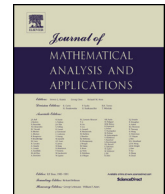




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The consistency of the nearest neighbor estimator of the density function based on WOD samples [☆]

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ABSTRACT

In this paper, the consistency of the nearest neighbor estimator of the density function based on widely orthant dependent (WOD, in short) samples is investigated. The convergence rate of strong consistency, the complete consistency, the uniformly complete consistency and uniformly strong consistency of the nearest neighbor estimator of the density function based on WOD samples are established. Our results established in the paper generalize or improve the corresponding ones for independent samples and some negatively dependent samples.

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1. Introduction

1.1. Brief review

Suppose that the population X has the unknown density function $f(x)$ and X_1, X_2, \dots, X_n are samples of X . Let $\{k_n, n \geq 1\}$ be a sequence of positive integers such that $1 \leq k_n \leq n$. For fixed x and n , denote

$$a_n(x) = \min\{a : \text{there exist at least } k'_n \text{ } i \text{ such that } X_i \in [x - a, x + a]\}.$$

Loftsgarden and Quesenberry [14] introduced the nearest neighbor estimator of $f(x)$ as follows:

$$f_n(x) = \frac{k_n}{2na_n(x)}. \quad (1.1)$$

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Since the concept of the nearest neighbor estimator of the density function was introduced by Loftsgarden and Quesenberry [14], many authors devoted to study the asymptotic properties of the nearest neighbor estimator of the density function. For independent samples, Loftsgarden and Quesenberry [14] and Wagner [19] established the weak consistency and strong consistency, respectively; Moore and Henrichon [15] and Devroye and Wagner [6] obtained the uniform consistency and strong uniform consistency, respectively; Chen [3] established the convergence rate of the consistency of the nearest neighbor density estimator, and so on. For dependent samples, Boente and Fraiman [1] studied the strong consistency of the nearest neighbor density estimator based on φ -mixing and α -mixing samples; Chai [2] obtained the strong consistency, weak consistency, uniformly strong consistency and convergence rate of the nearest neighbor density estimator based on φ -mixing stationary processes; Yang [26] investigated the weak consistency, strong consistency, uniformly strong consistency of the nearest neighbor density estimator based on negatively associated (NA, in short) samples; Liu and Zhang [13] established the consistency and asymptotic normality of nearest neighbor density estimator based on φ -mixing samples, and so on.

Recently, Yang [26] established the following weak consistency and strong consistency for NA samples.

Theorem A. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables and $k_n \rightarrow \infty$, $\frac{k_n}{n} \rightarrow 0$ as $n \rightarrow \infty$.*

(i) *If $\frac{k_n^2}{n} \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$f_n(x) \xrightarrow{P} f(x). \quad (1.2)$$

(ii) *If for any $\gamma > 0$,*

$$\sum_{n=1}^{\infty} \exp \left\{ -\frac{\gamma k_n^2}{n} \right\} < \infty, \quad (1.3)$$

then

$$f_n(x) \rightarrow f(x) \quad \text{a.s., as } n \rightarrow \infty. \quad (1.4)$$

The main purpose of this article is to generalize and improve the result of Theorem A for NA random variables to the case of widely orthant dependent (WOD) random variables, which includes NA as a special case. In addition, we will present the convergence rate of strong consistency, uniformly complete consistency and uniformly strong consistency for the nearest neighbor estimator of density function based on WOD samples. Our results generalize or improve the corresponding ones of Yang [26] for NA random variables to the case of WOD random variables. The key techniques used in the paper are the Bernstein type inequality and the truncated method.

1.2. Concepts of widely orthant dependence

In this subsection, we will recall the definition of WOD random variables, which was introduced by Wang et al. [23] as follows.

Definition 1.1. For the random variables $\{X_n, n \geq 1\}$, if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leq g_U(n) \prod_{i=1}^n P(X_i > x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^n P(X_i \leq x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{X_n, n \geq 1\}$ are widely orthant dependent (WOD, in short), and $g_U(n)$, $g_L(n)$, $n \geq 1$, are called dominating coefficients.

Recall that when $g_L(n) = g_U(n) = M$ for some positive constant M , the random variables $\{X_n, n \geq 1\}$ are called extended negatively upper orthant dependent (ENUOD, in short) and extended negatively lower orthant dependent (ENLOD, in short), respectively. If they are both ENUOD and ENLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are extended negatively orthant dependent (END, in short). The concept of END random variables was proposed by Liu [11] and further promoted by Chen et al. [4], Shen [16,18], Wu and Guan [25], Wang and Wang [22], Yang and Wang [27], and so forth. When $g_L(n) = g_U(n) = 1$ for any $n \geq 1$, the random variables $\{X_n, n \geq 1\}$ are called negatively upper orthant dependent (NUOD, in short) and negatively lower orthant dependent (NLOD, in short), respectively. If they are both NUOD and NLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are negatively orthant dependent (NOD, in short). The concept of negative dependence was introduced by Ebrahimi and Ghosh [7] and carefully studied by Joag-Dev and Proschan [10]. Joag-Dev and Proschan [10] pointed out that negatively associated (NA, in short) random variables are NOD. Hu [9] introduced the concept of negatively superadditive dependence (NSD, in short) and gave an example illustrating that NSD does not imply NA. Christofides and Vaggelatos [5] indicated that NA implies NSD. In addition, Hu [9] pointed out that NSD implies NOD. From the statements above, we can see that the class of WOD random variables includes END random variables, NOD random variables, NSD random variables, NA random variables and independent random variables as special cases. Hence, studying the limiting behavior of WOD random variables and its applications are of great interest.

