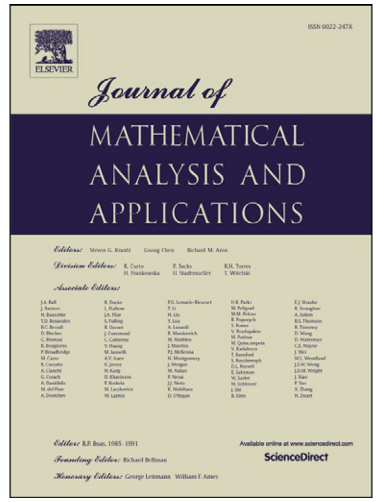

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Positive solutions to integral systems with weight and Bessel potentials

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Abstract

In this paper, we consider the integral system with weight and the Bessel potentials:

$$\begin{cases} u(x) = \int_{R^n} \frac{g_\alpha(x-y)u(y)^p v(y)^q}{|y|^\sigma} dy, \\ v(x) = \int_{R^n} \frac{g_\alpha(x-y)v(y)^p u(y)^q}{|y|^\sigma} dy, \end{cases}$$

where $u, v > 0,\ \sigma \geq 0,\ 0 < \alpha < n,\ p + q = \gamma \geq 2$ and $g_\alpha(x)$ is the Bessel potential of order α. First, we get the integrability by regularity lifting lemma. Then we also establish the regularity of the positive solutions. Afterwards, by the method of moving planes in integral forms, we show that the positive solutions are radially symmetric and monotone decreasing about the origin. Finally, by an extension of the idea of Lei[14] and analytical techniques, we get the decay rates of solutions when $|x| \to \infty$.

Keywords: Bessel potential; Integral system; Integrability; Regularity lifting; Radial symmetry; Method of moving planes; Decay rates.

1 Introduction

In this paper, we consider the following integral system with weight and the Bessel potentials:

$$\begin{cases} u(x) = \int_{R^n} \frac{g_\alpha(x-y)u(y)^p v(y)^q}{|y|^\sigma} dy, \\ v(x) = \int_{R^n} \frac{g_\alpha(x-y)v(y)^p u(y)^q}{|y|^\sigma} dy, \end{cases} \tag{1.1}$$

where $u, v > 0,\ \sigma \geq 0,\ 0 < \alpha < n,\ p + q = \gamma \geq 2$ and $g_\alpha(x)$ is the Bessel potential of order α. Here

$$g_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{(-\frac{\pi}{t}|x|^2 - \frac{t}{4\pi})} t^{(\alpha-n)/2} \frac{dt}{t}.$$

Integral system (1.1) is associated with the following partial differential equations(PDEs)

$$\begin{cases} (I-\Delta)^{\alpha/2} u = \frac{u^p v^q}{|y|^\sigma}, u > 0, \\ (I-\Delta)^{\alpha/2} v = \frac{v^p u^q}{|y|^\sigma}, v > 0. \end{cases} \tag{1.2}$$

When $\sigma = 0$, (1.2) becomes the following PDEs(cf. [23])

$$\begin{cases} (I-\Delta)^{\alpha/2}u = u^p v^q, u > 0, \\ (I-\Delta)^{\alpha/2}v = v^p u^q, v > 0. \end{cases} \tag{1.3}$$

When $\alpha = 2$, PDEs (1.3) is associated with the nonlinear Klein-Gordon equations and the quintic Schrödinger system (see [1,13,19]).

Lei [14] studied the uniqueness of the positive solution of (1.3) under some assumptions. In addition, he proved the integrability and radial symmetry of positive solutions of integral system. By an iteration he also obtained the estimate of the exponential decay of those solutions near infinity.

Another integral system with weight and the Bessel potential is the following

$$\begin{cases} u(x) = \int_{R^n} \frac{g_\alpha(x-y)v(y)^q}{|y|^\beta} dy, \\ v(x) = \int_{R^n} \frac{g_\alpha(x-y)u(y)^p}{|y|^\beta} dy, \end{cases} \tag{1.4}$$

where $0 \le \beta < \alpha < n,\ 1 < p, q < \frac{n-\beta}{\beta}$ and

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-\alpha+\beta}{n}.$$

Chen and Yang [4] proved regularity and symmetry of this integral system and obtained that system was actually equivalent to indefinite fractional elliptic system

$$\begin{cases} (-\Delta+I)^{\alpha/2}u = \frac{v^q}{|y|^\beta}, u > 0, \\ (-\Delta+I)^{\alpha/2}v = \frac{u^p}{|y|^\beta}, v > 0. \end{cases} \tag{1.5}$$

If $\alpha = 2$ and $\beta = 0$, (1.5) is the Hamiltonian type system [6]. In the special case, when $p = q$ and $u = v$, system (1.5) becomes

$$(-\Delta+I)^{\frac{\alpha}{2}}u = \frac{u^p}{|y|^\beta}. \tag{1.6}$$

It was known from [17] and [18] that the dynamical behavior of bosons spin-0 particles in relativistic fields can be described by the Schrödinger-Klein-Gordon equation

$$i\frac{\partial\psi}{\partial t} = (-\Delta+I)^{\frac{1}{2}}\psi - \psi + f(x,\psi). \tag{1.7}$$

Equation (1.6) arises in finding the standing wave $e^{it}u(x)$ of the pseudo-relativistic wave equation (1.7) with special f. For more papers, please see [5,7,9,19-22,25,26].

In particular, when $\sigma = 0$, if the Bessel potential in (1.3) is replaced by Riesz type potential, then we have

$$\begin{cases} u(x) = |x|^{\alpha-n} * v^q(x), \\ v(x) = |x|^{\alpha-n} * u^p(x). \end{cases} \tag{1.8}$$

The solutions (u, v) of (1.8) are critical points of the functional associated with the well-known classical Hardy-Littlewood-Sobolev (HLS) inequality (see [11,22]):

$$\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy \le C_{r,\alpha,n}\|f\|_{L^r}\|g\|_{L^s}, \tag{1.9}$$

where $0<\alpha<n;\ r,s>1$, such that $\frac{1}{r}+\frac{1}{s}+\frac{n-\alpha}{n}=2;\ f\in L^r(R^n),\ g\in L^s(R^n)$.

The following weighted Hardy-Littlewood-Sobolev (WHLS) inequality was introduced by Stein and Weiss (see [24]):

$$\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy \le C_{\alpha,\beta,s,\lambda,n}\|f\|_{L^r}\|g\|_{L^s},$$

where

$$1-\frac{1}{r}-\frac{\lambda}{n}<\frac{\alpha}{n}<1-\frac{1}{r} \text{ and } \frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}=2.$$

We can also write the WHLS inequality in another form. Let $Tg(x)=\int_{R^n}\frac{g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dy$. Then

$$\|Tg(x)\|_{L^p}\le C_{\alpha,\beta,s,\lambda,n}\|g(y)\|_{L^s}, \tag{1.10}$$

where $1+\frac{1}{p}=\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n},\ 1<s,p<\infty,\ \alpha+\beta\ge 0,\ 0<\lambda<n,\ \frac{1}{p}-\frac{\lambda}{n}<\frac{\alpha}{n}<\frac{1}{p}$.

