

Explicit solutions for an optimal stock selling problem under a Markov chain model



Q. Zhang

Department of Mathematics, University of Georgia, Athens, GA 30602, United States

ARTICLE INFO

Article history:

Received 5 October 2013
Available online 23 June 2014
Submitted by H.-M. Yin

Keywords:

Markov chain asset
Optimal stopping
Variational inequalities

ABSTRACT

This paper is concerned with explicit solutions for a classical optimal stock selling problem. In contrast to almost all market models treated in the literature, the underlying market is solely determined by a two-state Markov chain. Such Markov chain model is strikingly simple and yet appears capable capturing various market movements ranging from close-to-Brownian motion to no-so-Brownian ones. The purpose of this paper is to study the optimal selling rule under such a model and develop a set of analysis techniques useful for related optimal stopping problems. In this paper, the goal of the problem under consideration is to find an optimal stopping time to sell the stock so as to maximize an expected return. Explicit solutions to the associated variational inequalities are obtained. These solutions are given in terms of a set of threshold levels. Verification theorems are provided to justify their optimality. Finally, numerical examples are provided to illustrate the results.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction

Most market models in the literature are Brownian motion based including geometric Brownian motion, diffusion with possible jumps and regime switching; see related books by Duffie [3], Hull [9], Elliott and Kopp [4], Fouque et al. [5], Karatzas and Shreve [10], and Musiela and Rutkowski [13] among others. An alternative is the binomial tree model introduced by Cox–Ross–Rubinstein. The BTM is widely used in option pricing thanks to its striking simplicity and clear advantage when pricing American type options. However, a main drawback of the BTM is its non-Markovian nature. The lack of Markovian property makes it difficult to work with mathematically, not to mention closed-form solutions. To preserve the BTM's simplicity and enhance its mathematical tractability, we propose a model solely determined by a two-state Markov chain. The Markov chain model appears to be natural for not frequently traded securities such as illiquid stocks. Its capability capturing this type of markets makes it preferable in related applications. In

E-mail address: qingz@math.uga.edu.

addition, it is closely related to the traditional geometric Brownian motion models when the jump rates are large.

Recently, several Markov chain based models are developed. For example, van der Hoek and Elliott [17] introduced a stock price model based on stock dividend rates and a Markov chain noise. Norberg [14] used a Markov chain to represent interest rate and considered a market model driven by a Markov chain. In particular, the market model in [14] resembles a GBM in which the ‘drift’ is approximated by the duration between jumps and the ‘diffusion’ is given in terms of jump times. An additional advantage of a Markov chain driven model is that its price is almost everywhere differentiable. Such differentiability is desirable in the optimal control type market analysis proposed by Barmish and Primbs [1]. In connection with dynamic programming problems, the corresponding Hamilton–Jacobi–Bellman (HJB) equations are of the first order, which are easier to analyze than those under traditional Brownian motion based models. Finally, the Markov chain model is not that far apart from a GBM because it can be used to approximate a GBM by varying its jump rates. In fact, it is shown in Example 1 that a properly scaled Markov chain model converges weakly to that of a GBM as the jump rates go to infinity.

When to sell a stock is a crucial component in stock trading. It determines when to take profits or to cut losses. It is probably the most emotional part for individual investors in the trading process. Selling rules in financial markets have been studied for many years. For example, Zhang [21] considered a selling rule determined by two threshold levels: a target price and a stop-loss limit. One makes a selling decision whenever the price reaches either levels. Under a switching GBM, the objective is to determine these threshold levels to maximize an expected discounted reward function. In [21], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. In Guo and Zhang [6], they considered the optimal selling rule under a GBM model with regime switching. Using a smooth-fit technique, they were able to convert the optimal stopping problem to a set of algebraic equations. These algebraic equations were used to determine the optimal target levels. In addition to these analytical results, various mathematical tools have been developed to compute these threshold levels. For example, a stochastic approximation technique was used in Yin, Liu and Zhang [18] and a linear programming approach was developed in Helmes [7]. In addition, Merhi and Zervos [12] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a GBM market model. Similar problem under a more general market model was treated by Løkka and Zervos [11]. In addition, Du Toit and Peskir [2] considered selling at the peak problem under a geometric Brownian motion and Henderson and Hobson [8] treated a risk-averse selling under a stopping/optimal control framework. General treatment in connection with optimal stopping and finance applications can be found the book by Peskir and Shiryaev [16].

The purpose of this paper is study a classical optimal stopping problem under a simple yet flexible Markovian model and to develop a set of mathematical analysis techniques for Markov chain problems. In this paper, the stock price is assumed to follow a Markov chain model. Under this model, the state of the Markov chain can be estimated based on the stock price increments. This makes the Markov chain observable. In addition to its simplicity, the Markov chain model is able to capture price movements of a broader range of stocks. In this paper, under the Markov chain model, we consider an optimal stock selling rule and obtain its solution in terms of a set of threshold levels. In particular, we solve the corresponding dynamic programming problem and obtain these threshold levels. We point out that the standard smooth-fit method that works in a GBM setting is not adequate in one of the cases in this paper because of the lack of enough equations for the unknown parameters. To solve the problem, we need to explore other convexity conditions to determine uniquely these parameters. We also provide a set of sufficient conditions that guarantee their optimality. Numerical examples are reported to illustrate these results.

This paper is organized as follows. In Section 2, we formulate the problem and make a few assumptions. In Section 3, we study properties of the value functions, the associate HJB equations, and their solutions. In Section 4, we provide a set of sufficient conditions that guarantee the optimality of our selling rule. We

also include three numerical examples in this section. Some concluding remarks are given in Section 5. Some technical results are provided in Appendix A.

2. Problem formulation

Let $\{\alpha_t: t \geq 0\}$ denote a two-state Markov chain with state space $\mathcal{M} = \{1, 2\}$ and generator $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, for given $\lambda_1 > 0$ and $\lambda_2 > 0$. Let S_t denote the stock price at time t given by the equation

$$\frac{dS_t}{S_t} = f(\alpha_t)dt, \quad S_0 = x \geq 0, \quad t \geq 0,$$

where $f(1) = f_1 > 0$ represents the uptick return rate and $f(2) = f_2 < 0$ the downtick return rate. Let $\mathcal{F}_t = \{S_r: r \leq t\}$ denote the filtration generated by S_t . Note that α_t is observable and $\mathcal{F}_t = \{\alpha_r: r \leq t\}$.

Let K denote the fixed transaction cost. Given $S_0 = x$ and $\alpha_0 = i \in \mathcal{M}$, the objective of the problem is to choose an $\mathcal{F}_t$ stopping time τ so as to maximize

$$J(x, i, \tau) = E(e^{-\rho\tau}(S_\tau - K)I_{\{\tau < \infty\}}),$$

where $\rho > 0$ is the discount factor and I_A is the indicator function of event A .

Let $V(x, i) = \sup_\tau J(x, i, \tau)$ be the value function. Then it is easy to see that $V(x, i) \geq 0$, $V(0, i) = 0$, $i = 1, 2$. Moreover, $V(x, i)$ is convex in x for fixed $i = 1, 2$.