The concept of WOD random variables was introduced by Wang et al. [23] and many applications have been found. See, for example, Wang and Chen [20] presented some basic renewal theorems for random walks with widely dependent increments. Wang et al. [21] studied the uniform asymptotics for the finite-time ruin probability of a new dependent risk model with a constant interest rate. Liu et al. [12] gave the asymptotically equivalent formula for the finite-time ruin probability under a dependent risk model with constant interest rate. He et al. [8] provided the asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables. Shen [17] established the Bernstein type inequality for WOD random variables and gave some applications, Wang et al. [24] studied the complete convergence for WOD random variables and gave its applications in nonparametric regression models, and so forth.

This work is organized as follows: main results are presented in Section 2. Some important lemmas are provided in Section 3 and the proofs of the main results are provided in Section 4, respectively.

Throughout this article, let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with the dominating coefficients $g_U(n)$, $g_L(n)$, $n \geq 1$. Denote $g(n) = \max\{g_U(n), g_L(n)\}$. The symbol C denotes a positive constant which is not necessarily the same one in each appearance. $a_n = o(b_n)$ and $a_n = O(b_n)$ stand for $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $a_n \leq Cb_n$, respectively. Let $I(A)$ be the indicator function of the event A . Denote $\log x = \ln \max(x, e)$. $[x]$ denotes the integer part of x . Let $F(x)$ be the corresponding distribution function of the density function $f(x)$ and $F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i < x)$ be the empirical distribution function of X_1, X_2, \dots, X_n .

2. Main results

In this section, let $c(f)$ denote all continuity points of the density function f . Our main results are as follows.

2.1. Weak consistency

Theorem 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables and $\frac{k_n}{n} \rightarrow 0$, $\frac{k_n^2}{n} \rightarrow \infty$ as $n \rightarrow \infty$. If for any $\gamma > 0$,

$$\lim_{n \rightarrow \infty} g(n) \exp \left\{ -\frac{\gamma k_n^2}{n} \right\} = 0, \quad (2.1)$$

then for any $x \in c(f)$, (1.2) holds.

2.2. Complete consistency and strong consistency

Theorem 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables and $\frac{k_n}{n} \rightarrow 0$ as $n \rightarrow \infty$. If for any $\gamma > 0$,

$$\sum_{n=1}^{\infty} g(n) \exp \left\{ -\frac{\gamma k_n^2}{n} \right\} < \infty, \quad (2.2)$$

then for any $x \in c(f)$,

$$f_n(x) \rightarrow f(x) \quad \text{completely, as } n \rightarrow \infty, \quad (2.3)$$

and thus (1.4) holds.

Remark 2.1. We point out that the condition $k_n \rightarrow \infty$ as $n \rightarrow \infty$ is not needed in Theorem 2.1 and Theorem 2.2. If $g(n) = M$, where M is a positive constant, then (2.1) is obvious, and (2.2) is equivalent to (1.3). In this case, $\{X_n, n \geq 1\}$ is a sequence of END random variables, which includes independent random variables, NA random variables, NOD random variables as special cases. Hence, Theorem 2.1 and Theorem 2.2 hold for independent random variables, NA random variables, NOD random variables and END random variables. In addition, Borel–Cantelli lemma yields that complete convergence implies strong convergence. Hence, the results of Theorem 2.1 and Theorem 2.2 generalize and improve the corresponding ones of (i) and (ii) in Theorem A for NA random variables, respectively.

By using Theorem 2.2, we can get the following corollary.

Corollary 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $g(n) = O(n^\delta)$ for some $\delta \geq 0$ and $\frac{k_n}{n} \rightarrow 0$ as $n \rightarrow \infty$. If

$$\frac{k_n}{\sqrt{n \log n}} \rightarrow \infty, \quad \text{as } n \rightarrow \infty, \quad (2.4)$$

then for any $x \in c(f)$, (2.3) holds, and thus (1.4) holds.

The following result presents the complete consistency and strong convergence rate for the nearest neighbor estimator of the density function.

Theorem 2.3. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $g(n) = O(n^\delta)$ for $\delta \geq 0$. Let $f(x)$ satisfy the local Lipschitz condition at x and $f(x) > 0$. If there exists a sequence $\{q_n, n \geq 1\}$ of positive numbers such that

$$q_n \rightarrow 0, \quad \frac{k_n}{nq_n} \rightarrow 0, \quad \frac{k_n q_n}{\sqrt{n \log n}} \rightarrow \infty, \quad \text{as } n \rightarrow \infty, \quad (2.5)$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|f_n(x) - f(x)| > \varepsilon q_n) < \infty, \quad (2.6)$$

and thus,

$$|f_n(x) - f(x)| = o(q_n) \quad \text{a.s., as } n \rightarrow \infty. \quad (2.7)$$

Remark 2.2. We point out that the condition (2.5) yields that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Actually,

$$k_n = \frac{k_n q_n}{\sqrt{n \log n}} \cdot \frac{\sqrt{n \log n}}{q_n} \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

Taking $k_n = \lfloor n^{3/4} \log^{1/4} n \rfloor$ and $q_n = n^{-1/4} (\log n)^{1/4} \log \log n$ in Theorem 2.3, we can get the following result on strong convergence rate.

