J. Math. Anal. Appl. 421 (2015) 51–58

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The Bishop–Phelps–Bollobás property for operators from $\mathcal{C}(K)$ to uniformly convex spaces ☆

Sun Kwang Kim [a], Han Ju Lee [b,*]

[a] *Department of Mathematics, Kyonggi University, Suwon 443-760, Republic of Korea*
[b] *Department of Mathematics Education, Dongguk University – Seoul, 100-715 Seoul, Republic of Korea*

ARTICLE INFO

Article history:
Received 19 April 2014
Available online 2 July 2014
Submitted by Richard M. Aron

Keywords:
Banach space
Approximation
Bishop–Phelps–Bollobás theorem

ABSTRACT

We show that the pair $(C(K), X)$ has the Bishop–Phelps–Bollobás property for operators if K is a compact Hausdorff space and X is a uniformly convex space.

1. Introduction

In this paper, we deal with strengthening of the famous Bishop–Phelps theorem. In 1961, Bishop and Phelps [8] showed that the set of all norm attaining functionals on a Banach space X is dense in its dual space X^* which is now called Bishop–Phelps theorem. This theorem has been extended to operators between Banach spaces X and Y. In general, the set of norm attaining operators $\mathcal{NA}(X,Y)$ is not dense in the space of linear operators $\mathcal{L}(X,Y)$. However, it is true for some pair of Banach spaces (X,Y). One of very well-known examples is the pair of every reflexive Banach space X and every Banach space Y, which was shown by Lindenstrauss [24]. After that, this was generalized by Bourgain to Banach space X with Radon–Nikodým property [10], and also there have been many efforts to find other positive examples [12,13,15,17,19,26,27].

Meanwhile, Bollobás sharpened Bishop–Phelps theorem as follows. From now on, the unit ball and the unit sphere of a Banach space X will be denoted by B_X and S_X, respectively.

☆ The second named author partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A1006869).

* Corresponding author.
E-mail addresses: sunkwang@kgu.ac.kr (S.K. Kim), hanjulee@dongguk.edu (H.J. Lee).

http://dx.doi.org/10.1016/j.jmaa.2014.06.081
0022-247X/

Theorem 1.1. *(See [9].) For an arbitrary $\epsilon > 0$, if $x^* \in S_{X^*}$ satisfies $|1 - x^*(x)| < \frac{\epsilon^2}{4}$ for $x \in B_X$, then there are both $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \epsilon$ and $\|y^* - x^*\| < \epsilon$.*

This Bishop–Phelps–Bollobás theorem shows that if a functional almost attains its norm at a point, then it is possible to approximate simultaneously both the functional and the point by norm attaining functionals and their corresponding norm attaining points. Clearly, Bishop–Phelps–Bollobás theorem implies Bishop–Phelps theorem.

Similarly to the case of Bishop–Phelps theorem, Acosta, Aron, García and Maestre [1] started to extend this theorem to bounded linear operators between Banach spaces and introduced the new notion *Bishop–Phelps–Bollobás property*.

Definition 1.2. (See [1, Definition 1.1].) Let X and Y be Banach spaces. We say that the pair (X, Y) has the Bishop–Phelps–Bollobás property for operators (*BPBp*) if, given $\epsilon > 0$, there exists $\eta(\epsilon) > 0$ such that if there exist both $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist both an operator $S \in S_{\mathcal{L}(X,Y)}$ and $u_0 \in S_X$ such that

$$\|Su_0\| = 1, \qquad \|x_0 - u_0\| < \epsilon \quad \text{and} \quad \|T - S\| < \epsilon.$$

Acosta et al. showed [1] that the pair (X, Y) has the *BPBp* for finite dimensional Banach spaces X and Y, and that the pair (ℓ_∞^n, Y) has the *BPBp* for every n if Y is a uniformly convex space. In the same paper, they asked if the pairs (c_0, Y) and (ℓ_∞, Y) have the BPBp for uniformly convex spaces Y. The first author solved the c_0 case and proved [20] that (c_0, Y) has the Bishop–Phelps–Bollobás property for all uniformly convex spaces Y.

