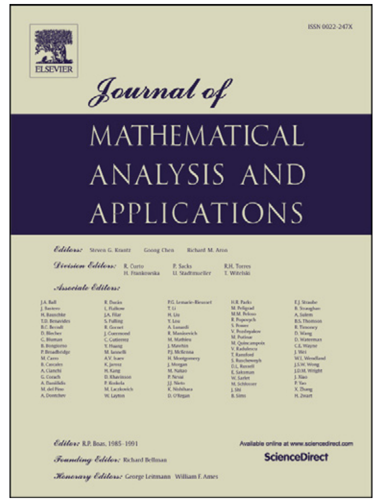

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Properties of the zeros of generalized hypergeometric polynomials

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Abstract

We define the *generalized hypergeometric polynomial* of degree N as follows:

$$P_N\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;z\right)=\sum_{m=0}^{N}\left[\frac{(-N)_m\,(\alpha_1)_m\cdots(\alpha_p)_m\ \ z^{N-m}}{m!\ \left(\beta_1\right)_m\cdots\left(\beta_q\right)_m}\right]$$
$$=\ \ z^N\ {}_{p+1}F_q\left(-N,\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;1/z\right)\ .$$

Here N is an arbitrary *positive* integer, p and q are arbitrary *nonnegative* integers, the $p+q$ parameters α_j and β_k are arbitrary ("generic", possibly complex) numbers, $(\alpha)_m$ is the Pochhammer symbol and ${}_{p+1}F_q\left(\alpha_0,\alpha_1,...\alpha_p;\beta_1,...,\beta_q;z\right)$ is the generalized hypergeometric function. In this paper we obtain a set of N *nonlinear algebraic equations* satisfied by the N zeros ζ_n of this polynomial. We moreover manufacture an $N\times N$ matrix $\underline{L}$ in terms of the $1+p+q$ parameters N, α_j, β_k characterizing this polynomial, and of its N zeros ζ_n, and we show that it features the N eigenvalues $\lambda_m=m\ \prod_{k=1}^{q}\left(-\beta_k+1-m\right)\ ,\quad m=1,...,N\ .$ These N eigenvalues depend only on the q parameters β_j, implying that the $N\times N$ matrix $\underline{L}$ is *isospectral* for variations of the p parameters α_j; and they clearly are *integer* (or *rational*) numbers if the q parameters β_k are themselves *integer* (or *rational*) numbers: a nontrivial *Diophantine* property.

Keywords: hypergeometric polynomials, diophantine properties, Jacobi polynomials, isospectral matrices, special functions.

1 Introduction

The investigation of the properties of the zeros of polynomials has a long history going back for several centuries, yet new approaches and findings have also emerged in relatively recent times. A class of such findings extended the results pioneered by G. Szëgo (see in particular Section 6.7 of [1]), by identifying additional sets of *nonlinear algebraic relations* satisfied by the zeros of the classical polynomials and, more generally, of polynomials belonging to the Askey scheme, as well as $N\times N$ matrices, constructed with the N zeros of these polynomials (of degree N), whose eigenvalues could be explicitly identified and in many cases feature *Diophantine* properties: see for instance [2] [3] [4] [5] [6] [8] [9]. The present paper reports analogous results for *generalized hypergeometric polynomials*. These findings are displayed in the following Section 2, and proven in the subsequent Section 3; certain polynomial identities which are essential for obtaining and reporting these results—and are themselves remarkable—are proven and displayed in Appendix A. A terse Section 4 ("Outlook") outlines possible future developments.

2 Results

The generalized hypergeometric function ${}_{p+1}F_q\left(\alpha_0,\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;z\right)$ is defined as follows (see for instance [10]):

$$ {}_{p+1}F_q\left(\alpha_0,\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;z\right)=\sum_{j=0}^{\infty}\left[\frac{(\alpha_0)_j\ \ (\alpha_1)_j\cdots(\alpha_p)_j\ \ z^j}{j!\ \left(\beta_1\right)_j\cdots\left(\beta_q\right)_j}\right]\ . \tag{1a}$$

Above and throughout, the Pochhammer symbol $(\alpha)_j$ is defined as follows:

$$(\alpha)_0 = 1 \ ; \quad (\alpha)_j = \alpha \ (\alpha+1) \cdots (\alpha+j-1) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \quad \text{for} \quad j = 1, 2, 3, ... \ . \tag{1b}$$

Clearly if one of the $p+1$ parameters α_j is a *negative* integer, say $\alpha_0 = -N$, and all the other $p+q$ parameters α_j and β_k have *generic* (possibly complex) values, the series in the right-hand side of the definition (1a) of the generalized hypergeometric function terminates at $j = N$ (since $(-N)_j = 0$ for $j = N+1, \ N+2, ...$) . Hereafter we call *generalized hypergeometric polynomial* the resulting polynomial (of degree N in z, and conveniently defined as follows, so that it is *monic*):

$$P_N\left(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z\right) = \sum_{m=0}^{N} \left[\frac{(-N)_m (\alpha_1)_m \cdots (\alpha_p)_m \ z^{N-m}}{m! \ (\beta_1)_m \cdots (\beta_q)_m} \right] ; \tag{2a}$$

and we denote its N zeros as ζ_n,

$$P_N\left(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; \zeta_n\right) = 0 \ , \quad n = 1, ..., N \ . \tag{2b}$$

Hence the values of the N numbers ζ_n depend on the $1+p+q$ parameters N, α_j, β_k.

Notation 2.1. Above and hereafter N is an (arbitrarily assigned) *positive* integer, p and q are two (arbitrarily assigned) *nonnegative* integers, and indices such as n, m, ℓ (but not necessarily j, k) run over the N integers from 1 to N (unless otherwise indicated). The N zeros ζ_n are of course defined up to permutations. In the following we always assume the *same* assignment to be made for the correlation of the values of the N zeros of a polynomial with the values of the index n labeling them. And below underlined lower-case letters denote N-vectors (hence, for instance, $\underline{\zeta} \equiv (\zeta_1, ..., \zeta_N)$); and underlined upper-case letters denote $N \times N$ matrices (hence for instance the matrix $\underline{L}$ has the N^2 elements L_{nm}). Finally: we always adopt the standard convention according to which a sum containing no terms vanishes, and a product containing no terms equals unity: for instance, $\sum_{j,k=1;j\neq k}^{1} = 0$, $\prod_{j,k=1;j\neq k}^{1} = 1$. $\square$

The first result of this paper consists of the following

Proposition 2.1. The (set of) N zeros ζ_n of the *generalized hypergeometric polynomial* $P_N\left(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z\right)$, see (2a) and (2b), satisfy the following system of N algebraic equations:

$$\sum_{k=1}^{q+1} \left[b_k \ \ f_n^{(k)}\left(\underline{\zeta}\right) \right] - \sum_{j=0}^{p} \left[a_j \ g_n^{(j)}\left(\underline{\zeta}\right) \right] = 0 \ , \quad n = 1, ..., N \ . \tag{3}$$

Here the $q+1$ coefficients b_k, respectively the $p+1$ coefficients a_j, are defined in terms of the q parameters β_k respectively the p parameters α_j so that

$$x \prod_{k=1}^{q} (\beta_k - 1 - x) = \sum_{k=1}^{q+1} \left(b_k \ x^k \right) \ , \tag{4a}$$

hence

$$b_1 = \prod_{k=1}^{q} (\beta_k - 1 \ ) \ , \tag{4b}$$

$$b_2 = -\sum_{j=1}^{q} \left[\prod_{k=1, \ k\neq j}^{q} (\beta_k - 1) \right] \ , \tag{4c}$$

$$b_3 = \frac{1}{2} \sum_{\ell,j=1;\ell\neq j}^{q} \left[\prod_{k=1, \ k\neq \ell,j}^{q} (\beta_k - 1) \right] \ , \tag{4d}$$

and so on up to

$$b_{q+1} = (-1)^q \ ; \tag{4e}$$

respectively

$$\prod_{j=1}^{p} (\alpha_j - x) = \sum_{j=0}^{p} a_j \ x^j \ , \tag{5a}$$

hence

$$a_0 = \prod_{j=1}^{p} (\alpha_j) \ , \tag{5b}$$

$$a_1 = -\sum_{k=1}^{p} \left[\prod_{j=1;\; j\neq k}^{p} (\alpha_j) \right] \;, \tag{5c}$$

$$a_2 = \frac{1}{2} \sum_{\ell,k=1;\ell\neq k}^{p} \left[\prod_{j=1;\; j\neq \ell,k}^{p} (\alpha_j) \right] \;, \tag{5d}$$

and so on up to

$$a_p = (-1)^p \;. \tag{5e}$$

As for the functions $f_n^{(j)}\left(\underline{\zeta}\right)$ of the N zeros ζ_m, they are defined recursively as follows:

$$f_n^{(j+1)}\left(\underline{\zeta}\right) = -f_n^{(j)}\left(\underline{\zeta}\right) + \sum_{\ell=1;\; \ell\neq n}^{N} \left[\frac{\zeta_n\; f_\ell^{(j)}\left(\underline{\zeta}\right) + \zeta_\ell\; f_n^{(j)}\left(\underline{\zeta}\right)}{\zeta_n - \zeta_\ell} \right] \;, \quad j = 1,2,3,... \;, \tag{6a}$$

with

$$f_n^{(1)}\left(\underline{\zeta}\right) = \zeta_n \;, \tag{6b}$$

implying the expressions of $f_n^{(j)}\left(\underline{\zeta}\right)$ with $j = 1,2,3,...$ reported in the Appendix, see (59).