When $p=q=\frac{n+\alpha}{n-\alpha}$ and $u(x)=v(x)$, the integral system (1.8) becomes (cf. [2] and [15])

$$u(x)=\int_{R^n}\frac{u^\gamma(y)}{|x-y|^{n-\alpha}}dy,\ u>0,\ in\ R^n. \tag{1.11}$$

(1.11) arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequalities (see [16]). In [16], Lieb classified all the maximizers of functional (1.9) under the constraints $\|f\|_r=\|g\|_s=1$ in the critical case where $\gamma=\frac{n+\alpha}{n-\alpha}$, and thus obtained the sharp constant in the HLS inequalities in that case, he then posed the classification of all the critical points of the functional, the solutions of the integral equation (1.11) as an open problem. In [3], Chen, Li, and Ou solved the open problem by using the method of moving planes. In particular, the corresponding PDE becomes

$$-\triangle u=u^{\frac{n+\alpha}{n-\alpha}},\ u>0,\ in\ R^n. \tag{1.12}$$

(1.12) is also of practical interest and importance. The classification of solutions of (1.12) has provided an important ingredient in the study of the well-known Yamabe problem and the prescribing scalar curvature problem.

Han and Lu [10] considered the following system without weight:

$$\begin{cases} (-\Delta+I)^{\alpha/2}u=v^p, u>0,\\ (-\Delta+I)^{\beta/2}v=u^q, v>0, \end{cases} \tag{1.13}$$

where $\alpha,\ \beta>0$ and $p,\ q>1$. By virtue of the integral form and the decay estimate for the Bessel kernel both at the origin and the infinity, they proved the L^∞ and Lipschitz continuity of the positive solutions to the system (1.13). Obviously, (1.2) is very different from (1.13).

To the best of our knowledge, there are no results about (1.1). In this paper, we will prove the integrability, the regularity, radial symmetry and decay rates of the positive solutions of (1.1).

Before we state our main results, we give two propositions which will be useful in the proof of main results.

Proposition 1 (See [23]). Let $0 < \alpha < n$. The kernel g_α satisfies

$$g_\alpha(x) = \begin{cases} C|x|^{-n+\alpha} + o(|x|^{-n+\alpha}), & \text{when } |x| \to 0, \\ O(e^{-\frac{|x|}{2}}), & \text{when } |x| \to \infty. \end{cases} \tag{1.14}$$

Here C is a constant.

Proposition 2 (See [14]). For $\beta \in (0, \alpha]$, then we can get

$$\mathcal{B}_\alpha(f)(x) \leq CI_\beta(f)(x), \ x \in R^n, \tag{1.15}$$

where $C > 0$, $\mathcal{B}_\alpha(f)(x) = (g_\alpha * f)(x) = \int_{R^n} g_\alpha(x-y)f(y)dy$ and $I_\beta(f)(x) := \frac{\Gamma((n-\beta)/2)}{2^\beta \pi^{n/2}\Gamma(\beta/2)}|x|^{\beta-n} * f(x)$ are the Bessel potential and the Riesz potential of a positive function $f \in L^p(R^n)$ $(1 \leq p \leq \infty)$ respectively.

Now we state the main results as follows:

Theorem 1.1. Let $\sigma \geq 0,\ 0 < \alpha < n,\ p+q = \gamma \geq 2$. For all $\beta \in (\sigma, \min\left\{\frac{n+\sigma}{2}, \alpha\right\})$, assume $(u(x), v(x)) \in L^{\frac{n(\gamma-1)}{\beta-\sigma}}(R^n) \times L^{\frac{n(\gamma-1)}{\beta-\sigma}}(R^n)$ is a pair of positive solutions of (1.1), then

(R-1) $u, v \in L^r(R^n)$ for any $r \in [1, \infty)$.

(R-2) $(u(x), v(x))$ is uniformly bounded in R^n.

(R-3) $(u(x), v(x))$ is continuous.

(R-4) $u(x)$ and $v(x)$ are radially symmetric and monotone decreasing about some point $\overline{x} \in R^n$.

(R-5) We can get the decay rates

$$\lim_{|x|\to\infty} \frac{1}{g_\alpha(x)} u(x) = \int_{R^n} \frac{u^p v^q}{|y|^\sigma} dy,$$

$$\lim_{|x|\to\infty} \frac{1}{g_\alpha(x)} v(x) = \int_{R^n} \frac{v^p u^q}{|y|^\sigma} dy.$$

Remark 1.1. Since the Bessel kernel does not have singularity at infinity, we can extend the integrability interval to the whole $[1, \infty]$, the result is the same as the case $\sigma = 0$. Note that $u(x)$ is monotone decreasing about some point $\overline{x} \in R^n$ means that $u(x)$ is monotone decreasing about $|x - \bar{x}|$.

Remark 1.2. When $\sigma \geq 0$, both $u(x)$ and $v(x)$ are radially symmetric and decreasing about $\overline{x} \in R^n$. In particular, when $\sigma \neq 0$, we can prove that $\overline{x}$ must be the origin.

Remark 1.3. We only treat the case $p+q \geq 2$, the case $1 < p+q < 2$ can not be handled in the same way, we will exploit the corresponding questions in the future.

2 Integrability

In this section, we use the method of regularity lifting lemma to prove (R-1) of Theorem 1.1.

Proof of (R-1). Step 1. We first show $u, v \in L^r(R^n)$, for all $r > \frac{n}{n-\beta}$.

Let $w = u + v$, for $A > 0$, define

$$w_A(x) = \begin{cases} w(x), & if\ |w(x)| > A\ or\ |x| > A, \\ 0, & elsewhere. \end{cases}$$

From (1.1) and the definition of $B_\alpha(f)$, we know that

$$w(x) = \int_{R^n} g_\alpha(x-y)\left(\frac{u^p(y)v^q(y)}{|y|^\sigma} + \frac{v^p(y)u^q(y)}{|y|^\sigma}\right)dy \le c\int_{R^n} g_\alpha(x-y)\frac{w^\gamma(y)}{|y|^\sigma}dy = cB_\alpha\left(\frac{w^\gamma}{|y|^\sigma}\right)(x).$$

Now set $R(x) = \frac{w(x)}{B_\alpha(\frac{w^\gamma}{|y|^\sigma})(x)}$, then $0 < R(x) \le c$, and

$$w(x) = R(x)\int_{R^n} g_\alpha(x-y)\frac{w^\gamma(y)}{|y|^\sigma}dy. \tag{2.1}$$

Let $f \in L^r(R^n)$ for all $r > \frac{n}{n-\beta}$, define an operator T by

$$(Tf)(x) := R(x)\int_{R^n} g_\alpha(x-y)\frac{w_A^{\gamma-1}(y)}{|y|^\sigma}f(y)dy$$

and write

$$F = R(x)\int_{R^n} g_\alpha(x-y)\frac{(w-w_A)^\gamma(y)}{|y|^\sigma}dy.$$

Clearly, w is a solution of the following equation

$$f = Tf + F. \tag{2.2}$$

By (1.10), (1.14) and the Hölder inequality, we have

$$\|Tf\|_{L^r} \le c\|I_\beta\left(\frac{w_A^{\gamma-1}f}{|y|^\sigma}\right)\|_{L^r} \le c\|w_A^{\gamma-1}f\|_{L^m} \le c\|w_A\|^{\gamma-1}_{L^{\frac{n(\gamma-1)}{\beta-\sigma}}}\|f\|_{L^r},$$

where $\frac{1}{r} = \frac{1}{m} - \frac{\beta-\sigma}{n}$.