Let $X = L_\infty(\mu)$ or $X = c_0(\Gamma)$ for a set Γ. Very recently, Lin and authors [23] proved that (X, Y) has the BPBp for every uniformly convex space Y. So $(L_\infty(\mu), L_p(\nu))$ has the BPBp for all $1 < p < \infty$ and for all measures ν. They also proved that (X, Y), as a pair of complex spaces, has the BPBp for every uniformly complex convex space Y. In particular, $(L_\infty(\mu), L_1(\nu))$, as a pair of complex spaces, has the BPBp, since $L_1(\nu)$ is uniformly complex convex [18].

On the other hand, there have been several researches about the BPBp for operators into $C(K)$ spaces (or uniform algebras). Even though Schachermayer showed [26] that the set of norm attaining operators is not dense in $\mathcal{L}(L_1[0,1], C[0,1])$, there are some positive results about the BPBp. It is shown [5] that $(X, C(K))$ has the BPBp if X is an Asplund space. This result was extended so that (X, A) has the BPBp if X is Asplund and A is a uniform algebra [11]. The authors also proved [21] that $(X, C(K))$ has the BPBp if X^* admits a uniformly simultaneously continuous retractions. It is also worthwhile to remark that the pair $(C(K), C(L))$ of the spaces of real-valued continuous functions has the BPBp for every compact Hausdorff spaces K and L [2]. Concerning the results about L_∞ spaces, it is shown [7] that $(L_1(\mu), L_\infty[0,1])$ has the BPBp and this was generalized [14] so that $(L_1(\mu), L_\infty(\nu))$ has the BPBp if μ is any measure and ν is a localizable measure. These are the strengthening of the results that the set of norm-attaining operators is dense in $\mathcal{L}(L_1(\mu), L_\infty(\nu))$ [17,25] for every measure μ and every localizable measure ν. Finally we remark that if X is uniformly convex, then (X, Y) has the BPBp for every Banach space Y [3,6,22].

Throughout this paper, we consider only real Banach spaces. It is the main result of this paper that $(C(K), X)$ has the *BPBp* for every compact Hausdorff space K and for every uniformly convex space X. Recall that Schachermayer showed [26] that every weakly compact operator from $C(K)$ into a Banach space can be approximated by norm attaining weakly compact operators (cf. [4, Theorem 2]). So the set of all norm attaining operators is dense in $\mathcal{L}(C(K), Y)$ for every reflexive space Y. Notice that the reflexivity of Y is not sufficient to prove that $(C(K), Y)$ has the BPBp. Indeed, if we take a reflexive strictly convex space Y_0 which is not uniformly convex, then $(\ell_1^{(2)}, Y_0)$ does not have the BPBp [1,6]. If we take K_0 as the set consisting of only two points, then $C(K_0)$ is isometrically isomorphic to 2-dimensional $\ell_1^{(2)}$ space.

Hence $(C(K_0), Y_0)$ does not have the BPBp. However, if X is uniformly convex, then it will be shown that $(C(K), X)$ has the BPBp.

2. Main result

Given a Banach space X, the modulus of convexity $\delta_X(\epsilon)$ of the unit ball B_X is defined by for $0 < \varepsilon < 1$,

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X,\ \|x-y\| \geqslant \epsilon\right\}.$$

A Banach space X is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon < 1$. It is well known that every uniformly convex space is reflexive.

In [20], the following result was shown: Let $1 > \epsilon > 0$ be given and X be a reflexive Banach space and Y be a uniformly convex Banach space with modulus of convexity $\delta_X(\epsilon) > 0$. If $T \in S_{\mathcal{L}(X,Y)}$ and $x_1 \in S_X$ satisfy

$$\|Tx_1\| > 1 - \frac{\epsilon}{2^5}\delta_X\left(\frac{\epsilon}{2}\right),$$

then there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_2 \in S_X$ such that $\|Sx_2\| = 1$, $\|S - T\| < \epsilon$ and $\|Tx_1 - Sx_2\| < \epsilon$.

This says that for a reflexive space X and a uniformly convex space Y, the pair (X, Y) has a little weaker property than *BPBp*. The only difference from the *BPBp* and the above is approximating the image of a point if the given operator almost attains its norm. Since the set of all norm attaining operators is dense in $\mathcal{L}(X, Y)$ for every Y if X is reflexive, the following result generalize the result mentioned above [20].