Let $\Phi(\rho) = (\rho + \lambda_1 - f_1)(\rho + \lambda_2 - f_2) - \lambda_1\lambda_2$. Then the bigger root of $\Phi(\rho) = 0$ is given by

$$B_0 = \frac{1}{2}(f_1 - \lambda_1 + f_2 - \lambda_2 + \sqrt{((f_1 - \lambda_1) - (f_2 - \lambda_2))^2 + 4\lambda_1\lambda_2}).$$

Note that if $\rho \leq B_0$, then we can show, as in Guo and Zhang [6], that it is optimal not to sell at all. In the rest of this paper, we only consider the case when $\rho > B_0$, which implies $\Phi(\rho) > 0$. We summarize the conditions to be imposed in the rest of this paper:

(A1) $f_1 > 0$ and $f_2 < 0$;

(A2) $\Phi(\rho) > 0$.

Let $(\nu_1, \nu_2) = (\lambda_2/(\lambda_1 + \lambda_2), \lambda_1/(\lambda_1 + \lambda_2))$ denote the stationary distribution of α_t and let $\mu = \nu_1 f_1 + \nu_2 f_2$. Then, $f_2 < \mu < f_1$. Moreover, it is easy to see that $\Phi(\mu) = (\mu - f_1)(\mu - f_2) < 0$. This implies $B_0 > \mu$. Therefore, $\rho > \mu$.

Note that, for any $\mathcal{F}_t$ stopping time τ ,

$$J(x, i, \tau) = xE\left(e^{-\rho\tau} \exp \int_0^\tau f(\alpha_s)ds\right)I_{\{\tau < \infty\}} - KEe^{-\rho\tau}.$$

In order to have finite $V(x, i)$, necessarily

$$\sup_\tau E\left(e^{-\rho\tau} \exp \int_0^\tau f(\alpha_s)ds\right)I_{\{\tau < \infty\}} < \infty.$$

In view of this, $V(x, i)$ needs to be at most linear growth in x . In addition, note that the stock price S_t is differentiable and the value of α_t can be given in terms of the derivative of $\log(S_t)$.

3. HJB equations

Let $\mathcal{A}$ denote the generator of (S_t, α_t) , i.e., for any differentiable functions $h(x, i)$, $i = 1, 2$,

$$\begin{cases} \mathcal{A}h(x, 1) = xf_1h'(x, 1) + \lambda_1(h(x, 2) - h(x, 1)), \\ \mathcal{A}h(x, 2) = xf_2h'(x, 2) + \lambda_2(h(x, 1) - h(x, 2)), \end{cases}$$

where h' denotes the derivative of h with respect to x . The associated HJB equations should have the form:

$$\begin{cases} \min\{\rho v(x, 1) - \mathcal{A}v(x, 1), v(x, 1) - (x - K)\} = 0, \\ \min\{\rho v(x, 2) - \mathcal{A}v(x, 2), v(x, 2) - (x - K)\} = 0. \end{cases} \quad (1)$$

In this section, we solve these HJB equations. First, if the price S_t is small, then one should hold the position because the price is not attractive regardless $\alpha_t = 1$ or 2 . In view of this, we expect the existence of x^* such that no selling is $S_t < x^*$. The corresponding interval $(0, x^*)$ gives a continuation region. Note that $V(x, i) \geq 0$ implies $x^* \geq K$. On this interval, the equalities $\rho v(x, i) - \mathcal{A}v(x, i) = 0$, $i = 1, 2$, must hold. Using the generator $\mathcal{A}$, we can write

$$\begin{cases} (\rho + \lambda_1)v(x, 1) = xf_1v'(x, 1) + \lambda_1v(x, 2), \\ (\rho + \lambda_2)v(x, 2) = xf_2v'(x, 2) + \lambda_2v(x, 1). \end{cases}$$

Using the first equation, we write

$$v(x, 2) = \frac{1}{\lambda_1}((\rho + \lambda_1)v(x, 1) - xf_1v'(x, 1)).$$

Substitute this into the second equation and simplify to obtain

$$x^2f_1f_2v''(x, 1) + x(f_1f_2 - D_1)v'(x, 1) + D_2v(x, 1) = 0, \quad (2)$$

where

$$\begin{cases} D_1 = (\rho + \lambda_1)f_2 + (\rho + \lambda_2)f_1, \\ D_2 = (\rho + \lambda_1)(\rho + \lambda_2) - \lambda_1\lambda_2. \end{cases} \quad (3)$$

Let $\beta_1 < 0$ and $\beta_2 > 0$ denote the roots of

$$f_1f_2\beta^2 - D_1\beta + D_2 = 0. \quad (4)$$

Then,

$$\begin{cases} \beta_1 = \frac{D_1 + \sqrt{D_1^2 - 4f_1f_2D_2}}{2f_1f_2} < 0, \\ \beta_2 = \frac{D_1 - \sqrt{D_1^2 - 4f_1f_2D_2}}{2f_1f_2} > 0. \end{cases} \quad (5)$$

The general solution to (2) can be given as

$$v(x, 1) = A_1x^{\beta_1} + A_2x^{\beta_2},$$

for some constants A_1 and A_2 .

On $(0, x^*)$, the convexity condition implies that $v(x, 1)$ is bounded. Necessarily, $A_1 = 0$. Therefore, $v(x, 1) = A_2 x^{\beta_2}$. Substitute this back into the first equation to obtain $v(x, 2) = \kappa_2 A_2 x^{\beta_2}$, where $\kappa_2 = (\rho + \lambda_1 - f_1 \beta_2)/\lambda_1$.

Recall that α_t is observable. One should hold the position longer under the condition $\alpha_t = 1$ (uptick) than that under $\alpha_t = 2$ (downtick). In view of this, we consider the HJB equations on (x^*, x_0^*) for some $x_0^* > x^*$. The idea is to sell if $(S_t, \alpha_t) \in [x^*, \infty) \times \{2\}$ and hold if $(S_t, \alpha_t) \in (0, x_0^*) \times \{1\}$ till S_t reaches x_0^* . Clearly, $v(x, 2) = x - K$ and $\rho v(x, 1) - \mathcal{A}v(x, 1) = 0$, on (x^*, x_0^*) . Using this, we solve the equation

$$\rho v(x, 1) - \mathcal{A}v(x, 1) = 0,$$

which gives

$$(\rho + \lambda_1)v(x, 1) = x f_1 v'(x, 1) + \lambda_1(x - K).$$

It is easy to see a particular solution

$$\phi_0(x) = A_0 x + B_0, \quad \text{where } A_0 = \frac{\lambda_1}{\rho + \lambda_1 - f_1} \text{ and } B_0 = -\frac{\lambda_1 K}{\rho + \lambda_1}. \quad (6)$$

Let $\gamma_1 = (\rho + \lambda_1)/f_1$. Then, the general solution can be given by

$$v(x, 1) = C_1 x^{\gamma_1} + \phi_0(x),$$

for any constant C_1 . Note that $\gamma_1 > 1$ following Assumption (A2). In fact, $\Phi(\rho) > 0$ implies $\rho + \lambda_1 - f_1 > \lambda_1 \lambda_2 / (\rho + \lambda_2 - f_2) > 0$.

