 0$, having set $v = \alpha_\sigma \circ d$, we have that

$$\begin{aligned} \Delta_\varphi v &= \varphi'\big(\alpha'_\sigma\big)\alpha''_\sigma |\nabla_0 d|^2 + \varphi\big(\alpha'_\sigma |\nabla_0 d|\big)|\nabla_0 d| \frac{2m+1}{d} \\ &= \varphi\big(\sigma|\nabla_0 d|\big)|\nabla_0 d| \frac{2m+1}{d}. \end{aligned}$$

Setting $|\nabla_0 d| = s$ for ease of notation, we need to show that

$$\varphi(\sigma s) s \frac{2m+1}{d} \leqslant f\big(\sigma(d - t_1) + k_1\big)\ell(\sigma s),$$

that is

$$\frac{\varphi(\sigma s)s}{\ell(\sigma s)} \leqslant \frac{d}{2m+1} f\big(\sigma(d-t_1)+k_1\big).$$

But, using (7), we know that for sufficiently small σ,

$$\frac{\varphi(\sigma s)}{\ell(\sigma s)} \leqslant \frac{t_0}{2m+1} f(k_0).$$

Therefore we have

$$\frac{\varphi(\sigma s)s}{\ell(\sigma s)} \leqslant \frac{\varphi(\sigma s)}{\ell(\sigma s)} \leqslant \frac{t_0}{2m+1} f(k_0) \leqslant \frac{d}{2m+1} f\big(\sigma(d-t_1)+k_1\big),$$

showing that v is a supersolution.

Again, the only thing left to show is that we can choose a value of σ such that the function α_σ satisfies all of the above requirements and the conditions

$$\alpha'_\sigma\big(t^{\frac{1}{2}}\big) \neq h_1(t) \quad \text{and} \quad \alpha'_\sigma\big(t^{\frac{1}{2}}\big) \neq h_2(t).$$

To this end we observe that, with our choice of α_σ, condition

$$\alpha'_\sigma\big(t^{\frac{1}{2}}\big) = h_1(t) \quad \text{or} \quad \alpha'_\sigma\big(t^{\frac{1}{2}}\big) = h_2(t) \tag{18}$$

becomes

$$\sigma = h_1(t) \quad \text{or} \quad \sigma = h_2(t)$$

and, since the other requests on α_σ are satisfied for every σ in a sufficiently small neighborhood of 0, what we need to show is that the union of the images of the set E under the maps h_i cannot contain a whole neighborhood of 0. But this is certainly true under the assumptions of both Proposition 3.4 and Proposition 3.5, indeed in the former case h_1 and h_2 are, by assumption, maps of class C^β and therefore

$$\dim\big(h_i(E)\big) \leqslant \frac{1}{\beta} \dim E < 1$$

(see Proposition A.1 in Appendix A for the details). Hence the set of values of σ for which (18) holds for every $t \in E$ has Hausdorff dimension strictly less than 1 and therefore is a null set with respect to the Lebesgue measure. In particular, it cannot contain any interval.

The latter case is even simpler, since we have that the set of inadmissible values of σ is at most countable, which trivially implies the claim.

Thus we can choose a value of σ that satisfies all of the previous requests and such that (18) cannot be satisfied for any value of $t \in E$. Having fixed such a value of σ, we rename α the corresponding α_σ and the claim is proved. □

4. Maximum principle

In this section we shall prove a maximum principle for non-negative solutions of (2). To prove such a result we will be interested in the set of non-stationary points where the horizontal gradient vanishes. In the next proposition we prove that such a set is small in a suitable sense.

Proposition 4.1. *Let $u \in C^2_H(\overline{\Omega})$ and consider the set*

$$C = \{p \in \Omega\colon\ X_j u(p) = Y_j u(p) = 0,\ \forall j = 1, \dots, m,\ Tu(p) \neq 0\},$$

and for $z_0 \in \mathbb{C}^m$ define $E_{z_0} = \{(z_0, t) \in C\}$. If $m = 1$, then for a.e. $z_0 \in \mathbb{C}$, E_{z_0} has Hausdorff dimension, with respect to the Euclidean metric,

$$\dim_E(E_{z_0}) \leqslant \frac{1}{2}.$$

If $m > 1$, then for a.e. $z_0 \in \mathbb{C}^m$, E_{z_0} is at most countable and discrete.

Proof. Set, for ease of notation, $Z_j = X_j$ and $Z_{j+m} = Y_j$ for $j = 1, \dots, m$. Let $p \in C$ and consider the matrix

$$A(p) = \begin{bmatrix} Z_1 Z_1 u(p) & \cdots & Z_1 Z_{2m} u(p) \\ \vdots & \ddots & \vdots \\ Z_{2m} Z_1 u(p) & \cdots & Z_{2m} Z_1 u(p) \end{bmatrix}.$$

Observe that due to the commutation relations

$$A(p) - A(p)^T = -4Tu(p) \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$$

where A^T denotes the transpose matrix and I_m the identity matrix. In particular since $Tu(p) \neq 0$, $A(p) - A(p)^T$ is a $2m$ non-singular matrix, so that

$$2m = \operatorname{rk}\bigl(A(p) - A(p)^T\bigr) \leqslant \operatorname{rk}\bigl(A(p)\bigr) + \operatorname{rk}\bigl(A(p)^T\bigr) = 2\operatorname{rk}\bigl(A(p)\bigr)$$

hence $\operatorname{rk}(A(p)) \geqslant m$. It follows that for every $p \in C$ there exist indices $j_1, j_2, \dots, j_m$ such that the horizontal vector fields $\nabla_0 Z_{j_k} u$ are linearly independent. Since the functions $Z_i Z_j u$ are continuous, this extends to a neighborhood $\mathcal{U}$ of p. Let

$$C_0 = \{p \in \mathcal{U}\colon\ Z_{j_k} u(p) = 0 \text{ for } k = 1, \dots, m\}.$$

Since in $\mathcal{U}$, $\nabla_0 Z_{j_1} u \wedge \cdots \wedge \nabla_0 Z_{j_m} u$ does not vanish, C_0 is an m-codimensional $\mathbb{H}$-regular surface according to the definition given in [13] and by Corollary 4.4 therein it is $\mathcal{H}^{m+2}$-measurable and has Hausdorff dimension equal to $m+2$. Thus C is also $\mathcal{H}^{m+2}$-measurable (since it is an F_σ set) and $\dim(C) \leqslant m+2$.

