



# On the spectrum of positive linear operators with a partition of unity property



Johannes Nagler

Fakultät für Informatik und Mathematik, Universität Passau, Germany

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## ABSTRACT

We characterize the spectrum of positive linear operators between Banach function spaces having finite rank and a partition of unity property. Our main result states that all the points in the spectrum are eigenvalues and 1 is the only eigenvalue on the unit circle. Finally, we show that the iterates converge in the uniform operator topology to a projection operator that reproduces constant functions and we provide a simple criterion to obtain the limiting projection operator.

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We study positive linear operators that have finite rank on a general infinite-dimensional complex Banach function space  $X$  that contains the constant function 1 with norm equal to one. In addition, we assume that the associated basis functions of the positive finite-rank operator form a partition of unity. Operators of this kind are used in many applications to approximate functions where only a finite number of samples are available. The partition of unity property guarantees the exact reconstruction of constant functions. Of our interest here is the asymptotic behaviour of iterative applications of the operator and the question whether the limit exists.

The asymptotic behaviour of the iterates of positive linear operators has extensively been discussed by many authors. Kelisky and Rivlin [12] have first been considering the limit of iterates of the classical Bernstein operator on the space  $C([0, 1])$ . This result has been extended by Karlin and Ziegler [10] to a more general setting. In [15,16], J. Nagel has examined the asymptotic behaviour of the Bernstein and the Kantorovič operators. Using a contraction principle, Rus [19] has shown an alternative way to prove the convergence of the iterates of the Bernstein operator. The iterates of the Bernstein operators have been also revisited by Badea [2] using spectral properties. Recently, contributions have been made by Gavrea and Ivan [4–7] and by Altomare [1] using methods based on Korovkin-type approximation theory. However, all these results are restricted to the space of continuous functions, i.e., are not applicable for the  $L^p$  spaces, and there is no general theory that guarantees the existence of the limit of the iterates.

E-mail address: johannes.nagler@uni-passau.de.

Here, we provide a functional analysis based approach using spectral properties that guarantees the existence of the limit of these iterates. We generalize some results of the manuscript [17], where spectral properties of this kind have been shown concretely for the variation-diminishing Schoenberg operator in order to show the limit of the iterates and to prove lower bounds for their approximation error in terms of different moduli of smoothness. If these spectral properties have been established, we apply the famous Theorem of Katznelson and Tzafriri [11] that states that the iterates converge in the operator norm if and only if the spectrum of  $T$  has either no points on the unit circle or 1 is the only spectral value on the unit circle. This article is devoted to give an application of this beautiful result in the field of approximation theory.

Finally, we remark that our results are applicable without explicit knowledge on spectral theory and additionally, we provide a simple criterion to derive the limiting projection operator.

## 1. Main results and examples

First, we introduce the notation used throughout this article. The general setting where our results are applicable is described next, followed by the presentation of our main results. We conclude this section with two examples to demonstrate the simplicity of our method.

### 1.1. Notation

Given two Banach spaces  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  and their topological duals  $(X^*, \|\cdot\|_{X^*})$ ,  $(Y^*, \|\cdot\|_{Y^*})$  respectively, we denote the space of bounded linear operators from  $X$  to  $Y$  by  $\mathcal{B}(X, Y)$  equipped with the usual operator norm  $\|\cdot\|_{op}$ . With  $I$  we denote the identity operator on  $\mathcal{B}(X, Y)$ . For  $T \in \mathcal{B}(X, Y)$ , we denote by  $\sigma(T)$  the spectrum of  $T$  and by  $\sigma_p(T)$  the point spectrum of  $T$ , i.e., the set of all eigenvalues of  $T$ . We denote by  $\mathcal{R}(T)$  the range and by  $\mathcal{N}(T)$  the null space of  $T$ . The open unit disk in the complex plane will be denoted by  $\mathbb{D}$  and its closure by  $\overline{\mathbb{D}}$ .