Next we consider two separate cases to continue solving the HJB equations.

Case I: $\rho \leq f_1$

Assuming $\rho \leq f_1$, we first show that $x_0^* = \infty$. If not, we must have $v(x, 1) = v(x, 2) = x - K$ for $x > x_0^*$. In order to satisfy the HJB equations (1), $v(x, 1)$ has to satisfy the inequality $\rho v(x, 1) - \mathcal{A}v(x, 1) \geq 0$, for $x > x_0^*$. Plugging $v(x, 1) = v(x, 2) = x - K$ in this inequality, we have

$$\rho(x - K) \geq x f_1.$$

Therefore, $(\rho - f_1)x \geq \rho K$, for $x > x_0^*$. This contradicts $\rho \leq f_1$. Hence $x_0^* = \infty$.

In view of these, on (x^*, ∞) , $v(x, 1) = C_1 x^{\gamma_1} + \phi_0(x)$ and $v(x, 2) = x - K$. This means never sell when $\alpha_t = 1$. Recall the linear growth property and nonnegativity of $v(x, i)$. It follows that $C_1 = 0$ because $\gamma_1 > 1$.

Next, we determine the values of x^* and A_2 . Recall that $v(x, 1)$ and $v(x, 2)$ are convex on $(0, \infty)$. Necessarily, they are continuous. In particular, they are continuous at $x = x^*$. Therefore,

$$\begin{cases} A_2 (x^*)^{\beta_2} = A_0 x^* + B_0, \\ \kappa_2 A_2 (x^*)^{\beta_2} = x^* - K. \end{cases}$$

Solving these equations, we have

$$x^* = -\frac{K + \kappa_2 B_0}{\kappa_2 A_0 - 1}, \quad (7)$$

and

$$A_2 = \frac{A_0 x^* + B_0}{(x^*)^{\beta_2}}. \quad (8)$$

It is elementary to check that

$$x^* = \left(\frac{\rho + \lambda_1 - f_1}{\rho + \lambda_1} \right) \left(\frac{K\beta_2}{\beta_2 - 1} \right). \quad (9)$$

Note that x^* above is well defined because $(\beta_2 - 1) > 0$ following [Lemma A \(Appendix A\)](#).

The solutions to the HJB equations (1) should have the form:

$$\begin{cases} v(x, 1) = \begin{cases} A_2 x^{\beta_2} & \text{if } 0 \leq x \leq x^*, \\ A_0 x + B_0 & \text{if } x > x^*, \end{cases} \\ v(x, 2) = \begin{cases} \kappa_2 A_2 x^{\beta_2} & \text{if } 0 \leq x \leq x^*, \\ x - K & \text{if } x > x^*. \end{cases} \end{cases} \quad (10)$$

Theorem 1. Assume $\rho \leq f_1$. Then the functions $v(x, i)$, $i = 1, 2$, given above are continuous on $(0, \infty)$ and differentiable on $(0, \infty) - \{x^*\}$. They satisfy the HJB equations (1). In particular, the following inequalities hold

$$\begin{cases} A_2 x^{\beta_2} > x - K & \text{on } (0, x^*), \\ \kappa_2 A_2 x^{\beta_2} > x - K & \text{on } (0, x^*), \\ v(x, 1) > x - K & \text{on } (x^*, \infty), \\ \rho v(x, 2) - \mathcal{A}v(x, 2) \geq 0 & \text{on } (x^*, \infty). \end{cases} \quad (11)$$

Proof. It is sufficient to show these four inequalities. First, note that $\Phi(\rho) > 0$ implies $\rho + \lambda_1 - f_1 > 0$. Under the condition $\rho \leq f_1$, we have $A_0 \geq 1$. The third inequality in (11) follows from $B_0 > -K$. In addition, the first inequality follows from the second one because $0 < \kappa_2 < 1$ as shown in [Appendix A \(Lemma B\)](#). To show the second inequality, we claim that $A_2 > 0$ and

$$\beta_2 \kappa_2 A_2 (x^*)^{\beta_2 - 1} < 1. \quad (12)$$

To see $A_2 > 0$, notice that (9) implies

$$x^* > \frac{(\rho + \lambda_1 - f_1)K}{\rho + \lambda_1},$$

because $\beta_2 > 1$ (see [Lemma A in Appendix A](#)). Therefore $A_0 x^* + B_0$ is positive, so is A_2 . To show (12), use again (9), which yields

$$x^* < \frac{K\beta_2}{\beta_2 - 1}.$$

This is equivalent to (12) because $\kappa_2 A_2 (x^*)^{\beta_2} = x^* - K$. Let $\phi(x) = \kappa_2 A_2 x^{\beta_2} - (x - K)$. In view of the above claim and the definition of x^* , it follows that, on $(0, x^*)$,

$$\begin{aligned}\phi(x^*) &= \kappa_2 A_2 (x^*)^{\beta_2} - (x^* - K) = 0, \\ \phi'(x^*) &= \beta_2 \kappa_2 A_2 (x^*)^{\beta_2-1} - 1 < 0, \\ \phi''(x) &= \beta_2(\beta_2 - 1) \kappa_2 A_2 x^{\beta_2-2} > 0.\end{aligned}$$

Consequently, $\phi'(x)$ is increasing on $(0, x^*)$, which implies $\phi'(x) < 0$. Hence, $\phi(x)$ is decreasing. Therefore, $\phi(x) > 0$ on $(0, x^*)$, which implies the second inequality in (11).

It remains to show the last inequality $\rho v(x, 2) - \mathcal{A}v(x, 2) \geq 0$ in (11). This is equivalent to

$$\rho(x - K) \geq x f_2 + \lambda_2 (\phi_0(x) - (x - K)).$$

It follows that

$$(\rho + \lambda_2 - f_2 - \lambda_2 A_0)x \geq (\rho + \lambda_2)K + \lambda_2 B_0.$$

Using the notation $\Phi(\rho)$ and D_2 , we have

$$\left(\frac{\Phi(\rho)}{\rho + \lambda_1 - f_1} \right) x \geq \frac{K D_2}{\rho + \lambda_1}.$$

Therefore, we need

$$x \geq \frac{K(\rho + \lambda_1 - f_1) D_2}{(\rho + \lambda_1) \Phi(\rho)},$$

for $x \geq x^*$. It suffices to show this inequality when $x = x^*$. Using the expression in (9), we only have to show

$$\frac{\beta_2}{\beta_2 - 1} \geq \frac{D_2}{\Phi(\rho)}.$$

Rewrite this to obtain

$$\frac{1}{\beta_2} \geq 1 - \frac{\Phi(\rho)}{D_2}. \quad (13)$$

Now, if $1 - \Phi(\rho)/D_2 \leq 0$ (i.e., $D_2 \leq \Phi(\rho)$), then we are done because $\beta_2 > 1$. Otherwise, $D_2 > \Phi(\rho)$. Under this condition, we rewrite (13) as

$$\beta_2 \leq \frac{D_2}{D_2 - \Phi(\rho)},$$

which is equivalent to

$$\sqrt{D_1^2 - 4f_1 f_2 D_2} \leq D_1 - \frac{2f_1 f_2 D_2}{D_2 - \Phi(\rho)}. \quad (14)$$