Moreover, by Theorem 4.1 in [13] and since the second derivatives of u are continuous on $\overline{\Omega}$, if B_R is any closed ball of radius R in $\mathbb{H}^m$, then we have

$$\mathcal{H}^{m+2}(C \cap B_R) < +\infty.$$

Let Z be the center of $\mathbb{H}^m$, that is

$$Z = \{(0, t) \in \mathbb{H}^m\colon\ t \in \mathbb{R}\},$$

and observe that the quotient group $\mathbb{H}^m / Z$ is just $\mathbb{C}^m$ and that the projection onto the quotient

$$\pi : \mathbb{H}^m \to \mathbb{C}^m,$$
$$(z,t) \mapsto z,$$

is trivially a Lipschitz map.

Assume now $m \geqslant 2$. Applying Eilenberg's inequality (see Theorem 2.10.25 of [10] or Theorem 13.3.1 of [6]), we find that

$$\int_{\mathbb{C}^m}^{*} \mathcal{H}^0\big(C \cap B_R \cap \pi^{-1}(z)\big)\, d\mathcal{H}^{2m}(z) \leqslant (\operatorname{Lip}\pi)^{2m}\mathcal{H}^{2m}(C \cap B_R),$$

where $\int^*$ denotes the upper Lebesgue integral.

Since $2m \geqslant m+2$, the RHS is finite and we can deduce that the integrand function on the LHS must be finite for a.e. $z \in \mathbb{C}^m$. But $\mathcal{H}^0$ is just the counting measure, so the set $\{(z_0,t) \in C \cap B_R\}$ has to be finite for a.e. $z_0 \in \mathbb{C}^m$. Set $E^R_{z_0} = E_{z_0} \cap B_R$ for $R > 0$ and observe that

$$\mathcal{H}^{2m}\bigg(\bigcup_{n\in\mathbb{N}}\big\{z_0 \in \mathbb{C}^m\colon\ \mathcal{H}^0\big(E^n_{z_0}\big) = +\infty\big\}\bigg) = 0$$

so that for a.e. $z_0 \in \mathbb{C}^m$ the set E_{z_0} is the union of countably many nested finite sets and is therefore at most countable. E_{z_0} is also trivially discrete because if it had a limit point, this would eventually fall into one of the $E^n_{z_0}$, which is absurd.

We now assume $m = 1$ and apply again Eilenberg's inequality to get

$$\int_{\mathbb{C}}^{*} \mathcal{H}^1\big(C \cap B_R \cap \pi^{-1}(z)\big)\, d\mathcal{H}^2(z) \leqslant \frac{\omega_1\omega_2}{\omega_3}(\operatorname{Lip}\pi)^2\mathcal{H}^3(C \cap B_R),$$

where ω_k is the volume of the Euclidean unit ball in $\mathbb{R}^k$.

Since the RHS is finite, we deduce that the integrand function on the LHS must be finite for a.e. $z \in \mathbb{C}$, that is, the set $\{(z_0,t) \in C \cap B_R\}$ has finite $\mathcal{H}^1$-measure for a.e. $z_0 \in \mathbb{C}$. Set $E^R_{z_0} = E_{z_0} \cap B_R$ for $R > 0$ and observe that

$$\mathcal{H}^2\bigg(\bigcup_{n\in\mathbb{N}}\big\{z_0 \in \mathbb{C}\colon\ \mathcal{H}^1\big(E^n_{z_0}\big) = +\infty\big\}\bigg) = 0,$$

so we can conclude that, for a.e. $z_0 \in \mathbb{C}$, $\mathcal{H}^1(E^R_{z_0}) < +\infty$ for every $R > 0$.

Fix now such a value of z_0 and denote by $\dim_E$ (resp. $\dim_\infty$) the Hausdorff dimension with respect to the Euclidean metric (resp. the metric d_∞) in $\mathbb{H}^1$. We want to prove that, if $\mathcal{H}^1(E^R_{z_0}) < +\infty$, then $\dim_E(E^R_{z_0}) \leqslant \frac{1}{2}$. To this end we recall that the image of an s-dimensional set under a C^α map has dimension at most $\frac{s}{\alpha}$ (see Proposition A.1 in Appendix A for the details), a result that we apply in the following way: consider $\mathbb{R}$ equipped with two distinct metrics: the usual Euclidean metric, denoted d_E and the metric $d_\infty(t_1,t_2) = |t_1 - t_2|^{\frac{1}{2}}$ and observe that the map $(z_0,t) \mapsto t$ realizes an isometry between E and a subset of $\mathbb{R}$, once we equip them with the appropriate metrics. Indeed

$$d_\infty\big((z_0,t_1),(z_0,t_2)\big) = \max\big\{|z_0 - z_0|, |t_1 - t_2|^{\frac{1}{2}}\big\} = d_\infty(t_1,t_2).$$

We also note that the identity map $id : (\mathbb{R}, d_\infty) \to (\mathbb{R}, d_E)$ is trivially of Hölder class C^α with $\alpha = 2$, indeed

$$d_E(t_1,t_2) = |t_1 - t_2| = \big(d_\infty(t_1,t_2)\big)^2.$$

If $\mathcal{H}^1(E^R_{z_0}) < +\infty$, then by definition $\dim_\infty(E^R_{z_0}) \leqslant 1$ and since this is true for every $R > 0$ and $E_{z_0} = \bigcup_{n\in\mathbb{N}} E^n_{z_0}$, then

$$\dim_\infty(E_{z_0}) = \dim_\infty\left(\bigcup_{n\in\mathbb{N}} E^n_{z_0}\right) = \sup_{n\in\mathbb{N}} \dim_\infty\left(E^n_{z_0}\right) \leqslant 1.$$

Hence, applying Proposition A.1, we deduce that

$$\dim_E(E_{z_0}) \leqslant \frac{1}{2}\dim_\infty(E_{z_0}) \leqslant \frac{1}{2},$$

which proves the claim. □

The proof of the maximum principle requires a comparison argument that, for the sake of completeness, we state below.

Proposition 4.2 *(Comparison theorem). Let $\Omega \subset\subset \mathbb{H}^m$ be a relatively compact domain. Let $u, v \in C^0(\overline{\Omega}) \cap C^1_H(\Omega)$ satisfy*

$$\begin{cases} \Delta_\varphi u \geqslant \Delta_\varphi v & \text{on } \Omega, \\ u \leqslant v & \text{on } \partial\Omega. \end{cases} \tag{19}$$

Then $u \leqslant v$ on Ω.

