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Blow-up criterion for the 3D compressible non-isentropic Navier–Stokes equations without thermal conductivity

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ABSTRACT

In this paper, we prove a blow-up criterion in terms of the density ρ and the pressure P for the strong solutions with vacuum to Cauchy problem of the 3D compressible non-isentropic Navier–Stokes equations without thermal conductivity. More precisely, we show that the strong solution exists globally if the norm $\|(\rho, P)\|_{L^\infty([0,T]\times\mathbb{R}^3)}$ is bounded from above.

1. Introduction

The motion of a viscous compressible fluid in $\mathbb{R}^3$ can be described by the compressible Navier–Stokes equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} \mathbb{T}, \\ (\rho e)_t + \operatorname{div}(\rho e u) + P \operatorname{div} u - \kappa \triangle \theta = \operatorname{div}(u\mathbb{T}) - u \operatorname{div} \mathbb{T}. \end{cases} \tag{1.1}$$

In this system, $x \in \mathbb{R}^3$ is the spatial coordinate, $t \geq 0$ is the time, ρ is the mass density, $u = (u^1, u^2, u^3) \in \mathbb{R}^3$ is the velocity vector of fluids, e is the specific internal energy, the constant κ is the thermal conductivity coefficient, P is the pressure satisfying

$$P = (\gamma - 1)\rho e = R\rho\theta, \tag{1.2}$$

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where θ is the absolute temperature, γ is the adiabatic exponent and R is a positive constant; $\mathbb{T}$ is the viscous stress tensor given by

$$\mathbb{T} = 2\mu D(u) + \lambda \operatorname{div} u \mathbb{I}_3, \tag{1.3}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix, $D(u) = \frac{\nabla u + (\nabla u)^\top}{2}$ is the deformation tensor, μ is the shear viscosity coefficient, $\lambda + \frac{2}{3}\mu$ is the bulk viscosity coefficient, μ and λ are both real constants satisfying

$$\mu > 0, \qquad 3\lambda + 2\mu \geq 0, \tag{1.4}$$

which ensure the ellipticity of the Lamé operator L defined by

$$Lu = -\operatorname{div} \mathbb{T} = -\mu \triangle u - (\lambda + \mu)\nabla \operatorname{div} u. \tag{1.5}$$

When $\kappa = 0$, from (1.2), system (1.1) can be written as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} \mathbb{T}, \\ P_t + u \cdot \nabla P + \gamma P \operatorname{div} u = (\gamma - 1)(\operatorname{div}(u\mathbb{T}) - u \operatorname{div} \mathbb{T}). \end{cases} \tag{1.6}$$

This paper is aimed at giving a blow-up criterion of strong solutions to the Cauchy problem of system (1.6) with the following initial data

$$(\rho, u, P)|_{t=0} = (\rho_0(x), u_0(x), P_0(x)), \quad x \in \mathbb{R}^3 \tag{1.7}$$

and the far field behavior

$$(\rho, u, P)(t, x) \to (0, 0, 0) \quad \text{as } |x| \to +\infty,\ t > 0. \tag{1.8}$$

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$\begin{aligned} &\|f\|_s = \|f\|_{H^s(\mathbb{R}^3)}, \qquad |f|_p = \|f\|_{L^p(\mathbb{R}^3)}, \qquad \|(f,g)\|_X = \|f\|_X + \|g\|_X, \\ &D^{k,r} = \{f \in L^1_{loc}(\mathbb{R}^3) : |\nabla^k f|_r < +\infty\}, \qquad D^k = D^{k,2}, \qquad |f|_{D^{k,r}} = \|f\|_{D^{k,r}(\mathbb{R}^3)}, \\ &D^1_0 = \{f \in L^6(\mathbb{R}^3) : |\nabla f|_2 < \infty\}, \qquad |f|_{D^1_0} = \|f\|_{D^1_0(\mathbb{R}^3)}. \end{aligned}$$

A detailed study on homogeneous Sobolev spaces can be found in [4].