From $w \in L^{\frac{n(\gamma-1)}{\beta-\sigma}}(R^n)$, we can find a large constant A, such that

$$\|Tf\|_{L^r} \le \frac{1}{2}\|f\|_{L^r}.$$

Then for $r > \frac{n}{n-\beta}$, the operator T is a contracting map from $L^r(R^n)$ to itself. Similar to the argument above, for any $r > \frac{n}{n-\beta}$, there holds

$$\|F\|_{L^r} \le C\|(w-w_A)^\gamma\|_{L^{\frac{nr}{n+r(\beta-\sigma)}}}.$$

In view of the definition of $w_A(x)$, we have $F \in L^r(R^n)$. Using (2.2) and the regularity lifting lemma [11], we can get

$$w \in L^r(R^n),\ \forall r > \frac{n}{n-\beta}. \tag{2.3}$$

Step 2. In this step, we need to prove

$$\int_{R^n} \frac{w^\gamma(y)}{|y|^\sigma} dy \leq C. \tag{2.4}$$

Clearly,

$$\int_{R^n} \frac{w^\gamma(y)}{|y|^\sigma} dy = \int_{B_R} \frac{w^\gamma(y)}{|y|^\sigma} dy + \int_{R^n \setminus B_R} \frac{w^\gamma(y)}{|y|^\sigma} dy := I_1 + I_2.$$

Take $k_1 = \frac{n}{n-\beta}$, it follows that $\gamma k_1 \geq \frac{2n}{n-\beta} > \frac{n}{n-\beta}$, then by using the Hölder inequality, we obtain

$$I_1 = \int_{B_R} \frac{w^\gamma(y)}{|y|^\sigma} dy \leq \left(\int_{B_R} w^{\gamma k_1}(y) dy \right)^{\frac{1}{k_1}} \left(\int_{B_R} \frac{1}{|y|^{\frac{\sigma k_1}{k_1 - 1}}} dy \right)^{1 - \frac{1}{k_1}} \leq C.$$

For all $\beta \in (\sigma, min\left\{\frac{n+\sigma}{2}, \alpha\right\})$, there exists $\frac{n}{2(n-\beta)} < k_2 < \frac{n}{n-\sigma}$, then by Hölder inequality, we can get

$$I_2 = \int_{R^n \setminus B_R} \frac{w^\gamma(y)}{|y|^\sigma} dy \leq \left(\int_{R^n \setminus B_R} w^{\gamma k_2}(y) dy \right)^{\frac{1}{k_2}} \left(\int_{R^n \setminus B_R} \frac{1}{|y|^{\frac{\sigma k_2}{k_2 - 1}}} dy \right)^{1 - \frac{1}{k_2}} \leq C.$$

Step 3. In the final step, we improve the integrability of u and v from (2.3) to

$$u, v \in L^r(R^n), \;\; for \;\; r \geq 1.$$

Since $\int_{R^n} e^{-\pi|x|^2/t} dx = t^{n/2}$, Fubini's theorem applied to $g_\alpha(x)$ shows

$$\int_{R^n} g_\alpha(x) dx = \frac{1}{(4\pi)^{\alpha/2} \Gamma(\alpha/2)} \int_0^\infty e^{-t/4\pi} t^{\alpha/2} \frac{dt}{t} = 1.$$

Now by virtue of (2.4), we can get

$$\begin{aligned} \int_{R^n} w(x) dx &= \int_{R^n} R(x) \int_{R^n} g_\alpha(x-y) \frac{w^\gamma(y)}{|y|^\sigma} dy dx \\ &\leq c \int_{R^n} \left(\int_{R^n} g_\alpha(x-y) dx \right) \frac{w^\gamma(y)}{|y|^\sigma} dy \\ &= c \int_{R^n} g_\alpha(x) dx \int_{R^n} \frac{w^\gamma(y)}{|y|^\sigma} dy \\ &= c \int_{R^n} \frac{w^\gamma(y)}{|y|^\sigma} dy < \infty. \end{aligned}$$

So

$$u, v \in L^1(R^n). \tag{2.5}$$

Combining (2.3) with (2.5), for all $m \in (1, \frac{n}{n-\beta}]$, we have

$$\begin{aligned} \int_{R^n} w^m dx &\leq \int_{0<w<1} w dx + \int_{w>1} w^{\frac{n}{n-\beta} + \epsilon} dx \\ &\leq \int_{R^n} w dx + \int_{R^n} w^{\frac{n}{n-\beta} + \epsilon} dx < \infty. \end{aligned}$$

Thus we get

$$u, v \in L^r(R^n), \;\; \forall r \geq 1. \tag{2.6}$$

3 Regularity

Based on the integrability of solutions obtained in the previous section, we will prove (R-2) and (R-3) of Theorem 1.1.

To prove the conclusions, we need the following lemma.

Lemma 3.1. For all $x \in R^n$, we have

$$\int_{B_t(x)} \frac{1}{|y|^\beta} dy \leq ct^{n-\beta}. \tag{3.1}$$

Proof. When $x \in R^n \setminus B_{2t}(0)$, for $y \in B_t(x)$, we have $|y| \geq \frac{|x|}{2}$, and hence $\frac{1}{|y|^\beta} \leq \frac{c}{t^\beta}$.

So

$$\int_{B_t(x)} \frac{1}{|y|^\beta} dy \leq c \int_{B_t(x)} \frac{1}{t^\beta} dy = ct^{n-\beta}.$$

When $x \in B_{2t}(0)$, for $y \in B_t(x)$, we have $y \in B_t(x) \subset B_{3t}(0)$.

Therefore,

$$\int_{B_t(x)} \frac{1}{|y|^\beta} dy \leq \int_{B_{3t}(0)} \frac{1}{|y|^\beta} dy \leq ct^{n-\beta}.$$

Thus, Lemma 3.1 is proved.