Corollary 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $g(n) = O(n^\delta)$ for $\delta \geq 0$. Let $f(x)$ satisfy the local Lipschitz condition at x and $f(x) > 0$. If $k_n = \lfloor n^{3/4} \log^{1/4} n \rfloor$, then

$$|f_n(x) - f(x)| = o\left(n^{-1/4} (\log n)^{1/4} \log \log n\right) \quad \text{a.s., as } n \rightarrow \infty.$$

2.3. Uniformly complete consistency and uniformly strong consistency

In this subsection, we will provide some results on uniformly complete consistency and uniformly strong consistency for the nearest neighbor estimator of the density function.

Theorem 2.4. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $g(n) = O(n^\delta)$ for some $\delta \geq 0$ and $f(x)$ be uniformly continuous. If (2.4) holds and $\frac{k_n}{n} \rightarrow 0$ as $n \rightarrow \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\sup_x |f_n(x) - f(x)| > \varepsilon\right) < \infty, \quad (2.8)$$

and thus,

$$\lim_{n \rightarrow \infty} \sup_x |f_n(x) - f(x)| = 0 \quad \text{a.s.} \quad (2.9)$$

Theorem 2.5. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $g(n) = O(n^\delta)$ for some $\delta \geq 0$ and $f(x)$ satisfy the Lipschitz condition on \mathbf{R} . If there exists a sequence $\{q_n, n \geq 1\}$ of positive numbers such that

$$k_n \rightarrow \infty, \quad q_n \rightarrow 0, \quad \frac{k_n}{nq_n^2} \rightarrow 0, \quad \frac{k_n q_n}{\sqrt{n \log n}} \rightarrow \infty, \quad \text{as } n \rightarrow \infty, \quad (2.10)$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\sup_x |f_n(x) - f(x)| > \varepsilon q_n\right) < \infty, \quad (2.11)$$

and thus,

$$\sup_x |f_n(x) - f(x)| = o(q_n) \quad a.s., \quad \text{as } n \rightarrow \infty. \quad (2.12)$$

Taking $k_n = \lfloor n^{2/3} \log^{1/3} n \rfloor$ and $q_n = n^{-1/6} (\log n)^{1/6} \log \log n$ in [Theorem 2.5](#), we can get the following strong convergence rate for the nearest neighbor estimator of the density function.

Corollary 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $g(n) = O(n^\delta)$ for some $\delta \geq 0$ and $f(x)$ satisfy the Lipschitz condition on \mathbf{R} . If $k_n = \lfloor n^{2/3} \log^{1/3} n \rfloor$, then*

$$\sup_x |f_n(x) - f(x)| = o\left(n^{-1/6} (\log n)^{1/6} \log \log n\right) \quad a.s., \quad \text{as } n \rightarrow \infty.$$

3. Some lemmas

To prove the main results of the paper, we need the following important lemmas.

The first one is the basic property for WOD random variables, which comes from Wang et al. [\[23\]](#).

Lemma 3.1. *Let $\{X_n, n \geq 1\}$ be WOD.*

- (i) *If $\{f_n(\cdot), n \geq 1\}$ are all nondecreasing (or nonincreasing), then $\{f_n(X_n), n \geq 1\}$ are still WOD.*
- (ii) *For each $n \geq 1$ and any $s \in \mathbf{R}$,*

$$E \exp \left\{ s \sum_{i=1}^n X_i \right\} \leq g(n) \prod_{i=1}^n E \exp \{ s X_i \}. \quad (3.1)$$

The next one is the Bernstein type inequality for WOD random variables, which was established by Shen [\[17\]](#).

Lemma 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_i = 0$ and $|X_i| \leq b$ for each $i \geq 1$, where b is a positive constant. Denote $B_n^2 = \sum_{i=1}^n EX_i^2$ for each $n \geq 1$. Then for any $\varepsilon > 0$,*

$$P \left(\left| \sum_{i=1}^n X_i \right| \geq \varepsilon \right) \leq 2g(n) \exp \left\{ -\frac{\varepsilon^2}{2B_n^2 + \frac{2}{3}b\varepsilon} \right\}. \quad (3.2)$$

The last one is a basic property for empirical distribution function, which has been proved by Yang [\[26\]](#).

Lemma 3.3. *Let $F(x)$ be a continuous distribution function. For $n \geq 3$, assume that $x_{n,k}$ satisfy*

$$F(x_{n,k}) = \frac{k}{n}, \quad k = 1, 2, \dots, n-1.$$

Then

$$\sup_x |F_n(x) - F(x)| \leq \max_{1 \leq k \leq n-1} |F_n(x_{n,k}) - F(x_{n,k})| + \frac{2}{n}, \quad (3.3)$$

where $F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i < x)$ is the empirical distribution function of X_1, X_2, \dots, X_n .

4. Proofs of the main results

In this section, we will present the proofs of the main results.