Proposition 2.1. *Let X be a Banach space and Y be a uniformly convex space. Suppose that the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$. Then, given $0 < \epsilon < 1$, there exists $\eta(\varepsilon) > 0$ such that if $T \in S_{\mathcal{L}(X,Y)}$ and $x_1 \in S_X$ satisfy $\|Tx_1\| > 1 - \eta(\varepsilon)$, then there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_2 \in S_X$ such that $\|Sx_2\| = 1$, $\|S - T\| < \varepsilon$ and $\|Tx_1 - Sx_2\| < \varepsilon$.*

Proof. Let $\delta_Y(\cdot)$ be the modulus of convexity of Y and $0 < \varepsilon_1 < \varepsilon$. Choose $\varepsilon_2 > 0$ such that $(1 - \varepsilon_2^2)^3 - 2\varepsilon_2 - \varepsilon_2^3 > 1 - \delta_Y(\varepsilon_1)$ and $\varepsilon_2^2 + 2\varepsilon_2 + \varepsilon_1 < \varepsilon$.

We show that $\eta(\varepsilon) = \varepsilon_2^2$ is a suitable number. Assume $\|Tx_1\| > 1 - \varepsilon_2^2$. Choose $y^* \in S_{Y^*}$ such that $y^*Tx_1 = \operatorname{Re} y^*Tx_1 > 1 - \varepsilon_2^2$ and define an operator $\tilde{T}_1$ by

$$\tilde{T}_1 x = Tx + \varepsilon_2 y^*(Tx)Tx_1 \quad \text{for every } x \in X.$$

It is easy to see that $1 - \varepsilon_2 < (1 - \varepsilon_2^2)(1 + \varepsilon_2(1 - \varepsilon_2^2)) \leqslant \|\tilde{T}_1 x_1\| \leqslant \|\tilde{T}_1\| \leqslant 1 + \varepsilon_2$.

Let $T_1 = \tilde{T}_1/\|\tilde{T}_1\|$. Since the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$, there exist an operator S and $z \in S_X$ such that $\|T_1 - S\| < \varepsilon_2^2$ and $\|Sz\| = \|S\| = 1$. Since $\|Sz - T_1 z\| < \varepsilon_2^2$, we see that $\|T_1 z\| > 1 - \varepsilon_2^2$, which means that

$$\left\|Tz + \varepsilon_2 y^*(Tz)Tx_1\right\| > \left(1 - \varepsilon_2^2\right)\|\tilde{T}_1\| > \left(1 - \varepsilon_2^2\right)\left(1 - \varepsilon_2^2\right)\left(1 + \varepsilon_2\left(1 - \varepsilon_2^2\right)\right).$$

Hence, we have $|y^*(Tz)| > (1 - \varepsilon_2^2)^3 - 2\varepsilon_2 - \varepsilon_2^3 > 1 - \delta_Y(\varepsilon_1)$. Choose $\alpha = \pm 1$ satisfying $y^*T(\alpha z) = |y^*T(z)|$ and let $x_2 = \alpha z$. Then

$$\left\|\frac{Tx_1 + Tx_2}{2}\right\| \geqslant \frac{y^*Tx_1 + y^*Tx_2}{2} > 1 - \delta_Y(\epsilon_1).$$

Hence, we see that $\|Tx_1 - Tx_2\| < \varepsilon_1$. Moreover,

$$\begin{aligned}\|Sx_2 - Tx_1\| &\leqslant \|Sx_2 - T_1x_2\| + \|T_1x_2 - \tilde{T}_1x_2\| + \|\tilde{T}_1x_2 - Tx_2\| + \|Tx_2 - Tx_1\| \\ &\leqslant \|S - T_1\| + \big|\|\tilde{T}_1\| - 1\big| + \varepsilon_2 + \varepsilon_1 \\ &< \varepsilon_2^2 + \varepsilon_2 + \varepsilon_2 + \varepsilon_1 < \varepsilon.\end{aligned}$$

This completes the proof. □

Now we state the main theorem of this paper.