For the proof of this proposition we refer the reader to [17]. Note that in [17] the domain Ω is assumed to have C^1 boundary, however a careful reading of the proof shows that this assumption is not necessary.

In the next theorem we prove a strong maximum principle for non-negative solutions of (2). Similar results were proved in [17] and [4]. In particular, in [17] the authors proved a maximum principle on the Heisenberg group for φ-subharmonic functions under quite restrictive conditions on φ. As for [4], in that paper a maximum principle was proved for solutions of (2) on Carnot groups. The assumptions on φ and ℓ, though, are much more restrictive than those in the present paper. In particular, the assumptions on φ imply that Δ_φ reduces essentially to a p-Laplacian.

Let $\Omega \subseteq \mathbb{H}^1$ be an open set, denote by $\pi : \mathbb{H}^1 \to \mathbb{C}$ the natural projection and fix $0 < \beta \leqslant 1$. We define the functional space

$$\mathcal{M}^\beta(\Omega) = \left\{u \in C^2_H(\Omega)\colon \text{ for a.e. } z_0 \in \mathbb{C} \cap \pi(\Omega),\ Tu(z_0, \cdot) \in C^\beta(\Omega_{z_0})\right\}, \tag{20}$$

where $\Omega_{z_0} = \{t \in \mathbb{R}\colon (z_0, t) \in \Omega\}$ and, for $U \subset \mathbb{R}$, $C^\beta(U)$ denotes the space of Hölder continuous functions on U with respect to the Euclidean metric.

Theorem 4.3 *(Maximum principle). Assume the validity of (3), (4), (5) and (7). Let $\Omega \subset \mathbb{H}^m$ be a domain and let $D = \{(\tilde{z}, t)\colon t \in \mathbb{R}\}$ for some $\tilde{z} \in \mathbb{C}^m$ and, for $m \geqslant 2$, let $u \in C^0(\overline{\Omega}) \cap C^1_H(\Omega) \cap C^2_H(\Omega \setminus D)$ while, for $m = 1$, let $u \in C^0(\overline{\Omega}) \cap C^1_H(\Omega) \cap \mathcal{M}^\beta(\Omega \setminus D)$ for some $\frac{1}{2} < \beta \leqslant 1$. Assume u is a non-negative solution of*

$$\Delta_\varphi u \geqslant f(u)\ell\big(|\nabla_0 u|\big) \quad \text{in } \Omega \tag{21}$$

and let $u^* = \sup_\Omega u$. If $u(q_M) = u^*$ for some $q_M \in \Omega$, then $u \equiv u^*$.

Proof. By contradiction, assume there exist a solution u of (21) and $q_M = (z_M, t_M) \in \Omega$ such that $u(q_M) = u^*$, but $u \not\equiv u^*$.

Set $\Gamma = \{q \in \Omega\colon\ u(q) = u^*\}$ and assume first that $\Gamma \subseteq D$. Let $r_0 = d(q_M, \partial\Omega)$ and let

$$S = \left\{q \in \mathbb{H}^m\colon\ d(q, D) \leqslant \frac{1}{4} r_0\right\}.$$

We apply Proposition 4.1 to the function u on the set $\Omega \setminus S$ where the assumptions on the regularity of u are satisfied.

Let $q' = (z_0, t_M) \notin S$ such that $d(q', q_M) < \frac{r_0}{2}$ (hence $q' \in \Omega$) and such that z_0 satisfies the condition in the thesis of Proposition 4.1 and (when $m = 1$) such that $Tu(z_0, \cdot) \in C^\beta$ (this is possible, since the latter two conditions are satisfied for a.e. $z_0 \in \mathbb{C}^m$). Now we construct an auxiliary function by means of Propositions 3.4 and 3.5. Towards this aim, let $R = d(q', \Gamma)$ and consider the annular region

$$E_R(q') = \overline{B_R(q')} \setminus B_{R/2}(q') \tag{22}$$

and define a radial function $v = \alpha \circ d_{q'}$ such that

$$\begin{cases} \Delta_\varphi v \leqslant f(v)\ell(|\nabla_0 v|) & \text{in } E_R(q'), \\ v \geqslant \max\limits_{\partial B_{R/2}(q')} u & \text{on } \partial B_{R/2}(q'), \\ v = u^* & \text{on } \partial B_R(q'). \end{cases} \tag{23}$$

We also choose E as the set E_{z_0} of Proposition 4.1 (which satisfies the dimensional request of Proposition 3.4). We finally choose

$$h_1(t) = 2t^{\frac{1}{2}} Tu(z_0, t + t_M) \quad \text{and} \quad h_2(t) = -2t^{\frac{1}{2}} Tu(z_0, t_M - t)$$

and observe that, when $m = 1$, these functions are trivially of class C^β, since so is $Tu(z_0, t)$ by assumption and $t^{\frac{1}{2}}$ is C^1 on the interval $[R/2, R]$. We also remark that for $m > 1$ the functions h_i are in general only continuous, but this is enough to apply Proposition 3.5.

We finally point out that the function α is strictly increasing on the interval $[R/2, R]$, since it is a line of slope σ.

Let us now assume that the maximum of $u - v$ on E_R be positive. Then it has to be internal, and therefore there must exist p_0 in the interior of E_R such that $u(p_0) > v(p_0)$ and $\nabla_0 u(p_0) = \nabla_0 v(p_0)$. Since f is strictly increasing, we deduce that

$$f\big(u(p_0)\big)\ell\big(\big|\nabla_0 u(p_0)\big|\big) \geqslant f\big(v(p_0)\big)\ell\big(\big|\nabla_0 v(p_0)\big|\big). \tag{24}$$

But we can actually prove that, for our choice of the supersolution v, the previous inequality is strict. In fact, the equality case in the last inequality can only be realized when $|\nabla_0 u| = |\nabla_0 v| = 0$ since, by definition, ℓ can only vanish at the origin. But $\nabla_0 v = \alpha' \nabla_0 d$ and

$$|\nabla_0 d|^2 = \frac{|z - z_0|^2}{d^2},$$

which only vanishes on the line $\{(z_0, t)\colon\ t \in \mathbb{R}\}$, while α' never vanishes.