Proof of Theorem 2.1. Let $x \in c(f)$. For any $\varepsilon > 0$ and $n \geq 1$, denote

$$b_n(x) = \frac{k_n}{2n(f(x) + \varepsilon)}, \quad c_n(x) = \frac{k_n}{2n(f(x) - \varepsilon/2)} \quad (f(x) > \varepsilon \text{ is needed}).$$

It follows by the definition of $f_n(x)$ that

$$\begin{aligned} A_x &\doteq \{|f_n(x) - f(x)| > \varepsilon\} \\ &= \{f_n(x) > f(x) + \varepsilon\} \cup \{f_n(x) < f(x) - \varepsilon\} \\ &= \{f_n(x) > f(x) + \varepsilon\} \cup \{f_n(x) < f(x) - \varepsilon, f(x) > \varepsilon\} \\ &\subset \{f_n(x) > f(x) + \varepsilon\} \cup \{f_n(x) < f(x) - \varepsilon/2, f(x) > \varepsilon\} \\ &= \{a_n(x) < b_n(x)\} \cup \{a_n(x) > c_n(x), f(x) > \varepsilon\} \\ &\subset \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) \geq \frac{k_n}{n} \right\} \\ &\quad \cup \left\{ F_n(x + c_n(x)) - F_n(x - c_n(x)) \leq \frac{k_n}{n}, f(x) > \varepsilon \right\} \\ &\doteq A_{1x} \cup A_{2x}. \end{aligned} \tag{4.1}$$

Noting that $b_n(x) \rightarrow 0$, $c_n(x) \rightarrow 0$, and $F'(x) = f(x)$, we have for $n \rightarrow \infty$ that

$$\frac{F(x + b_n(x)) - F(x - b_n(x))}{2b_n(x)} \rightarrow f(x)$$

and

$$\frac{F(x + c_n(x)) - F(x - c_n(x))}{2c_n(x)} \rightarrow f(x),$$

which imply that for all n large enough,

$$F(x + b_n(x)) - F(x - b_n(x)) < 2b_n(x)(f(x) + \varepsilon/2) = \frac{k_n}{n} \frac{f(x) + \varepsilon/2}{f(x) + \varepsilon} \tag{4.2}$$

and

$$F(x + c_n(x)) - F(x - c_n(x)) > 2c_n(x)(f(x) - \varepsilon/4) = \frac{k_n}{n} \frac{f(x) - \varepsilon/4}{f(x) - \varepsilon/2}, \tag{4.3}$$

respectively.

Denote $e(x) = \frac{\varepsilon}{8(f(x)+\varepsilon)} \leq \frac{1}{8}$, it follows by (4.1) and (4.2) that

$$\begin{aligned}
 A_{1x} &= \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) \geq \frac{k_n}{n} \right\} \\
 &\subset \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) - F(x + b_n(x)) + F(x - b_n(x)) \geq \frac{k_n}{n} - \frac{k_n}{n} \frac{f(x) + \varepsilon/2}{f(x) + \varepsilon} \right\} \\
 &= \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) - F(x + b_n(x)) + F(x - b_n(x)) \geq \frac{4k_n}{n} e(x) \right\} \\
 &\subset \left\{ |F_n(x + b_n(x)) - F(x + b_n(x))| \geq \frac{k_n}{n} e(x) \right\} \\
 &\quad \cup \left\{ |F_n(x - b_n(x)) - F(x - b_n(x))| \geq \frac{k_n}{n} e(x) \right\} \\
 &\doteq E_{1x} \cup E_{2x}.
 \end{aligned} \tag{4.4}$$

Similarly, it follows by (4.1) and (4.3) that

$$\begin{aligned}
 A_{2x} &= \left\{ F_n(x + c_n(x)) - F_n(x - c_n(x)) \leq \frac{k_n}{n}, f(x) > \varepsilon \right\} \\
 &\subset \{ F_n(x + c_n(x)) - F_n(x - c_n(x)) - F(x + c_n(x)) + F(x - c_n(x)) \\
 &\quad \leq \frac{k_n}{n} - \frac{k_n}{n} \frac{f(x) - \varepsilon/4}{f(x) - \varepsilon/2}, f(x) > \varepsilon \} \\
 &= \{ F_n(x + c_n(x)) - F_n(x - c_n(x)) - F(x + c_n(x)) + F(x - c_n(x)) \\
 &\quad \leq -\frac{k_n}{4n} \frac{\varepsilon}{f(x) - \varepsilon/2}, f(x) > \varepsilon \} \\
 &\subset \left\{ F_n(x + c_n(x)) - F_n(x - c_n(x)) - F(x + c_n(x)) + F(x - c_n(x)) \leq -\frac{k_n}{4n} \frac{\varepsilon}{f(x) + \varepsilon} \right\} \\
 &= \left\{ F_n(x + c_n(x)) - F_n(x - c_n(x)) - F(x + c_n(x)) + F(x - c_n(x)) \leq -\frac{2k_n}{n} e(x) \right\} \\
 &\subset \left\{ |F_n(x + c_n(x)) - F(x + c_n(x))| \geq \frac{k_n}{n} e(x) \right\} \\
 &\quad \cup \left\{ |F_n(x - c_n(x)) - F(x - c_n(x))| \geq \frac{k_n}{n} e(x) \right\} \\
 &\doteq E_{3x} \cup E_{4x}.
 \end{aligned} \tag{4.5}$$

By (4.1), (4.4) and (4.5), we have

$$A_x \subset E_{1x} \cup E_{2x} \cup E_{3x} \cup E_{4x}. \tag{4.6}$$

For fixed x and $n \geq 1$, denote

$$X_i^{(n)} = I(X_i < x + b_n(x)) - EI(X_i < x + b_n(x)), \quad 1 \leq i \leq n.$$