### 1.2. Setting

Let  $K$  be a compact Hausdorff space and let  $(X, \|\cdot\|_X)$  be a complex infinite-dimensional Banach function space on  $K$  that contains the constant function 1 with  $\|1\|_X = 1$ . Given an integer  $n > 0$  and linearly independent positive functions  $e_1, \dots, e_n \in X$  that form a partition of unity, i.e.,

$$\sum_{k=1}^n e_k = 1, \quad (1)$$

we set  $Y := \text{span}\{e_1, \dots, e_n\}$ . Clearly,  $Y$  is a finite-dimensional subspace of  $X$  with  $1 \in Y$ . Equipped with a norm  $\|\cdot\|_Y$  that satisfies  $\|1\|_Y = 1$ , the space  $Y$  becomes a Banach space. Consider, e.g.,  $Y$  equipped with the inherited norm of  $X$ .

Then we define the positive finite-rank operator  $T : X \rightarrow Y$  by

$$Tf = \sum_{k=1}^n \alpha_k^*(f) e_k, \quad f \in X, \quad (2)$$

where  $\alpha_k^*$  are positive linear functionals satisfying  $\|\alpha_k^*\|_{X^*} = \alpha_k^*(1) = 1$  and  $\alpha_k^*(e_k) > 0$  for  $k \in \{1, \dots, n\}$ . There are many operators that match this definition, consider e.g., the Bernstein operator, Schoenberg's variation-diminishing spline operator, that arise in many applications in approximation theory and CAGD.

Finally, we remark that these operators mentioned previously are usually defined on a real Banach function space  $X$ . In this case we consider its complexification  $X_{\mathbb{C}} = X \oplus iX$  equipped with the norm

$$\|f + ig\|_{\mathbb{C}} = \sup_{0 \leq \varphi \leq 2\pi} \|f \cos \varphi + g \sin \varphi\|_X, \quad f, g \in X.$$

Then  $X_{\mathbb{C}}$  is a Banach space and  $X$  is continuously embedded in  $X_{\mathbb{C}}$ . The corresponding complex extension  $T_{\mathbb{C}}$  of  $T$  is defined for all  $f, g \in X$  by

$$T_{\mathbb{C}}(f + ig) = Tf + iTg.$$

In this way, the operator norm is consistent, i.e.,  $\|T\|_{op} = \|T_{\mathbb{C}}\|_{op}$  holds. The spectrum  $\sigma(T)$  of  $T$  is defined as  $\sigma(T_{\mathbb{C}})$ . Note that the set  $\sigma_p(T_{\mathbb{C}}) \cap \mathbb{R}$  consists of the eigenvalues of  $T : X \rightarrow Y$ . For more details on the complexification of real Banach spaces we refer to [20, pp. 7–16] and [14].

### 1.3. Main results

We first give a characterization of the spectrum of the operator  $T$ , then we show the existence of the limit of the iterates in the uniform operator topology.

**Theorem 1** (The spectrum). *The spectrum of the operator  $T \in \mathcal{B}(X, Y)$ , defined above by (2), consists only of the point spectrum and is characterized by*

$$\{0\} \subset \sigma(T) = \sigma_p(T) \subset \mathbb{D} \cup \{1\}.$$

**Corollary 1.** *The positive finite-rank operator  $T \in \mathcal{B}(X, Y)$  given by (2) has the following properties:*

1.  $1 \in \mathcal{N}(T - I)$ , i.e., 1 is an (isolated) eigenvalue of  $T$ ,
2.  $\sigma(T) \subset \mathbb{D} \cup \{1\}$ , and finally,
3.  $\dim(\mathcal{N}(T)) = \infty$ .

*I.e., the only eigenvalue on the peripheral spectrum is 1 and all spectral values are eigenvalues. Moreover, 0 is always an eigenvalue of  $T$  corresponding to an infinite-dimensional eigenspace of  $T$ .*

Using the spectral properties shown here, the existence of a limit of the iterates is guaranteed by the Theorem of Katznelson and Tzafriri [11] and the limiting operator preserves the partition of unity property.

**Theorem 2** (The existence of the limit of the iterates). *Let the operator  $T \in \mathcal{B}(X, Y)$  be a finite-rank operator with a partition of unity property as in (2). Then*

$$\lim_{m \rightarrow \infty} T^m = P$$

*uniformly in the operator norm, where  $P$  is a compact operator on  $X$  such that  $P1 = 1$  holds. Moreover,  $P$  is the unique projection operator onto  $\mathcal{N}(T - I)$  that satisfies  $TP = PT = P$ .*

The next corollary provides a sufficient (and necessary) criterion to obtain the limiting operator  $P$ . As far as we know, there is at the time of writing no existing method to prove the uniform convergence for finite-rank operators by a general simple criterion.