We will show that, for our choice of the supersolution v, the difference $u - v$ cannot have a stationary point on the line $\{(z_0, t)\colon\ t \in \mathbb{R}\}$. Indeed, if $(z_0, \tilde{t})$ were a stationary point for $u - v$, then at $(z_0, \tilde{t})$ we would have $|\nabla_0 u| = 0$ and $Tu = Tv$. But a straightforward computation shows that

$$Tv(z_0, \tilde{t}) = \alpha'\big(|\tilde{t} - t_M|^{\frac{1}{2}}\big) Td(z_0, \tilde{t}) = \frac{\operatorname{sgn}(\tilde{t} - t_M)}{2|\tilde{t} - t_M|^{\frac{1}{2}}}\sigma,$$

which in particular does not vanish, implying $Tu(z_0,\tilde{t}) \neq 0$. In other words $(z_0,\tilde{t})$ would be a point of the set E_{z_0}, defined in the statement of Proposition 4.1, and

$$Tu(z_0,\tilde{t}) = \frac{\operatorname{sgn}(\tilde{t}-t_M)}{2|\tilde{t}-t_M|^{\frac{1}{2}}}\sigma,$$

that is

$$\sigma = 2\operatorname{sgn}(\tilde{t}-t_M)|\tilde{t}-t_M|^{\frac{1}{2}}Tu(z_0,\tilde{t}), \tag{25}$$

which is impossible for our choice of α, both when $m=1$ and when $m>1$.

Now set $\mu = \max_{E_R}(u-v)$ and let Λ_μ be the connected component of

$$\{q \in E_R\colon\ u(q)-v(q)=\mu\}$$

containing p_0. Observe that, by continuity, (24), which holds in its strict version at every point of Λ_μ, implies that

$$\Delta_\varphi u \geqslant \Delta_\varphi v$$

on a neighborhood U of Λ_μ. Fix $0<\rho<\mu$ and let Ω_ρ be the connected component containing p_0 of

$$\{q \in E_R\colon\ u(q) > v(q)+\rho\}.$$

We observe that Ω_ρ is a nested sequence as ρ tends to μ. We claim that if ρ is close to μ, then $\overline{\Omega}_\rho \subset U$. This can be shown by a compactness argument such as the following: since Λ_μ is closed and bounded, there exists $\varepsilon>0$ such that $d(U^c,\Lambda_\mu) \geqslant \varepsilon$. Suppose, by contradiction, that there exist sequences $\rho_n \uparrow \mu$ and $\{q_n\}$ such that $q_n \in \Omega_{\rho_n}$ and $q_n \notin U$, therefore $d(q_n,\Lambda_\mu) > \varepsilon$. Then, we can assume that the sequence is contained in Ω_{ρ_0} which, by construction, has compact closure; passing to a subsequence converging to some $\bar{q}$, we have by continuity

$$d(\bar{q},\Lambda_\mu) \geqslant \varepsilon, \tag{26}$$

but, on the other hand, $(u-v)(\bar{q}) = \lim_n (u-v)(q_n) \geqslant \lim_n \rho_n = \mu$, hence $\bar{q} \in \Lambda_\mu$ and this contradicts (26). Therefore, $d(\partial\Omega_\rho,\Lambda_\mu) \to 0$ as $\rho \to \mu$, and the claim is proved.

Therefore, on Ω_ρ we have

$$\Delta_\varphi u \geqslant \Delta_\varphi v = \Delta_\varphi(v+\rho)$$

and $u = v+\rho$ on $\partial\Omega_\rho$ which, by the comparison principle, implies that $u \leqslant v+\rho$ on Ω_ρ, a contradiction since $u(p_0) = v(p_0)+\mu$. This shows that the maximum of $u-v$ on E_R has to be non-positive, that is, $u-v \leqslant 0$ on E_R.

Now, we point out that, while the horizontal gradient of the homogeneous norm may vanish out of the origin, its Euclidean gradient does not, indeed

$$|\nabla d|^2 = \frac{1}{d^6}\left(|z|^6 + \frac{t^2}{4}\right).$$

In the light of this, there exists a positive constant $\lambda>0$ such that

$$\langle \nabla v, \nabla d\rangle = \alpha'(d)|\nabla d|^2 \geqslant \lambda > 0 \quad \text{on } \partial E_R(q'). \tag{27}$$

Going back to the function $v-u$, we found that it satisfies $v-u \geqslant 0$ on $E_R(q')$ and $v(q_M)-u(q_M) = u^*-u^* = 0$, so that $\langle \nabla(v-u), \nabla d\rangle(q_M) \leqslant 0$. Therefore

$$0 = \langle \nabla u, \nabla d\rangle(q_M) \geqslant \langle \nabla v, \nabla d\rangle(q_M) > 0, \tag{28}$$

a contradiction.

This concludes the proof in the case $\Gamma \subseteq D$. Assume now that there exists $q_M \in \Gamma$ such that $q_M \notin D$. Then q_M has positive distance from D. Choose an open neighborhood of D, say $U(D)$, so that $\Gamma \nsubseteq U(D)$ and u is not constant on $\Omega \setminus U(D)$. We will consider the set

$$\Omega \setminus \overline{U(D)}$$

which, from now on, we will rename simply Ω for the sake of brevity and we apply Proposition 4.1 to this new Ω.

We now pick a point $q' \in \Omega$ such that $d(q',\Gamma) < d(q',\partial\Omega)$ (this is possible provided q' is sufficiently close to q_M) and such that its projection onto $\mathbb{C}^m$ is a point z_0 that satisfies the condition in the thesis of Proposition 4.1 and (when $m=1$) such that $Tu(z_0,\cdot) \in C^\beta$ (again, this is possible, since the latter two conditions are satisfied for a.e. $z_0 \in \mathbb{C}^m$). The proof now proceeds as in the previous case. □

5. Non-existence results

In this section we prove our main result, a Liouville-type theorem for inequality (2). We recall that the space $\mathcal{M}^\beta$ was defined in (20).