As has been observed in Theorem 3 of Cho–Kim [3], in which the existence of unique local strong solution for system (1.6) was proved, in order to make sure that the Cauchy problem (1.6)–(1.8) with vacuum is well-posed, some compatibility condition on the initial data (ρ_0, u_0, P_0) was proposed to compensate the lack of a positive lower bound of the initial mass density ρ_0.

Theorem 1.1. *(See [3].) If the initial data (ρ_0, u_0, P_0) satisfy*

$$(\rho_0, P_0) \in H^1 \cap W^{1,q}, \qquad \rho_0 \geq 0, \qquad P_0 \geq 0, \qquad u_0 \in D^1_0 \cap D^2, \tag{1.9}$$

and the compatibility condition

$$Lu_0 + \nabla P_0 = \sqrt{\rho_0} h, \quad \textit{for some } h \in L^2, \tag{1.10}$$

then there exist a positive time T_* and a unique solution (ρ, u, P) to Cauchy problem (1.6)–(1.8) satisfying

$$\begin{aligned}
&\rho \geq 0, \qquad P \geq 0, \qquad (\rho, P) \in C([0, T_*]; H^1 \cap W^{1,q}),\\
&u \in C([0, T_*]; D_0^1 \cap D^2) \cap L^2([0, T_*]; D^{2,q}),\\
&u_t \in L^2([0, T_*]; D_0^1), \qquad \sqrt{\rho} u_t \in L^\infty([0, T_*]; L^2).
\end{aligned} \tag{1.11}$$

The well-posedness of global classical solutions with small energy but possibly large oscillations and vacuum for isentropic flow has been obtained for Cauchy problem or for initial–boundary value problem in Huang–Li–Xin [8,9]. However, for the non-isentropic flow, when the thermal conductivity κ vanishes, the finite time blow-up of classical or strong solutions has been studied in Xin [14] and Xin–Yan [15] for both Cauchy problem and initial–boundary value problem if the initial density is compactly supported or vanishes in some local domain.

Then these motivate us to consider the mechanism of breakdown for the strong solutions and the structure of singularities. The similar question has been studied for the incompressible Euler equation by Beale–Kato–Majda (BKM) in their pioneering work [2] which showed that the L^∞-bound of vorticity $\operatorname{rot} u$ must blow up if the life span of the corresponding strong solution is assumed to be finite. Later, Ponce [11] rephrased the BKM-criterion in terms of $D(u)$. The same conclusion as [11] has been extended to compressible isentropic Navier–Stokes equations in Huang–Li–Xin [6], which can be shown as follows. If $0 < \overline{T} < +\infty$ is the maximum existence time for the strong solution, then

$$\limsup_{T \to \overline{T}} \int_0^T |D(u)|_{L^\infty(\mathbb{R}^3)} \mathrm{d}t = +\infty. \tag{1.12}$$

Later on, also for the strong solutions to the compressible isentropic Navier–Stokes equations, Sun–Wang–Zhang [12] proved that

$$\limsup_{T \to \overline{T}} |\rho|_{L^\infty([0,T] \times \mathbb{R}^3)} = +\infty,$$

under the physical assumption (1.4) and $\lambda < 7\mu$, which has been extended to non-isentropic flow with $\kappa > 0$ in Wen–Zhu [13] that

$$\limsup_{T \to \overline{T}} \Big(|\rho|_{L^\infty([0,T] \times \mathbb{R}^3)} + |\theta|_{L^\infty([0,T] \times \mathbb{R}^3)} \Big) = +\infty, \tag{1.13}$$

under the physical assumption (1.4) and $\lambda < 3\mu$.

So it is interesting to ask what the blow-up criterion is for the case $\kappa = 0$ compared with [13] for the non-isentropic flow with $\kappa > 0$. Via introduce some new arguments and more accurate estimates, under the assumption

$$\mu > 0, \qquad 2\mu + 3\lambda \geq 0, \qquad \lambda < 3\mu, \tag{1.14}$$

our main result in the following theorem shows that the upper bounds of (ρ, P) control the possible blow-up for strong solutions, which means that if a solution is initially smooth and loses its regularity at some later time, then the formation of singularity must be caused by losing the upper bound of ρ or P as the blow-up time approaches.