Theorem 2.2. *Let X be a uniformly convex space and K be a compact Hausdorff space. Then the pair $(C(K), X)$ has the BPBp.*

Before we present the proof of the main result, we begin with preliminary comments on vector measure and two lemmas. Recall that a vector measure $G : \Sigma \to X$ on a σ-algebra Σ is said to be countably additive if, for every mutually disjoint sequence of Σ-measurable subsets $\{A_i\}_{i=1}^{\infty}$, we have

$$G\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} G(A_i).$$

For a Σ-measurable subset A, the semi-variation $\|G\|(A)$ of G is defined by

$$\|G\|(A) = \sup\big\{\big|x^*G\big|(A) : x^* \in B_{X^*}\big\},$$

where $|x^*G|(A)$ is the total variation of the scalar-valued countably additive measure x^*G on A. The vector measure G on a Borel σ-algebra is said to be regular if for each Borel subset E and $\epsilon > 0$ there exist a compact subset K and an open set O such that

$$K \subset E \subset O \quad \text{and} \quad \|G\|(O \setminus K) < \epsilon.$$

It is well known that if X is reflexive, each operator T in $\mathcal{L}(C(K), X)$ has an X-valued countably additive representing Borel measure G and the measure is regular (see [16, VI. Theorems 1, 5 and Corollary 14] for a reference). That is, for all $f \in C(K)$ and $x^* \in X^*$, we have

$$Tf = \int_K f\,dG, \qquad x^*T(f) = \int_K f\,d^*G \quad \text{and} \quad \|T\| = \|G\|(K).$$

If G is a countably additive representing measure for an operator T in $\mathcal{L}(C(K), X)$, then it is easy to see that for any bounded Borel measurable function $h : K \to \mathbb{R}$, the mapping S, defined by $Sf = \int fh\,dG$, is a bounded linear operator and $\|S\| \leqslant \|T\| \cdot \|h\|_\infty$, where $\|h\|_\infty = \sup\{|h(k)| : k \in K\}$.

Lemma 2.3. *Let G be a countably additive, Borel regular X-valued vector measure on a compact Hausdorff space K with $\|G\|(K) = 1$ and let $0 < \eta, \gamma < 1$. Assume that $f \in S_{C(K)}$ and $x^* \in S_{X^*}$ satisfy*

$$\int_K f\,dx^*G > 1 - \eta.$$

Then, we have

$$\left|x^*G\right|\left(K \setminus \left(A_\gamma^+ \cup A_\gamma^-\right)\right) < 2\frac{\eta}{\gamma} + \eta,$$

where $A_\gamma^+ = \{t \in K \mid f(t) \geqslant 1 - \gamma\}$ *and* $A_\gamma^- = \{t \in K \mid f(t) \leqslant -1 + \gamma\}$. *Moreover, there exist mutually disjoint compact sets* F^+, F^- *such that* x^*G *is positive on* F^+, *negative on* F^- *and*

$$\int\limits_{(F^+\cap A_\gamma^+)\cup(F^-\cap A_\gamma^-)} f\,dx^*G > 1 - 4\frac{\eta}{\gamma}.$$

Proof. The Hahn decomposition of x^*G and the regularity of G show that there exist mutually disjoint compact sets F^+, F^- such that x^*G is positive on F^+, negative on F^- and $\|G\|(K \setminus (F^+ \cup F^-)) < \eta$.

$$\begin{aligned}
1 - \eta &\leqslant \int\limits_K f\,dx^*G = \int\limits_{F^+} f\,dx^*G + \int\limits_{F^-} f\,dx^*G + \int\limits_{K\setminus(F^+\cup F^-)} f\,dx^*G \\
&= \int\limits_{F^+\cap A_\gamma^+} f\,dx^*G + \int\limits_{F^+\setminus A_\gamma^+} f\,dx^*G + \int\limits_{F^-\cap A_\gamma^-} f\,dx^*G + \int\limits_{F^-\setminus A_\gamma^-} f\,dx^*G + \int\limits_{K\setminus(F^+\cup F^-)} f\,dx^*G \\
&\leqslant x^*G\left(F^+ \cap A_\gamma^+\right) + (1-\gamma)x^*G\left(F^+ \setminus A_\gamma^+\right) - x^*G\left(F^- \cap A_\gamma^-\right) - (1-\gamma)x^*G\left(F^- \setminus A_\gamma^-\right) + \eta \\
&= x^*G\left(F^+\right) - x^*G\left(F^-\right) - \gamma\left(x^*G\left(F^+ \setminus A_\gamma^+\right) - x^*G\left(F^- \setminus A_\gamma^-\right)\right) + \eta.
\end{aligned}$$