Theorem 5.1. *Let φ, f, ℓ satisfy (3), (4), (5), (6), (7) and (10). Assume also the validity of the generalized Keller–Osserman condition (9). Let $D = \{(\tilde z, t)\colon t \in \mathbb{R}\}$ for some $\tilde z \in \mathbb{C}^m$. If u is a non-negative solution of*

$$\Delta_\varphi u \geqslant f(u)\ell\big(|\nabla_0 u|\big) \quad \text{on } \mathbb{H}^m \tag{29}$$

such that $u \in C^1_H(\mathbb{H}^m) \cap C^2_H(\mathbb{H}^m \setminus D)$ for $m>1$, or $u \in C^1(\mathbb{H}^1) \cap \mathcal{M}^\beta(\mathbb{H}^1 \setminus D)$ for some $\frac{1}{2} < \beta \leqslant 1$ for $m=1$, then u is constant.

Actually, we can prove that inequality (2) does not possess any non-negative entire **bounded** solution regardless of whether the Keller–Osserman condition be satisfied or not. This is stated in the next

Theorem 5.2. *Let φ, f, ℓ satisfy (3), (4), (5), (6), (7) and (10). Let $D = \{(\tilde z, t)\colon t \in \mathbb{R}\}$ for some $\tilde z \in \mathbb{C}^m$. If u is a non-negative, bounded solution of (29) such that $u \in C^1_H(\mathbb{H}^m) \cap C^2_H(\mathbb{H}^m \setminus D)$ for $m>1$, or $u \in C^1(\mathbb{H}^1) \cap \mathcal{M}^\beta(\mathbb{H}^1 \setminus D)$ for some $\frac{1}{2} < \beta \leqslant 1$ when $m=1$, then u is constant.*

The proof of these theorems is based on the same ideas as in [17] and [4]: we assume by contradiction the existence of a non-negative non-constant solution u and compare it with the supersolution v that was constructed in Section 3. In order to be able to perform this comparison and, at the same time, avoid the quite restrictive assumption $\ell(0) > 0$, we want to make sure that the function $u-v$ does not attain a maximum at a point where $\nabla_0 d$ vanishes. The reason of this request will become apparent in the proof of the theorem.

Proof of Theorems 5.1 and 5.2. We first prove Theorem 5.2 under the assumptions (3), (4), (5), (6), (7) and (10). Later on, under the additional hypothesis (9), we will also prove the constancy of possibly unbounded solutions u of (29).

Therefore, we denote $u^* = \sup u$ and we first assume that $u^* < +\infty$. We reason by contradiction and assume $u \not\equiv u^*$; by Theorem 4.3, $u < u^*$ on $\mathbb{H}^m$.

For a fixed $r > 0$, let $S = \{q \in \mathbb{H}^m\colon\ d(q, D) \leqslant r\}$. We now apply Proposition 4.1 to the set $\mathbb{H}^m \setminus S$, we fix z_0 as in the thesis of such proposition and we choose $(z_0, 0)$ as the center of the Korányi balls and distance.

Choose $r_0 > 0$ and define

$$u_0^* = \sup_{\overline{B}_{r_0}} u < u^*.$$

Fix $\eta > 0$ sufficiently small such that $u^* - u_0^* > 2\eta$ and choose $\tilde{q} \in \mathbb{H}^m \setminus \overline{B}_{r_0}$ such that $u(\tilde{q}) > u^* - \eta$. Choose also $0 < \varepsilon < \eta$ and A in such a way that $A > 2\eta + \varepsilon$. We then set $r_1 = d(\tilde{q})$ and, for our choice of $r_0, r_1, A, \varepsilon, \eta$ we construct a radial function $v(q) = \alpha(d(q))$ on $B_T \setminus B_{r_0}$ as in Propositions 3.2 and 3.3, so that

$$\begin{cases} \Delta_\varphi v \leqslant f(v)\ell\bigl(|\nabla_0 v|\bigr) & \text{on } B_T \setminus B_{r_0}, \\ v \equiv \varepsilon \quad \text{on } \partial B_{r_0}; \quad v = A \quad \text{on } \partial B_T, \\ \varepsilon \leqslant v \leqslant \eta \quad \text{on } B_{r_1} \setminus B_{r_0}. \end{cases}$$

We also choose E as the set E_{z_0} of Proposition 4.1 (which again satisfies the dimensional request of Proposition 3.2 when $m = 1$). We finally choose

$$h_1(t) = 2t^{\frac{1}{2}} T u(z_0, t) \quad \text{and} \quad h_2(t) = -2t^{\frac{1}{2}} T u(z_0, -t)$$

and observe that, when $m = 1$, these functions are trivially of class C^β. We stress that h_1 and h_2 are in general only continuous for $m > 1$, but we will not be needing any assumptions on the Hölder continuity of the functions h_i in this case.

Using the properties of v we deduce that

$$u(\tilde{q}) - v(\tilde{q}) > u^* - \eta - \eta = u^* - 2\eta,$$

and, on ∂B_{r_0},

$$u(q) - v(q) \leqslant u_0^* - \varepsilon < u^* - 2\eta - \varepsilon.$$

Since also

$$u(q) - v(q) \leqslant u^* - A < u^* - 2\eta - \varepsilon \quad \text{for } q \in \partial B_T,$$

the difference $u - v$ attains a positive maximum μ in $B_T \setminus \overline{B}_{r_0}$.

Let Λ_μ be a connected component of

$$\bigl\{q \in B_T \setminus \overline{B}_{r_0}\colon\ u(q) - v(q) = \mu\bigr\}.$$

Let $p_0 \in \Lambda_\mu$ and note that $u(p_0) > v(p_0)$ and $|\nabla_0 u(p_0)| = |\nabla_0 v(p_0)|$. As a consequence, since f is strictly increasing,

$$f\bigl(u(p_0)\bigr)\ell\bigl(\bigl|\nabla_0 u(p_0)\bigr|\bigr) \geqslant f\bigl(v(p_0)\bigr)\ell\bigl(\bigl|\nabla_0 v(p_0)\bigr|\bigr).$$

But we can actually prove that, for our choice of the supersolution v, the previous inequality is strict. In fact, the equality case in the last inequality can only be realized when $|\nabla_0 u| = |\nabla_0 v| = 0$ since, by definition, ℓ can only vanish at the origin. But $\nabla_0 v = \alpha' \nabla_0 d$ and

$$|\nabla_0 d|^2 = \frac{|z - z_0|^2}{d^2},$$

which only vanishes on the line $\{(z_0, t)\colon\ t \in \mathbb{R}\}$, while α' never vanishes.