Theorem 1.2. *Let (1.14) hold. Assume that (ρ, u, P) is the unique strong solution to Cauchy problem (1.6)–(1.8) with the initial data (ρ_0, u_0, P_0) satisfying (1.9)–(1.10). If $0 < \overline{T} < +\infty$ is the maximal time of existence, then*

$$\limsup_{T \to \overline{T}} \left(|\rho|_{L^\infty([0,T]\times\mathbb{R}^3)} + |P|_{L^\infty([0,T]\times\mathbb{R}^3)} \right) = +\infty. \tag{1.15}$$

In other words, the strong solution (ρ, u, P) exists globally if (ρ, P) is bounded from above.

Remark 1.1. It is worth mentioning that, compared with [13] for the non-isentropic flow with $\kappa > 0$, according to the criterion obtained in Theorem 1.2, the L^∞ bound of θ is not the key point to make sure that the solution (ρ, u, P) is a global one, and it may go to infinity in the vacuum region within the life span of our strong solution.

Remark 1.2. Due to the appearance of the quadratic term

$$\begin{aligned} Q(u) &= \operatorname{div}(u\mathbb{T}) - u \operatorname{div} \mathbb{T} \\ &= 2\mu \sum_{i=1}^{3} (\partial_i u_i)^2 + \lambda(\operatorname{div} u)^2 + \mu \sum_{i\neq j}^{3} (\partial_i u_j)^2 + 2\mu \sum_{i>j} (\partial_i u_j)(\partial_j u_i) \end{aligned}$$

in $(1.6)_3$ and the lack of smooth mechanism for the regularity of the pressure P, the arguments used in [12] or [13] cannot be applied to our case directly. For example, if we want to control the norm $|\nabla P|_q$, unlike the estimate for $|\nabla \rho|_q$ which can be totally determined by $\|\operatorname{div} u\|_{L^1(0,T;L^\infty(\mathbb{R}^3))}$ due to the scalar hyperbolic structure of the continuity equation $(1.6)_1$, here we need the upper bound of $\||\nabla u||\nabla^2 u|\|_{L^2([0,T];L^q(\mathbb{R}^3))}$. However, the latter is not easy to be obtained before we get the upper bound of $|\nabla P|_q$ because of the strong coupling between u and P in the momentum equations $(1.6)_2$.

The rest of this paper is organized as follows. In Section 2, we give some important lemmas which will be used frequently in our proof. In Section 3, we give the proof for the blow-up criterion (1.15).

2. Preliminary

In this section, we show some important lemmas that will be frequently used in our proof. The first one is the well-known Gagliardo–Nirenberg inequality.

Lemma 2.1. *(See [10].) Let $p \in [2, 6]$, $l \in (1, +\infty)$, and $r \in (3, +\infty)$. Then there exists some constant $C > 0$ that may depend on l and r such that for*

$$f \in H^1(\mathbb{R}^3), \quad \textit{and} \quad g \in L^l(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3),$$

we have

$$\begin{aligned} &|f|_p^p \le C|f|_2^{(6-p)/2} |\nabla f|_2^{(3p-6)/2}, \\ &|g|_\infty \le C|g|_l^{l(r-3)/(3r+l(r-3))} |\nabla g|_r^{3r/(3r+l(r-3))}. \end{aligned} \tag{2.1}$$

Some commonly used versions of this inequality can be written as

$$|u|_6 \le C|u|_{D^1}, \qquad |u|_\infty \le C|u|_6^{\frac{1}{2}} |\nabla u|_6^{\frac{1}{2}}, \qquad |u|_\infty \le C\|u\|_{W^{1,r}}. \tag{2.2}$$