Proof of (R-2). By (1.14), we can get

$$\begin{aligned} w(x) &= R(x) B_\alpha(\frac{w^\gamma}{|y|^\sigma})(x) \leq C I_\beta(\frac{w^\gamma}{|y|^\sigma})(x) \\ &= C(n-\beta) \int_{R^n} \frac{w(y)^\gamma}{|y|^\sigma} (\int_{|x-y|}^\infty t^{\beta-n} \frac{dt}{t}) dy \\ &= C(n-\beta) \int_0^\infty (\frac{\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}) \frac{dt}{t}. \end{aligned}$$

Thus,

$$\begin{aligned} w(x) &\leq C(n-\beta) \int_0^d (\frac{\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}) \frac{dt}{t} + C(n-\beta) \int_d^\infty (\frac{\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}) \frac{dt}{t} \\ &=: C(n-\beta)(J_1 + J_2). \end{aligned}$$

By the Hölder inequality, we get

$$\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy \leq \left(\int_{B_t(x)} w(y)^{\gamma k} dy \right)^{\frac{1}{k}} \left(\int_{B_t(x)} \left(\frac{1}{|y|^\sigma} \right)^{\frac{k}{k-1}} dy \right)^{1-\frac{1}{k}}. \tag{3.2}$$

Here $k > \frac{n}{\beta-\sigma}$. From (3.1), (3.2) and $w \in L^r, \ \forall r \geq 1$, we have

$$J_1 = \int_0^d (\frac{\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}) \frac{dt}{t} \leq c \int_0^d t^{\beta-\sigma-\frac{n}{k}} \frac{dt}{t} \leq c.$$

By (2.4), we can obtain

$$J_2 = \int_d^\infty (\frac{\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}) \frac{dt}{t} \leq c \int_d^\infty \frac{1}{t^{n-\beta}} \frac{dt}{t} \leq c.$$

From the estimates of J_1 and J_2, we can get $w(x) < c$ for all x. Hence, $u(x)$ and $v(x)$ are uniformly bounded in R^n.

Based on (R-2) of Theorem 1.1, we can deduce the following corollary.

Corollary 3.1. $(u(x), v(x))$ converges to 0 when $|x| \to \infty$.

Proof. Since u and v are bounded, $\forall \epsilon > 0$, there exists $d \in (0,1)$ such that

$$\int_0^d \left(\frac{\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}\right)\frac{dt}{t} \le \epsilon,$$

and for $z \in B_d(x)$, by virtue of $B_t(x) \subset B_{t+d}(z)$ with $t \ge d$, we get

$$\begin{aligned}\int_d^\infty \left(\frac{\int_{B_t(x)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}\right)\frac{dt}{t} &\le \int_d^\infty \frac{\int_{B_{t+d}(z)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{(t+d)^{n-\beta}} \left(\frac{t+d}{t}\right)^{n-\beta+1} \frac{d\,(t+d)}{t+d} \\ &\le C\int_0^\infty \left(\frac{\int_{B_t(z)} \frac{w(y)^\gamma}{|y|^\sigma} dy}{t^{n-\beta}}\right)\frac{dt}{t} \\ &\le CI_\beta\left(\frac{w^\gamma}{|y|^\sigma}\right)(z).\end{aligned}$$

Then from (R-2) of Theorem 1.1, we get $w(x) < C\epsilon + CI_\beta(\frac{w^\gamma}{|y|^\sigma})(z)$. When $r \ge 1$, we have

$$w^r(x) \le C\epsilon^r + C\left(I_\beta\left(\frac{w^\gamma}{|y|^\sigma}\right)(z)\right)^r.$$

Integrating on $B_d(x)$ and multiplying by $|B_d(x)|^{-1}$, we can obtain

$$w^r(x) \le C\epsilon^r + C|B_d(x)|^{-1}\|I_\beta\left(\frac{w^\gamma}{|y|^\sigma}\right)\|^r_{L^r(B_d(x))}. \tag{3.3}$$

From $w \in L^r(R^n)$ and $\frac{nr\gamma}{n+r(\beta-\sigma)} > 1$ for all $r \ge 1$, we have

$$\|I_\beta\left(\frac{w^\gamma}{|y|^\sigma}\right)\|_{L^r(B_d(x))} \le C\|w^\gamma\|_{\frac{nr}{n+r(\beta-\sigma)}} < \infty.$$

Therefore,

$$\lim_{|x|\to\infty} \int_{B_d(x)} \left[I_\beta\left(\frac{w^\gamma}{|y|^\sigma}\right)\right]^r (z)dz = 0 \tag{3.4}$$

By (3.3) and (3.4), we have

$$\lim_{|x|\to\infty} w^r(x) = 0.$$

Thus $(u(x), v(x))$ converges to 0. Corollary 3.1 is proved.

Proof of (R-3). Let $u(x) - u(z) = K_1 + K_2$, here

$$K_1 = \int_{B_R(x)} [g_\alpha(x-y) - g_\alpha(z-y)]\frac{u(y)^p v(y)^q}{|y|^\sigma} dy,$$

$$K_2 = \int_{R^n \setminus B_R(x)} [g_\alpha(x-y) - g_\alpha(z-y)]\frac{u(y)^p v(y)^q}{|y|^\sigma} dy.$$

When $y \in B_R(x)$ and $|x-z| \to 0$, we have $g_\alpha(x-y) - g_\alpha(z-y) \to 0$.

Now we estimate K_1 :

$$\lim_{|x-z|\to 0} K_1 = \lim_{|x-z|\to 0} \int_{B_R(x)} [g_\alpha(x-y) - g_\alpha(z-y)]\frac{u(y)^p v(y)^q}{|y|^\sigma} dy$$

$$= \int_{B_R(x)} \lim_{|x-z|\to 0} [g_\alpha(x-y) - g_\alpha(z-y)] \frac{u(y)^p v(y)^q}{|y|^\sigma} dy.$$

Based on the result of (2.4), we can get

$$\lim_{|x-z|\to 0} K_1 = 0.$$

As $|x-y| > R$, by virtue of (1.14) and (2.4), we can get

$$\begin{aligned} K_2 &= \int_{R^n \setminus B_R(x)} [g_\alpha(x-y) - g_\alpha(z-y)] \frac{u^p(y)v^q(y)}{|y|^\sigma} dy \\ &\leq \int_{R^n \setminus B_R(x)} g_\alpha(x-y) \frac{u^p(y)v^q(y)}{|y|^\sigma} dy \\ &\leq c\frac{1}{e^{\frac{R}{2}}} \int_{R^n \setminus B_R(x)} \frac{w^\gamma(y)}{|y|^\sigma} dy \\ &\leq c\frac{1}{e^{\frac{R}{2}}} \int_{R^n} \frac{w^\gamma(y)}{|y|^\sigma} dy \\ &\leq c\frac{1}{e^{\frac{R}{2}}}. \end{aligned}$$

Letting $R \to \infty$, we obtain

$$\lim_{|x-z|\to 0} K_2 = 0.$$

Therefore, $u(x)$ is continuous. Similarly, we can get another conclusion about $v(x)$.

4 Radial symmetry and monotonicity of solutions

In this section, we prove (R-4) of Theorem 1.1 by using the method of moving planes in integral forms which was established by Chen et al.[3].

For any real number λ, define

$$\Sigma_\lambda = \{x = (x_1, x_2, \cdots, x_n) \in R^n | x_1 > \lambda\};\ x^\lambda = (2\lambda - x_1, x_2, ..., x_n);$$

$$u_\lambda(x) = u(x^\lambda);\ T_\lambda = \{x = (x_1, x_2, \cdots, x_n) \in R^n | x_1 = \lambda\}.$$

Firstly, we give a lemma which will be useful in the proof of (R-4) of Theorem 1.1.