**Corollary 2.** *Let  $T \in \mathcal{B}(X, Y)$  be a positive linear operator as in (2). If there exists an idempotent operator, i.e.,  $P^2 = P$ , that commutes with  $T$  such that  $TP = PT = P$  holds with range  $\mathcal{R}(P) = \mathcal{N}(T - I)$ , then*

$$\lim_{m \rightarrow \infty} \|T^m - P\|_{op} = 0.$$

#### 1.4. Examples in $C([0, 1])$ and $L^1([0, 1])$

Before we prove these statements, we first give two examples where our main result can be applied. First the space of continuous functions is considered where an operator evaluates a continuous function on finitely many given points. In the other example we discuss the Kantorovič operator on  $L^1([0, 1])$  that yields an extension of the classical Bernstein operator to the space of integrable functions.

**Example 1** ( *$C([0, 1])$  and point evaluations*). The Riesz representation theorem gives a characterization of positive linear functionals on  $C([0, 1])$ . Namely, for every positive linear functional  $\alpha^* : C([0, 1]) \rightarrow \mathbb{R}$ , there is a unique positive Radon measure  $\nu$  such that

$$\alpha^*(f) = \int_0^1 f d\nu \quad \text{for every } f \in C([0, 1]).$$

A classical example of a positive linear functional on  $C([0, 1])$  is the Dirac measure at a point  $x \in [0, 1]$  defined for  $f \in C([0, 1])$  by

$$\delta_x(f) = f(x).$$

Given a partition  $\Delta_n = \{x_k\}_{k=1}^n$  of  $[0, 1]$  satisfying

$$0 = x_1 < x_2 < \dots < x_n = 1,$$

then a popular choice for the functionals are  $\alpha_k^* = \delta_{x_k}$  for  $k \in \{1, \dots, n\}$ . In this case, the positive finite-rank operator can be written for  $x \in [0, 1]$  as

$$Tf(x) = \sum_{k=1}^n f(x_k) e_k(x), \quad e_k \in C([0, 1]).$$

Operators of this kind are often used to approximate continuous functions by only a finite number of samples. To apply [Theorem 1](#), we also need the property that  $e_k(x_k) > 0$  for all  $k \in \{1, \dots, n\}$ . There are many examples where this property holds, see e.g., the Bernstein polynomials or the variation-diminishing Schoenberg splines. Let us suppose this property holds for  $T$ , then we obtain that 1 is the only spectral value on the unit circle and hence,  $\lim_{m \rightarrow \infty} T^m f$  exists uniformly for all  $f \in C([0, 1])$  by [Theorem 2](#). It has already been shown by Badea [\[2\]](#) that the operator  $Lf(x) = f(0) + (f(1) - f(0))x$  for  $f \in C([0, 1])$  satisfies the conditions of [Corollary 2](#) for the Bernstein operator. Thus, we provide here an additional simple criterion to obtain the limit of the iterates of the Bernstein operator.

In the next example we consider the Kantorovič operator on the space of integrable functions  $L^1([0, 1])$  and prove the limit of the iterates. Additionally, we illustrate the criterion given in [Corollary 2](#) to derive the limiting operator.

**Example 2** (*Kantorovič operator on  $L^1([0, 1])$* ). The Weierstrass approximation theorem says that every continuous function can be uniformly approximated by polynomials. An often used technique to prove this theorem is the classical Bernstein polynomials and the corresponding Bernstein operator. This result can be transferred to the space of integrable functions using the so called Kantorovič operators. In 1930, Kantorovič [\[9\]](#) has defined a sequence of operators  $K_n : L^1([0, 1]) \rightarrow C([0, 1])$  by

$$K_n f(x) = (n+1) \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt, \quad f \in L^1([0, 1]), \quad x \in [0, 1].$$