It is easily seen that $X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)}$ are still WOD random variables by Lemma 3.1 (i), and $EX_i^{(n)} = 0$, $|X_i^{(n)}| \leq 1$. Applying Lemma 3.2 with $X_i = X_i^{(n)}$, $b = 1$ and $B_n^2 \doteq \sum_{i=1}^n E(X_i^{(n)})^2 \leq n$, we have by (3.2) that,

$$\begin{aligned}
P(E_{1x}) &= P\left(|F_n(x + b_n(x)) - F(x + b_n(x))| \geq \frac{k_n}{n}e(x)\right) \\
&= P\left(\left|\sum_{i=1}^n X_i^{(n)}\right| \geq k_n e(x)\right) \\
&\leq 2g(n) \exp\left\{-\frac{k_n^2 e^2(x)}{2B_n^2 + \frac{2}{3}k_n e(x)}\right\} \\
&\leq 2g(n) \exp\left\{-\frac{k_n^2 e^2(x)}{2n + \frac{1}{12}n}\right\} \quad (\text{since } k_n \leq n \text{ and } e(x) \leq \frac{1}{8}) \\
&= 2g(n) \exp\left\{-\frac{12e^2(x)k_n^2}{25n}\right\}. \tag{4.7}
\end{aligned}$$

Similarly, we have

$$P(E_{jx}) \leq 2g(n) \exp\left\{-\frac{12e^2(x)k_n^2}{25n}\right\}, \quad j = 2, 3, 4. \tag{4.8}$$

Therefore, we have by (4.6)–(4.8) and (2.1) that

$$\begin{aligned}
P(|f_n(x) - f(x)| > \varepsilon) &= P(A_x) \leq \sum_{j=1}^4 P(E_{jx}) \\
&\leq 8g(n) \exp\left\{-\frac{12e^2(x)k_n^2}{25n}\right\} \\
&\rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{4.9}
\end{aligned}$$

This completes the proof of the theorem. \square

Proof of Theorem 2.2. We use the same notations as those in Theorem 2.1. The desired result (2.3) follows by (4.9) and (2.2), and (1.4) follows by (2.3) and Borel–Cantelli lemma, respectively. The proof is completed. \square

Proof of Corollary 2.1. It suffices to prove (2.2). In fact, for any $\gamma > 0$, we have by $g(n) = O(n^\delta)$ and (2.4) that

$$\begin{aligned}
\sum_{n=1}^{\infty} g(n) \exp\left\{-\frac{\gamma k_n^2}{n}\right\} &= \sum_{n=1}^{\infty} g(n) \exp\left\{-\frac{\gamma k_n^2}{n \log n} \log n\right\} \\
&\leq C \sum_{n=1}^{\infty} n^{-2} < \infty.
\end{aligned}$$

The proof is completed. \square

Proof of Theorem 2.3. For any $\varepsilon > 0$ and $n \geq 1$, denote

$$u_n(x) = \frac{k_n}{2n(f(x) + \varepsilon q_n)}, \quad v_n(x) = \frac{k_n}{2n(f(x) - \varepsilon q_n/2)} \quad (f(x) > \varepsilon q_n \text{ is needed}).$$

Similar to the proof of (4.1), we have

$$\begin{aligned}
 B_x &\doteq \{|f_n(x) - f(x)| > \varepsilon q_n\} \\
 &\subset \{f_n(x) > f(x) + \varepsilon q_n\} \cup \{f_n(x) < f(x) - \varepsilon q_n/2, f(x) > \varepsilon q_n\} \\
 &\subset \left\{F_n(x + u_n(x)) - F_n(x - u_n(x)) \geq \frac{k_n}{n}\right\} \\
 &\quad \cup \left\{F_n(x + v_n(x)) - F_n(x - v_n(x)) \leq \frac{k_n}{n}, f(x) > \varepsilon q_n\right\} \\
 &\doteq B_{1x} \cup B_{2x}.
 \end{aligned} \tag{4.10}$$

By the Differential Mean Value Theorem, we can see that there exist $\theta_1 \in (x - u_n(x), x + u_n(x))$ and $\theta_2 \in (x - v_n(x), x + v_n(x))$ such that

$$F(x + u_n(x)) - F(x - u_n(x)) = 2u_n(x)f(\theta_1) \tag{4.11}$$

and

$$F(x + v_n(x)) - F(x - v_n(x)) = 2v_n(x)f(\theta_2). \tag{4.12}$$

If $\omega \in B_{1x}$, then we have by (4.11) that

$$\begin{aligned}
 &F_n(x + u_n(x)) - F_n(x - u_n(x)) - F(x + u_n(x)) + F(x - u_n(x)) \\
 &\geq \frac{k_n}{n} - 2u_n(x)f(\theta_1) = \frac{k_n}{n} \cdot \frac{f(x) - f(\theta_1) + \varepsilon q_n}{f(x) + \varepsilon q_n}.
 \end{aligned} \tag{4.13}$$

Similarly, If $\omega \in B_{2x}$, then we have by (4.12) that

$$\begin{aligned}
 &F_n(x + v_n(x)) - F_n(x - v_n(x)) - F(x + v_n(x)) + F(x - v_n(x)) \\
 &\leq \frac{k_n}{n} - 2v_n(x)f(\theta_2) = \frac{k_n}{n} \cdot \frac{f(x) - f(\theta_2) - \varepsilon q_n/2}{f(x) - \varepsilon q_n/2}.
 \end{aligned} \tag{4.14}$$