Lemma 4.1. We have

$$\begin{aligned} u_\lambda(x) - u(x) = &\int_{\Sigma_\lambda} (g_\alpha(x-y) - g_\alpha(x^\lambda - y)) \frac{1}{|y^\lambda|^\sigma} ({u_\lambda}^p {v_\lambda}^q - u^p v^q) dy \\ &- \int_{\Sigma_\lambda} (g_\alpha(x-y) - g_\alpha(x^\lambda - y)) (\frac{1}{|y|^\sigma} - \frac{1}{|y^\lambda|^\sigma}) u^p v^q dy, \\ v_\lambda(x) - v(x) = &\int_{\Sigma_\lambda} (g_\alpha(x-y) - g_\alpha(x^\lambda - y)) \frac{1}{|y^\lambda|^\sigma} ({v_\lambda}^p {u_\lambda}^q - v^p u^q) dy \\ &- \int_{\Sigma_\lambda} (g_\alpha(x-y) - g_\alpha(x^\lambda - y)) (\frac{1}{|y|^\sigma} - \frac{1}{|y^\lambda|^\sigma}) v^p u^q dy. \end{aligned}$$

Proof. By $g_\alpha(x-y^\lambda)=g_\alpha(x^\lambda-y)$,

$$u(x)=\int_{\Sigma_\lambda}\frac{g_\alpha(x-y)u^p(y)v^q(y)}{|y|^\sigma}dy+\int_{\Sigma_\lambda}\frac{g_\alpha(x^\lambda-y){u_\lambda}^p(y){v_\lambda}^q(y)}{|y^\lambda|^\sigma}dy,$$

$$u_\lambda(x)=\int_{\Sigma_\lambda}\frac{g_\alpha(x^\lambda-y)u^p(y)v^q(y)}{|y|^\sigma}dy+\int_{\Sigma_\lambda}\frac{g_\alpha(x-y){u_\lambda}^p(y){v_\lambda}^q(y)}{|y^\lambda|^\sigma}dy,$$

then we have

$$\begin{aligned}u_\lambda(x)-u(x)=&\int_{\Sigma_\lambda}(g_\alpha(x-y)-g_\alpha(x^\lambda-y))\frac{1}{|y^\lambda|^\sigma}({u_\lambda}^p{v_\lambda}^q-u^pv^q)dy\\&-\int_{\Sigma_\lambda}(g_\alpha(x-y)-g_\alpha(x^\lambda-y))(\frac{1}{|y|^\sigma}-\frac{1}{|y^\lambda|^\sigma})u^pv^qdy.\end{aligned}$$

Similarly, v has the same property. Lemma 4.1 is proved.

Proof of (R-4). Step 1. One can claim that there exists an $N\geq 0$ such that for $\lambda<-N$,

$$u(x)\geq u_\lambda(x),\ \ v(x)\geq v_\lambda(x). \tag{4.1}$$

Define

$$\Sigma_\lambda^u=\{x\in\Sigma_\lambda|u(x)\leq u_\lambda(x)\},\quad \Sigma_\lambda^v=\{x\in\Sigma_\lambda|v(x)\leq v_\lambda(x)\},\quad \Sigma_\lambda^-=\Sigma_\lambda\setminus(\Sigma_\lambda^u\cup\Sigma_\lambda^v).$$

By virtue of $g_\alpha(x-y)>g_\alpha(x^\lambda-y)$ on Σ_λ, we get

$$(g_\alpha(x-y)-g_\alpha(x^\lambda-y))(\frac{1}{|y|^\sigma}-\frac{1}{|y^\lambda|^\sigma})\geq 0, \tag{4.2}$$

then

$$\begin{aligned}u_\lambda(x)-u(x)\ \ &\leq\ \ \int_{\Sigma_\lambda}(g_\alpha(x-y)-g_\alpha(x^\lambda-y))\frac{1}{|y^\lambda|^\sigma}({u_\lambda}^p{v_\lambda}^q-u^pv^q)dy\\&=\ \ \int_{\Sigma_\lambda^u}(g_\alpha(x-y)\frac{1}{|y^\lambda|^\sigma}v_\lambda^q({u_\lambda}^p-u^p)dy+\int_{\Sigma_\lambda^v}(g_\alpha(x-y)\frac{1}{|y^\lambda|^\sigma}u^p({v_\lambda}^q-v^q)dy+I,\end{aligned}$$

here

$$\begin{aligned}I=&\int_{\Sigma_\lambda^-}(g_\alpha(x-y)-g_\alpha(x^\lambda-y))\frac{1}{|y^\lambda|^\sigma}v_\lambda^q({u_\lambda}^p-u^p)dy+\int_{\Sigma_\lambda^-}(g_\alpha(x-y)-g_\alpha(x^\lambda-y))\frac{1}{|y^\lambda|^\sigma}u^p({v_\lambda}^q-v^q)dy\\&-\int_{\Sigma_\lambda^u}g_\alpha(x^\lambda-y)\frac{1}{|y^\lambda|^\sigma}v_\lambda^q({u_\lambda}^p-u^p)dy-\int_{\Sigma_\lambda^v}g_\alpha(x^\lambda-y)\frac{1}{|y^\lambda|^\sigma}u^p({v_\lambda}^q-v^q)dy\leq 0.\end{aligned}$$

From (1.14) and above,

$$\begin{aligned}u_\lambda(x)-u(x)\leq&\int_{\Sigma_\lambda^u}g_\alpha(x-y)\frac{1}{|y^\lambda|^\sigma}v_\lambda^q({u_\lambda}^p-u^p)(y)dy+\int_{\Sigma_\lambda^v}g_\alpha(x^\lambda-y)\frac{1}{|y^\lambda|^\sigma}u^p({v_\lambda}^q-v^q)(y)dy\\\leq&\ c\int_{\Sigma_\lambda^u}g_\alpha(x-y)\frac{1}{|y^\lambda|^\sigma}v_\lambda^q{u_\lambda}^{p-1}(u_\lambda-u)(y)dy+c\int_{\Sigma_\lambda^v}g_\alpha(x^\lambda-y)\frac{1}{|y^\lambda|^\sigma}u^pv_\lambda^{q-1}(v_\lambda-v)(y)dy\\\leq&\ c\int_{\Sigma_\lambda^u}\frac{1}{|y^\lambda|^\sigma|x-y|^{n-\beta}}v_\lambda^qu_\lambda^{p-1}(u_\lambda-u)(y)dy+c\int_{\Sigma_\lambda^v}\frac{1}{|y^\lambda|^\sigma|x^\lambda-y|^{n-\beta}}u^p{v_\lambda}^{q-1}(v_\lambda-v)(y)dy.\end{aligned}$$

Let $m = \frac{n(\gamma-1)}{\beta-\sigma}$, by (1.10), (2.6) and the Hölder inequality, we can deduce that

$$\begin{aligned}\|u_\lambda(x) - u(x)\|_{L^m} &\le c\|v_\lambda^q u_\lambda^{p-1}(u_\lambda - u)\|_{L^{\frac{mn}{n+m(\beta-\sigma)}}(\Sigma_\lambda^u)} + c\|u^p v_\lambda^{q-1}(v_\lambda - v)\|_{L^{\frac{mn}{n+m(\beta-\sigma)}}(\Sigma_\lambda^v)} \\ &\le c\|v_\lambda\|_{L^m}^q \|u_\lambda\|_{L^m}^{p-1} \|u_\lambda - u\|_{L^m(\Sigma_\lambda^u)} + c\|u\|_{L^m}^p \|v_\lambda\|_{L^m}^{q-1} \|v_\lambda - v\|_{L^m(\Sigma_\lambda^v)}. \end{aligned} \tag{4.3}$$