Since $x^*G(F^+) - x^*G(F^-) = |x^*G|(F^+ \cup F^-) \leqslant \|G\|(K) = 1$, we get

$$\left|x^*G\right|\left(\left(F^+ \setminus A_\gamma^+\right) \cup \left(F^- \setminus A_\gamma^-\right)\right) = x^*G\left(F^+ \setminus A_\gamma^+\right) - x^*G\left(F^- \setminus A_\gamma^-\right) \leqslant 2\frac{\eta}{\gamma}.$$

This shows that

$$\begin{aligned}
\left|x^*G\right|\left(K \setminus \left(A_\gamma^+ \cup A_\gamma^-\right)\right) &\leqslant \left|x^*G\right|\left(K \setminus \left(F^+ \cup F^-\right)\right) + \left|x^*G\right|\left(\left(F^+ \cup F^-\right) \setminus \left(A_\gamma^+ \cup A_\gamma^-\right)\right) \\
&\leqslant \|G\|\left(K \setminus \left(F^+ \cup F^-\right)\right) + \left|x^*G\right|\left(\left(F^+ \setminus A^+\right) \cup \left(F^- \setminus A_\gamma^-\right)\right) \\
&< 2\frac{\eta}{\gamma} + \eta
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_{(F^+\cap A_\gamma^+)\cup(F^-\cap A_\gamma^-)} f\,dx^*G &= \int\limits_{F^+\cup F^-} f\,dx^*G - \int\limits_{(F^+\setminus A_\gamma^+)\cup(F^-\setminus A_\gamma^-)} f\,dx^*G \\
&\geqslant \int\limits_K f\,dx^*G - \|G\|\left(K \setminus \left(F^+ \cup F^-\right)\right) - \left|x^*G\right|\left(\left(F^+ \setminus A_\gamma^+\right) \cup \left(F^- \setminus A_\gamma^-\right)\right) \\
&> 1 - 2\eta - 2\frac{\eta}{\gamma} > 1 - 4\frac{\eta}{\gamma}.
\end{aligned}$$

This completes the proof. □

Lemma 2.4. *Let* X *be a uniformly convex space with the modulus of convexity* δ_X *and* $T \in S_{\mathcal{L}(C(K),X)}$ *be an operator represented by the countably additive, Borel regular vector measure* G. *Let* $0 < \epsilon < 1$ *and* A *be a Borel set of* K. *Suppose that an operator* S, *defined by* $Sf = \int_A f\,dG$, *satisfies* $\|S\| > 1 - \delta_X(\epsilon)$. *Then*

$$\|T - S\| = \sup_{f \in B_{C(K)}} \left\| \int\limits_{K\setminus A} f\,dG \right\| < \epsilon.$$

Proof. Choose $x^* \in S_{X^*}$, $f_0 \in S_{C(K)}$ such that $\|Sf_0\| = x^*Sf_0 > 1 - \delta_X(\varepsilon)$. By the regularity of G, we may choose a compact set $A_1 \subset A$ such that

$$\int\limits_{A_1} f_0\, dx^*G > 1 - \delta_X(\epsilon).$$

Fix a closed set $B \subset K \setminus A$ and $g \in B_{C(B)}$. Then, choose $g_+, g_- \in B_{C(K)}$ satisfying

$$\begin{aligned} &g_+(t) = g_-(t) = f_0(t) \quad \text{for } t \in A_1 \quad \text{and} \\ &g_+(t) = -g_-(t) = g(t) \quad \text{for } t \in B. \end{aligned}$$

So, we have

$$1 - \delta_X(\epsilon) < \int\limits_{A_1} f_0\, dx^*G \leqslant \left\| \int\limits_{A_1} f_0\, dG \right\| = \frac{1}{2}\left\| \int\limits_{A_1 \cup B} g_+\, dG + \int\limits_{A_1 \cup B} g_-\, dG \right\|.$$

Note that $\| \int_{A_1 \cup B} g_+\, dG\|, \| \int_{A_1 \cup B} g_+\, dG\| \leqslant 1$. Thus, from the uniform convexity of X, we get that

$$\left\| 2\int\limits_{B} g\, dG \right\| = \left\| \int\limits_{A_1 \cup B} g_+\, dG - \int\limits_{A_1 \cup B} g_-\, dG \right\| < \epsilon.$$