We will show that, for our choice of the supersolution v, the difference $u - v$ cannot have a stationary point on the line $\{(z_0, t)\colon\ t \in \mathbb{R}\}$. Indeed, if $(z_0, \tilde{t})$ were a stationary point for $u - v$, then at $(z_0, \tilde{t})$ we would have $|\nabla_0 u| = 0$ and $Tu = Tv$. But a straightforward computation shows that

$$Tv(z_0, \tilde{t}) = \alpha'\big(|\tilde{t}|^{\frac{1}{2}}\big) Td(z_0, \tilde{t}) = \frac{\operatorname{sgn}(\tilde{t})}{2|\tilde{t}|^{\frac{1}{2}}} \alpha'\big(|\tilde{t}|^{\frac{1}{2}}\big),$$

which in particular does not vanish, implying $Tu(z_0, \tilde{t}) \neq 0$. In other words $(z_0, \tilde{t})$ would be a point of the set E_{z_0}, defined in the statement of Proposition 4.1, and

$$Tu(z_0, \tilde{t}) = \frac{\operatorname{sgn}(\tilde{t})}{2|\tilde{t}|^{\frac{1}{2}}} \alpha'\big(|\tilde{t}|^{\frac{1}{2}}\big),$$

that is

$$\alpha'\big(|\tilde{t}|^{\frac{1}{2}}\big) = 2 \operatorname{sgn}(\tilde{t}) |\tilde{t}|^{\frac{1}{2}} Tu(z_0, \tilde{t}),$$

which is impossible for our choice of α.

By virtue of these considerations, we can conclude that

$$f\big(u(p_0)\big)\ell\big(\big|\nabla_0 u(p_0)\big|\big) > f\big(v(p_0)\big)\ell\big(\big|\nabla_0 v(p_0)\big|\big)$$

which, by continuity, in turn assures that

$$\Delta_\varphi u \geqslant \Delta_\varphi v$$

on an open set $V \supset \Lambda_\mu$.

Arguing as in the final part of the proof of Theorem 4.3 one obtains a contradiction. This shows that u is constant.

Assume now the validity of the Keller–Osserman condition (9), and suppose that u is a solution of (29). By the previous arguments, if u is not constant then necessarily $u^* = +\infty$. Again, fix z_0 as before and $r_0 > 0$ such that $u \not\equiv 0$ on B_{r_0}, and define $u_0^* = \sup_{\overline{B}_{r_0}} u$. Choose $\tilde{q}$, η, ε in such a way that $u(\tilde{q}) > 2u_0^*$, $0 < \varepsilon < \eta < u_0^*$, and consider the function α defined as before with $A = +\infty$. Then, $v(q) = \alpha(d(q))$ is a supersolution of (29) and

$$\begin{gathered} u(q) - v(q) \leqslant u_0^* - \varepsilon \quad \text{on } \partial B_{r_0}, \\ u(\tilde{q}) - v(\tilde{q}) > 2u_0^* - \eta > u_0^*, \\ u(q) - v(q) \to -\infty \quad \text{as } r(q) \to T^-. \end{gathered}$$

Hence, $u - v$ attains a positive maximum in $B_T \setminus \overline{B}_{r_0}$. The proof now proceeds in the same way as in the previous case. $\square$

Remark 5.3. As the reader will have noticed, the statements of the theorems and their proofs are more involved for $\mathbb{H}^1$ than they are for $\mathbb{H}^m$ with $m > 1$ and even require more demanding assumptions on the regularity of the solution in order to prove the non-existence result when $\ell(0) = 0$. This quite unusual fact appears to be a merely technical difficulty and the proofs of the results for $m = 1$ can be made considerably simpler under a different, slightly more restrictive and less intrinsic set of hypotheses on the regularity of the solution u, i.e. by assuming that u is a solution of class C^2 in the traditional Euclidean sense instead of assuming $u \in \mathcal{M}^\alpha(\mathbb{H}^1)$.

Since the former set of hypotheses implies the latter, the validity of the result under these more restrictive assumptions is trivial but, as it is apparent, besides being less complex to state, the "Euclidean" set of assumptions has the advantage of allowing to proceed with essentially the same proof as in the case $m > 1$. Indeed, under these assumptions, the set C in the statement of Proposition 4.1 has Hausdorff dimension at most 2 with respect to the Euclidean metric on $\mathbb{H}^1$. Therefore, the Euclidean version of Eilenberg's inequality can be used to deduce that the set E_{z_0} is at most countable and discrete.

On the other hand, this "Euclidean" set of assumptions on u appears to be less natural for the setting of the Heisenberg group, since it involves requests on the regularity of u as a function on $\mathbb{R}^3$, thus completely ignoring the group structure and the other peculiar features of the Heisenberg group.

The regularity assumptions on the solution required by Theorem 5.1 and Theorem 5.2 are more restrictive than those of their analogues in [17] and [4]. This is a technical issue due to the fact that Proposition 4.1 does not even make sense for functions less regular than C^2_H. It is not clear to us if the assumption C^1_H alone is enough to prove a non-existence result. It is probably possible to weaken such regularity assumptions at the expense of the legibility of the statement and the simplicity of the proof. We will not pursue this here.

A simple result with a weaker set of assumptions is the following.

Theorem 5.4. *Let φ, f, ℓ satisfy (3), (4), (5), (6), (7) and (10). Assume also the validity of the generalized Keller–Osserman condition (9) and let $G = \{(z,t)\colon\ t \in \mathbb{R},\ |z - \tilde{z}| < r\}$ for some $\tilde{z} \in \mathbb{C}^m$ and $r > 0$. If u is a non-negative solution of*

$$\Delta_\varphi u \geqslant f(u)\ell\big(|\nabla_0 u|\big) \quad \text{on } \mathbb{H}^m \tag{30}$$

such that $u \in C^1_H(\mathbb{H}^m) \cap C^2_H(G)$ for $m > 1$, or $u \in C^1(\mathbb{H}^1) \cap \mathcal{M}^\beta(G)$ for some $\frac{1}{2} < \beta \leqslant 1$ for $m = 1$, then u has an absolute maximum at a point $q_M \in \mathbb{H}^m \setminus G$. In particular u cannot be unbounded.