And the functions $g_n^{(j)}\left(\underline{\zeta}\right)$ of the N zeros ζ_n are defined as follows:

$$g_n^{(0)}\left(\underline{\zeta}\right) = 1 \;, \tag{7a}$$

$$g_n^{(j)}\left(\underline{\zeta}\right) = \sum_{\ell=1;\; \ell\neq n}^{N} \left[\frac{f_n^{(j)}\left(\underline{\zeta}\right) + f_\ell^{(j)}\left(\underline{\zeta}\right)}{\zeta_n - \zeta_\ell} \right] \;, \quad j = 1,2,... \;, \tag{7b}$$

implying

$$g_n^{(1)}\left(\underline{\zeta}\right) = \sum_{\ell=1;\; \ell\neq n}^{N} \left(\frac{\zeta_n + \zeta_\ell}{\zeta_n - \zeta_\ell} \right) \;, \tag{7c}$$

and the expressions of $g_n^{(j)}\left(\underline{\zeta}\right)$ with $j = 1,2,3,...$ reported in the Appendix, see (61). $\square$

Remark 2.1. The functions $f_n^{(j)}\left(\underline{\zeta}\right)$ and $g_n^{(j)}\left(\underline{\zeta}\right)$ are *universal*: they do not depend on the generalized hypergeometric polynomial under consideration. But of course their arguments do, being the N zeros ζ_n of the polynomial $P_N\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;\zeta\right)$, see (2). $\square$

The second (and main) result of this paper consist of the following

Proposition 2.2. Let the (unordered) set of N numbers ζ_n denote the N zeros of the *generalized hypergeometric polynomial* $P_N\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;z\right)$, see (2a) and (2b); and let the $N\times N$ matrix $\underline{L}\left(\underline{\zeta}\right)$ be defined componentwise as follows, in terms of these N zeros and the $1+p+q$ parameters N, α_j, β_k characterizing the *generalized hypergeometric polynomial* $P_N\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;z\right)$:

$$L_{nm}\left(\underline{\zeta}\right) = \sum_{k=1}^{q+1} \left[b_k\; f_{n,m}^{(k)}\left(\underline{\zeta}\right) \right] - \sum_{j=1}^{p} \left[a_j\; g_{n,m}^{(j)}\left(\underline{\zeta}\right) \right] \;, \tag{8}$$

where of course the coefficients b_k and a_j are defined as above, see (5) and (4), while $f_{n,m}^{(k)}\left(\underline{\zeta}\right)$ respectively $g_{n,m}^{(j)}\left(\underline{\zeta}\right)$ are defined, in terms of the quantities $f_n^{(k)}\left(\underline{\zeta}\right)$ respectively $g_n^{(j)}\left(\underline{\zeta}\right)$ (see (6) respectively (7)), as follows:

$$f_{n,m}^{(k)}\left(\underline{\zeta}\right) = \left. \frac{\partial\; f_n^{(k)}\left(\underline{z}\right)}{\partial\; z_m} \right|_{\underline{z}=\underline{\zeta}} \;, \quad g_{n,m}^{(j)}\left(\underline{\zeta}\right) = \left. \frac{\partial\; g_n^{(j)}\left(\underline{z}\right)}{\partial\; z_m} \right|_{\underline{z}=\underline{\zeta}} \;. \tag{9}$$

Expressions of $f_{n,m}^{(j)}\left(\underline{\zeta}\right)$ and $g_{n,m}^{(j)}\left(\underline{\zeta}\right)$ with $j = 1,2,3,...$ are reported in the Appendix, see (64) and (65).

Then the N eigenvalues λ_m of the $N\times N$ matrix $\underline{L}\left(\underline{\zeta}\right)$,

$$\underline{L}\left(\underline{\zeta}\right)\; \underline{v}^{(m)}\left(\underline{\zeta}\right) = \lambda_m\; \underline{v}^{(m)}\left(\underline{\zeta}\right) \;, \quad m = 1,...,N \tag{10a}$$

—hence the N roots λ_m of the following polynomial equation (of degree N in λ):

$$\det\left[\underline{L}\left(\underline{\zeta}\right) - \lambda\right] = 0 \tag{10b}$$

—are given by the formula

$$\lambda_m\left(\beta_1,...,\beta_q\right)=m\ \prod_{k=1}^{q}\left(\beta_k-1+m\right)\ ,\quad m=1,...,N\ .\ \square \tag{10c}$$

Remark 2.2. The functions $f_{n,m}^{(j)}\left(\underline{\zeta}\right)$ and $g_{n,m}^{(j)}\left(\underline{\zeta}\right)$ are *universal*: they do not depend on the generalized hypergeometric polynomial under consideration (see *Remark 2.1*). But of course their arguments do, being the N zeros ζ_n of the polynomial $P_N\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;\zeta\right)$, see (2). □

Remark 2.3. The N eigenvalues λ_m of the $N\times N$ matrix $\underline{L}$ (see (8)) depend only on the q parameters β_k (see (10c)), while the matrix $\underline{L}$ depends itself on the $q+p$ parameters β_k and α_j—via the dependence of the parameters b_k respectively a_j on β_k respectively α_j (see (4) respectively (5)) and the dependence of the N zeros ζ_n on the parameters β_k and α_j, see (2b) or, equivalently, on b_k and a_j, see (3). Hence the $N\times N$ matrix $\underline{L}$ is *isospectral* for variations of the p parameters α_j. And note moreover that the N eigenvalues λ_m are *integer* (or *rational*) numbers if the q parameters β_k are themselves *integer* (or *rational*) numbers: a nontrivial *Diophantine* property of the $N\times N$ matrix $\underline{L}$. □

Remark 2.4. All the above results are of course true as written only provided the N zeros ζ_n are all different among themselves; but they clearly remain valid by taking appropriate limits whenever this restriction does not hold. □

Remark 2.5. Immediate generalizations—whose explicit formulations can be left to the interested reader—of *Propositions 2.1* and *2.2* obtain from these two propositions via the special assignment $\alpha_{\hat{q}+j}=\beta_{\hat{p}+j}$ for $j=1,...r$ with r an arbitrary *nonnegative integer* such that both $\hat{q}=q-r$ and $\hat{p}=p-r$ are *positive integers*. These propositions refer then to the N zeros of the generalized hypergeometric polynomial $P_N\left(\alpha_1,...,\alpha_{\hat{p}};\beta_1,...,\beta_{\hat{q}};z\right)$—which depend only on the $1+\hat{p}+\hat{q}=1+p+q-2r$ parameters N, α_j with $j=1,...,\hat{p}=p-r$ and b_k with $k=1,...,\hat{q}=q-r$, but feature quantities b_k and a_j (see (3) and (8)) that depend on the $1+p+q$ parameters N, α_j with $j=1,...,p$ and b_k with $k=1,...,q$. □

The two *Propositions 2.1* and *2.2* are proven in the following Section 3; some comments and prospects of future developments are outlined in the last Section 4.

Let us end this Section 2 by displaying explicitly the above results for small values of the integers p, q and (of course) r (see *Remark 2.5*).

2.1 The case $p=q=1,\ r=0$

For $p=q=1$, $r=0$ (for the definition of r see *Remark 2.5*) implying (see (5) and (4))

$$a_0=\alpha_1,\ a_1=-1,\quad b_1=\beta_1-1,\ b_2=-1\ , \tag{11}$$

let the N numbers ζ_n be the N zeros of the hypergeometric polynomial

$$P_N\left(\alpha_1;\beta_1;z\right)=\sum_{m=0}^{N}\left[\frac{\left(-N\right)_m\left(\alpha_1\right)_m\ z^{N-m}}{m!\ \left(\beta_1\right)_m}\right]\ . \tag{12}$$

Then *Proposition 2.1* implies that these N zeros ζ_n satisfy the following system of N nonlinear algebraic equations

$$N-1-\alpha_1+\beta_1\ \zeta_n-2\ \left(\zeta_n-1\right)\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)=0\ ,\quad n=1,...,N\ . \tag{13}$$

Notation 2.2. Above and hereafter the notation $\sigma_n^{(r,\rho)}\left(\underline{\zeta}\right)$ is defined by (52). □

As for *Proposition 2.2,* it implies in this case that the following $N\times N$ matrix $\underline{L}\left(\underline{\zeta}\right)$, defined componentwise as follows (see (8) with (64), (65) and (52)),

$$\begin{aligned} L_{nm}\left(\underline{\zeta}\right) &= \delta_{nm}\left\{\beta_1+2\sum_{\ell=1;\ell\neq n}^{N}\left[\frac{\zeta_\ell\ \left(\zeta_\ell-1\right)}{\left(\zeta_n-\zeta_\ell\right)^2}\right]\right\} \\ &\quad -2\ \left(1-\delta_{nm}\right)\ \frac{\zeta_n\ \left(\zeta_n-1\right)}{\left(\zeta_n-\zeta_m\right)^2}\ , \end{aligned} \tag{14a}$$

features the N eigenvalues

$$\lambda_m=m\ \left(\beta_1-1+m\right)\ ,\quad m=1,...,N\ . \tag{14b}$$

2.2 The case $p = 2,\ q = 1,\ r = 0$

For $p = 2$, $q = 1$, $r = 0$ (for the definition of r see *Remark 2.5*), implying (see (5) and (4))

$$a_0 = \alpha_1\ \alpha_2,\ \ a_1 = -\left(\alpha_1 + \alpha_2\right),\ \ a_2 = 1\ ,\quad b_1 = \beta_1 - 1,\ \ b_2 = -1, \tag{15}$$

let the N numbers ζ_n be the N zeros of the hypergeometric polynomial

$$P_N\left(\alpha_1, \alpha_2; \beta_1; z\right) = \sum_{m=0}^{N} \left[\frac{(-N)_m\,(\alpha_1)_m\ \ (\alpha_2)_m\ \ z^{N-m}}{m!\ \ (\beta_1)_m}\right]\ . \tag{16}$$