We now show that this is a finite-rank operator of the form (2). To this end, let us denote the Bernstein polynomials by  $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$  and the functionals by  $\alpha_{n,k}(f) := \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt$  for  $f \in L^1([0, 1])$ . Then the Kantorovič operator can be represented for  $f \in L^1([0, 1])$  by

$$K_n f(x) = (n+1) \sum_{k=0}^n \alpha_{n,k}(f) p_{n,k}(x).$$

In fact, the Bernstein polynomials form a partition of unity,  $\sum_{k=0}^n p_{n,k}(x) = 1$ , and  $\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} p_{n,k}(t) dt > 0$  as  $p_{n,k}(x) > 0$  on the open interval  $(\frac{k}{n+1}, \frac{k+1}{n+1})$  for all  $k \in \{1, \dots, n\}$ . Thus, each of the operators  $K_n$  is positive and has finite-rank. Using Theorem 1, we can characterize the spectrum of these operators by  $\sigma(K_n) \subset \mathbb{D} \cup \{1\}$  and 1 is an isolated eigenvalue of  $K_n$ . Hence, for fixed  $n$  the iterates  $K_n^m$  converge in the operator norm if  $m$  tends to infinity to an operator that preserves the ability to reproduce constants. As this operator only preserves constant functions, we demonstrate an application of Corollary 2.

Let us consider the operator  $L : L^1([0, 1]) \rightarrow C([0, 1])$ ,

$$L(f; x) = \int_0^1 f(t) dt \cdot 1,$$

i.e., each  $f$  is mapped to the constant value of the integral over  $[0, 1]$ . This operator  $L$  is idempotent, as

$$L^2(f; x) = \int_0^1 \left( \int_0^1 f(t) dt \right) ds = \int_0^1 f(t) dt \cdot 1.$$

Also  $K_n \circ L = L$  holds, as for all  $f \in L^1([0, 1])$  we obtain for  $x \in [0, 1]$ :

$$\begin{aligned} (K_n \circ L)(f; x) &= (n+1) \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} L(f; t) dt \\ &= (n+1) \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_0^1 f(s) ds dt \\ &= (n+1) \int_0^1 f(s) ds \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \frac{1}{n+1} \\ &= \int_0^1 f(s) ds \cdot 1. \end{aligned}$$

In the last step we used the partition of unity property of the Kantorovič polynomials, namely that  $K_n(1; x) = 1$  holds for all  $x \in [0, 1]$ . Also  $L \circ K_n = L$  holds. To prove this let  $f \in C([0, 1])$  and  $x \in [0, 1]$ . Then

$$\begin{aligned}
(L \circ K_n)(f; x) &= \int_0^1 K_n(f; t) dt = (n+1) \int_0^1 \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s) ds dt \\
&= (n+1) \sum_{k=0}^n \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s) ds \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dt \\
&= (n+1) \sum_{k=0}^n \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s) ds \frac{1}{n+1} \\
&= \int_0^1 f(s) ds = L(f; x).
\end{aligned}$$

Here, we note that  $\int_0^1 t^k (1-t)^{n-k} dt$  is the value of the Beta function  $\beta(k+1, n+1-k) = \frac{k!(n-k)!}{(n+1)!}$ . Therefore, the integral of the Bernstein polynomials is constant,  $\int_0^1 \binom{n}{k} t^k (1-t)^{n-k} dt = \frac{1}{n+1}$ . This has been shown, e.g., by Kreyszig [13]. The preceding results have shown that  $TL = LT = L$  holds and that  $L^2 = L$ . Therefore, all conditions of Corollary 2 are satisfied and we conclude that

$$\lim_{m \rightarrow \infty} K_n^m(f; x) = L(f; x) \quad \text{uniformly on } [0, 1] \text{ for all } f \in L^1([0, 1]).$$

Indeed, this result has first been shown by Nagel [16] but only for  $f \in L^2([0, 1])$ . In contrast, our result not only extends the uniform convergence to the space  $L^1([0, 1])$  we also derive the convergence in the uniform operator norm.

## 2. Basic properties of positive finite-rank operators

This section discusses properties that characterize the positive finite-rank operator  $T$ . The next lemma states the positivity of  $T$  and the ability to reconstruct constants.