One can choose a sufficiently large $N > 0$, such that for $\lambda \le -N < 0$,

$$c\|v_\lambda\|_{L^m}^q \|u_\lambda\|_{L^m}^{p-1} \le \frac{1}{4},$$

and

$$C\|u\|_{L^m}^p \|v_\lambda\|_{L^m}^{q-1} \le \frac{1}{4}.$$

Then (4.3) implies that

$$\|u_\lambda(x) - u(x)\|_{L^m(\Sigma_\lambda^u)} \le \frac{1}{4}\|u_\lambda(x) - u(x)\|_{L^m(\Sigma_\lambda^u)} + \frac{1}{4}\|v_\lambda(x) - v(x)\|_{L^m(\Sigma_\lambda^v)}. \tag{4.4}$$

Similarly, we have

$$\|v_\lambda(x) - v(x)\|_{L^m(\Sigma_\lambda^v)} \le \frac{1}{4}\|v_\lambda(x) - v(x)\|_{L^m(\Sigma_\lambda^v)} + \frac{1}{4}\|u_\lambda(x) - u(x)\|_{L^m(\Sigma_\lambda^u)}. \tag{4.5}$$

Combining (4.4) with (4.5), we can get $\|u_\lambda(x) - u(x)\|_{L^m(\Sigma_\lambda^u)} = 0$. So the measure of $\Sigma_\lambda^u, \Sigma_\lambda^v$ must be zero. This means (4.1).

Step 2. Now we start from this neighborhood of $x_1 = -\infty$ and move the plane to the right as long as (4.1) holds to the limiting position and argue that the solution w must be symmetric about the limiting plane. More precisely, define

$$\lambda_0 = \sup\{\mu | (4.1) \text{ holds for any } \lambda \le \mu\}.$$

One can see that $\lambda_0 < +\infty$ by using the similar to step 1 and starting the plane T_λ near $x_1 = +\infty$. We will show that u and v are symmetric about the plane T_{λ_0}:

$$u(x) \equiv u_{\lambda_0}(x) \text{ and } \; v(x) \equiv v_{\lambda_0}(x) \;\; a.e. \;\; \forall x \in \Sigma_{\lambda_0}. \tag{4.6}$$

Suppose for such $\lambda_0 < 0$, we have, on Σ_{λ_0},

$$u(x) \ge u_{\lambda_0}(x) \text{ and } \; v(x) \ge v_{\lambda_0}(x), \text{ but } \; u(x) \not\equiv u_{\lambda_0}(x) \text{ or } \; v(x) \not\equiv v_{\lambda_0}(x), \; a.e. \; \forall x \in \Sigma_{\lambda_0}; \tag{4.7}$$

we show that the plane can be moved to the right. More precisely, there exists an $\epsilon > 0$ such that

$$u(x) \ge u_\lambda(x) \text{ and } \; v(x) \ge v_\lambda(x) \; a.e. \; \forall x \in \Sigma_\lambda, \; \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon). \tag{4.8}$$

This contradicts to the definition of λ_0.

In the case that $u(x) \not\equiv u_{\lambda_0}(x)$ in Σ_{λ_0}, we can get that $u(x) \geq u_\lambda(x)$ in the interior of Σ_{λ_0}. Let

$$\Phi_{\lambda_0}^{\tilde{u}} = \{x \in \Sigma_{\lambda_0} | u(x) \leq u_{\lambda_0}(x)\} \text{ and } \Phi_{\lambda_0}^{\tilde{v}} = \{x \in \Sigma_{\lambda_0} | v(x) \leq v_{\lambda_0}(x)\}.$$

Then obviously, $\Phi_{\lambda_0}^{\tilde{u}}$ has measure zero, and $\lim_{\lambda \to \lambda_0} \Sigma_\lambda^u \subset \Phi_{\lambda_0}^{\tilde{u}}$. The same is true for that of v. Again the integrability conditions $u,\ v \in L^m(R^n)$ ensure that one can choose ϵ sufficiently small, so that for all λ in $[\lambda_0, \lambda_0 + \epsilon)$, by the step 1, we have

$$\|u_\lambda - u\|_{L^m(\Sigma_\lambda^u)} = 0 \text{ and } \|v_\lambda - v\|_{L^m(\Sigma_\lambda^v)} = 0,$$

hence the measure of Σ_λ^u and Σ_λ^v must be zero. This verifies (4.1) and hence (4.6).

Since x_1 direction can be chosen arbitrarily, we deduce that $u(x)$ and $v(x)$ must be radially symmetric and monotone decreasing about some point $\overline{x} \in R^n$.

If $\sigma \neq 0$, we claim that $\overline{x} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n)$ must be the origin. Otherwise, there exists $k \in \{1, 2, \cdots, n\}$ such that $\overline{x}_k \neq 0$. Without loss of generality, suppose $\overline{x}_k < 0$. For any x satisfying $x_k > \overline{x}_k$, we denote the reflection point of x about the plane $x = \overline{x}$ by x^*. Thus, by Lemma 4.1 and the symmetry, we obtain

$$0 = u(x^*) - u(x) = -\int\limits_{\{y;\ y_k > \overline{x}_k\}} (g_\alpha(x-y) - g_\alpha(x^*-y))(\frac{1}{|y|^\sigma} - \frac{1}{|y^*|^\sigma})u^p v^q dy < 0.$$

It is impossible, so the claim is verified.

5 Decay rate

In this section, we will prove (R-5) of Theorem 1.1.

We first need a lemma which makes it possible to get the the decay rates of u and v.

Lemma 5.1. Under the same conditions of Theorem 1.1, there exist $c, C > 0$, such that when $|x| \to \infty$,

$$cg_\alpha(x) \leq u, v \leq Cg_\alpha(x).$$

Proof. Step 1. We first prove that $u, v \geq cg_\alpha(x)$.

When $|x| > 2$, for $u, v > 0$, we have

$$\int_{B_1(0) \cap B_{|x|}(x)} \frac{u^p(y)v^q(y)}{|y|^\sigma} dy \geq c > 0.$$

Therefore, for sufficiently large $|x|$,

$$\begin{aligned} u(x) &= \int_{R^n} \frac{g_\alpha(x-y)u(y)^p v(y)^q}{|y|^\sigma} dy \\ &= c \int_0^\infty \frac{\int_{R^n} e^{(-\pi|x-y|^2/t)} \frac{u^p(y)v^q(y)}{|y|^\sigma} dy}{e^{(t/(4\pi))} t^{(n-\alpha)/2}} \frac{dt}{t} \end{aligned}$$

$$\begin{aligned}
&\geq \quad c\int_0^\infty \frac{\int_{B_1(0)\cap B_{|x|}(x)} \frac{u^p(y)v^q(y)}{|y|^\sigma}dy}{e^{(\pi|x|^2/4+t/(4\pi))}t^{(n-\alpha)/2}}\frac{dt}{t} \\
&\geq \quad cg_a(x).
\end{aligned}$$

Similarly, for sufficiently large $|x|$, we can get $v(x) \geq cg_\alpha(x)$.