Note that

$$\Phi(\rho) = D_2 - D_1 + f_1 f_2. \quad (15)$$

Therefore, $D_2 > \Phi(\rho)$ implies that $D_1 > f_1 f_2$. Under this condition, it is easy to check

$$D_1 - \frac{2f_1 f_2 D_2}{D_2 - \Phi(\rho)} > f_1 f_2 - \frac{2f_1 f_2 D_2}{D_2 - \Phi(\rho)} = -f_1 f_2 \left(\frac{D_2 + \Phi(\rho)}{D_2 - \Phi(\rho)} \right) > 0.$$

Square both sides of (14) to obtain

$$D_1^2 - 4f_1 f_2 D_2 \leq D_1^2 - \frac{4f_1 f_2 D_1 D_2}{D_2 - \Phi(\rho)} + \frac{4f_1^2 f_2^2 D_2^2}{(D_2 - \Phi(\rho))^2}.$$

Simplify this inequality to have

$$D_1(D_2 - \Phi(\rho)) - f_1 f_2 D_2 \geq (D_2 - \Phi(\rho))^2.$$

Furthermore, using (15), we have $D_2 - \Phi(\rho) = D_1 - f_1 f_2$. Substitute this into the above inequality to obtain

$$D_1(D_1 - f_1 f_2) - f_1 f_2 D_2 \geq (D_1 - f_1 f_2)^2.$$

This is equivalent to

$$D_2 - D_1 \geq -f_1 f_2,$$

which leads $\Phi(\rho) \geq 0$, which holds under the assumption $\Phi(\rho) > 0$. Therefore, $\rho v(x, 2) - \mathcal{A}v(x, 2) \geq 0$ on (x^*, ∞) . The proof is complete. $\square$

Case II: $\rho > f_1$

We consider the second case when $\rho > f_1$. Note that a large ρ encourages selling sooner. Naturally, we expect $x_0^* < \infty$. The solutions to the HJB equations (1) should have the form:

$$\begin{cases} v(x, 1) = \begin{cases} A_2 x^{\beta_2} & \text{if } 0 \leq x \leq x^*, \\ C_1 x^{\gamma_1} + \phi_0(x) & \text{if } x^* < x \leq x_0^*, \\ x - K & \text{if } x > x_0^*, \end{cases} \\ v(x, 2) = \begin{cases} \kappa_2 A_2 x^{\beta_2} & \text{if } 0 \leq x \leq x^*, \\ x - K & \text{if } x > x^*. \end{cases} \end{cases} \quad (16)$$

We need to determine the values of A_2 , C_1 , x_0^* , and x^* . Again, following the continuity of the value functions at x^* and x_0^* , we have

$$\begin{cases} A_2 (x^*)^{\beta_2} = C_1 (x^*)^{\gamma_1} + \phi_0(x^*), \\ \kappa_2 A_2 (x^*)^{\beta_2} = x^* - K, \\ C_1 (x_0^*)^{\gamma_1} + \phi_0(x_0^*) = x_0^* - K. \end{cases} \quad (17)$$

Note that there are only three equations, which are not adequate to determine uniquely the values of the four unknowns. We need to find further conditions. Note that to satisfy the HJB equations (1), the following inequalities have to hold

$$\begin{cases} A_2 x^{\beta_2} > x - K, \\ \kappa_2 A_2 x^{\beta_2} > x - K, \end{cases} \quad \text{on } (0, x^*); \quad (18)$$

$$\begin{cases} C_1 x^{\gamma_1} + \phi_0(x) > x - K, \\ \rho v(x, 2) - \mathcal{A}v(x, 2) \geq 0, \end{cases} \quad \text{on } (x^*, x_0^*); \quad (19)$$

$$\begin{cases} \rho v(x, 1) - \mathcal{A}v(x, 1) \geq 0, \\ \rho v(x, 2) - \mathcal{A}v(x, 2) \geq 0, \end{cases} \quad \text{on } (x_0^*, \infty). \quad (20)$$

First, we consider (18). Note that convexity of $v(x, 2)$ at $x = x^*$ implies

$$\beta_2 \kappa_2 A_2 (x^*)^{\beta_2 - 1} \leq 1. \quad (21)$$

Recall that $\kappa_2 > 0$ as shown in Lemma B (Appendix A), $x^* \geq K$, and the second equality in (17). It follows that $A_2 \geq 0$. In addition, using (21) and following from similar argument used to prove the second inequality in (11) with $\phi(x) = \kappa_2 A_2 x^{\beta_2} - (x - K)$ for possibly different x^* , we can show the second inequality in (18) holds, so does the first one. Therefore, the inequalities in (18) are equivalent to (21), which can be simplified and written as:

$$x^* \leq \frac{K \beta_2}{\beta_2 - 1}. \quad (22)$$

Next, we consider (20). The first inequality implies

$$\rho(x - K) \geq x f_1, \quad \text{for } x > x_0^*.$$

It follows that

$$x_0^* \geq \frac{\rho K}{\rho - f_1}. \quad (23)$$

The second inequality in (20) is automatically satisfied because $f_2 < 0$.

Finally, go back to (19). Again, the convexity of $v(x, 1)$ at $x = x_0^*$ yields

$$\gamma_1 C_1 (x_0^*)^{\gamma_1 - 1} + A_0 \leq 1. \quad (24)$$

It follows from the third equality in (17) that

$$C_1 = \frac{x_0^* - K - \phi_0(x_0^*)}{(x_0^*)^{\gamma_1}}. \quad (25)$$

Under the condition $x_0^* \geq \rho K / (\rho - f_1)$, it is easy to see $C_1 > 0$. Note that $0 < A_0 < 1$ under $\rho > f_1$. Let $\phi(x) = C_1 x^{\gamma_1} + \phi_0(x) - (x - K)$. Then, it is direct to check that $\phi(x_0^*) = 0$, $\phi'(x_0^*) \leq 0$, $\phi''(x) = \gamma_1(\gamma_1 - 1)C_1 x^{\gamma_1 - 2} > 0$. Therefore, $\phi'(x)$ is increasing on (x^*, x_0^*) . Thus, $\phi'(x) < 0$ on (x^*, x_0^*) , which implies $\phi(x)$ is decreasing. Therefore, $\phi(x) > 0$ on (x^*, x_0^*) . The first inequality in (19) follows from (24).