Then *Proposition 2.1* implies that these N zeros ζ_n satisfy the following system of N nonlinear algebraic equations

$$\begin{aligned} &-\alpha_1\ \alpha_2 + (N-1)\ (\alpha_1 + \alpha_2 + 1) + \beta_1\ \zeta_n \\ &+2\ (3 - N + \alpha_1 + \alpha_2 - \zeta_n)\ \sigma_n^{(1,1)}(\underline{\zeta}) + 3\ \sigma_n^{(2,2)}(\underline{\zeta}) + 3\ [\sigma_n^{(1,1)}(\underline{\zeta})]^2 \ = \ 0\ , \\ &n = 1, ..., N\ . \end{aligned} \tag{17}$$

As for *Proposition 2.2,* it implies in this case that the following $N \times N$ matrix $\underline{L}\left(\underline{\zeta}\right)$, defined componentwise as follows (see (8) with (64), (65) and (52)),

$$\begin{aligned} L_{nn}\left(\underline{\zeta}\right) \ &= \ 2\ \left[\frac{\beta_1}{2} + (N - 3 - \alpha_1 - \alpha_2)\ \ \sigma_n^{(1,2)}\left(\underline{\zeta}\right) + \sigma_n^{(2,2)}\left(\underline{\zeta}\right) - 3\ \sigma_n^{(2,3)}\left(\underline{\zeta}\right)\right. \\ &\left. +3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)\ \ \sigma_n^{(1,2)}\left(\underline{\zeta}\right)\right]\ ,\quad n = 1, 2, ..., N\ , \end{aligned} \tag{18a}$$

$$\begin{aligned} L_{nm}\left(\underline{\zeta}\right) \ &= \ 2\ \zeta_n\ \left[\frac{\alpha_1 + \alpha_2 - N - \zeta_n - 3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)}{\left(\zeta_n - \zeta_m\right)^2}\right. \\ &\left. + \frac{3\ \zeta_n}{\left(\zeta_n - \zeta_m\right)^3}\right]\ ,\quad n, m = 1, ..., N\ ,\quad n \neq m\ , \end{aligned} \tag{18b}$$

features (again) the N eigenvalues

$$\lambda_m = m\ \ (\beta_1 - 1 + m)\ \ ,\quad m = 1, ..., N\ . \tag{18c}$$

Note the *isospectral* character of this matrix $\underline{L}\left(\underline{\zeta}\right)$, which depends explicitly on the 2 parameters $\alpha = \alpha_1 + \alpha_2$ and β_1 and implicitly on the 3 parameters α_1, α_2 and β_1 via the dependence on these 3 parameters of the N zeros ζ_n of the polynomial $P_N\left(\alpha_1, \alpha_2; \beta_1; z\right)$, while its eigenvalues λ_m only depend on the single parameter β_1.

2.3 The case $p = q = 2,\ r = 0$

For $p = q = 2$, $r = 0$ (for the definition of r see *Remark 2.5*), implying (see (5) and (4))

$$\begin{aligned} a_0 \ &= \ \alpha_1\ \alpha_2\ ,\quad a_1 = -\left(\alpha_1 + \alpha_2\right)\ ,\quad a_2 = 1\ , \\ b_1 \ &= \ (1 - \beta_1)\ \ (1 - \beta_2)\ ,\quad b_2 = 2 - \beta_1 - \beta_2\ ,\quad b_3 = 1\ , \end{aligned} \tag{19}$$

let the N numbers ζ_n be the N zeros of the hypergeometric polynomial

$$P_N\left(\alpha_1, \alpha_2; \beta_1, \beta_2; z\right) = \sum_{m=0}^{N} \left[\frac{(-N)_m\,(\alpha_1)_m\ \ (\alpha_2)_m\ \ z^{N-m}}{m!\ \ (\beta_1)_m\ \ (\beta_2)_m}\right]\ . \tag{20}$$

Then *Proposition 2.1* implies that these N zeros ζ_n satisfy the following system of N nonlinear algebraic equations

$$\begin{aligned} &-\alpha_1\alpha_2 + (N-1)\ (\alpha_1 + \alpha_2 + 1) + \beta_1\ \beta_2\ \zeta_n \\ &-2\ \ [(1 + \beta_1 + \beta_2)\zeta_n - \alpha_1 - \alpha_2 + N - 3]\ \ \sigma_n^{(1,1)}(\underline{\zeta}) \\ &+3\ (\zeta_n - 1)\ \left\{\left[\sigma_n^{(1,1)}(\underline{\zeta})\right]^2 - \sigma_n^{(2,2)}(\underline{\zeta})\right\} = 0\ ,\quad n = 1, ..., N\ . \end{aligned} \tag{21}$$

As for *Proposition 2.2,* it implies in this case that the following $N \times N$ matrix $\underline{L}\left(\underline{\zeta}\right)$, defined componentwise as follows (see (8) with (64), (65) and (52)),

$$L_{nn}\left(\underline{\zeta}\right) = \beta_1\ \beta_2 + [5 + 2\ \ (\beta_1 + \beta_2)]\ \ \sigma_n^{(2,2)}\left(\underline{\zeta}\right) + 6\ \sigma_n^{(3,3)}\left(\underline{\zeta}\right)$$
$$+2\ \left[(N - 3 - \alpha_1 - \alpha_2)\ \ \sigma_n^{(1,2)}\left(\underline{\zeta}\right) - 3\ \sigma_n^{(2,3)}\left(\underline{\zeta}\right) + 3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)\ \ \sigma_n^{(1,2)}\left(\underline{\zeta}\right)\right]$$
$$-3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)\ \ \left[\sigma_n^{(1,1)}\left(\underline{\zeta}\right) + 2\ \sigma_n^{(2,2)}\left(\underline{\zeta}\right)\right]\ ,\quad n = 1, 2, ..., N\ , \tag{22a}$$

$$L_{nm}\left(\underline{\zeta}\right) \quad = \quad 2\ \zeta_n\ \left[\frac{\alpha_1 + \alpha_2 - N + (2 - \beta_1 - \beta_2)\ \ \zeta_n + 3\ \ (\zeta_n - 1)\ \ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)}{(\zeta_n - \zeta_m)^2}\right.$$
$$\left. - \frac{3\ \zeta_n\ \ (\zeta_n - 1)}{(\zeta_n - \zeta_m)^3}\right]\ ,\quad n, m = 1, ..., N\ ,\quad n \neq m\ , \tag{22b}$$

features the N eigenvalues

$$\lambda_m = m\ \ (\beta_1 - 1 + m)\ \ (\beta_2 - 1 + m)\ \ ,\quad m = 1, ..., N\ . \tag{22c}$$

Note the isospectral character of this matrix $\underline{L}\left(\underline{\zeta}\right)$, which depends explicitly on the 3 parameters $\alpha = \alpha_1 + \alpha_2$, β_1 and β_2 and implicitly on the 4 parameters α_1, α_2, β_1 and β_2 via the dependence on these 4 parameters of the N zeros ζ_n of the polynomial $P_N\left(\alpha_1, \alpha_2; \beta_1, \beta_2; z\right)$, while its eigenvalues λ_m only depend on the 2 parameter β_1 and β_2.

2.4 The case $p = q = 2$, $r = 1$

For $p = q = 2$, $r = 1$ (for the definition of r see *Remark 2.5*) we have (as above, see (19), but now with $\beta_2 = \alpha_2$)

$$\begin{aligned} a_0 &= \alpha_1\ \alpha_2\ ,\quad a_1 = -\left(\alpha_1 + \alpha_2\right),\quad a_2 = 1\ , \\ b_1 &= (1 - \beta_1)\ \ (1 - \alpha_2)\ \ ,\quad b_2 = 2 - \beta_1 - \alpha_2\ \ ,\quad b_3 = 1\ , \end{aligned} \tag{23}$$

and we now define the N numbers ζ_n as the N zeros of the polynomial (12) (rather than (20); so that these N zeros do not depend on the *arbitrary* parameter $\beta_2 = \alpha_2$). Then *Proposition 2.1* implies that these N zeros ζ_n satisfy the following system of N nonlinear algebraic equations

$$-\alpha_1\ \alpha_2 + (N - 1)\ \ (\alpha_1 + \alpha_2 + 1) + \beta_1\ \alpha_2\ \zeta_n$$
$$+2\ \ [3 - N + \alpha_1 + \alpha_2 - (1 + \beta_1 + \alpha_2)\ \zeta_n]\ \ \sigma_n^{(1,1)}(\underline{\zeta})$$
$$+3(\zeta_n - 1)\left\{\left[\sigma_n^{(1,1)}(\underline{\zeta})\right]^2 - \sigma_n^{(2,2)}(\underline{\zeta})\right\} = 0\ ,\quad n = 1, ..., N\ , \tag{24a}$$

—which is *different* from (13), although satisfied by the *same* zeros. But since this system of N equations must hold for arbitrary values of the parameters α_2, it amounts in fact to the following two separate systems of N equations:

$$-\alpha_1 + N - 1 + \beta_1\ \zeta_n + 2\ \ (1 - \zeta_n)\ \ \sigma_n^{(1,1)}(\underline{\zeta}) = 0\ ,\quad n = 1, ..., N\ , \tag{25a}$$

$$(N - 1)\ \ (\alpha_1 + 1) + 2\ \ [3 - N + \alpha_1 - (1 + \beta_1)\ \zeta_n]\ \ \sigma_n^{(1,1)}(\underline{\zeta})$$
$$+3(\zeta_n - 1)\left\{\left[\sigma_n^{(1,1)}(\underline{\zeta})\right]^2 - \sigma_n^{(2,2)}(\underline{\zeta})\right\} = 0\ ,\quad n = 1, ..., N\ , \tag{25b}$$

which must both be satisfied by the N zeros ζ_n of the polynomial (12). Indeed the first of these two systems coincides with (13); while the second is new (but in fact both these systems of N equations are not quite new, see Subsection 2.5).