**Lemma 1.** *The linear operator  $T$ , defined by (2), is positive and reproduces constants.*

**Proof.** As the  $\alpha_k^*$  are linear positive functionals and  $e_k \geq 0$ , we conclude for  $f \in X$ ,  $f \geq 0$ ,

$$Tf = \sum_{k=1}^n \alpha_k^*(f) e_k \geq 0.$$

And we obtain by applying the preconditions on  $T$  that

$$T1 = \sum_{k=1}^n \alpha_k^*(1) e_k = \sum_{k=1}^n e_k = 1. \quad \square$$

**Lemma 2.** *The operator  $T : X \rightarrow Y$  is bounded and  $\|T\|_{op} = 1$ .*

**Proof.** Let  $f \in X$  such that  $\|f\|_X = 1$ . Then

$$\|Tf\|_Y = \left\| \sum_{k=1}^n \alpha_k^*(f) e_k \right\|_Y \leq \max_k |\alpha_k^*(f)| \cdot \left\| \sum_{k=1}^n e_k \right\|_Y \leq \|f\|_X \cdot \max_k \|\alpha_k^*\|_{X^*} = 1,$$

where we used the partition of unity (1) and that  $\|\alpha_k^*\|_{X^*} = 1$ . Using that  $T1 = 1$ , we conclude that  $\|T\|_{op} = 1$ .  $\square$

Now we will prove that the operator  $T$  is indeed a finite-rank operator and give additional basic properties.

**Lemma 3.** *The linear operator  $T$  has finite rank. Thus, the operator  $T$  is compact.*

**Proof.** As  $\mathcal{R}(T) = \text{span}\{\sum_{k=1}^n \alpha_k^*(f)e_k : f \in X\}$ , the range can be written as a linear combination of the  $n$  basis functions  $e_k$  and hence,  $\dim(\mathcal{R}(T)) \leq n$ . Therefore, the linear operator  $T$  has finite rank and each finite-rank operator is compact.  $\square$

Let us denote the topological duals of  $X$  and  $Y$  by  $X^*$  and  $Y^*$  respectively. Then we can give a representation of the adjoint of  $T$ .

**Theorem 3.** *The adjoint  $T^* : Y^* \rightarrow X^*$  of  $T$  is a finite-rank operator. It is given for  $x^* \in Y^*$  as*

$$T^*x^*(f) = \sum_{k=1}^n x^*(e_k)\alpha_k^*(f), \quad f \in X.$$

**Proof.** We calculate

$$x^*(Tf) = x^*\left(\sum_{k=1}^n \alpha_k^*(f) \cdot e_k\right) = \sum_{k=1}^n \alpha_k^*(f)x^*(e_k) = T^*x^*(f). \quad \square$$

### 3. Proof of the main results

We will now prove the main results. First we show that the spectrum of the finite-rank operator is contained in  $\mathbb{D} \cup \{1\}$  and that it consists only of eigenvalues. Finally, we show the existence of the limit of the iterates.

Let us recall that we consider two complex Banach spaces  $X$  and  $Y$  with their topological duals  $X^*$  and  $Y^*$ . The range and the null space of  $T \in \mathcal{B}(X, Y)$  are denoted by  $\mathcal{R}(T)$  and  $\mathcal{N}(T)$ , respectively. For  $M \subset X$  we denote by

$$M^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ for every } x \in M\} \subset X^*,$$

the *annihilator* of  $M$ , whereas the *pre-annihilator* of a set  $A \subset X^*$  will be denoted by

$$A_\perp = \{x \in X : x^*(x) = 0 \text{ for every } x^* \in A\} \subset X.$$

The annihilator set  $M^\perp$  contains all continuous linear functionals on  $X$  that vanish on  $M$ , while  $A_\perp$  is the subset of  $X$  on which every bounded functional from  $A$  is zero. For more properties of annihilators we refer to Rudin [18].

#### 3.1. Proof of Theorem 1

Since  $\|T\|_{op} = 1$ , the inequality  $|\lambda| \leq \|T\|_{op} = 1$  holds for each  $\lambda \in \sigma(T)$ . Therefore,

$$\sigma(T) \subset \overline{\mathbb{D}}.$$

In the following, we show that  $\sigma(T) \subset \mathbb{D} \cup \{1\}$ , i.e., if  $\lambda \in \sigma(T)$  with  $|\lambda| = 1$  then  $\lambda = 1$  and all the spectral values are eigenvalues of  $T$ .

Note that for compact operators it is known that every  $\lambda \neq 0$  in the spectrum is contained in the point spectrum. This classical result is stated, e.g., in Rudin [18, Theorem 4.25]. Therefore, if  $0 \in \sigma_p(T)$ , then it follows already that

$$\sigma(T) = \sigma_p(T).$$

The proof is organized as follows: in the first step, we prove that  $0 \in \sigma_p(T)$ . Then, we will show that  $1 \in \sigma_p(T)$ . Finally, we consider eigenvalues  $\lambda \in \sigma_p(T) \setminus \{0, 1\}$  and we show that in this case  $|\lambda| < 1$  holds. Here we will use the well-known result of Gershgorin [8] to describe the spectrum.