The ellipticity of the Lamé operator L is very important in our analysis. Consider the following boundary value problem:

$$\begin{cases} -\mu\triangle V - (\mu+\lambda)\nabla \operatorname{div} V = F, \quad \text{in } \mathbb{R}^3, \\ V(t,x) \to 0 \quad \text{as } |x| \to +\infty, \end{cases} \tag{2.3}$$

where $V = (V^1, V^2, V^3)$, $F = (F^1, F^2, F^3)$. It is well known that $(2.3)_1$ is a strongly elliptic system under the assumption (1.4). If $F \in W^{-1,2}(\mathbb{R}^3)$, then there exists a unique weak solution $V \in D_0^1(\mathbb{R}^3)$. Moreover, we have the following estimates for this system in $L^p(\mathbb{R}^3)$ spaces, which can be found in [1].

Lemma 2.2. *Let $p \in (1, +\infty)$ and u be a solution of (2.3). Then there exists a constant C depending only on λ, μ and p such that the following estimates hold:*

(1) *if $F \in L^p(\mathbb{R}^3)$, then we have*

$$|V|_{D^{2,p}} \le C|F|_p; \tag{2.4}$$

(2) *if $F \in W^{-1,p}(\mathbb{R}^3)$ (i.e., $F = \operatorname{div} f$ with $f = (f_{ij})_{3\times 3}$, $f_{ij} \in L^p(\mathbb{R}^3)$), then we have*

$$|V|_{D^{1,p}} \le C|f|_p. \tag{2.5}$$

The following lemma is about the positiveness of the quadratic term $Q(u)$. The proof can be found in [14].

Lemma 2.3. *(See [14].) If $u(x) \in D_0^1 \cap D^2$, then $Q(u) \ge 0$.*

Finally, we state the following Beal–Kato–Majda inequality which will be used later to prove the upper bound of $|(\nabla\rho, \nabla P)|_q$. The proof can be found in [7].

Lemma 2.4. *(See [7].) Let $p \in (3, +\infty)$. Then there exists a constant $C = C(p)$ such that the following inequality holds for any $\nabla u \in L^2 \cap D^{1,p}(\mathbb{R}^3)$:*

$$|\nabla u|_{L^\infty(\mathbb{R}^3)} \le C\left((|\operatorname{div} u|_\infty + |\operatorname{rot} u|_\infty)\ln(e + |\nabla^2 u|_p) + |\nabla u|_2 + 1\right). \tag{2.6}$$

3. Proof of blow-up criterion (1.15)

For any given $T \in (0, \overline{T}]$, we first show the classical energy estimates of the unique strong solution to (ρ, u, P) the Cauchy problem (1.6)–(1.8) obtained in Theorem 1.1. Hereinafter we use C to denote a generic constant which may vary each time when it appears.

Lemma 3.1. *For $0 \le t < T$, we have*

$$|\sqrt{\rho}u(t)|_2^2 + |P(t)|_1 \le C \quad \textit{and} \quad P(t) \ge 0,$$

where C depends only on (ρ_0, u_0, P_0), μ, λ, γ and T.

Proof. First, multiplying the momentum equations $(1.6)_2$ by u and the continuity equation $(1.6)_1$ by $\frac{|u|^2}{2}$, summing them together and integrating over $\mathbb{R}^3$ by parts, we get the classical energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \rho|u|^2 \mathrm{d}x - \int_{\mathbb{R}^3} u \operatorname{div} \mathbb{T} \mathrm{d}x = \int_{\mathbb{R}^3} P \operatorname{div} u \mathrm{d}x. \tag{3.1}$$

Second, from the pressure equation $(1.6)_3$, we have

$$P \operatorname{div} u = \frac{-1}{\gamma - 1}\big(P_t + \operatorname{div}(uP)\big) + \operatorname{div}(u\mathbb{T}) - u \operatorname{div} \mathbb{T},$$

which, together with (3.1) and Lemma 2.3, implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(\frac{1}{2}\rho|u|^2 + \frac{P}{\gamma-1}\Big)\mathrm{d}x = 0. \tag{3.2}$$