Proof. First of all note that our regularity assumptions do not allow to use Theorem 4.3 (the maximum principle). However, using the argument in the proof of Theorem 5.1 (which does not require the maximum principle) we can show that u must be bounded. To apply such argument in our case, it is enough to choose the center of balls inside G. In a similar way the first part of the proof can be used to show that u attains an absolute maximum at some point $q_M \in \mathbb{H}^m$. Suppose now that $q_M \in G$; then we can apply the maximum principle to show that u is constant in G, hence in $\overline{G}$. Thus u attains its maximum also on $\partial G \subset \mathbb{H}^m \setminus G$. $\square$

6. Existence

Theorem 6.1. *Assume the validity of (3), (4), (5) and (6). Assume also*

$$\frac{\varphi'(t)}{\ell(t)} \in L^1\big(0^+\big). \tag{31}$$

Then, if the generalized Keller–Osserman condition (9) is not satisfied, there exists a non-negative, non-constant solution $u \in C^1_H(\mathbb{H}^m) \cap C^2_H(\mathbb{H}^m \setminus \{z = 0\})$ of inequality

$$\Delta_\varphi u \geqslant f(u)l\big(|\nabla_0 u|\big). \tag{32}$$

Proof. Since inequality (32) has no non-negative non-constant bounded solutions regardless of the Keller–Osserman condition, our aim is to produce an unbounded, non-negative solution under the assumption that the Keller–Osserman condition is not satisfied. This will be achieved by pasting together two subsolutions defined on complementary sets. Such solutions will be "radial" in the variables of the first layer, that is, functions of the form $v(z,t) = w(|z|)$. Straightforward computation shows that

$$\left|\nabla_0 |z|\right| \equiv 1, \qquad \Delta_{\mathbb{H}^m} |z| = \frac{2m-1}{|z|}, \tag{33}$$

and thus the expression of the φ-Laplacian for such functions is

$$\Delta_\varphi v = \varphi'\big(\big|w'\big(|z|\big)\big|\big)w''\big(|z|\big) + \frac{2m-1}{|z|}\operatorname{sgn}\big(w'\big(|z|\big)\big)\varphi\big(\big|w'\big(|z|\big)\big|\big). \tag{34}$$

Assume first that

$$\frac{1}{K^{-1}(F(t))} \in L^1\big(0^+\big),$$

then we set $v(z,t) = w(|z|)$, where w is defined implicitly by

$$t = \int_0^{w(t)} \frac{ds}{K^{-1}(F(s))}. \tag{35}$$

Note that w is well defined, $w(0) = 0$ and, since the Keller–Osserman condition does not hold, $w(t) \to +\infty$ as $t \to +\infty$. Differentiating (35) yields

$$w'(t) = K^{-1}\big(F\big(w(t)\big)\big) \geqslant 0, \tag{36}$$

and a further differentiation gives

$$\varphi'\big(w'\big)w'' = f(w)l\big(w'\big), \tag{37}$$

so that

$$\Delta_\varphi v = \varphi'\big(w'\big(|z|\big)\big)w''\big(|z|\big) + \frac{2m-1}{|z|}\varphi\big(w'\big(|z|\big)\big) \geqslant f(v)l\big(|\nabla_0 v|\big).$$

Finally, v is trivially of class $C^2_H(\mathbb{H}^m \setminus \{z=0\})$ and is also $C^1_H(\mathbb{H}^m)$ since, by construction, $w'(0) = 0$.

Assume now that

$$\frac{1}{K^{-1}(F(t))} \notin L^1\big(0^+\big).$$

Fix $\varepsilon > 0$ and define implicitly the C^2-function w on $(0,+\infty)_0$ by setting

$$t = \int_\varepsilon^{w(t)} \frac{ds}{K^{-1}(F(s))}. \tag{38}$$

As before, w is well defined, $w(0) = \varepsilon$ and, since the Keller–Osserman condition does not hold, $w(t) \to +\infty$ as $t \to +\infty$. Note that (36) and (37) still hold. We fix $z_1 > 0$ to be specified later and set

$$A_{z_1} = \{(z,t) \in \mathbb{H}^m \colon |z| < z_1\},$$

and let u_1 be the function defined on $\mathbb{H}^m \setminus A_{z_1}$ by the formula $u_1(z,t) = w(|z|)$. Since, by (33), $|\nabla_0 u_1| = w'$, using (34) and (37) we conclude that u_1 satisfies

$$\Delta_\varphi u_1 = \varphi'\big(w'(|z|)\big)w''(|z|) + \frac{2m-1}{|z|}\varphi\big(w'(|z|)\big) \geqslant f(u_1)l\big(|\nabla_0 u_1|\big) \tag{39}$$

on $\mathbb{H}^m \setminus A_{z_1}$.

To produce a subsolution u_2 on A_{z_1} define a function Ω through

$$\int_0^{\Omega(s)} \frac{\varphi'(t)}{l(t)}\,dt = f(1)s. \tag{40}$$

Note that Ω is well defined by (31), solves the differential equation

$$\frac{\varphi'(\Omega(s))}{l(\Omega(s))}\Omega'(s) = f(1)$$

and is increasing and unbounded. We set

$$\beta(t) = \int_0^t \Omega(s)\,ds + \beta_0$$

where

$$\beta_0 = 1 - \frac{F(1)}{f(1)}$$

(note that $\beta_0 > 0$ since f is monotone increasing) and observe that, by a simple change of variable,

$$\beta(t) = \frac{1}{f(1)}\int_0^{\Omega(t)} \frac{u\varphi'(u)}{l(u)}\,du + \beta_0 = \frac{K(\Omega(t))}{f(1)} + \beta_0.$$

We define the function $u_2(z,t) = \beta(|z|)$ and we shall prove that we can choose the parameters z_1 and ε so that u_1 and u_2 join at ∂A_{z_1} together with their first and second derivatives. To this end, we choose z_1 so that it satisfies

$$\Omega(z_1) = K^{-1}\big(F(1)\big)$$

(this is possible since Ω is increasing and unbounded from above) and observe that

$$\beta(z_1) = \frac{F(1)}{f(1)} + \beta_0 = 1.$$

A simple computation then shows that, for every $t \in [0, z_1]$,

$$\varphi'\big(\beta'(t)\big)\beta''(t) = f(1)l\big(\beta'(t)\big) \geqslant f\big(\beta(t)\big)l\big(\beta'(t)\big),$$

so that

$$\Delta_\varphi u_2 = \varphi'\big(\beta'\big(|z|\big)\big)\beta''\big(|z|\big) + \frac{2m-1}{|z|}\varphi\big(\beta'\big(|z|\big)\big) \geqslant f(u_2)l\big(|\nabla_0 u_2|\big).$$