As for *Proposition 2.2,* it implies in this case that the matrix $\underline{L}\left(\underline{\zeta}\right)$ defined by (22a) and (22b) (of course with $\beta_2 = \alpha_2$) hence reading

$$L_{nn}\left(\underline{\zeta}\right) = \beta_1\ \alpha_2 + [5 + 2\ \ (\beta_1 + \alpha_2)]\ \ \sigma_n^{(2,2)}\left(\underline{\zeta}\right) + 6\ \sigma_n^{(3,3)}\left(\underline{\zeta}\right)$$
$$+2\ \left[(N - 3 - \alpha_1 - \alpha_2)\ \ \sigma_n^{(1,2)}\left(\underline{\zeta}\right) - 3\ \sigma_n^{(2,3)}\left(\underline{\zeta}\right) + 3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)\ \ \sigma_n^{(1,2)}\left(\underline{\zeta}\right)\right]$$
$$-3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)\ \ \left[\sigma_n^{(1,1)}\left(\underline{\zeta}\right) + 2\ \sigma_n^{(2,2)}\left(\underline{\zeta}\right)\right]\ ,\quad n = 1, 2, ..., N\ , \tag{26a}$$

$$L_{nm}\left(\underline{\zeta}\right) = 2\ \zeta_n\ \left[\frac{\alpha_1+\alpha_2-N+\left(2-\beta_1-\alpha_2\right)\ \zeta_n+3\ \left(\zeta_n-1\right)\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)}{\left(\zeta_n-\zeta_m\right)^2} - \frac{3\ \zeta_n\ \left(\zeta_n-1\right)}{\left(\zeta_n-\zeta_m\right)^3}\right]\ ,\quad n,m=1,...,N\ ,\quad n\neq m\ , \tag{26b}$$

—but now with the N zeros ζ_n in these two formulas being again those of the generalized hypergeometric polynomial (12) rather than (20), so that they only depend on N, α_1 and β_1 (but not on $\beta_2=\alpha_2$)—features the N eigenvalues

$$\lambda_m = m\ \left(\beta_1-1+m\right)\ \left(\alpha_2-1+m\right)\ ,\quad m=1,...,N\ . \tag{26c}$$

But again, since these properties must hold for arbitrary values of the parameter α_2, they amount to two separate statements, the first of which is easily seen to reproduce the statement that the matrix (14a) features the eigenvalues (14b), while the second states that the above matrix (26a), (26b) with $\alpha_2=0$, i. e.

$$L_{nn}\left(\underline{\zeta}\right) = \left(5+2\ \beta_1\right)\ \sigma_n^{(2,2)}\left(\underline{\zeta}\right)+6\ \sigma_n^{(3,3)}\left(\underline{\zeta}\right)$$
$$+2\ \left[\left(N-3-\alpha_1\right)\ \sigma_n^{(1,2)}\left(\underline{\zeta}\right)-3\ \sigma_n^{(2,3)}\left(\underline{\zeta}\right)+3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)\ \sigma_n^{(1,2)}\left(\underline{\zeta}\right)\right]$$
$$-3\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)\ \left[\sigma_n^{(1,1)}\left(\underline{\zeta}\right)+2\ \sigma_n^{(2,2)}\left(\underline{\zeta}\right)\right]\ ,\quad n=1,2,...,N\ , \tag{27a}$$

$$L_{nm}\left(\underline{\zeta}\right) = 2\ \zeta_n\ \left[\frac{\alpha_1-N+\left(2-\beta_1\right)\ \zeta_n+3\ \left(\zeta_n-1\right)\ \sigma_n^{(1,1)}\left(\underline{\zeta}\right)}{\left(\zeta_n-\zeta_m\right)^2} - \frac{3\ \zeta_n\ \left(\zeta_n-1\right)}{\left(\zeta_n-\zeta_m\right)^3}\right]\ ,\quad n,m=1,...,N\ ,\quad n\neq m\ , \tag{27b}$$

features the eigenvalues (26c) with $\alpha_2=0$, i. e.

$$\lambda_m = m\ \left(m-1\right)\left(\beta_1-1+m\right)\ ,\quad m=1,...,N\ . \tag{27c}$$

Note that, while these eigenvalues depend on the parameters β_1, they do *not* depend on the parameter α_1; hence in this case the matrix $\underline{L}\left(\underline{\zeta}\right)$, which itself depends on the 2 parameters α_1 and β_1, is *isospectral* for variations of the parameter α_1.

2.5 Results for Jacobi polynomials

For $p=q=1$—or, equivalently, for $\hat{p}=\hat{q}=1$ (for this notation see *Remark 2.5*)—the generalized hypergeometric polynomial is simply related to the Jacobi polynomial (see eq. 10.8(16) of [16]): the transformation (up to an irrelevant multiplicative constant) from the generalized hypergeometric polynomial $P_N\left(\alpha_1;\beta_1;z\right)$, see (12), to the standard Jacobi polynomial $P_N^{(\alpha,\beta)}\left(x\right)$ (see [16]) corresponds to the change of variables $\beta_1=\alpha+1$, $\alpha_1=N+\alpha+\beta+1$ and $z=2/\left(1-x\right)$. It can thereby be verified (with some labor) that the results—corresponding to *Proposition 2.1* with $p=q=1$, $r=0$ respectively $p=q=2$, $r=1$—reported above for these cases reproduce known results [2]: specifically (13) (or, equivalently, (25a)) respectively (25b) reproduce (up to appropriate notational changes) eqs. (5.2a) respectively (5.2b) of [2].

On the other hand *Proposition 2.2* with the above assignments of p, q and r seem to produce *new* results for the N zeros x_n of the Jacobi polynomial $P_N^{(\alpha,\beta)}\left(x\right)$, as displayed below. Indeed for $p=q=1$, $r=0$ it yields the following

Proposition 2.5.1. The $N\times N$ matrix $\underline{L}\left(\underline{x}\right)$ defined componentwise, in terms of the N zeros x_n of the Jacobi polynomial $P_N^{(\alpha,\beta)}\left(x\right)$ and the two parameters α and β as follows,

$$L_{nm}\left(\underline{x}\right) = \delta_{nm}\ \left\{\alpha+1+\sum_{\ell=1,\ \ell\neq n}^{N}\left[\frac{\left(1+x_\ell\right)\ \left(1-x_n\right)^2}{\left(x_n-x_\ell\right)^2}\right]\right\} - \left(1-\delta_{nm}\right)\ \left[\frac{\left(1+x_n\right)\ \left(1-x_m\right)^2}{\left(x_n-x_m\right)^2}\right]\ , \tag{28a}$$

has the N eigenvalues

$$\lambda_m = m\ \left(m+\alpha\right)\ ,\quad m=1,...,N\ .\ \square \tag{28b}$$

Let us again note the *isospectral* property of this matrix $\underline{L}(\underline{x})$, whose elements depend, via the N zeros $x_n \equiv x_n(\alpha,\beta)$, on the two parameters α and β, while its eigenvalues depend only on the parameter α.

Also note that Corollary 5.2.2 of [2] with $s = N-1$, $r = 0$, and of course $n = N$, identifies an $N \times N$ matrix G (see eq. (5.17) of [2] with C and X defined by eq. (5.4) and (5.3) of [2]) defined componentwise as follows:

$$G_{nm} = \delta_{nm} \sum_{\ell=1,\ell\neq n}^{N} \frac{(1-x_\ell^2)(1-x_n)}{(x_n-x_\ell)^2} - (1-\delta_{nm})\frac{(1-x_m)^2(1+x_m)}{(x_n-x_m)^2}, \tag{29a}$$

and states that its N eigenvalues g_m read as follows (see eq. (5.19) of [2], with $N-m$ replaced by $m-1$ since these numbers span the same set of values—from 0 to $N-1$—for $m = 1,...,N$):

$$g_m = (m-1)\ \ (m+\alpha-1)\ \ , \quad m = 1,...,N\ . \tag{29b}$$

These formulas, (29), are similar to, but different from, (28); although of course there must be a way to relate them, since they hold for the same set of $N+1$ numbers α and $x_n \equiv x_n(\alpha,\beta)$.

And likewise for $p = q = 2$, $r = 1$ it yields—via a treatment analogous to that of Section 2.4—the following

Proposition 2.5.2. The $N \times N$ matrix $\underline{L}(\underline{x})$ defined componentwise, in terms of the N zeros x_n of the Jacobi polynomial $P_N^{(\alpha,\beta)}(x)$ and the two parameters α and β as follows,

$$\begin{aligned} L_{nn} \;=\;& (7+2\alpha)\ \sigma_n^{(2,2)}(\underline{x}) + 6\ \sigma_n^{(3,3)}(\underline{x}) - 2\ \ (4+\alpha+\beta)\ \ \sigma_n^{(1,2)}(\underline{x}) - 6\ \sigma_n^{(2,3)}(\underline{x}) \\ & -3\ \left[\sigma_n^{(1,1)}(\underline{x})\right]^2 - 6\ \sigma_n^{(1,1)}(\underline{x})\ \ \sigma_n^{(2,2)}(\underline{x}) + 6\ \sigma_n^{(1,1)}(\underline{x})\ \ \sigma_n^{(1,2)}(\underline{x})\ \ , \end{aligned} \tag{30a}$$

$$\begin{aligned} L_{nm} \;=\;& \left[(\alpha+\beta+1)\ \ (1-x_n) + 2\ \ (1-\alpha) + 3\ \ (1+x_n)\ \ \sigma_n^{(1,1)}(\underline{x})\right]\ \left(\frac{1-x_m}{x_n-x_m}\right)^2 \\ & -3\ \ (1+x_n)\ \ \left(\frac{1-x_m}{x_n-x_m}\right)^3\ \ , \end{aligned} \tag{30b}$$

where (see (52))

$$\sigma_n^{(r,\rho)}(\underline{x}) == \sum_{\ell=1;\ell\neq n}^{N}\left[\left(\frac{2}{1-x_\ell}\right)^{r-\rho}\ \ \left(\frac{1-x_n}{x_n-x_\ell}\right)^{\rho}\right]\ , \tag{30c}$$

has the N eigenvalues

$$\lambda_m = m\ \ (m-1)\ \ (m+\alpha)\ \ , \quad m = 1,...,N\ .\ \square \tag{30d}$$

Note again the *isospectral* character of this matrix, which depends on the two parameters α and β, while its eigenvalues depend only on the parameter α.