Step 1: In order to prove that  $0 \in \sigma_p(T)$  we show  $\mathcal{N}(T) \neq \{0\}$ . Using [18, Theorem 4.12], we obtain that  $\mathcal{N}(T) = \mathcal{R}(T^*)^\perp$ . As  $T(X)$  is closed in  $Y$ , so is  $T^*(Y^*)$  weak\*-closed in  $X^*$ . Suppose now that  $\mathcal{N}(T) = \{0\}$ . It follows that  $\mathcal{R}(T^*)^\perp = \{0\}$  and therefore,  $(\mathcal{R}(T^*)^\perp)^\perp = X^*$ . This requires that  $\mathcal{R}(T^*)$  is weak\*-dense in  $X^*$ . This gives a contradiction as  $\mathcal{R}(T^*) = \text{span}\{\alpha_1^*, \dots, \alpha_n^*\}$  is weak\*-closed and  $X^* \neq \text{span}\{\alpha_1^*, \dots, \alpha_n^*\}$ , because  $X^*$  is infinite-dimensional.

We conclude that  $\mathcal{N}(T) \neq \{0\}$  and the finite-rank operator  $T$  is not one-to-one, i.e.,  $0 \in \sigma_p(T)$ .

Step 2: By definition of the operator  $T$  we have that  $1 \in \sigma(T)$ , because of the partition of unity property and the unit 1 is an eigenfunction of  $T$  corresponding to the eigenvalue 1,  $T1 = 1$ .

Step 3: We consider now all the other possible eigenvalues of  $T$  that are not equal to zero or one. We prove that for all these eigenvalues  $\lambda \in \sigma(T) \setminus \{0, 1\}$ , we have

$$|\lambda| < 1.$$

Let  $\lambda \in \sigma(T) \setminus \{0, 1\}$ . As the operator maps continuous functions to finite dimensional space  $\mathcal{R}(T)$ , the eigenfunctions have to be in this space, too. Let  $p \in \mathcal{R}(T)$ ,  $p = \sum_{k=1}^n c_k e_k$ , be such an eigenfunction for the eigenvalue  $\lambda$ . Then we get the following characterization:

$$\begin{aligned} Tp &= \lambda p \\ \iff \sum_{k=1}^n \sum_{j=1}^n c_j \alpha_k^*(e_j) e_k(x) &= \lambda \sum_{k=1}^n c_k e_k(x) \\ \iff \sum_{k=1}^n \left[ \sum_{j=1}^n c_j \alpha_k^*(e_j) - \lambda c_k \right] e_k(x) &= 0 \\ \iff \sum_{j=1}^n c_j \alpha_k^*(e_j) &= \lambda c_k, \quad \text{for all } k \in \{1, \dots, n\}. \end{aligned}$$

Thus,  $\lambda \neq 0$  is an eigenvalue of the operator  $T$ , if and only if  $\lambda$  is an eigenvalue of the matrix  $M \in \mathbb{R}^{n \times n}$ ,

$$M = \begin{pmatrix} \alpha_1^*(e_1) & \alpha_1^*(e_2) & \cdots & \alpha_1^*(e_n) \\ \alpha_2^*(e_1) & \alpha_2^*(e_2) & \cdots & \alpha_2^*(e_n) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^*(e_1) & \alpha_n^*(e_2) & \cdots & \alpha_n^*(e_n) \end{pmatrix}.$$

This matrix  $M$  is non-negative as  $e_k \geq 0$  and  $\alpha_k^*$  are positive linear functionals. Moreover, every row sums up to one because of the partition of unity property. To see this, let us calculate for some fixed row  $k \in \{1, \dots, n\}$  the following sum:



$$\sum_{i=1}^n \alpha_k^*(e_j) = \alpha_k^* \left( \sum_{j=1}^n e_j \right) = \alpha_k^*(1) = 1. \quad (3)$$

Hence, the underlying matrix of the finite-rank operator  $T$  is a stochastic matrix. We will show next, that 1 is the only spectral value on the unit circle.