Finally, again from equation $(1.6)_3$ and Lemma 2.3, P can be expressed by

$$\begin{aligned} P(t,x) = \exp\Big(-\int_0^t(\gamma-1)\operatorname{div} u(s,U(s,t,x))\mathrm{d}s\Big)\Big(P_0(U(0;t,x)) \\ + \int_0^t(\gamma-1)Q(s,U(s,t,x))\exp\Big(\int_0^s(\gamma-1)\operatorname{div} u(\tau,U(\tau,t,x))\mathrm{d}\tau\Big)\mathrm{d}s\Big) \geq 0, \end{aligned}$$

where $U \in C([0,T]\times[0,T]\times\mathbb{R}^3)$ is the solution to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}U(s;t,x) = u(s,U(s;t,x)), & 0 \leq s \leq T, \\ U(t;t,x) = x, & 0 \leq t \leq T,\ x \in \mathbb{R}^3. \end{cases} \qquad \square$$

Now we assume that the opposite of (1.15) holds, i.e.,

$$\limsup_{T \mapsto \overline{T}}\Big(|\rho|_{L^\infty([0,T]\times\mathbb{R}^3)} + |P|_{L^\infty([0,T]\times\mathbb{R}^3)}\Big) = C_0 < +\infty. \tag{3.3}$$

Then based on (3.3), we can improve the energy estimate obtained in Lemma 3.1.

Lemma 3.2. *Let (1.14) hold. Then we have*

$$\int_{\mathbb{R}^3}\rho|u(t)|^4\mathrm{d}x + \int_0^T\int_{\mathbb{R}^3}|u|^2|\nabla u|^2\mathrm{d}x\mathrm{d}t \leq C, \quad \textit{for } 0 \leq t < T, \tag{3.4}$$

where C depends only on (ρ_0, u_0, P_0), C_0, μ, λ, γ and T.

Proof. For any (λ,μ) satisfying (1.14), there exists a sufficiently small constant $\alpha_{\lambda\mu} > 0$ such that

$$\lambda < (3 - \alpha_{\lambda\mu})\mu. \tag{3.5}$$

So we only need to prove (3.4) under the assumption (3.5).

First, multiplying $(1.6)_2$ by $r|u|^{r-2}u$ ($r \geq 3$) and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|u|^r\mathrm{d}x + \int_{\mathbb{R}^3}H_r\mathrm{d}x \\ &= -r(r-2)(\mu+\lambda)\int_{\mathbb{R}^3}\operatorname{div} u|u|^{r-3}u\cdot\nabla|u|\mathrm{d}x + \int_{\mathbb{R}^3}rP\operatorname{div}(|u|^{r-2}u)\mathrm{d}x, \end{aligned} \tag{3.6}$$

where

$$H_r = r|u|^{r-2}\big(\mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div} u|^2 + \mu(r-2)\big|\nabla|u|\big|^2\big).$$

For any given $\epsilon_1 \in (0,1)$ and $\epsilon_0 \in (0,1/4)$, we define a nonnegative function which will be determined in *Case 2* as follows

$$\phi(\epsilon_0,\epsilon_1,r) = \begin{cases} \dfrac{\mu\epsilon_1(r-1)}{3\big(-\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\mu+\lambda)}{4(r-1)}\big)}, & \text{if } -\dfrac{(4-\epsilon_0)\mu}{3} - \lambda + \dfrac{r^2(\mu+\lambda)}{4(r-1)} > 0, \\ 0, \quad \text{otherwise.} \end{cases}$$

We prove (3.4) in two cases.