3 Proofs

In this section we prove the findings reported in the preceding Section 2.

Let the t-dependent monic polynomial, of degree N in z and characterized by its N zeros $z_n(t)$ and its N coefficients $c_m(t)$,

$$\begin{aligned} \psi_N(z,t) \;=\;& \prod_{n=1}^{N}[z-z_n(t)] = z^N + \sum_{m=1}^{N}\left[c_m(t)\ \ z^{N-m}\right] \\ =\;& \sum_{m=0}^{N}\left[c_m(t)\ \ z^{N-m}\right] \quad \text{with} \quad c_0 = 1\ , \end{aligned} \tag{31}$$

satisfy the *linear* Partial Differential Equation (PDE)

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)\ \psi_N(z,t) = -\Bigg\{ & \left(z\ \frac{\partial}{\partial z} - N\right)\prod_{j=1}^{q}\left[\beta_j - 1 - \left(z\ \frac{\partial}{\partial z} - N\right)\right] \\ & - \left(\frac{\partial}{\partial z}\right)\prod_{j=1}^{p}\left[\alpha_j - \left(z\ \frac{\partial}{\partial z} - N\right)\right]\Bigg\}\ \psi_N(z,t)\ \ . \end{aligned} \tag{32}$$

Clearly this implies that its coefficients $c_m(t)$ satisfy the following system of N *linear* Ordinary Differential Equations (ODEs):

$$\left(\frac{d}{d\,t}\right)\; c_m(t) = m\;\left[\prod_{j=1}^{q}\left(\beta_j - 1 + m\right)\right]\; c_m(t)$$

$$+\left(N+1-m\right)\;\left[\prod_{j=1}^{p}\left(\alpha_j - 1 + m\right)\right]\; c_{m-1}(t)\;,\quad m=1,...,N\;, \tag{33a}$$

with the conditions (see (31))

$$c_0(t) = 1\;, \tag{33b}$$

$$c_j(t) = 0 \quad \text{for} \quad j<0 \quad \text{and for} \quad j>N\;. \tag{33c}$$

Hence the general solution of this system of N *linear* ODEs reads as follows:

$$\underline{c}(t) = \sum_{m=1}^{N}\left[\tilde{\eta}_m\;\exp\left(\tilde{\lambda}_m\;t\right)\;\underline{\tilde{v}}^{(m)}\right]\;, \tag{34}$$

where the N (t-independent) parameters $\tilde{\eta}_m$ can be arbitrarily assigned (or adjusted to satisfy the N *initial* conditions $c_m(0)$), while the numbers $\tilde{\lambda}_m$ respectively the N-vectors $\underline{\tilde{v}}^{(m)}$ are clearly the N eigenvalues respectively the N eigenvectors of the algebraic eigenvalue problem

$$\underline{\Lambda}\;\underline{\tilde{v}}^{(m)} = \tilde{\lambda}_m\;\underline{\tilde{v}}^{(m)}\;,\quad m=1,...,N\;, \tag{35a}$$

with the $N\times N$ matrix $\underline{\Lambda}$ defined componentwise as follows:

$$\Lambda_{m,m} = m\;\prod_{j=1}^{q}\left(\beta_j - 1 + m\right)\;,\quad m=1,...,N\;, \tag{35b}$$

$$\Lambda_{m,m-1} = \left(N+1-m\right)\;\prod_{j=1}^{p}\left(\alpha_j - 1 + m\right)\;,\quad m=2,...,N\;, \tag{35c}$$

with all other elements vanishing, $\Lambda_{m,n}=0$ unless $n=m$ or $n=m-1$.

The (lower) *triangular* character of the matrix $\underline{\Lambda}$ implies that its N eigenvalues $\tilde{\lambda}_m$ can be explicitly evaluated:

$$\tilde{\lambda}_m = \Lambda_{m,m} = m\;\prod_{j=1}^{q}\left(\beta_j - 1 + m\right)\;,\quad m=1,...,N\;. \tag{36}$$

Let us now denote as $\bar{\psi}_N(z)$ the t-independent *monic polynomial* solution of (32),

$$\begin{aligned}\bar{\psi}_N(z) \;&=\; \prod_{n=1}^{N}\left[z-\zeta_n\right] = z^N + \sum_{m=1}^{N}\left[\gamma_m\;z^{N-m}\right]\\ &=\; \sum_{m=0}^{N}\left[\gamma_m\;z^{N-m}\right] \quad \text{with} \quad \gamma_0 = 1\;,\end{aligned} \tag{37}$$

hence the monic polynomial solution of the ODE

$$\left\{\left(z\,\frac{d}{d\,z} - N\right)\;\prod_{j=1}^{q}\left[\beta_j - 1 - \left(z\,\frac{d}{d\,z} - N\right)\right]\right.$$

$$\left.-\left(\frac{d}{d\,z}\right)\;\prod_{j=1}^{p}\left[\alpha_j - \left(z\,\frac{d}{d\,z} - N\right)\right]\right\}\;\bar{\psi}_N(z) = 0\;. \tag{38}$$

Note that we denote as γ_m its coefficients and as ζ_n its N zeros, see (37) (the fact that the notation for the zeros is identical to that used above, see (2b), is not accidental: see below). Clearly the t-independent coefficients γ_m correspond to the "equilibrium configuration" $c_m(t)=\gamma_m$ of the linear "dynamical system" (33a), hence they are characterized as the solutions of the system of N *linear algebraic* equations

$$\left(m+1\right)\;\left[\prod_{j=1}^{q}\left(\beta_j + m\right)\right]\;\gamma_{m+1} = \left(m-N\right)\;\left[\prod_{j=1}^{p}\left(\alpha_j + m\right)\right]\;\gamma_m\;,\quad m=0,...,N\;, \tag{39a}$$

with the conditions

$$\gamma_0 = 1 \ , \quad \gamma_{N+1} = 0 \ . \tag{39b}$$

The first of these conditions corresponds to (33b); the second (which via the recursion (39a) clearly implies $\gamma_m = 0$ for $m > N$) corresponds to (33c) and it is automatically satisfied because the right-hand side of the recursion (39a) vanishes for $m = N$ due to the factor $(m - N)$ (and note that no conditions need to be assigned on γ_m with $m < 0$ since no such values enter in the recursion (39a); in any case for $m = -1$ the recursion (39a) would imply $\gamma_{-1} = 0$ since its left-hand side vanishes due to the factor $(m + 1)$).

It is then plain from the two-term (hence explicitly solvable) recursion relation (39) that the $N+1$ parameters γ_m read as follows:

$$\gamma_m = \frac{(-N)_m \prod_{j=1}^{p} (\alpha_j)_m}{m! \prod_{j=1}^{q} (\beta_j)_m}, \quad m = 0, 1, ..., N \ . \tag{40}$$

This—besides implying that the ODE (38) does possess a polynomial solution $\bar{\psi}(z)$ of degree N in z—shows that this t-independent polynomial solution $\bar{\psi}(z)$ of the PDE (32) coincides with the *generalized hypergeometric polynomial* (2a) (compare (2a) to (37) with (40)); of course up to an overall multiplicative constant, which can be arbitrarily assigned due to the *linear* character of the ODE (38), and was chosen above so that the polynomial $\bar{\psi}(z)$ be *monic*, see (37), hence indeed coincide with the *generalized hypergeometric polynomial* (2a).

Our next task is to identify the "equations of motion" characterizing the t-evolution of the N zeros $z_n(t)$ of $\psi(z,t)$, see (31), implied by the PDE (32) and by the corresponding t-evolution of the N coefficients $c_m(t)$, see (33). To this end it is convenient to firstly reformulate the PDE (32) as follows:

$$\left(\frac{\partial}{\partial t}\right) \psi_N(z,t) = -\left\{\sum_{k=1}^{q+1}\left[b_k \left(z\frac{\partial}{\partial z} - N\right)^k\right] - \left(\frac{\partial}{\partial z}\right)\sum_{j=0}^{p}\left[a_j \left(z\frac{\partial}{\partial z} - N\right)^j\right]\right\} \psi_N(z,t) \ , \tag{41}$$

where of course the new parameters b_k respectively a_j are related to the parameters β_k respectively α_j as detailed above, see (4) and (5).

Then it is easily seen that the equations of motion characterizing the t-evolution of the N zeros $z_n(t)$ of $\psi(z,t)$, see (31), read as follows (of course below a superimposed dot denotes a t-differentiation):

$$\dot{z}_n = \sum_{k=1}^{q+1}\left[b_k \ f_n^{(k)}(\underline{z})\right] - \sum_{j=0}^{p}\left[a_j \ g_n^{(j)}(\underline{z})\right] \ , \quad n = 1, ..., N \ . \tag{42}$$

Indeed, this clearly follows from the PDE (41) via the identities (58a) and (60a) and the analogous identity corresponding to the logarithmic t-derivative of (31) hence reading (via the short-hand notation (48))

$$\left(\frac{\partial}{\partial t}\right) \psi_N(z,t) \Leftrightarrow -\dot{z}_n \ . \tag{43}$$

It is moreover plain from the developments reported above (see (37) and the sentence following (40)) that the coordinates ζ_n characterizing the *equilibrium* configuration $\underline{\zeta} \equiv (\zeta_1, ..., \zeta_n)$ of this system, —which of course (see (42)) satisfy the set of N algebraic (generally *nonlinear*) equations (3)—coincides with the N zeros of the generalized hypergeometric polynomial $P_N(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z)$, see (2b).