Since $f(x)$ satisfies the local Lipschitz condition at x and $f(x) > 0$, $q_n \rightarrow 0$, $\frac{k_n}{nq_n} \rightarrow 0$ as $n \rightarrow \infty$, we can see that there exists a positive constant $L(x)$ depending only on x such that for all n large enough,

$$|f(x) - f(\theta_1)| \leq L(x)|x - \theta_1| \leq L(x)u_n(x) = \frac{L(x)k_n}{2n(f(x) + \varepsilon q_n)} \leq \frac{L(x)k_n}{2nf(x)} \leq \frac{\varepsilon q_n}{2}, \tag{4.15}$$

$$|f(x) - f(\theta_2)| \leq L(x)|x - \theta_2| \leq L(x)v_n(x) = \frac{L(x)k_n}{2n(f(x) - \varepsilon q_n/2)} \leq \frac{L(x)k_n}{nf(x)} \leq \frac{\varepsilon q_n}{4}. \tag{4.16}$$

Note that the density function $f(x)$ is bounded. Denote $M = \sup_x f(x) < \infty$. By (4.15) and (4.16), we can see that for all n large enough,

$$\frac{k_n}{n} \cdot \frac{f(x) - f(\theta_1) + \varepsilon q_n}{f(x) + \varepsilon q_n} \geq \frac{k_n}{n} \cdot \frac{-\varepsilon q_n/2 + \varepsilon q_n}{f(x) + \varepsilon q_n} \geq \frac{k_n q_n}{n} \cdot \frac{\varepsilon}{4M} \tag{4.17}$$

and

$$\frac{k_n}{n} \cdot \frac{f(x) - f(\theta_2) - \varepsilon q_n/2}{f(x) - \varepsilon q_n/2} \leq \frac{k_n}{n} \cdot \frac{\varepsilon q_n/4 - \varepsilon q_n/2}{f(x) - \varepsilon q_n/2} \leq -\frac{k_n q_n}{n} \cdot \frac{\varepsilon}{4M}. \tag{4.18}$$

Denote $d = \frac{\varepsilon}{8M}$, we have by (4.13) and (4.17) that for all n large enough,

$$\begin{aligned}
 B_{1x} &\subset \left\{ F_n(x + u_n(x)) - F_n(x - u_n(x)) - F(x + u_n(x)) + F(x - u_n(x)) \geq \frac{k_n}{n} - 2u_n(x)f(\theta_1) \right\} \\
 &\subset \left\{ F_n(x + u_n(x)) - F_n(x - u_n(x)) - F(x + u_n(x)) + F(x - u_n(x)) \geq \frac{2k_n q_n d}{n} \right\} \\
 &\subset \left\{ |F_n(x + u_n(x)) - F(x + u_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
 &\quad \bigcup \left\{ |F_n(x - u_n(x)) - F(x - u_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
 &\doteq Q_{1x} \bigcup Q_{2x}.
 \end{aligned} \tag{4.19}$$

Similarly, we have by (4.14) and (4.18) that for all n large enough,

$$\begin{aligned}
 B_{2x} &\subset \left\{ |F_n(x + v_n(x)) - F(x + v_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
 &\quad \bigcup \left\{ |F_n(x - v_n(x)) - F(x - v_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
 &\doteq Q_{3x} \bigcup Q_{4x}.
 \end{aligned} \tag{4.20}$$

Therefore, we have by (4.10), (4.19) and (4.20) that for all n large enough,

$$B_x \subset Q_{1x} \bigcup Q_{2x} \bigcup Q_{3x} \bigcup Q_{4x}. \tag{4.21}$$

For fixed x and n , denote $X_i^{(n)} = I(X_i < x + u_n(x)) - EI(X_i < x + u_n(x))$, $1 \leq i \leq n$. It is easily seen that $X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)}$ are still WOD random variables by Lemma 3.1 (i) and $EX_i^{(n)} = 0$, $|X_i^{(n)}| \leq 1$. Applying Lemma 3.2 with $X_i = X_i^{(n)}$, $b = 1$ and $B_n^2 \doteq \sum_{i=1}^n E(X_i^{(n)})^2 \leq n$, we have by $g(n) = O(n^\delta)$ and $\frac{k_n q_n}{\sqrt{n \log n}} \rightarrow \infty$ that for all n large enough,

$$\begin{aligned}
 P(Q_{1x}) &= P\left(\left|\sum_{i=1}^n X_i^{(n)}\right| \geq k_n q_n d\right) \\
 &\leq 2g(n) \exp\left\{-\frac{k_n^2 q_n^2 d^2}{2B_n^2 + \frac{2}{3}k_n q_n d}\right\} \\
 &\leq 2g(n) \exp\left\{-\frac{k_n^2 q_n^2 d^2}{4n}\right\} \\
 &= 2g(n) \exp\left\{-\frac{d^2}{4} \cdot \frac{k_n^2 q_n^2}{n \log n} \cdot \log n\right\} \\
 &\leq 2n^{-2}.
 \end{aligned} \tag{4.22}$$

Similar to the proof of (4.22), we have that for all n large enough,

$$P(Q_{jx}) \leq 2n^{-2}, \quad j = 2, 3, 4. \tag{4.23}$$

Therefore, it follows by (4.21)–(4.23) that

$$\sum_{n=1}^{\infty} P(|f_n(x) - f(x)| > \varepsilon q_n) \leq C \sum_{n=1}^{\infty} n^{-2} < \infty, \quad (4.24)$$

which implies (2.6). The desired result (2.7) follows by (4.24) and the Borel–Cantelli lemma immediately. This completes the proof of the theorem. \square

Proof of Theorem 2.4. We use the same notations as those in the proof of Theorem 2.1.