This implies $\|T - S\| < \epsilon$ and the proof is done. □

Proof of Theorem 2.2. Let δ_X be the modulus of convexity for B_X. Fix $0 < \epsilon < \frac{1}{2^8}$ and let η be the function which appears in Proposition 2.1 for the pair $(C(K), X)$, and let $\gamma(t) = \min\{\eta(t), \delta_X(t), \frac{t}{3}\}$ for $t \in (0, 1)$. Assume that $T \in S_{\mathcal{L}(C(K),X)}$ and $f_0 \in S_{C(K)}$ satisfy that

$$\|Tf_0\| > 1 - \frac{\epsilon}{8}\gamma\left(\frac{\epsilon}{6}\delta_X\left(\frac{\epsilon}{6}\right)\right).$$

Let G be the representing vector measure for T which is countably additive Borel regular on K. Choose $x_1^* \in S_{X^*}$ such that $x_1^*Tf_0 > 1 - \frac{\epsilon}{8}\gamma(\frac{\epsilon}{6}\delta_X(\frac{\epsilon}{6}))$. By Lemma 2.3 there exist two mutually disjoint compact sets F^+, F^- such that x^*G is positive on F^+, negative on F^- and

$$\int\limits_{(F^+ \cap A^+_{\epsilon/2}) \cup (F^- \cap A^-_{\epsilon/2})} f\, dx^*G > 1 - \gamma\left(\frac{\epsilon}{6}\delta_X\left(\frac{\epsilon}{6}\right)\right),$$

where $A^+_{\epsilon/2} = \{t \in K \mid f_0(t) \geqslant 1 - \frac{\epsilon}{2}\}$ and $A^-_{\epsilon/2} = \{t \in K \mid f_0(t) \leqslant -1 + \frac{\epsilon}{2}\}$.

Let $A_1 = F^+ \cap A^+_{\epsilon/2}$, $A_2 = F^- \cap A^-_{\epsilon/2}$ and $A = A_1 \cup A_2$. Then, define $S_1 \in B_{\mathcal{L}(C(K),X)}$ by $S_1 f = \int_A f\, dG$ for every $f \in C(K)$. Then Lemma 2.4 shows that $\|T - S_1\| < \frac{\epsilon}{6}$. Choose $f_1 \in S_{C(K)}$ such that

$$\begin{aligned} &f_1(t) = 1 \quad \text{for } t \in A_1 \quad \text{and} \\ &f_1(t) = -1 \quad \text{for } t \in A_2. \end{aligned}$$

For $f \in C(K)$, the restriction of f to A will be denoted by $f|_A$. Now consider S_1 as an operator in $\mathcal{L}(C(A), X)$. Then we have

$$\|S_1(f_1|_A)\| > 1 - \gamma\left(\frac{\epsilon}{6}\delta_X\left(\frac{\epsilon}{6}\right)\right).$$

So Proposition 2.1 shows that there exist $S_2 \in S_{\mathcal{L}(C(A),X)}$ and $f_2 \in S_{C(A)}$ such that $\|S_2 f_2\| = 1$, $\|S_2 - \frac{S_1}{\|S_1\|}\| < \frac{\epsilon}{6}\delta_X(\frac{\epsilon}{6})$ and $\|S_2 f_2 - \frac{S_1(f_1|_A)}{\|S_1\|}\| < \frac{\epsilon}{6}\delta_X(\frac{\epsilon}{6})$. Let G' be the representing vector measure for S_2 which is countably additive Borel regular on A. Choose $x_2^* \in S_{X^*}$ so that $x_2^* S_2 f_2 = \|S_2 f_2\| = \int_A f_2\, dx_2^* G' = 1$.