Step 2. In this step, we show that $u, v \leq Cg_\alpha(x)$.

For $\eta \in (0, 1/3)$, take a sequence

$$\xi_0 = 1,\ \xi_k = \eta\sum_{j=1}^k \frac{1}{2^j},\ k = 1, 2, \cdots,$$

then

$$\lim_{|k|\to\infty} \xi_k = \eta. \tag{5.1}$$

Write $w = u + v$, then (2.1) leads to

$$w(x) \leq c\int_{R^n} \frac{g_\alpha(x-y)w^\gamma(y)}{|y|^\sigma}dy = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \int_{B_R} \frac{g_\alpha(x-y)w^\gamma(y)}{|y|^\sigma}dy, \\
I_2 &= \int_{(R^n\setminus B_R)\setminus B(x,\xi_1|x|)} \frac{g_\alpha(x-y)w^\gamma(y)}{|y|^\sigma}dy, \\
I_3 &= \int_{B(x,\xi_1|x|)} \frac{g_\alpha(x-y)w^\gamma(y)}{|y|^\sigma}dy.
\end{aligned}$$

Where $R > 0$.

For fixed $R > 0$, When $y \in B_R$, there holds

$$\lim_{|x|\to\infty} |\frac{g_\alpha(x-y)}{g_\alpha(x)} - 1| = 0.$$

Thus, by virtue of (2.4), we can get

$$I_1 \leq 2g_\alpha(x)\int_{B_R} \frac{w(y)^\gamma}{|y|^\sigma}dy \leq cg_\alpha(x).$$

Since g_α is decreasing, when $y \in (R^n \setminus B_R) \setminus B(x, \xi_1|x|)$ and $|x| \to \infty$, we have

$$g_\alpha(x-y) \leq g_\alpha(\xi_1|x|) \leq 1.$$

Therefore,

$$I_2 \leq \int_{R^n\setminus B_R} \frac{w(y)^\gamma}{|y|^\sigma}dy.$$

Based on Corollary 3.1 and the monotonicity of w, we can get, when $|x| \to \infty$,

$$w^{\gamma-1}((1-\xi_1)x) \leq \frac{1}{2}. \tag{5.2}$$

Thus, we have

$$I_3 \le \frac{w^\gamma((1-\xi_1)x)}{[(1-\xi_1)|x|]^\sigma}\int_{B(x,\zeta_1|x|)} g_\alpha(x-y)dy \le \frac{1}{2}\cdot\frac{w((1-\xi_1)x)}{[(1-\xi_1)|x|]^\sigma}\int_{R^n} g_\alpha(y)dy = \frac{1}{2}\cdot\frac{w((1-\xi_1)x)}{[(1-\xi_1)|x|]^\sigma}.$$

Combining the results above, we have

$$w(x) \le 2g_\alpha(x)\int_{B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy + \int_{R^n\setminus B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy + \frac{1}{2}\cdot\frac{w((1-\xi_1)x)}{[(1-\xi_1)|x|]^\sigma} \tag{5.3}$$

and furthermore,

$$\begin{aligned} w((1-\xi_k)x) \le& \int_{B_R} g_\alpha((1-\xi_k)x-y)\frac{w^\gamma(y)}{|y|^\sigma}dy + \int_{(R^n\setminus B_R)\setminus B(x,\xi_{k+1}|x|)} g_\alpha((1-\xi_k)x-y)\frac{w^\gamma(y)}{|y|^\sigma}dy \\ &+ \int_{B(x,\xi_{k+1}|x|)} g_\alpha((1-\xi_k)x-y)\frac{w^\gamma(y)}{|y|^\sigma}dy \\ \le& 2g_\alpha((1-\xi_k)x)\int_{B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy + \int_{R^n\setminus B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy + \frac{1}{2}\frac{w((1-\xi_{k+1})x)}{[(1-\xi_{k+1})|x|]^\sigma},\ k=1,2,\cdots \end{aligned}$$

Inserting these estimates into (5.3) we can get that

$$\begin{aligned} w(x) \le& [2g_\alpha(x) + \frac{2g_\alpha((1-\xi_1)x)}{2[(1-\xi_1)|x|]^\sigma} + \frac{2g_\alpha((1-\xi_2)x)}{2\cdot 2\cdot[(1-\xi_1)|x|]^\sigma[(1-\xi_2)|x|]^\sigma} \\ &+ \cdots + \frac{2g_\alpha((1-\xi_m)x)}{2^m\cdot[(1-\xi_1)|x|]^\sigma\cdots[(1-\xi_m)|x|]^\sigma}]\int_{B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy \\ &+[1+\frac{1}{2[(1-\xi_1)|x|]^\sigma} + \frac{1}{2^2[(1-\xi_1)|x|]^\sigma[(1-\xi_2)|x|]^\sigma} \\ &+\cdots+\frac{1}{2^m[(1-\xi_1)|x|]^\sigma[(1-\xi_2)|x|]^\sigma\cdots[(1-\xi_m)|x|]^\sigma}]\int_{R^n\setminus B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy \\ &+\left(\frac{1}{2}\right)^{m+1}\frac{w((1-\xi_{m+1})x)}{[(1-\xi_1)|x|]^\sigma\cdots[(1-\xi_{m+1})|x|]^\sigma}. \end{aligned}$$

Note that when $\xi_k \le \xi_m$, we have

$$g_\alpha((1-\xi_k)x) \le g_\alpha((1-\xi_m)x),$$

$$\frac{1}{[(1-\xi_k)|x|]^\sigma} \le \frac{1}{[(1-\xi_m)|x|]^\sigma}.$$

Thus we get

$$\begin{aligned} w(x) \le& 2g_\alpha((1-\xi_m)x)[1+\frac{1}{2[(1-\xi_m)|x|]^\sigma} + \frac{1}{2^2[(1-\xi_m)|x|]^{2\sigma}} + \cdots + \frac{1}{2^m[(1-\xi_m)|x|]^{m\sigma}}]\int_{B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy \\ &+[1+\frac{1}{2[(1-\xi_m)|x|]^\sigma} + \frac{1}{2^2[(1-\xi_m)|x|]^{2\sigma}} + \cdots + \frac{1}{2^m[(1-\xi_m)|x|]^{m\sigma}}]\int_{R^n\setminus B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy \\ &+\left(\frac{1}{2}\right)^{m+1}\frac{w((1-\xi_{m+1})x)}{[(1-\xi_{m+1})|x|]^{(m+1)\sigma}}. \end{aligned}$$

So

$$w(x) \le 2g_\alpha((1-\xi_m)x)\frac{2\,[(1-\xi_m)|x|]^\sigma}{2\,[(1-\xi_m)|x|]^\sigma - 1}\int_{B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy$$

$$+\frac{2\left[(1-\xi_m)|x|\right]^\sigma}{2\left[(1-\xi_m)|x|\right]^\sigma-1}\int_{R^n\setminus B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy+\left(\frac{1}{2}\right)^{m+1}\frac{w((1-\xi_{m+1})x)}{\left[(1-\xi_{m+1})|x|\right]^{(m+1)\sigma}}.$$