Use (25) and rewrite (24) to obtain

$$\gamma_1 (x_0^* - K - \phi_0(x_0^*)) \leq (1 - A_0)x_0^*,$$

which leads to

$$(\gamma_1 - 1)(1 - A_0)x_0^* \leq \gamma_1(K + B_0).$$

Recall that $\gamma_1 > 1$ and $A_0 < 1$. It follows that

$$x_0^* \leq \frac{\gamma_1(K + B_0)}{(\gamma_1 - 1)(1 - A_0)} = \frac{\rho K}{\rho - f_1}.$$

Combining the opposite inequality (23), we have

$$x_0^* = \frac{\rho K}{\rho - f_1}.$$

Next, we claim that the second inequality in (19) follows from

$$C_1(x^*)^{\gamma_1} + \phi_0(x^*) \leq \frac{(\rho + \lambda_2)(x^* - K) - x^* f_2}{\lambda_2}. \quad (26)$$

To see this, let

$$\phi(x) = C_1 x^{\gamma_1} + \phi_0(x) - \frac{(\rho + \lambda_2)(x - K) - x f_2}{\lambda_2}.$$

Then, using $v(x, 1) = C_1 x^{\gamma_1} + \phi_0(x)$ and $v(x, 2) = x - K$, the second inequality becomes $\phi(x) \leq 0$ on $x \in (x^*, x_0^*)$. Under (26), $\phi(x^*) \leq 0$. Note also that

$$\phi'(x_0^*) = \gamma_1 C_1 (x_0^*)^{\gamma_1} + A_0 - \frac{\rho + \lambda_2 - f_2}{\lambda_2} \leq 1 - \frac{\rho + \lambda_2 - f_2}{\lambda_2} < 0.$$

In addition, $\phi''(x) = \gamma_1(\gamma_1 - 1)C_1 x^{\gamma_1-2} > 0$. In view of this, $\phi'(x)$ is increasing. Therefore, $\phi'(x) < 0$ on (x^*, x_0^*) , which implies $\phi(x)$ is decreasing. So $\phi(x) \leq \phi(x^*) \leq 0$ on (x^*, x_0^*) .

Furthermore, using (17), we can rewrite (26) as

$$\frac{x^* - K}{\kappa_2} \leq \frac{(\rho + \lambda_2)(x^* - K) - x^* f_2}{\lambda_2},$$

which in turn gives

$$(\lambda_2 - \kappa_2(\rho + \lambda_2 - f_2))x^* \leq (\lambda_2 - \kappa_2(\rho + \lambda_2))K.$$

This inequality is equivalent to

$$x^* \leq \frac{(\lambda_2 - \kappa_2(\rho + \lambda_2))K}{\lambda_2 - \kappa_2(\rho + \lambda_2 - f_2)},$$

because $\lambda_2 - \kappa_2(\rho + \lambda_2 - f_2) > 0$ as shown in Lemma D in Appendix A.

In view of (22), x^* has to be bounded above by

$$X_0 := \min \left\{ x_0^*, \frac{K\beta_2}{\beta_2 - 1}, \frac{(\lambda_2 - \kappa_2(\rho + \lambda_2))K}{\lambda_2 - \kappa_2(\rho + \lambda_2 - f_2)} \right\}.$$

In view of Lemma C (Appendix A), we have

$$x_0^* = \frac{\rho K}{\rho - f_1} < \frac{K\beta_2}{\beta_2 - 1}.$$

Therefore, an upper bound for x^*

$$X_0 = \min \left\{ \frac{K\beta_2}{\beta_2 - 1}, \frac{(\lambda_2 - \kappa_2(\rho + \lambda_2))K}{\lambda_2 - \kappa_2(\rho + \lambda_2 - f_2)} \right\}.$$

To obtain x^* , we only need to solve the first two equations in (17). Eliminating A_2 , we obtain

$$C_1(x^*)^{\gamma_1} + \phi_0(x^*) = \frac{x^* - K}{\kappa_2}, \quad (27)$$

with $K \leq x^* \leq x_0^*$.

Let $\phi^*(x) = C_1 x^{\gamma_1} + \phi_0(x) - (x - K)/\kappa_2$. Then it is easy to check that $\phi_0(K)$ is positive, so is $\phi^*(K)$. In addition, $\phi^*(x_0^*) = (x_0^* - K)(1 - 1/\kappa_2) < 0$ because $0 < \kappa_2 < 1$. Furthermore, it can be shown that $(\phi^*)''(x) > 0$, which implies that $(\phi^*)'(x)$ is increasing. Using (24) to obtain $(\phi^*)'(x) < 0$, which implies $\phi^*(x)$ is decreasing. Therefore, $\phi^*(x)$ has a unique zero $K \leq x^* \leq x_0^*$.

Recall that

$$x_0^* = \frac{\rho K}{\rho - f_1} \quad \text{and} \quad C_1 = \frac{x_0^* - K - \phi_0(x_0^*)}{(x_0^*)^{\gamma_1}}.$$

Let x^* be the solution of (27) over $[K, x_0^*]$ and let

$$A_2 = \frac{x^* - K}{\kappa_2(x^*)^{\beta_2}}.$$

We have proved the following results.

Theorem 2. Assume $\rho > f_1$. Then the functions $v(x, i)$, $i = 1, 2$, given in (16) are continuous on $(0, \infty)$. Moreover, $v(x, 1)$ is differentiable on $(0, \infty) - \{x^*, x_0^*\}$ and $v(x, 2)$ is differentiable on $(0, \infty) - \{x^*\}$. If $x^* \leq X_0$, then they satisfy the HJB equations (1).

Remark 1. Note that $X_0 \in [K, x_0^*]$ and $\phi^*(x)$ is decreasing on $[K, x_0^*]$. A sufficient condition for $x^* \leq X_0$ is $\phi^*(X_0) \leq 0$.

4. Verification theorems and numerical examples

First, we give two verification theorems depending on $\rho \leq f_1$ and $\rho > f_1$. We only prove Theorem 3. The proof of Theorem 4 can be given in a similar way.

Theorem 3. Assume the conditions of Theorem 1. Then, $v(x, i) = V(x, i)$, $i = 1, 2$. Moreover, let $D = (0, \infty) \times \{1\} \cup (0, x^*) \times \{2\}$ denote the continuation region. Then

$$\tau^* = \inf \{t \geq 0; (S_t, \alpha_t) \notin D\}$$

is an optimal selling time.

Proof. We only sketch the proof because it is similar to that of Zhang and Zhang [20, Theorem 5]. For any given stopping time τ and $n = 1, 2, \dots$, we have

$$\begin{aligned} v(x, i) &\geq E(e^{-\rho(\tau \wedge n)} v(S_{\tau \wedge n}, \alpha_{\tau \wedge n})) \\ &= E(e^{-\rho\tau} v(S_\tau, \alpha_\tau) I_{\{\tau < n\}}) + E(e^{-\rho n} v(S_n, \alpha_n) I_{\{\tau \geq n\}}) \\ &\geq E(e^{-\rho\tau} (S_\tau - K) I_{\{\tau < n\}}) + E(e^{-\rho n} v(S_n, \alpha_n) I_{\{\tau \geq n\}}). \end{aligned} \quad (28)$$

Recall the linear growth of $v(x, i)$ and $E(e^{-\rho t} S_t) \rightarrow 0$ under (A2) as $t \rightarrow \infty$. The second term goes to 0 as $n \rightarrow \infty$. Note also that the first term converges to $E(e^{-\rho \tau} (S_\tau - K) I_{\{\tau < \infty\}}) = J(x, i, \tau)$. It follows that $v(x, i) \geq J(x, i, \tau)$.