Since

$$\begin{aligned} x_2^* S_2(f_1|_A + f_2) &\geqslant 2x_2^* S_2 f_2 - \|S_2 f_2 - S_2(f_1|_A)\| \\ &\geqslant 2 - \left\|S_2 f_2 - \frac{S_1(f_1|_A)}{\|S_1\|}\right\| - \left\|\frac{S_1(f_1|_A)}{\|S_1\|} - S_2(f_1|_A)\right\| \\ &> 2\left(1 - \frac{\epsilon}{6}\delta_X\left(\frac{\epsilon}{6}\right)\right), \end{aligned}$$

we get

$$\int_A \frac{f_1 + f_2}{2}\, dx^* G' > 1 - \frac{\epsilon}{6}\delta_X\left(\frac{\epsilon}{6}\right).$$

By applying Lemma 2.3 again, we get a compact subset F of A such that

$$F \subset \left\{t \in A : \left|f_1(t) + f_2(t)\right| > 2(1 - \epsilon)\right\}$$

and

$$\left\|\int_F \frac{f_1 + f_2}{2}\, dG'\right\| > 1 - \delta_X\left(\frac{\epsilon}{6}\right).$$

Let $B = \{t \in A : f_1(t) f_2(t) \geqslant 0\}$. Then, $F \subset B$ and

$$\sup_{f \in B_{C(A)}}\left\|\int_B f\, dG'\right\| \geqslant \left\|\int_F \frac{f_1 + f_2}{2}\, dG'\right\| > 1 - \delta_X\left(\frac{\epsilon}{6}\right).$$

By Lemma 2.4, we have

$$\sup_{f \in B_{C(K)}}\left\|\int_{A \setminus B} f\, dG'\right\| < \frac{\epsilon}{6}.$$

Define $S \in \mathcal{L}(C(A), X)$ by, for $f \in C(A)$,

$$Sf = \int_B f\, dG' - \int_{A \setminus B} f\, dG'$$

and let

$$f_3 = \begin{cases} |f_2| & \text{for } t \in A_1, \\ -|f_2| & \text{for } t \in A_2. \end{cases}$$

So $f_3 \in C(A)$ and $f_3 = f_2\chi_B - f_2\chi_{A\setminus B}$, where χ_S is the characteristic function on a set S. Hence we have $Sf_3 = S_2 f_2$, $\|Sf_3\| = \|S\| = 1$ and $\|S - S_2\| < \frac{\epsilon}{3}$. On the other hand, we have $\|2f_3 - f_1|_A\| \leqslant 1$. Since X is uniformly convex and we have $Sf_3 = \frac{S(f_1|_A) + S(2f_3 - f_1|_A)}{2}$, we get

$$Sf_3 = S(f_1|_A) = S(2f_3 - f_1|_A).$$

We now consider S_1, S_2, S as operators in $\mathcal{L}(C(K), X)$ using the canonical extension. That is, $S(f) = S(f|_A)$, $S_i(f) = S_i(f|_A)$ for all $f \in C(K)$ and for $i = 1, 2$. Let C be the compact subset defined by

$$C = \{t \in K : |f_1(t) - f_0(t)| \geqslant \epsilon\}.$$

Note that A and C are mutually disjoint. Indeed, if $t \in A$, then $|f_0(t) - f_1(t)| \leqslant \epsilon/2$. So there is $\phi \in C(K)$ such that $0 \leqslant \phi \leqslant 1$, $\phi(k) = 1$ for $k \in A$ and $\phi(k) = 0$ for $k \in C$. Let $g = \phi f_1 + (1 - \phi) f_0$. Then we see that $\|Sg\| = 1$,

$$\begin{aligned}\|S - T\| &\leqslant \|S - S_2\| + \left\|S_2 - \frac{S_1}{\|S_1\|}\right\| + \left\|\frac{S_1}{\|S_1\|} - S_1\right\| + \|S_1 - T\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon\end{aligned}$$

and $\|g - f_0\| = \sup_{k \in K \setminus C} |\phi(k)(f_1(k) - f_0(k))| < \epsilon$. This completes the proof. □

References

[1] M.D. Acosta, R.M. Aron, D. García, M. Maestre, The Bishop–Phelps–Bollobás theorem for operators, J. Funct. Anal. 254 (2008) 2780–2799.

[2] M.D. Acosta, J. Becerra-Guerrero, Y.S. Choi, M. Ciesielski, S.K. Kim, H.J. Lee, M.L. Lourenço, M. Martín, The Bishop–Phelps–Bollobás property for operators between spaces of continuous functions, Nonlinear Anal. 95 (2014) 323–332.

[3] M.D. Acosta, J. Becerra-Guerrero, D. García, M. Maestre, The Bishop–Phelps–Bollobás theorem for bilinear forms, Trans. Amer. Math. Soc. 365 (2013) 5911–5932.