In order to have $\beta(z_1) = w(z_1)$, we need to find a value of ε such that $w(z_1) = 1$, that is

$$z_1 = \int_\varepsilon^1 \frac{ds}{K^{-1}(F(s))}.$$

This is possible by virtue of the fact that, by assumption,

$$\frac{1}{K^{-1}(F(t))} \notin L^1\big(0^+\big).$$

With this choice of the parameters, we have that

$$w(z_1) = 1 = \beta(z_1),$$
$$w'(z_1) = K^{-1}\big(F(1)\big) = \beta'(z_1)$$

and, since

$$\varphi'\big(\beta'(z_1)\big)\beta''(z_1) = f(1)l\big(\beta'(z_1)\big) = \varphi'\big(w'(z_1)\big)w''(z_1),$$

we also have the equality of the second derivatives. This, together with the fact that $\beta'(0) = 0$ by construction, proves that the function

$$u(x) = \begin{cases} u_1(x) & \text{on } \mathbb{H}^m \setminus A_{z_1}, \\ u_2(x) & \text{on } A_{z_1} \end{cases} \tag{41}$$

is a solution of $\Delta_\varphi u \geqslant f(u)l(|\nabla u|)$ with the required regularity. □

Acknowledgment

The authors wish to thank Luciano Mari for some helpful conversations on the subject of the paper.

Appendix A

In this section, we state and prove a simple result concerning the Hausdorff dimension of the image of a set through a Hölder continuous map, which we have used to prove Proposition 3.2, Proposition 3.4 and Proposition 4.1. Although the proof of this proposition is quite straightforward, we reproduce it here for the sake of completeness.

First of all we recall that, if (X, d_1) and (Y, d_2) are metric spaces and $\beta \geqslant 0$, a map $f : X \to Y$ is said to be Hölder continuous of exponent β if there exists $C > 0$ such that for every $x, y \in X$

$$d_2\big(f(x), f(y)\big) \leqslant C d_1(x, y)^\beta.$$

We shall denote by $C^\beta(X, Y)$ the space of Hölder continuous maps with exponent β and observe that, if $\beta = 1$, $C^\beta(X, Y)$ is just the space of Lipschitz continuous maps.

Proposition A.1. *Let (X, d_1) and (Y, d_2) be metric spaces, let $f : X \to Y$ be a C^β map for some $\beta > 0$ and $A \subseteq X$. Then*

$$\dim_{(Y,d_2)}\bigl(f(A)\bigr) \leqslant \frac{1}{\beta}\dim_{(X,d_1)}(A).$$

Proof. Let C be the Hölder constant of f and let $s \geqslant 0$. Assume that $\mathcal{H}^s(A) < +\infty$ and recall that

$$\mathcal{H}^s(A) = \lim_{\delta\to 0}\inf_{\substack{\bigcup S_j \supseteq A\\ \operatorname{diam} S_j \leqslant \delta}}\sum_j \omega_s\left(\frac{\operatorname{diam} S_j}{2}\right)^s,$$

where

$$\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}.$$

The set A has finite s-dimensional Hausdorff measure if and only if there exist positive M and δ_0 such that for every $\delta < \delta_0$

$$\inf_{\substack{\bigcup S_j \supseteq A\\ \operatorname{diam} S_j \leqslant \delta}}\sum_j (\operatorname{diam} S_j)^s < M.$$

Denoting by $\{S_{j,\delta}^{(k)}\}$ a minimizing sequence of coverings of A with diameters smaller than δ which realizes the infimum in the definition of Hausdorff measure, then we have that there exist $M > 0$ and $\delta_0 > 0$ such that, for every $\delta < \delta_0$

$$\sum_j \bigl(\operatorname{diam} S_{j,\delta}^{(k)}\bigr)^s < M$$

for sufficiently large k.

Now set

$$K_{j,\delta}^{(k)} = f\bigl(S_{j,\delta}^{(k)}\bigr)$$

and observe that $K_{j,\delta}^{(k)}$ is a covering of $f(A)$ with the property that

$$\operatorname{diam} K_{j,\delta}^{(k)} \leqslant C\bigl(\operatorname{diam} S_{j,\delta}^{(k)}\bigr)^\beta.$$

In particular,

$$\operatorname{diam} K_{j,\delta}^{(k)} \leqslant C\delta^\beta$$

and, for δ small and k big enough,

$$\sum_j \bigl(\operatorname{diam} K_{j,\delta}^{(k)}\bigr)^{\frac{s}{\beta}} \leqslant C^{\frac{s}{\beta}}\sum_j \bigl(\operatorname{diam} S_{j,\delta}^{(k)}\bigr)^s < C^{\frac{s}{\beta}} M.$$

With this in mind we can prove that $\mathcal{H}^{\frac{s}{\beta}}(f(A)) < +\infty$: fix $M_1 = C^{\frac{s}{\beta}} M > 0$ and set $\delta_1 = C\delta_0^\beta$. Then for every $\delta < \delta_1$ we have

$$\inf_{\substack{\bigcup S_j \supseteq f(A) \\ \operatorname{diam} S_j \leqslant \delta}} \sum_j (\operatorname{diam} S_j)^{\frac{s}{\beta}} \leqslant \sum_j \big(\operatorname{diam} K^{(k)}_{j,(\frac{\delta}{C})^{\frac{1}{\beta}}}\big)^{\frac{s}{\beta}} < M_1$$

for k big enough, which proves the claim.

Now recall that, by definition,

$$\dim(A) = \inf\{p\colon\ \mathcal{H}^p(A) = 0\} = \sup\{p\colon\ \mathcal{H}^p(A) = +\infty\},$$

hence for every s such that $\mathcal{H}^s(A) < +\infty$,

$$\dim\big(f(A)\big) = \sup\big\{p\colon\ \mathcal{H}^p\big(f(A)\big) = +\infty\big\} \leqslant \frac{s}{\beta},$$

that is, $\beta \dim(f(A))$ is a lower bound for the set

$$\big\{p\colon\ \mathcal{H}^p(A) < +\infty\big\} \supseteq \big\{p\colon\ \mathcal{H}^p(A) = 0\big\},$$

hence

$$\beta \dim\big(f(A)\big) \leqslant \inf\big\{p\colon\ \mathcal{H}^p(A) = 0\big\} = \dim(A),$$

which completes the proof. □

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