Letting $m\to\infty$, we have (5.1) and

$$w(x)\le 2g_\alpha((1-\eta)x)\frac{2\left[(1-\eta)|x|\right]^\sigma}{2\left[(1-\eta)|x|\right]^\sigma-1}\int_{B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy+\frac{2\left[(1-\eta)|x|\right]^\sigma}{2\left[(1-\eta)|x|\right]^\sigma-1}\int_{R^n\setminus B_R}\frac{w^\gamma(y)}{|y|^\sigma}dy.$$

Letting $R\to\infty$, we can get

$$w(x)\le 2g_\alpha((1-\eta)x)\frac{2\left[(1-\eta)|x|\right]^\sigma}{2\left[(1-\eta)|x|\right]^\sigma-1}\int_{R^n}\frac{w^\gamma(y)}{|y|^\sigma}dy. \tag{5.4}$$

Letting $\eta\to 0$, we obtain the upper bound of w from (2.4).

Therefore,

$$u,v\le Cg_\alpha(x). \tag{5.5}$$

Combining Step 1 with Step 2, we prove Lemma 5.1.

Proof of (R-5). Clearly,

$$\frac{u(x)}{g_\alpha(x)}=\int_{R^n}\frac{g_\alpha(x-y)u^p(y)v^q(y)}{g_\alpha(x)|y|^\sigma}dy:=H_1+H_2+H_3,$$

here

$$H_1=\int_{B_R}\frac{g_\alpha(x-y)u^p(y)v^q(y)}{g_\alpha(x)|y|^\sigma}dy,$$
$$H_2=\int_{(R^n\setminus B_R)\setminus B(x,|x|/2)}\frac{g_\alpha(x-y)u^p(y)v^q(y)}{g_\alpha(x)|y|^\sigma}dy,$$
$$H_3=\int_{B(x,|x|/2)}\frac{g_\alpha(x-y)u^p(y)v^q(y)}{g_\alpha(x)|y|^\sigma}dy.$$

Where $R>0$.

For fixed $R>0$, when $y\in B_R$, we have

$$\lim_{|x|\to\infty}\left|\frac{g_\alpha(x-y)}{g_\alpha(x)}-1\right|=0,$$

and

$$\frac{u^p(y)v^q(y)}{|y|^\sigma}\left|\frac{g_\alpha(x-y)}{g_\alpha(x)}-1\right|\le 3\frac{u^p(y)v^q(y)}{|y|^\sigma}.$$

From (2.4), we have

$$\int_{R^n}\frac{u^p(y)v^q(y)}{|y|^\sigma}dy\le\int_{R^n}\frac{w^\gamma(y)}{|y|^\sigma}dy\le c.$$

Then when $|x|\to\infty$, by the Dominated convergence theorem, we can get

$$\left|\int_{B_R}\frac{u^pv^q}{|y|^\sigma}\left[\frac{g_\alpha(x-y)}{g_\alpha(x)}-1\right]dy\right|\to 0,\text{as }|x|\to\infty.$$

This result leads to

$$\lim_{R\to\infty}\lim_{|x|\to\infty}H_1=\int_{R^n}\frac{u^pv^q}{|y|^\sigma}dy.$$

Next we consider

$$H_2 = \int_{(R^n \setminus B_R)\setminus B(x,|x|/2)} \frac{g_\alpha(x-y)u^p v^q}{g_\alpha(x)|y|^\sigma} dy.$$

According to the Lemma 5.1, Proposition 1 and (5.2), when $|x| \to \infty$ and $|x-y| \geq \frac{|x|}{2}$, we can get

$$\begin{aligned} \int_{(R^n \setminus B_R)\setminus B(x,|x|/2)} \frac{g_\alpha(x-y)}{g_\alpha(x)} \frac{u^p v^q}{|y|^\sigma} dy &\leq c\frac{1}{|R|^\sigma} \int_{R^n \setminus B_R} \frac{e^{\frac{|x|}{2}} w^\gamma(y)}{e^{\frac{|x-y|}{2}}} dy \\ &\leq c\frac{1}{|R|^\sigma} \int_{R^n \setminus B_R} \frac{e^{\frac{|x|}{2}} g_\alpha(y)}{e^{\frac{|x-y|}{2}}} dy \\ &= c\frac{1}{|R|^\sigma} \int_{R^n \setminus B_R} \frac{e^{\frac{|x|}{2}}}{e^{\frac{|x-y|}{2}} \cdot e^{\frac{|y|}{2}}} dy. \end{aligned}$$

As $|x-y| + |y| > |x|$, then $\frac{e^{\frac{|x|}{2}}}{e^{\frac{|x-y|}{2}} \cdot e^{\frac{|y|}{2}}} < 1$.

Clearly, when $R \to \infty$,

$$H_2 = \int_{(R^n \setminus B_R)\setminus B(x,|x|/2)} \frac{g_\alpha(x-y)}{g_\alpha(x)} \frac{u^p v^q}{|y|^\sigma} dy \to 0.$$

Finally, we consider

$$H_3 = \int_{B(x,|x|/2)} \frac{g_\alpha(x-y)u^p(y)v^q(y)}{g_\alpha(x)|y|^\sigma} dy.$$

(R-4) of Theorem 1.1 implies w is radially symmetric and decreasing about origin o. Therefore, if we denote the point $\overline{ox} \cap \partial B(x,|x|/2)$ by x_0, then $|x_0| = |x|/2$. By the result, (1.14) and (5.5), we know

$$\begin{aligned} H_3 &= \int_{B(x,|x|/2)} \frac{g_\alpha(x-y)u^p(y)v^q(y)}{g_\alpha(x)|y|^\sigma} dy \\ &\leq c\frac{w^\gamma(x_0)}{|x_0|^\beta g_\alpha(x)} \int_{B(x,|x|/2)} g_\alpha(x-y) dy \\ &\leq c\frac{w^\gamma(x_0)}{|x|^\beta g_\alpha(x)} \leq c\frac{g_\alpha^\gamma(x_0)}{|x|^\beta g_\alpha(x)} \leq c\frac{e^{\frac{|x|}{2}}}{|x|^\beta e^{\frac{|x|}{4}\cdot\gamma}}. \end{aligned}$$

Hence, when $p+q=\gamma \geq 2$, and $|x| \to \infty$,

$$H_3 = \int_{B(x,|x|/2)} \frac{g_\alpha(x-y)u^p(y)v^q(y)}{g_\alpha(x)|y|^\sigma} dy \to 0.$$

Combining all the estimate of H_1, H_2 and H_3, we get

$$\lim_{|x|\to\infty} \frac{1}{g_\alpha(x)} u(x) = \int_{R^n} \frac{u^p v^q}{|y|^\sigma} dy.$$

Similarly,

$$\lim_{|x|\to\infty} \frac{1}{g_\alpha(x)} v(x) = \int_{R^n} \frac{v^p u^q}{|y|^\sigma} dy.$$

Thus, we complete the proof of Theorem 1.1.

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