To show the equality, first if $(S_0, \alpha_0) = (x, i) \notin D$, then $\tau^* = 0$. This implies $v(x, i) = x - K = J(x, i, \tau^*)$. If $(x, i) \in D$, then similarly as in (28), we have

$$\begin{aligned} v(x, i) &= E(e^{-\rho(\tau^* \wedge n)} v(S_{\tau^* \wedge n}, \alpha_{\tau^* \wedge n})) \\ &= E(e^{-\rho \tau^*} v(S_{\tau^*}, \alpha_{\tau^*}) I_{\{\tau^* < n\}}) + E(e^{-\rho n} v(S_n, \alpha_n) I_{\{\tau^* \geq n\}}) \\ &= E(e^{-\rho \tau^*} (S_{\tau^*} - K) I_{\{\tau^* < n\}}) + E(e^{-\rho n} v(S_n, \alpha_n) I_{\{\tau^* \geq n\}}) \\ &\rightarrow E(e^{-\rho \tau^*} (S_{\tau^*} - K) I_{\{\tau^* < \infty\}}) + 0 \\ &= J(x, i, \tau^*), \end{aligned}$$

as $n \rightarrow \infty$. $\square$

Theorem 4. Assume the conditions of Theorem 2. Then, $v(x, i) = V(x, i)$, $i = 1, 2$. Moreover, let $D = (0, x_0^*) \times \{1\} \cup (0, x^*) \times \{2\}$ denote the continuation region. Then

$$\tau^* = \inf\{t \geq 0; (S_t, \alpha_t) \notin D\}$$

is an optimal selling time.

Corollary 1. Let $\mathcal{T}$ denote the class of almost sure finite $\mathcal{F}_t$ stopping times. Then,

$$\sup_{\tau \in \mathcal{T}} E\left(e^{-\rho \tau} \exp \int_0^\tau f(\alpha_t) dt\right) < \infty.$$

Proof. Given $\alpha_0 = i$, we have

$$xE\left(e^{-\rho \tau} \exp \int_0^\tau f(\alpha_t) dt\right) I_{\{\tau < \infty\}} \leq J(x, i, \tau) + K \leq V(x, i) + K \leq \max_i V(x, i) + K.$$

Set $x = 1$ to obtain

$$\sup_{\tau \in \mathcal{T}} E\left(e^{-\rho \tau} \exp \int_0^\tau f(\alpha_t) dt\right) \leq \max_i V(1, i) + K < \infty. \quad \square$$

Example 1 (Convergence to a Brownian motion). In this example, given $\varepsilon > 0$, we consider

$$f_1 = \mu + \frac{\sigma}{\sqrt{\varepsilon}}, \quad f_2 = \mu - \frac{\sigma}{\sqrt{\varepsilon}}, \quad \lambda_1 = \lambda_2 = \frac{1}{\varepsilon}.$$

Using the asymptotic normality given in Yin and Zhang [19, Theorem 5.9], we can show that $S_t = S_t^\varepsilon$ converges weakly to

$$S_t^0 = S_0 e^{\mu t + \sigma W_t}, \quad \text{as } \varepsilon \rightarrow 0,$$

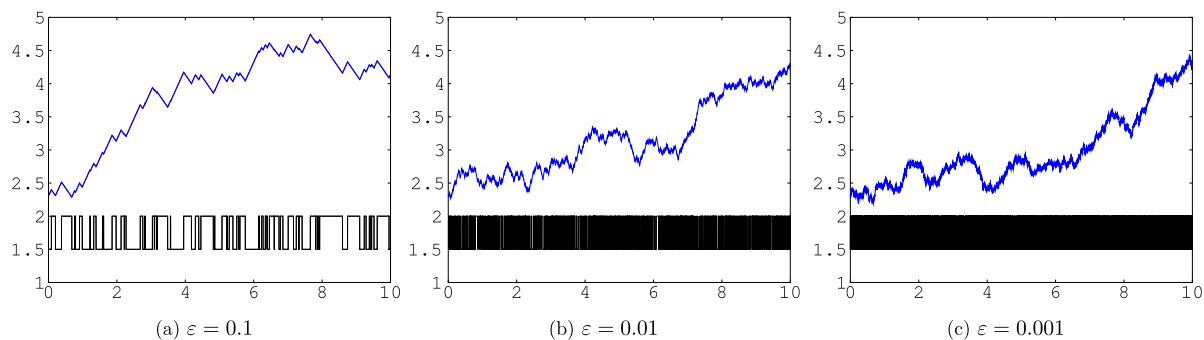


Fig. 1. Monte Carlo sample paths: $(\log(S_t^\varepsilon), \alpha_t^\varepsilon)$.

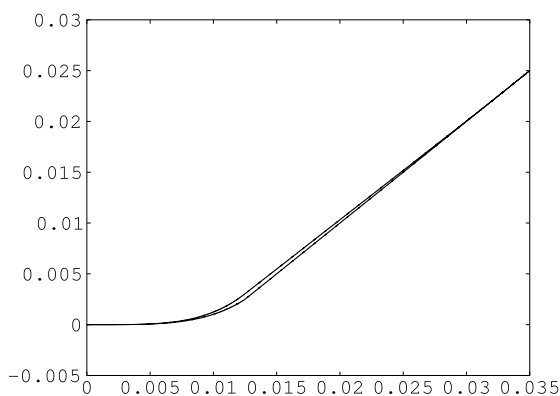


Fig. 2. Value functions $v(x, 1)$ and $v(x, 2)$.

where W_t is a standard Brownian motion. Such a limit is the solution to the stochastic differential equation

$$\frac{dS_t}{S_t} = \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

It is elementary to show that, as $\varepsilon \rightarrow 0$,

$$\beta_2 = \beta_2^\varepsilon \rightarrow \beta_0 = \frac{-\mu + \sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2}.$$

This implies that $x^* = x^{*,\varepsilon}$ defined in (9) converges to the selling threshold $x^0 = K\beta_0/(\beta_0 - 1)$ obtained in Øksendal [15, Example 10.2.2].

Taking $\mu = 0.2$ and $\sigma = 0.3$, we give sample paths of $\log(S_t^\varepsilon)$ and α_t^ε with varying ε in Fig. 1. It is clear from the pictures that as ε gets smaller and smaller, the fluctuation of α_t^ε is more and more rapidly and the corresponding S_t^ε approaches to a GBM.

Example 2 (Case II). In this example, we consider Case II with $\rho > f_1$ and use the following parameters

$$f_1 = 0.07, \quad f_2 = -0.03, \quad \lambda_1 = \lambda_2 = 1, \quad \rho = 0.10, \quad K = 0.01.$$

Solving Eq. (27) with $x_0^* = \rho K/(\rho - f_1)$, we have $(x^*, x_0^*) = (0.012478, 0.033333)$ and $X_0 = 0.013326$. The corresponding value functions are given in Fig. 2, in which $V(x, 1)$ is given by the upper curve and $V(x, 2)$ the lower one.