Since $f(x)$ is uniformly continuous, we have that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that when $|x - y| < \delta$,

$$|f(x) - f(y)| < \frac{\varepsilon}{4}. \quad (4.25)$$

It follows by $\frac{k_n}{n} \rightarrow 0$ that $\frac{k_n}{\varepsilon n} < \delta$ for all n large enough. Hence,

$$b_n(x) = \frac{k_n}{2n(f(x) + \varepsilon)} \leq \frac{k_n}{2\varepsilon n} < \delta, \quad \text{for all } x \quad (4.26)$$

and

$$c_n(x) = \frac{k_n}{2n(f(x) - \varepsilon/2)} \leq \frac{k_n}{2n(\varepsilon - \varepsilon/2)} = \frac{k_n}{\varepsilon n} < \delta, \quad \text{for all } x. \quad (4.27)$$

By the Differential Mean Value Theorem, we can see that there exist $\theta_1 \in (x - b_n(x), x + b_n(x))$ and $\theta_2 \in (x - c_n(x), x + c_n(x))$ such that

$$F(x + b_n(x)) - F(x - b_n(x)) = 2b_n(x)f(\theta_1) \quad (4.28)$$

and

$$F(x + c_n(x)) - F(x - c_n(x)) = 2c_n(x)f(\theta_2). \quad (4.29)$$

It follows by (4.26) and (4.27) that $|x - \theta_1| < \delta$ and $|x - \theta_2| < \delta$, respectively, which together with (4.25) yield that

$$|f(x) - f(\theta_1)| < \frac{\varepsilon}{4}, \quad |f(x) - f(\theta_2)| < \frac{\varepsilon}{4}. \quad (4.30)$$

Denote

$$M = \sup_x f(x) < \infty, \quad d = \frac{\varepsilon}{8(M + \varepsilon)} \quad \text{and} \quad B = \left\{ \sup_x |F_n(x) - F(x)| \geq \frac{dk_n}{n} \right\}.$$

By (4.28) and (4.30), we can see that

$$\begin{aligned} A_{1x} &= \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) - F(x + b_n(x)) + F(x - b_n(x)) \geq \frac{k_n}{n} - 2b_n(x)f(\theta_1) \right\} \\ &= \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) - F(x + b_n(x)) + F(x - b_n(x)) \geq \frac{k_n}{n} \cdot \frac{f(x) - f(\theta_1) + \varepsilon}{f(x) + \varepsilon} \right\} \\ &\subset \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) - F(x + b_n(x)) + F(x - b_n(x)) \geq \frac{k_n}{n} \cdot \frac{-\varepsilon/4 + \varepsilon}{f(x) + \varepsilon} \right\} \end{aligned}$$

$$\begin{aligned}
& \subset \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) - F(x + b_n(x)) + F(x - b_n(x)) \geq \frac{k_n}{n} \cdot \frac{\varepsilon}{4(M + \varepsilon)} \right\} \\
& \subset \left\{ F_n(x + b_n(x)) - F_n(x - b_n(x)) - F(x + b_n(x)) + F(x - b_n(x)) \geq \frac{2dk_n}{n} \right\} \\
& \subset \left\{ |F_n(x + b_n(x)) - F(x + b_n(x))| \geq \frac{dk_n}{n} \right\} \cup \left\{ |F_n(x - b_n(x)) - F(x - b_n(x))| \geq \frac{dk_n}{n} \right\} \\
& \subset B.
\end{aligned} \tag{4.31}$$

Similar to the proof of (4.31), we have by (4.29) and (4.30) that

$$\begin{aligned}
A_{2x} &= \left\{ F_n(x + c_n(x)) - F_n(x - c_n(x)) - F(x + c_n(x)) + F(x - c_n(x)) \leq \frac{k_n}{n} \cdot \frac{f(x) - f(\theta_2) - \varepsilon/2}{f(x) - \varepsilon/2} \right\} \\
&\subset \left\{ F_n(x + c_n(x)) - F_n(x - c_n(x)) - F(x + c_n(x)) + F(x - c_n(x)) \leq \frac{k_n}{n} \cdot \frac{\varepsilon/4 - \varepsilon/2}{f(x) + \varepsilon} \right\} \\
&\subset \left\{ F_n(x + c_n(x)) - F_n(x - c_n(x)) - F(x + c_n(x)) + F(x - c_n(x)) \leq -\frac{2dk_n}{n} \right\} \\
&\subset \left\{ |F_n(x + c_n(x)) - F(x + c_n(x))| \geq \frac{dk_n}{n} \right\} \cup \left\{ |F_n(x - c_n(x)) - F(x - c_n(x))| \geq \frac{dk_n}{n} \right\} \\
&\subset B.
\end{aligned} \tag{4.32}$$

Hence, we have by (4.1), (4.31) and (4.32) that $A_x \subset B$ for all x , which together with Lemma 3.3 yields that for all n large enough,

$$\begin{aligned}
P\left(\sup_x |f_n(x) - f(x)| > \varepsilon\right) &= P\left(\bigcup_x A_x\right) \\
&\leq P\left(\sup_x |F_n(x) - F(x)| \geq \frac{dk_n}{n}\right) \\
&\leq P\left(\max_{1 \leq k \leq n-1} |F_n(x_{n,k}) - F(x_{n,k})| + \frac{2}{n} \geq \frac{dk_n}{n}\right) \\
&\leq P\left(\max_{1 \leq k \leq n-1} |F_n(x_{n,k}) - F(x_{n,k})| \geq \frac{dk_n}{2n}\right) \\
&\leq \sum_{k=1}^{n-1} P\left(|F_n(x_{n,k}) - F(x_{n,k})| \geq \frac{dk_n}{2n}\right).
\end{aligned} \tag{4.33}$$