[4] J. Alaminos, Y.S. Choi, S.G. Kim, R. Payá, Norm attaining bilinear forms on spaces of continuous functions, Glasg. Math. J. 40 (1998) 467–482.

[5] R.M. Aron, B. Cascales, O. Kozhushkina, The Bishop–Phelps–Bollobás theorem and Asplund operators, Proc. Amer. Math. Soc. 139 (2011) 3553–3560.

[6] R.M. Aron, Y.S. Choi, S.K. Kim, H.J. Lee, M. Martín, The Bishop–Phelps–Bollobás version of Lindenstrauss properties A and B, preprint.

[7] R.M. Aron, Y.S. Choi, D. García, M. Maestre, The Bishop–Phelps–Bollobás theorem for $L(L_1(\mu), L_\infty[0,1])$, Adv. Math. 228 (2011) 617–628.

[8] E. Bishop, R.R. Phelps, A proof that every Banach space is subreflexive, Bull. Amer. Math. Soc. 67 (1961) 97–98.

[9] B. Bollobás, An extension to the theorem of Bishop and Phelps, Bull. Lond. Math. Soc. 2 (1970) 181–182.

[10] J. Bourgain, Dentability and the Bishop–Phelps property, Israel J. Math. 28 (1977) 265–271.

[11] B. Cascales, A.J. Guirao, V. Kadets, A Bishop–Phelps–Bollobás type theorem for uniform algebras, Adv. Math. 240 (2013) 370–382.

[12] Y.S. Choi, S.K. Kim, The Bishop–Phelps–Bollobás property and lush spaces, J. Math. Anal. Appl. 390 (2013) 549–555.

[13] Y.S. Choi, S.K. Kim, The Bishop–Phelps–Bollobás theorem for operators from $L_1(\mu)$ to Banach spaces with the Radon–Nikodým property, J. Funct. Anal. 261 (2013) 1446–1456.

[14] Y.S. Choi, S.K. Kim, H.J. Lee, M. Martín, The Bishop–Phelps–Bollobás theorem for operators on $\boldsymbol{L_1(\mu)}$, J. Funct. Anal. 267 (1) (2014) 214–242.

[15] Y.S. Choi, H.J. Lee, H.G. Song, Denseness of norm-attaining mappings on Banach spaces, Publ. Res. Inst. Math. Sci. 46 (2010) 171–182.

[16] J. Diestel, J.J. Uhl Jr., Vector Measures, Amer. Math. Soc. Math. Surveys, vol. 15, 1977.

[17] C. Finet, R. Payá, Norm attaining operators from L_1 into L_∞, Israel J. Math. 108 (1998) 139–143.

[18] J. Globevnik, On complex strict and uniform convexity, Proc. Amer. Math. Soc. 47 (1975) 175–178.

[19] J. Johnson, J. Wolfe, Norm attaining operators, Studia Math. 65 (1998) 7–19.

[20] S.K. Kim, The Bishop–Phelps–Bollobás theorem for operators from c_0 to uniformly convex spaces, Israel J. Math. 197 (2013) 425–435.

[21] S.K. Kim, H.J. Lee, Simultaneously continuous retraction and Bishop–Phelps–Bollobás type theorem, J. Math. Anal. Appl. 420 (1) (2014) 758–771.

[22] S.K. Kim, H.J. Lee, Uniform convexity and Bishop–Phelps–Bollobás property, Canad. J. Math. 66 (2) (2014) 373–386.

[23] S.K. Kim, H.J. Lee, P.K. Lin, The Bishop–Phelps–Bollobás theorem for operators from $L_\infty(\mu)$ to uniformly convex spaces, preprint.

[24] J. Lindenstrauss, On operators which attain their norm, Israel J. Math. 1 (1963) 139–148.

[25] R. Payá, Y. Saleh, Norm attaining operators from $L_1(\mu)$ into $L_\infty(\nu)$, Arch. Math. 75 (2000) 380–388.

[26] W. Schachermayer, Norm attaining operators on some classical Banach spaces, Pacific J. Math. 105 (1983) 427–438.

[27] J.J. Uhl Jr., Norm attaining operators on $L_1[0,1]$ and the Radon–Nykodým property, Pacific J. Math. 63 (1976) 293–300.