Example 3 (*Model calibration and a market test*). First we give a model calibration method. We consider

$$f_1 = \mu + \sigma_1 \quad \text{and} \quad f_2 = \mu + \sigma_2,$$

with $\nu_1\sigma_1 - \nu_2\sigma_2 = 0$, where $(\nu_1, \nu_2) = (\lambda_2/(\lambda_1 + \lambda_2), \lambda_1/(\lambda_1 + \lambda_2))$. Given T , let $Y_t = \log(S_t/S_0) = \int_0^t f(\alpha_s)ds$, $0 \leq t \leq T$. Then, μ can be approximated by Y_T/T . To estimate σ_1 and σ_2 , given step size $\delta > 0$, let $n\delta = T$,

$$\Delta Z_k = \log(S_{(k+1)\delta}) - \log(S_{k\delta}), \quad \text{for } k = 0, 1, 2, \dots, n,$$

and $\bar{Z} = (\sum_{k=0}^{n-1} \Delta Z_k)/n$. Then, $\bar{Z} \approx \delta\mu$. In addition, using Yin and Zhang [19, Theorem 5.9], we can show

$$E(\Delta Z_k - \bar{Z})^2 \approx \delta \left(\frac{2\lambda_1\lambda_2(\sigma_1 + \sigma_2)^2}{(\lambda_1 + \lambda_2)^3} \right).$$

Let

$$\sigma_0^2 = \frac{\sum_{k=0}^{n-1} (\Delta Z_k - \bar{Z})^2}{n-1}.$$

Then, by the Law of Large Numbers, we have

$$\sigma_0^2 \approx \delta \left(\frac{2\lambda_1\lambda_2(\sigma_1 + \sigma_2)^2}{(\lambda_1 + \lambda_2)^3} \right).$$

Using $\nu_1\sigma_1 = \nu_2\sigma_2$, we have

$$\sigma_1 = \frac{\sigma_0}{\sqrt{\delta}} \sqrt{\frac{\lambda_1(\lambda_1 + \lambda_2)}{2\lambda_2}} \quad \text{and} \quad \sigma_2 = \frac{\sigma_0}{\sqrt{\delta}} \sqrt{\frac{\lambda_2(\lambda_1 + \lambda_2)}{2\lambda_1}}.$$

Finally, we estimate λ_1 and λ_2 . Let $R = \nu_1/\nu_2$. Then, $\lambda_2 = R\lambda_1$.

$$R_1 = \#\{k: \Delta Z_k < 0 \text{ and } \Delta Z_{k+1} \geq 0\},$$

$$R_2 = \#\{k: \Delta Z_k > 0 \text{ and } \Delta Z_{k+1} \leq 0\}.$$

Then, it follows that

$$\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2} = T.$$

Therefore, the jump rates are given by

$$\begin{cases} \lambda_1 = \frac{1}{T} \left(R_1 + \frac{R_2}{R} \right), \\ \lambda_2 = \frac{1}{T} (RR_1 + R_2). \end{cases}$$

We test our selling rules using Apple Inc. (AAPL) daily closing prices during 2009/1/2 and 2013/3/28, see Fig. 3(a). Suppose we owned 100 AAPL shares at the beginning of 2009. We evaluate at the end of each half year during this period based on that half year stock prices to determine if we should sell the shares in the near future.

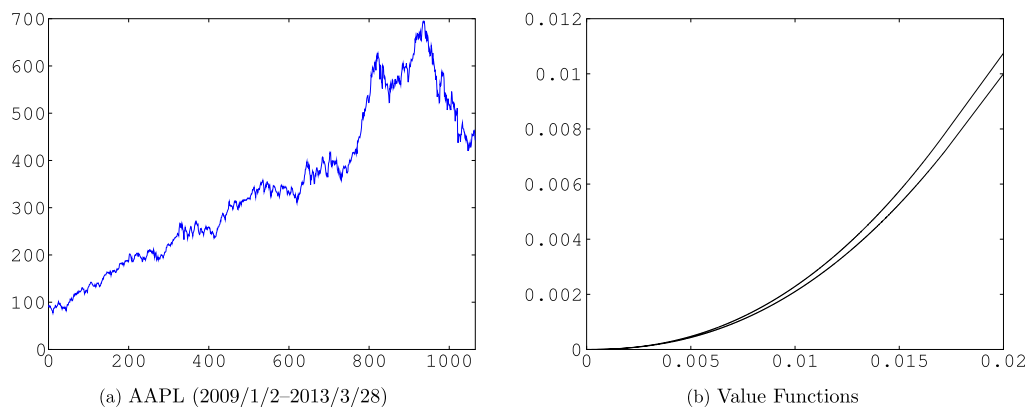


Fig. 3. Apple Inc. daily closes and value functions $v(x, 1)$ and $v(x, 2)$.

Table 1
Parameter values at the end of each period.

Periods	f_1	f_2	λ_1	λ_2	$\Phi(0.03)$
1st half of 2009	10.45	−10.61	100.48	124.23	−336.06
2nd half of 2009	3.21	−2.32	102.15	141.44	−217.41
1st half of 2010	3.06	−3.15	97.98	127.02	−83.18
2nd half of 2010	2.27	−1.92	103.57	134.25	−103.09
1st half of 2011	1.80	−1.85	117.19	125.00	−5.02
2nd half of 2011	3.01	−2.72	97.98	107.95	−60.56
1st half of 2012	5.39	−4.80	108.21	127.19	−185.32
2nd half of 2012	4.89	−5.13	135.25	130.95	35.79

We assume the risk free rate to be $\rho = 0.03$ and transaction cost $K = 0.01$. We use the calibration method discussed earlier and obtain the following results.

Note that in all periods $\rho \leq f_1$. Therefore, only Case I applies in this example. In Table 1, we should hold through the next half year if $\Phi(0.03) \leq 0$ and sell (following our selling rule) if $\Phi(0.03) > 0$. Clearly, a selling decision has to be made at the end of 2012. Using the parameter values $f_1 = 4.89$, $f_2 = -5.13$, $\lambda_1 = 135.25$, and $\lambda_2 = 130.95$, we obtain $x^* = 0.017213$ and the corresponding value functions $v(x, i) = V(x, i)$, $i = 1, 2$, which are plotted in Fig. 3(b). Therefore, one should sell as soon as α_t turns to 2 after the new year. This occurs on the second trading day (January 3) of 2013. The shares should be sold at the close of that day at \$542.10/share. As can be seen in this example, our selling rule helps to achieve the goal of letting your profits run and cutting your losses short.

5. Conclusion

In this paper, we have considered an optimal stock selling rule under a Markov chain model. The model is attractive for market analysis because of its remarkable simplicity and yet mathematical tractability. In addition, the optimal selling rules obtained are intuitive and easy to implement.

It would be interesting to consider more general models with multi-scale structure as treated in Yin and Zhang [19] so as to capture both long-term and short-term market movements. Such extension and related optimization problems could be subjects of future studies.

Acknowledgments

This research is supported in part by the Simons Foundation (235179). We thank the referee and the editors for their valuable comments and suggestions, which led to improvements of the paper.

Appendix A

Lemma A. Under the assumption $\Phi(\rho) > 0$, the bigger root of (4) $\beta_2 > 1$.