Proposition 2.1 is thereby proven.

Our next step is to consider the behavior of the dynamical system (42) in the infinitesimal vicinity of its equilibrium configuration $\underline{z}(t) = \underline{\zeta}$. To this end we set

$$\underline{z}(t) = \underline{\zeta} + \varepsilon \ \underline{x}(t) \ ; \quad z_n(t) = \zeta_n + \varepsilon \ x_n(t) \ , \quad n = 1, .., N \ , \tag{44}$$

with ε infinitesimal. We thereby *linearize* the equations of motion (42), getting

$$\underline{\dot{x}} = \underline{L} \ \underline{x} \ ; \quad \dot{x}_n = \sum_{m=1}^{N}(L_{nm} \ x_m) \ , \quad n = 1, ..., N \ , \tag{45}$$

with the $N \times N$ matrix $\underline{L}$ defined componentwise by (8).

The *general* solution of the system of *linear* ODEs (45) reads of course then as follows:

$$\underline{x}(t) = \sum_{m=1}^{N} \left[\eta_m \ \exp(\lambda_m \ t) \ \underline{v}^{(m)} \right] \ , \tag{46}$$

where the N (t-independent) parameters η_m can be arbitrarily assigned (or adjusted to satisfy the N *initial* conditions $x_n(0)$), while the numbers λ_m respectively the N-vectors $\underline{v}^{(m)}$ are clearly the N eigenvalues respectively the N eigenvectors of the algebraic eigenvalue problem (10a). But the behavior of the dynamical system (42) in the *immediate vicinity* of its *equilibria* cannot differ from its *general* behavior, which is characterized by the N exponentials $\exp\left(\tilde{\lambda}_m \ t\right)$, as implied by the relation between the N zeros $z_n(t)$ and the coefficients $c_m(t)$ of the monic polynomial (of degree N in z) $\psi_N(z,t)$, see (31), and by the explicit formula (34) with (36) detailing the time evolution of the N coefficients $c_m(t)$. Hence the (set of) eigenvalues λ_m of the matrix $\underline{L}$, see (8), must coincide with the (set of) eigenvalues $\tilde{\lambda}_m$, see (36), of the matrix $\underline{\Lambda}$, see (35).

Proposition 2.2 is thereby proven.

4 Outlook

A follow-up to the present paper shall take advantage of the new polynomial identities reported in Appendix A in order to identify *new* classes of *solvable* N-body problems of "goldfish" type [11] [12] [13] [14] involving *many-body* interactions and featuring several free parameters, by extending in a fairly obvious manner (indeed, see (42)) the approach and findings reported in [15].

Two possible directions of further investigation shall try and extend the approach and findings, reported in this paper for the N zeros of hypergeometric polynomials of order N, to the N zeros of basic hypergeometric polynomials of order N and to the, generally infinite, zeros of (nonpolynomial) generalized hypergeometric functions.

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6 Appendix A: Polynomial identities

Several *polynomial identities* are reported in Appendix A of [14] (see in particular the paperback version of this monograph, where the identities (A.8k) and (A.8l) are corrected) and in Appendix A of [15]. In this appendix we report, and then prove, two additional classes of analogous identities. These findings have an interest of their own: for instance they open the way to the identification of new classes of solvable N-body problems of "goldfish" type (as mentioned in Section 4).

To make this Appendix self-consistent we firstly introduce the notation we use below, even if this might entail a bit of repetition (see [14]).

Let $\psi(z)$ be a *monic* polynomial of degree N in z, and denote as z_n its N zeros,

$$\psi(z) = \prod_{n=1}^{N} (z - z_n) \ , \tag{47}$$

and as $\underline{z}$ the N-vector of components z_n, $\underline{z} \equiv (z_1, ..., z_N)$.

We then introduce the following short-hand notation: the formula

$$D \ \psi(z) \Longleftrightarrow f_n(\underline{z}) \tag{48a}$$

stands for the identity

$$D\ \psi\left(z\right)=\sum_{n=1}^{N}\left[\left(z-z_n\right)^{-1}\ f_n\left(\underline{z}\right)\right]\ . \tag{48b}$$

Here D is a differential operator with polynomial coefficients acting on the variable z, say

$$D=\sum_{j=0}^{J}\left[p^{(j)}\left(z\right)\ \left(\frac{d}{d\ z}\right)\right]\ , \tag{48c}$$

with the coefficients $p^{(j)}\left(z\right)$ polynomial in z. The simpler examples of such identities are

$$\left(\frac{d}{d\ z}\right)\ \psi\left(z\right)\Longleftrightarrow 1\ , \tag{49a}$$

$$\left(z\ \frac{d}{d\ z}-N\right)\ \psi\left(z\right)\Longleftrightarrow z_n\ ; \tag{49b}$$

clearly the first obtains by logarithmic differentiation of (47), and the second from the first via the identity $z/\left(z-z_n\right)=1+z_n/\left(z-z_n\right)$ (these identities, (49a) and 49b), are reported in [14] as (A.4) and (A.6a)).

Finally, to write more compactly some of the formulas obtained below and reported in Section 2 we introduce the following convenient notations:

$$\sum_{\ell=1;(n)}^{N}\equiv\sum_{\ell=1;\ell\neq n}^{N}\ , \tag{50a}$$

$$\sum_{\ell,m=1;(n)}^{N}\equiv\sum_{\ell,m=1;\ell\neq n,m\neq n,\ell\neq m}^{N}\ , \tag{50b}$$

$$\sum_{\ell,k,m=1;(n)}^{N}\equiv\sum_{\ell,k,m=1;\ell\neq n,k\neq n,m\neq n,\ell\neq k,\ell\neq m,k\neq m}^{N}\ ; \tag{50c}$$

$$d^{(3)}_{n(\ell m)}\left(\underline{z}\right)=\left(z_n-z_\ell\right)\ \left(z_n-z_m\right)\ \left(z_\ell-z_m\right)\ , \tag{51a}$$

$$d^{(4)}_{n(\ell km)}\left(\underline{z}\right)=\left(z_n-z_\ell\right)\ \left(z_n-z_k\right)\ \left(z_n-z_m\right)\ \left(z_\ell-z_m\right)\ \left(z_\ell-z_k\right)\ \left(z_k-z_m\right)\ . \tag{51b}$$

Remark A.1. The upper label on the quantities $d^{(3)}_{n(\ell m)}\left(\underline{z}\right)$ and $d^{(4)}_{n(\ell km)}\left(\underline{z}\right)$ is a reminder that $d^{(3)}_{n(\ell m)}\left(\underline{z}\right)$ involves the coordinates of the 3 different zeros z_n, z_ℓ, z_m, and $d^{(4)}_{n(\ell km)}\left(\underline{z}\right)$ involves the coordinates of the 4 different zeros z_n, z_ℓ, z_k, z_m. It is moreover plain that these quantities—that shall play the role of denominators in sums over the zeros, see below—are antisymmetric under the exchange of any pair of these zeros: this entails that, by taking advantage of these (anti)symmetries, the arguments of the sums reported below could be rewritten in many different ways by appropriate exchanges of the dummy indices being summed upon. We use this property to also obtain convenient expressions based on the quantities $\sigma_n^{(r,\rho)}\left(\underline{z}\right)$, introduced immediately below. □

It is now convenient to introduce the quantities

$$\sigma_n^{(r,\rho)}\left(\underline{z}\right)=\sum_{\ell=1;\ell\neq n}^{N}\left[\frac{z_\ell^r}{\left(z_n-z_\ell\right)^\rho}\right]\ . \tag{52}$$

Here and hereafter r and ρ are *arbitrary nonnegative integers* (unless otherwise indicated). Clearly this definition implies

$$\frac{d\ \sigma_n^{(r,\rho)}\left(\underline{z}\right)}{d\ z_m}=-\rho\ \delta_{nm}\ \sigma_n^{(r,\rho+1)}\left(\underline{z}\right)+\left(1-\delta_{nm}\right)\ \frac{\left[r\ z_n+\left(\rho-r\right)\ z_m\right]\ z_m^{r-1}}{\left(z_n-z_m\right)^{\rho+1}} \tag{53a}$$

hence

$$\frac{d\ \sigma_n^{(r,r)}\left(\underline{z}\right)}{d\ z_m}=-r\ \delta_{nm}\ \sigma_n^{(r,r+1)}\left(\underline{z}\right)+\left(1-\delta_{nm}\right)\ \frac{r\ z_n\ z_m^{r-1}}{\left(z_n-z_m\right)^{r+1}}\ ; \tag{53b}$$

and it is also obvious (and used below) that

$$\sigma_n^{(0,0)}\left(\underline{z}\right)=N-1\ . \tag{54}$$

Remark A.2. The quantities $\sigma_n^{(r,\rho)}\left(\underline{z}\right)$, see (52), are related to each other by trivial identities, for instance clearly the replacement of z_ℓ^r with $z_\ell^{r-1}\left[z_n+\left(z_\ell-z_n\right)\right]$ in the numerator of their definition (52) implies

$$\sigma_n^{(r,\rho)}\left(\underline{z}\right)=z_n\ \sigma_n^{(r-1,\rho)}\left(\underline{z}\right)-\sigma_n^{(r-1,\rho-1)}\left(\underline{z}\right)\ ,\quad r,\rho=1,2,3,... \tag{55}$$

yielding (by iteration)

$$\sigma_n^{(r,\rho)}\left(\underline{z}\right)=\sum_{k=0}^{s}\left[\left(-1\right)^k\ \binom{s}{k}\ z_n^{s-k}\ \sigma_n^{(r-s,\rho-k)}\left(\underline{z}\right)\right]\ ,\quad s=0,1,2,...,\min\left(r,\rho\right)\ , \tag{56a}$$

hence in particular

$$\sigma_n^{(r,\rho)}\left(\underline{z}\right)=\sum_{k=0}^{r}\left[\left(-1\right)^k\ \binom{r}{k}\ z_n^{r-k}\ \sigma_n^{(0,\rho-k)}\left(\underline{z}\right)\right]\quad \text{if}\quad r\leq\rho\ , \tag{56b}$$

$$\sigma_n^{(r,\rho)}\left(\underline{z}\right)=\sum_{k=0}^{\rho}\left[\left(-1\right)^k\ \binom{\rho}{k}\ z_n^{\rho-k}\ \sigma_n^{(r-\rho,\rho-k)}\left(\underline{z}\right)\right]\quad \text{if}\quad r\geq\rho\ .\ \square \tag{56c}$$

To obtain the second version of the identities reported below we also used the identity

$$z_n\ \sigma_n^{(r,\rho)}\left(\underline{z}\right)=\sigma_n^{(r+1,\rho)}\left(\underline{z}\right)+\sigma_n^{(r,\rho-1)}\left(\underline{z}\right)\ ,\quad r=0,1,2,...,\ \rho=1,2,3,... \tag{57}$$

(which correspond clearly to (55)), and the property (54).