By the famous Theorem of Gershgorin [8], the eigenvalues of  $M$  are contained in the union of circles,

$$\lambda \in \bigcup_{j=-k}^{n-1} D_j,$$

where

$$D_k := \left\{ \lambda \in \mathbb{C} : |\lambda - \alpha_k^*(e_k)| \leq \sum_{j=1, j \neq k}^n \alpha_k^*(e_j) \right\}.$$

Applying (3) yields

$$D_k = \{ \lambda \in \mathbb{C} : |\lambda - \alpha_k^*(e_k)| \leq 1 - \alpha_k^*(e_k) \}.$$

We conclude the proof noting that  $\alpha_k^*(e_k) > 0$  holds for all  $k \in \{1, \dots, n\}$  and thus,

$$\bigcup_{k=1}^n D_k \cap \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} = \{1\}.$$

Finally, we have proved that all spectral values of  $T$  are in particular eigenvalues and the spectrum of  $T$  is contained in  $\mathbb{D} \cup \{1\}$ . Hence, the only spectral point on the unit circle is the eigenvalue one.  $\square$

### 3.2. Proof of Theorem 2 and Corollary 2

Now we will show the existence of the limit of the iterates of the finite-rank operator that has the partition of unity property.

Katznelson and Tzafriri [11] have shown that for every linear operator  $T$  on a Banach space  $X$  with  $\|T\|_{op} \leq 1$  the limit

$$\lim_{m \rightarrow \infty} \|T^{m+1} - T^m\|_{op} = \lim_{m \rightarrow \infty} \|T^m(T - I)\|_{op} = 0$$

holds if and only if either there is no spectral value on the unit circle or the unit circle contains the single value 1. As Corollary 1 states, a finite-rank operator with a partition of unity property contains 1 as an eigenvalue and this is the only spectral value on the unit circle,  $\sigma(T) \subset \mathbb{D} \cup \{1\}$ .

Now we will prove Theorem 2. To this end, let us consider the sequence  $(T^m)_{m \in \mathbb{N}}$ . Combining the result of Katznelson and Tzafriri [11] with the classical work of Dunford [3, Thm. 3.18 on p. 216] this series converges and has a limit  $P$  in the Banach algebra  $\mathcal{B}(X, Y)$  with  $P^2 = P$  and  $\mathcal{R}(P) = \mathcal{N}(T - I)$ . As  $T^m 1 = 1$  for any positive integer  $m$  we obtain due to the uniform convergence in the operator norm the result  $P1 = 1$ . As a limit of the finite-rank operators  $T^m$ , the operator  $P$  is compact. Clearly,  $TP = PT = P$  holds.

Applying Dunford [3, Thm. 3.16 on p. 216] we conclude that the Banach space  $X$  can be decomposed into

$$X = \mathcal{N}(T - I) \oplus \mathcal{R}(T - I) = \mathcal{R}(P) \oplus \mathcal{N}(P) \quad (4)$$

We now prove that  $\mathcal{N}(P) = \mathcal{R}(T - I)$ . As  $P(T - I) = PT - P = 0$ ,  $\mathcal{R}(T - I) \subset \mathcal{N}(P)$  holds. We show that the converse holds also true. Let  $0 \neq x \in \mathcal{N}(P)$ , then  $x \notin \mathcal{N}(T - I)$ . By (4), there are  $x_{\mathcal{N}} \in \mathcal{N}(T - I)$  and  $x_{\mathcal{R}} \in \mathcal{R}(T - I)$  such that  $x = x_{\mathcal{N}} + x_{\mathcal{R}}$ . Using that  $x \in \mathcal{N}(P)$ , we obtain

$$0 = Px = Px_{\mathcal{N}} + Px_{\mathcal{R}} = Px_{\mathcal{N}},$$

as  $x_{\mathcal{R}} \in \mathcal{R}(T - I) \subset \mathcal{N}(P)$ . From  $x_{\mathcal{N}} \in \mathcal{N}(P)$  we conclude that  $x_{\mathcal{N}} = 0$ . Therefore,  $x = x_{\mathcal{R}}$  and we get the final statement  $x \in \mathcal{R}(T - I)$ . Hence,  $\mathcal{R}(P) = \mathcal{N}(T - I)$  and  $\mathcal{N}(P) = \mathcal{R}(T - I)$ . As direct consequence we obtain Corollary 2 using the uniqueness of the limiting projection operator that satisfies  $TP = PT = P$ .  $\square$

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