Proof. Recall the definition of D_1 and D_2 given in (3). It is easy to check $\Phi(\rho) > 0$ implies

$$D_2 > D_1 - f_1 f_2.$$

This leads to

$$\sqrt{D_1^2 - 4f_1 f_2 D_2} > D_1 - 2f_1 f_2.$$

Therefore, we have $\beta_2 > 1$. $\square$

Lemma B. Let $\kappa_2 = (\rho + \lambda_1 - f_1 \beta_2)/\lambda_1$, where β_2 is given in (5). Then, $0 < \kappa_2 < 1$.

Proof. To see $\kappa_2 > 0$, it suffices to show $\rho + \lambda_1 > f_1 \beta_2$. Recall that $f_2 < 0$. We only need to show $2(\rho + \lambda_1)f_2 < D_1 - \sqrt{D_1^2 - 4f_1 f_2 D_2}$, with D_1 and D_2 given in (3).

This is equivalent to

$$\sqrt{D_1^2 - 4f_1 f_2 D_2} < D_1 - 2(\rho + \lambda_1)f_2. \quad (29)$$

It is easy to check $D_1 - 2(\rho + \lambda_1)f_2 = (\rho + \lambda_2)f_1 - (\rho + \lambda_1)f_2 > 0$. Square both sides of (29) to obtain

$$D_1^2 - 4f_1 f_2 D_2 < D_1^2 - 4(\rho + \lambda_1)f_2 D_1 + 4((\rho + \lambda_1)f_2)^2.$$

Simplify this inequality to obtain

$$D_2 < (\rho + \lambda_1)(\rho + \lambda_2).$$

This clearly holds. Therefore, $\kappa_2 > 0$.

Similarly, to show $\kappa_2 < 1$, it suffices to show $\rho < f_1 \beta_2$. This is equivalent to

$$\sqrt{D_1^2 - 4f_1 f_2 D_2} > D_1 - 2\rho f_2.$$

Square and simplify to obtain $\rho \lambda_1 f_1 > \rho \lambda_1 f_2$. This holds because $f_1 > 0$ and $f_2 < 0$. $\square$

Lemma C. Under $\rho > f_1$, we have

$$\frac{K\beta_2}{\beta_2 - 1} < \frac{\rho K}{\rho - f_1}.$$

Proof. It is easy to see this inequality is equivalent to $f_1 \beta_2 > \rho$. Using the definition of β_2 and noting that $f_2 < 0$, we have

$$\sqrt{D_1^2 - 4f_1 f_2 D_2} > D_1 - 2\rho f_2.$$

If $D_1 - 2\rho f_2 < 0$, then we are done. Otherwise, square both sides to obtain

$$D_1^2 - 4f_1 f_2 D_2 > D_1^2 - 4\rho f_2 D_1 + 4\rho^2 f_2^2.$$

Simplify this inequality to have

$$\rho\lambda_1 f_1 > \rho\lambda_1 f_2,$$

which clearly holds because $f_1 > 0$ and $f_2 < 0$. $\square$

Lemma D. Under $\rho > f_1$, we have

$$\lambda_2 - \kappa_2(\rho + \lambda_2 - f_2) > 0, \quad (30)$$

which implies $\lambda_2 - \kappa_2(\rho + \lambda_2) > 0$.

Proof. Let $H_0 = \rho + \lambda_2 - f_2 > 0$. Using the definition of κ_2 , (30) is equivalent to

$$H_0(\rho + \lambda_1 - f_1\beta_2) < \lambda_1\lambda_2.$$

Therefore, we have

$$(\rho + \lambda_1)H_0 - \left(\frac{D_1 - \sqrt{D_1^2 - 4f_1f_2D_2}}{2f_2} \right) H_0 < \lambda_1\lambda_2.$$

Multiply both sides by $(2f_2)$ and rearrange the terms to obtain

$$H_0\sqrt{D_1^2 - 4f_1f_2D_2} > H_0D_1 - 2f_2[(\rho + \lambda_1)H_0 - \lambda_1\lambda_2].$$

Square both sides and simplify to have

$$(\rho + \lambda_1 - f_1)H_0 - \lambda_1\lambda_2 > 0,$$

which is exactly the assumption $\Phi(\rho) > 0$. This completes the proof. $\square$

References

- [1] B.R. Barmish, J.A. Primbs, On market-neutral stock trading arbitrage via linear feedback, in: Proc. American Control Conference, IEEE, Montreal, 2012.
- [2] J. Du Toit, G. Peskir, Selling a stock at the ultimate maximum, Ann. Appl. Probab. 19 (2009) 983–1014.
- [3] D. Duffie, Dynamic Asset Pricing Theory, second ed., Princeton University Press, Princeton, NJ, 1996.
- [4] R.J. Elliott, P.E. Kopp, Mathematics of Financial Markets, Springer-Verlag, New York, 1998.
- [5] J.P. Fouque, G. Papanicolaou, R.K. Sircar, Derivatives in Financial Markets with Stochastic Volatility, Cambridge University Press, 2000.
- [6] X. Guo, Q. Zhang, Optimal selling rules in a regime switching model, IEEE Trans. Automat. Control 50 (2005) 1450–1455.
- [7] K. Helmes, Computing optimal selling rules for stocks using linear programming, in: G. Yin, Q. Zhang (Eds.), Mathematics of Finance, in: Contemp. Math., Amer. Math. Soc., 2004, pp. 187–198.
- [8] V. Henderson, D. Hobson, An explicit solution for an optimal stopping/optimal control problem which models an asset sale, Ann. Appl. Probab. 18 (2008) 1681–1705.
- [9] J.C. Hull, Options, Futures, and Other Derivatives, third ed., Prentice Hall, Upper Saddle River, NJ, 1997.
- [10] I. Karatzas, S.E. Shreve, Methods of Mathematical Finance, Springer, New York, 1998.
- [11] A. Løkka, M. Zervos, Long-term optimal real investment strategies in the presence of adjustment costs, preprint, 2007.
- [12] A. Merhi, M. Zervos, A model for reversible investment capacity expansion, SIAM J. Control Optim. 46 (2007) 839–876.
- [13] M. Musiela, M. Rutkowski, Martingale Methods in Financial Modeling, Springer, New York, 1997.
- [14] R. Norberg, The Markov chain market, Astin Bull. 33 (2003) 265–287.
- [15] B. Øksendal, Stochastic Differential Equations, sixth ed., Springer-Verlag, New York, 2003.
- [16] G. Peskir, A. Shiryaev, Optimal Stopping and Free-Boundary Problems, Birkhäuser, Berlin, 2006.
- [17] J. van der Hoek, R.J. Elliott, American option prices in a Markov chain market model, Appl. Stoch. Models Bus. Ind. 28 (2012) 35–59.

- [18] G. Yin, R.H. Liu, Q. Zhang, Recursive algorithms for stock liquidation: a stochastic optimization approach, *SIAM J. Optim.* 13 (2002) 240–263.
- [19] G. Yin, Q. Zhang, *Continuous-Time Markov Chains and Applications, a Two-Time-Scale Approach*, second ed., Springer, New York, 2013.
- [20] H. Zhang, Q. Zhang, Trading a mean-reverting asset: buy low and sell high, *Automatica* 44 (2008) 1511–1518.
- [21] Q. Zhang, Stock trading: an optimal selling rule, *SIAM J. Control Optim.* 40 (2001) 64–87.