The first class of new identities reads as follows:

$$\left(z\ \frac{d}{d\ z}-N\right)^j\ \psi\left(z\right)\Longleftrightarrow f_n^{(j)}\left(\underline{z}\right)\ ,\quad j=1,2,...\ , \tag{58a}$$

with the N quantities $f_n^{(j)}\left(\underline{z}\right)$ defined recursively as follows:

$$f_n^{(j+1)}\left(\underline{z}\right)=-f_n^{(j)}\left(\underline{z}\right)+\sum_{\ell=1;(n)}^{N}\left[\frac{z_n\ f_\ell^{(j)}\left(\underline{z}\right)+z_\ell\ f_n^{(j)}\left(\underline{z}\right)}{z_n-z_\ell}\right]\ ,\quad j=1,2,...\ , \tag{58b}$$

with

$$f_n^{(1)}\left(\underline{z}\right)=z_n\ , \tag{59a}$$

implying

$$\begin{aligned}f_n^{(2)}\left(\underline{z}\right)&=-z_n+2\sum_{\ell=1;(n)}^{N}\left(\frac{z_n\ z_\ell}{z_n-z_\ell}\right)\ ,\\ &=z_n\ \left[-1+2\ \sigma_n^{(1,1)}\left(\underline{z}\right)\right]\ ,\end{aligned} \tag{59b}$$

$$\begin{aligned}f_n^{(3)}\left(\underline{z}\right)&=z_n-6\sum_{\ell=1;(n)}^{N}\left(\frac{z_n\ z_\ell}{z_n-z_\ell}\right)+6\sum_{\ell,m=1;(n)}^{N}\left(\frac{z_n\ z_\ell^2\ z_m}{d_{n(\ell m)}^{(3)}\left(\underline{z}\right)}\right)\\ &=z_n\left\{1-6\ \sigma_n^{(1,1)}\left(\underline{z}\right)-3\ \sigma_n^{(2,2)}\left(\underline{z}\right)+3\ \left[\sigma_n^{(1,1)}\left(\underline{z}\right)\right]^2\right\}\ ,\end{aligned} \tag{59c}$$

$$\begin{aligned}f_n^{(4)}\left(\underline{z}\right)&=-z_n+14\sum_{\ell=1;(n)}^{N}\left(\frac{z_n\ z_\ell}{z_n-z_\ell}\right)-36\sum_{\ell,m=1;(n)}^{N}\left(\frac{z_n\ z_\ell^2\ z_m}{d_{n\ell m}^{(3)}\left(\underline{z}\right)}\right)\\ &+24\sum_{\ell,k,m=1;(n)}^{N}\left(\frac{z_n\ z_\ell\ z_k^3\ z_m^2}{d_{n\ell km}^{(4)}\left(\underline{z}\right)}\right)\\ &=z_n\ \left\{-1+14\ \sigma_n^{(1,1)}(\underline{z})+18\ \sigma_n^{(2,2)}(\underline{z})+8\ \sigma_n^{(3,3)}(\underline{z})\right.\\ &\left.-18\ \left[\sigma_n^{(1,1)}(\underline{z})\right]^2-12\ \sigma_n^{(1,1)}(\underline{z})\ \sigma_n^{(2,2)}(\underline{z})+4\ \left[\sigma_n^{(1,1)}(\underline{z})\right]^3\right\}\ ,\end{aligned} \tag{59d}$$

and so on (via standard, but increasingly tedious, computations).

The second class of new identities reads as follows:

$$\left(\frac{d}{d\,z}\right)\ \left(z\ \frac{d}{d\,z}-N\right)^{j}\ \psi\left(z\right)\Longleftrightarrow g_{n}^{(j)}\left(\underline{z}\right)\ ,\quad j=1,2,... \tag{60a}$$

(for the case $j=0$ see (49a) implying $g_{n}^{(0)}\left(\underline{z}\right)=1$), with the N quantities $g_{n}^{(j)}\left(\underline{z}\right)$ defined as follows:

$$g_{n}^{(j)}\left(\underline{z}\right)=\sum_{\ell=1;(n)}^{N}\left[\frac{f_{n}^{(j)}\left(\underline{z}\right)+f_{\ell}^{(j)}\left(\underline{z}\right)}{z_{n}-z_{\ell}}\right]\ ,\quad j=1,2,...\ , \tag{60b}$$

implying (for the second version of these formulas via (54))

$$\begin{aligned} g_{n}^{(1)}\left(\underline{z}\right)&=z_{n}\ \sum_{\ell=1;(n)}^{N}\left(\frac{1}{z_{n}-z_{\ell}}\right)+\sum_{\ell=1;(n)}^{N}\left(\frac{z_{\ell}}{z_{n}-z_{\ell}}\right)\\ &=\ N-1+2\ \sigma_{n}^{(1,1)}\left(\underline{z}\right)\ , \end{aligned}\tag{61a}$$

$$\begin{aligned} g_{n}^{(2)}\left(\underline{z}\right)&=-\sum_{\ell=1;(n)}^{N}\left(\frac{z_{n}+z_{\ell}}{z_{n}-z_{\ell}}\right)-2\sum_{\ell,m=1;(n)}^{N}\left[\frac{z_{m}^{2}\ \left(z_{n}+z_{\ell}\right)}{d_{n(\ell m)}^{(3)}\left(\underline{z}\right)}\right]\\ &=\ 1-N+2\ \left(N-3\right)\ \sigma_{n}^{(1,1)}\left(\underline{z}\right)-3\ \sigma_{n}^{(2,2)}\left(\underline{z}\right)+3\ \left[\sigma_{n}^{(1,1)}\left(\underline{z}\right)\right]^{2}\ , \end{aligned}\tag{61b}$$

$$\begin{aligned} g_{n}^{(3)}\left(\underline{z}\right)&=\sum_{\ell=1;(n)}^{N}\left(\frac{z_{n}+z_{\ell}}{z_{n}-z_{\ell}}\right)+6\sum_{\ell,m=1;(n)}^{N}\left[\frac{\left(z_{n}+z_{\ell}\right)\ z_{m}^{2}}{d_{n(\ell m)}^{(3)}\left(\underline{z}\right)}\right]\\ &\quad+6\sum_{\ell,k,m=1;(n)}^{N}\left[\frac{\left(z_{n}+z_{\ell}\right)\ z_{m}^{2}\ z_{k}^{3}}{d_{n(\ell km)}^{(4)}\left(\underline{z}\right)}\right]\\ &=\ N-1-2\ \left(3\ N-7\right)\sigma_{n}^{(1,1)}(\underline{z})-\frac{9}{4}\ \left(N-7\right)\ \sigma_{n}^{(2,2)}(\underline{z})+6\ \sigma_{n}^{(3,3)}(\underline{z})\\ &\quad+\frac{9}{4}\ \left(N-7\right)\ \left[\sigma_{n}^{(1,1)}(\underline{z})\right]^{2}-9\ \sigma_{n}^{(1,1)}(\underline{z})\ \sigma_{n}^{(2,2)}(\underline{z})+3\ \left[\sigma_{n}^{(1,1)}(\underline{z})\right]^{3}\ , \end{aligned}\tag{61c}$$

and so on (via standard, but increasingly tedious, computations).