For fixed k and n , denote $X_i^{(nk)} = I(X_i < x_{n,k}) - EI(X_i < x_{n,k})$, $1 \leq i \leq n$. It is easily seen that $X_1^{(nk)}, X_2^{(nk)}, \dots, X_n^{(nk)}$ are still WOD random variables by Lemma 3.1 (i) and $EX_i^{(nk)} = 0$, $|X_i^{(nk)}| \leq 1$. Applying Lemma 3.2 with $X_i = X_i^{(nk)}$, $b = 1$ and $B_n^2 \doteq \sum_{i=1}^n E(X_i^{(nk)})^2 \leq n$, we have by (4.33), $g(n) = O(n^\delta)$ and $\frac{k_n}{\sqrt{n \log n}} \rightarrow \infty$ that for all n large enough,

$$\begin{aligned}
P\left(\sup_x |f_n(x) - f(x)| > \varepsilon\right) &\leq \sum_{k=1}^{n-1} P\left(\left|\sum_{i=1}^n X_i^{(nk)}\right| \geq \frac{dk_n}{2}\right) \\
&\leq 2g(n) \sum_{k=1}^{n-1} \exp\left\{-\frac{d^2 k_n^2 / 4}{2B_n^2 + \frac{1}{3}dk_n}\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq 2g(n) \sum_{k=1}^{n-1} \exp \left\{ -\frac{Ck_n^2}{n} \right\} \\
&= 2g(n) \sum_{k=1}^{n-1} \exp \left\{ -\frac{Ck_n^2}{n \log n} \cdot \log n \right\} \\
&\leq 2n^{-2},
\end{aligned} \tag{4.34}$$

which implies (2.8). Combining (2.8) with Borel–Cantelli lemma, we can get the desired result (2.9) immediately. This completes the proof of the theorem. \square

Proof of Theorem 2.5. We used the same notations as those in the proof of Theorem 2.3.

Since $f(x)$ satisfies Lipschitz condition on \mathbf{R} , we have by $\frac{k_n}{nq_n^2} \rightarrow 0$ that there exists a constant $L > 0$ such that for all n large enough,

$$|f(x) - f(\theta_1)| \leq L|x - \theta_1| \leq Lu_n(x) = \frac{Lk_n}{2n(f(x) + \varepsilon q_n)} \leq \frac{Lk_n}{2n\varepsilon q_n} \leq \frac{\varepsilon q_n}{2} \tag{4.35}$$

and

$$|f(x) - f(\theta_2)| \leq L|x - \theta_2| \leq Lv_n(x) \leq \frac{Lk_n}{2n(\varepsilon q_n - \varepsilon q_n/2)} = \frac{Lk_n}{n\varepsilon q_n} \leq \frac{\varepsilon q_n}{4} \tag{4.36}$$

hold for all x .

Denote

$$M = \sup_x f(x) < \infty, \quad d = \frac{\varepsilon}{8M} \quad \text{and} \quad D = \left\{ \sup_x |F_n(x) - F(x)| \geq \frac{k_n q_n d}{n} \right\}.$$

Similar to the proof of (4.17)–(4.20), we have by (4.35) and (4.36) that

$$\begin{aligned}
B_{1x} &\subset \left\{ |F_n(x + u_n(x)) - F(x + u_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
&\quad \cup \left\{ |F_n(x - u_n(x)) - F(x - u_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
&\subset D
\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}
B_{2x} &\subset \left\{ |F_n(x + v_n(x)) - F(x + v_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
&\quad \cup \left\{ |F_n(x - v_n(x)) - F(x - v_n(x))| \geq \frac{k_n q_n d}{n} \right\} \\
&\subset D.
\end{aligned} \tag{4.38}$$

Hence, by (4.10), (4.37) and (4.38), we obtain that $B_x \subset D$ for all x , which together with (4.33) implies that for all n large enough,

$$\begin{aligned}
P \left(\sup_x |f_n(x) - f(x)| > \varepsilon q_n \right) &\leq P \left(\sup_x |F_n(x) - F(x)| \geq \frac{k_n q_n d}{n} \right) \\
&\leq \sum_{k=1}^{n-1} P \left(|F_n(x_{n,k}) - F(x_{n,k})| \geq \frac{k_n q_n d}{2n} \right).
\end{aligned} \tag{4.39}$$

Similar to the proof of (4.34), we have by (4.39) and (2.10) that for all n large enough,

$$\begin{aligned} P\left(\sup_x |f_n(x) - f(x)| > \varepsilon q_n\right) &\leq \sum_{k=1}^{n-1} P\left(|F_n(x_{n,k}) - F(x_{n,k})| \geq \frac{k_n q_n d}{2n}\right) \\ &\leq 2g(n) \sum_{k=1}^{n-1} \exp\left\{-\frac{C k_n^2 q_n^2}{n \log n} \cdot \log n\right\} \\ &\leq 2n^{-2}, \end{aligned}$$

which implies (2.11). Combining (2.11) with Borel–Cantelli lemma, we can get the desired result (2.12) immediately. This completes the proof of the theorem. \square

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