To prove the identities (58) we note first of all that (59a) coincides with (49b) (via (48)). Next—to prove (58a) with (58b)—we apply the operator $(z\ d/dz-N)$ to (the long-hand version of) (58a), getting thereby the following chain of equations:

$$\begin{aligned} &\left(z\ \frac{d}{d\,z}-N\right)^{j+1}\ \psi\left(z\right)\\ &=\ \left(z\ \frac{d}{d\,z}-N\right)\ \left\{\psi\left(z\right)\ \sum_{n=1}^{N}\left[\left(z-z_{n}\right)^{-1}\ f_{n}^{(j)}\left(\underline{z}\right)\right]\right\} \end{aligned}\tag{62a}$$

$$\begin{aligned} &=\ \left[\left(z\ \frac{d}{d\,z}-N\right)\ \psi\left(z\right)\right]\ \sum_{n=1}^{N}\left[\left(z-z_{n}\right)^{-1}\ f_{n}^{(j)}\left(\underline{z}\right)\right]\\ &\quad-\psi\left(z\right)\ \sum_{n=1}^{N}\left[z\ \left(z-z_{n}\right)^{-2}\ f_{n}^{(j)}\left(\underline{z}\right)\right] \end{aligned}\tag{62b}$$

$$= \psi(z) \left\{ \sum_{n,\ell=1}^{N} \left[(z-z_n)^{-1} \; (z-z_\ell)^{-1} \; z_\ell \; f_n^{(j)}(\underline{z}) \right] - \sum_{n=1}^{N} \left[z \; (z-z_n)^{-2} \; f_n^{(j)}(\underline{z}) \right] \right\} \tag{62c}$$

$$= \psi(z) \left\{ \sum_{n,\ell=1;\; \ell\neq n}^{N} \left[(z-z_n)^{-1} \; (z-z_\ell)^{-1} \; z_\ell \; f_n^{(j)}(\underline{z}) \right] - \sum_{n=1}^{N} \left[(z-z_n)^{-1} \; f_n^{(j)}(\underline{z}) \right] \right\} \tag{62d}$$

$$= \psi(z) \left\{ \sum_{n,\ell=1;\; \ell\neq n}^{N} \left[(z-z_n)^{-1} \; (z_n-z_\ell)^{-1} \; z_\ell \; f_n^{(j)}(\underline{z}) \right] - \sum_{n,\ell=1;\; \ell\neq n}^{N} \left[(z-z_\ell)^{-1} \; (z_n-z_\ell)^{-1} \; z_\ell \; f_n^{(j)}(\underline{z}) \right] - \sum_{n=1}^{N} \left[(z-z_n)^{-1} \; f_n^{(j)}(\underline{z}) \right] \right\} \tag{62e}$$

$$= \psi(z) \left\{ \sum_{n=1}^{N} \left[(z-z_n)^{-1} \; \left\{ -f_n^{(j)}(\underline{z}) + \sum_{\ell=1;\; \ell\neq n}^{N} (z_n-z_\ell)^{-1} \; \left[z_\ell \; f_n^{(j)}(\underline{z}) + z_n \; f_\ell^{(j)}(\underline{z}) \right] \right\} \right] \right\} . \tag{62f}$$

The first equality is implied by (the long-hand version of) (58a); the second by standard differentiation; the third, by (the long-hand version of) (49b); the fourth obtains by taking advantage of the cancellation of the terms featuring double poles; the fifth, by using the elementary identity $(z-z_n)^{-1} \; (z-z_\ell)^{-1} = \left[(z-z_n)^{-1} - (z-z_\ell)^{-1} \right] \; (z_n-z_\ell)^{-1}$; and the sixth and last, by exchanging the two dummy indices n and ℓ in the second sum. The last equality is clearly the long-hand version of (58a) (with j replaced by $j+1$ and with (58b)); the recursion (58b) is thereby proven.

To prove (60a) with (60b) we apply the operator d/dz to (the long-hand version of) (58a), getting thereby the following chain of equations:

$$\left(\frac{d}{d\,z} \right) \; \left(z \; \frac{d}{d\,z} - N \right)^{j} \; \psi(z) = \left(\frac{d}{d\,z} \right) \; \left\{ \psi(z) \; \sum_{n=1}^{N} \left[(z-z_n)^{-1} \; f_n^{(j)}(\underline{z}) \right] \right\} \tag{63a}$$

$$= \left[\frac{d}{d\,z} \; \psi(z) \right] \; \sum_{n=1}^{N} \left[(z-z_n)^{-1} \; f_n^{(j)}(\underline{z}) \right] - \psi(z) \; \sum_{n=1}^{N} \left[(z-z_n)^{-2} \; f_n^{(j)}(\underline{z}) \right] \tag{63b}$$

$$= \psi(z) \; \sum_{n,\ell=1}^{N} \left\{ \left[(z-z_n)^{-1} \; (z-z_\ell)^{-1} \right] \; f_n^{(j)}(\underline{z}) \right\} - \psi(z) \; \sum_{n=1}^{N} \left[(z-z_n)^{-2} \; f_n^{(j)}(\underline{z}) \right] \tag{63c}$$

$$= \psi(z) \sum_{n,\ell=1;\ \ell\neq n}^{N} \left[(z-z_n)^{-1}\ \ (z-z_\ell)^{-1}\ \ f_n^{(j)}(\underline{z})\right] \tag{63d}$$

$$= \psi(z) \sum_{n,\ell=1;\ \ell\neq n}^{N} \left\{\left[(z-z_n)^{-1}-(z-z_\ell)^{-1}\right]\ \ (z_n-z_\ell)^{-1}\ \ f_n^{(j)}(\underline{z})\right\} \tag{63e}$$

$$= \psi(z) \sum_{n,\ell=1;\ \ell\neq n}^{N} \left\{(z-z_n)^{-1}\ \ (z_n-z_\ell)^{-1}\ \ \left[f_n^{(j)}(\underline{z})+f_\ell^{(j)}(\underline{z})\right]\right\}\ . \tag{63f}$$

This sequence, (63), of equalities is sufficiently analogous to that reported above, see (62), not to require any repetition of the explanations provided above (after (62)). And clearly the last equality proves the identity (60a) with (60b).

Finally, let us report explicit expressions of the quantities $f_{n,m}^{(j)}(\underline{\zeta})$ and $g_{n,m}^{(j)}(\underline{\zeta})$, see (9) (conveniently obtained via (53) from the expressions of $f_n^{(j)}(\underline{\zeta})$ and $g_n^{(j)}(\underline{\zeta})$ in terms of the quantities $\sigma_n^{(r,\rho)}(\underline{\zeta})$, see (59) and (61)):

$$f_{n,m}^{(1)}(\underline{\zeta}) = \delta_{nm}\ , \tag{64a}$$

$$\begin{aligned} f_{n,m}^{(2)}(\underline{\zeta}) \quad &= \quad -\delta_{nm}\ \left[1+2\ \sigma_n^{(2,2)}(\underline{\zeta})\right] \\ &\quad +(1-\delta_{nm})\ 2\ \left(\frac{\zeta_n}{\zeta_n-\zeta_m}\right)^2 , \end{aligned} \tag{64b}$$

$$\begin{aligned} & f_{n,m}^{(3)}(\underline{\zeta}) = \delta_{nm}\ \left\{1+9\ \sigma_n^{(2,2)}(\underline{\zeta})+6\ \sigma_n^{(3,3)}(\underline{\zeta})\right. \\ & \left.-3\ \sigma_n^{(1,1)}(\underline{\zeta})\ \ \left[\sigma_n^{(1,1)}(\underline{\zeta})+2\ \sigma_n^{(2,2)}(\underline{\zeta})\right]\right\} \\ & -6\ \ (1-\delta_{nm})\ \ \frac{\zeta_n^2\ \left[\zeta_n-(\zeta_n-\zeta_m)\ \ \sigma_n^{(1,1)}(\underline{\zeta})\right]}{(\zeta_n-\zeta_m)^3}\ , \end{aligned} \tag{64c}$$

and so on (via standard, but increasingly tedious, computations); respectively (see (7) and (60))

$$g_{n,m}^{(1)}(\underline{\zeta}) = -2\ \delta_{nm}\ \sigma_n^{(1,2)}(\underline{\zeta})+(1-\delta_{nm})\ \ \frac{2\ \zeta_n}{(\zeta_n-\zeta_m)^2}\ , \tag{65a}$$

$$\begin{aligned} & g_{n,m}^{(2)}(\underline{\zeta}) = \delta_{nm}\ \left\{-2\ \ (N-3)\ \ \sigma_n^{(1,2)}(\underline{\zeta})+6\ \sigma_n^{(2,3)}(\underline{\zeta})\right. \\ & \left.-6\ \sigma_n^{(1,1)}(\underline{\zeta})\ \ \sigma_n^{(1,2)}(\underline{\zeta})\right\} \\ & +(1-\delta_{nm})\ \ \left\{\frac{2\ \zeta_n\ \ [(N-3)\ \ \zeta_n-N\ \zeta_m]}{(\zeta_n-\zeta_m)^3}+\frac{6\ \zeta_n\ \sigma_n^{(1,1)}(\underline{\zeta})}{(\zeta_n-\zeta_m)^2}\right\}\ , \end{aligned} \tag{65b}$$

$$\begin{aligned} & g_{n,m}^{(3)}(\underline{\zeta}) = \delta_{nm}\ \left\{2\ \ (3\ N-7)\ \ \sigma_n^{(1,2)}(\underline{\zeta})\right. \\ & +\frac{9}{2}\ \ (N-7)\ \ \sigma_n^{(2,3)}(\underline{\zeta})-18\ \sigma_n^{(3,4)}(\underline{\zeta}) \\ & -\frac{9}{2}\ \ (N-7)\ \ \sigma_n^{(1,1)}(\underline{\zeta})\ \ \sigma_n^{(1,2)}(\underline{\zeta})+9\ \sigma_n^{(1,2)}(\underline{\zeta})\ \ \sigma_n^{(2,2)}(\underline{\zeta}) \\ & \left.+18\ \sigma_n^{(1,1)}(\underline{\zeta})\ \ \sigma_n^{(2,3)}(\underline{\zeta})-9\ \left[\sigma_n^{(1,1)}(\underline{\zeta})\right]^2\ \ \sigma_n^{(1,2)}(\underline{\zeta})\right\} \\ & +(1-\delta_{nm})\ \ \zeta_n\ \ \left\{\frac{18\ \zeta_m^2}{(\zeta_n-\zeta_m)^4}-\frac{9\ \zeta_m\ \left[(N-7)/2\ +2\ \sigma_n^{(1,1)}(\underline{\zeta})\right]}{(\zeta_n-\zeta_m)^3}\right. \\ & \left.\frac{-2\ \ (3\ N-7)+(9/2)\ \ (N-7)\ \ \sigma_n^{(1,1)}(\underline{\zeta})-9\ \sigma_n^{(2,2)}(\underline{\zeta})+9\ \left[\sigma_n^{(1,1)}(\underline{\zeta})\right]^2}{(\zeta_n-\zeta_m)^2}\right\}\ , \end{aligned} \tag{65c}$$

and so on (via standard, but increasingly tedious, computations).

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