Case 1: we assume that

$$\int\limits_{\mathbb{R}^3\cap\{|u|>0\}} |u|^r\Big|\nabla\Big(\frac{u}{|u|}\Big)\Big|^2 \mathrm{d}x > \phi(\epsilon_0,\epsilon_1,r) \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} |u|^{r-2}\big|\nabla|u|\big|^2 \mathrm{d}x. \tag{3.7}$$

A direct calculation shows for $|u|>0$ that

$$|\nabla u|^2 = |u|^2\left|\nabla\frac{u}{|u|}\right|^2 + \big|\nabla|u|\big|^2, \tag{3.8}$$

which plays an important role in the proof. By (3.6) and Cauchy's inequality, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} \rho|u|^r \mathrm{d}x + \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} H_r \mathrm{d}x \\
&= -r(r-2)(\mu+\lambda)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}} \operatorname{div} u|u|^{\frac{r-2}{2}}|u|^{\frac{r-4}{2}} u\cdot\nabla|u|\mathrm{d}x + \int\limits_{\mathbb{R}^3} rP\operatorname{div}(|u|^{r-2}u)\mathrm{d}x \\
&\le r(\mu+\lambda)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}} |u|^{r-2}|\operatorname{div} u|^2\mathrm{d}x + \frac{r(r-2)^2(\mu+\lambda)}{4}\int\limits_{\mathbb{R}^3\cap\{|u|>0\}} |u|^{r-2}\big|\nabla|u|\big|^2\mathrm{d}x \\
&\quad + \int\limits_{\mathbb{R}^3} rP\operatorname{div}(|u|^{r-2}u)\mathrm{d}x.
\end{aligned} \tag{3.9}$$

From Hölder's inequality, Lemmas 3.1 and 2.1, and Young's inequality, we have

$$\begin{aligned}
J_1 &= \int\limits_{\mathbb{R}^3} rP\operatorname{div}(|u|^{r-2}u)\mathrm{d}x \\
&\le Cr(r-1)\left(\int\limits_{\mathbb{R}^3} |u|^{r-2}|\nabla u|^2\mathrm{d}x\right)^{\frac12}\left(\int\limits_{\mathbb{R}^3} |u|^{r-2}P^2\mathrm{d}x\right)^{\frac12} \\
&\le Cr(r-1)|P|_{\frac{12r}{4r+4}}\left(\int\limits_{\mathbb{R}^3} |u|^{r-2}|\nabla u|^2\mathrm{d}x\right)^{\frac12}\left(\int\limits_{\mathbb{R}^3} (|u|^{\frac{r}{2}})^6\mathrm{d}x\right)^{\frac{2(r-2)}{12r}} \\
&\le Cr(r-1)\left(\int\limits_{\mathbb{R}^3} |u|^{r-2}|\nabla u|^2\mathrm{d}x\right)^{\frac12}\left(\int\limits_{\mathbb{R}^3} \big|\nabla|u|^{\frac{r}{2}}\big|^2\mathrm{d}x\right)^{\frac{r-2}{2r}}
\end{aligned}$$

$$\leq \frac{1}{2}\mu r\epsilon_0 \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} |u|^{r-2}|\nabla u|^2 \mathrm{d}x + C(C_0,\mu,r,\epsilon_0), \tag{3.10}$$

where $\epsilon_0 \in (0,1/4)$ is independent of r. Combining (3.8)–(3.10), it is obvious that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} \rho|u|^r \mathrm{d}x + \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} \mu r(1-\epsilon_0)|u|^{r-2}|\nabla|u||^2 \mathrm{d}x \\
&\quad + \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} \mu r(1-\epsilon_0)|u|^r \left|\nabla\frac{u}{|u|}\right|^2 \mathrm{d}x + \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} \mu r(r-2)|u|^{r-2}\big|\nabla|u|\big|^2 \mathrm{d}x \\
&\leq \frac{r(r-2)^2(\mu+\lambda)}{4} \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} |u|^{r-2}\big|\nabla|u|\big|^2 \mathrm{d}x + C(C_0,\mu,r,\epsilon_0).
\end{aligned} \tag{3.11}$$

According to (3.7) and (3.11), we then obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} \rho|u|^r \mathrm{d}x + rf(\epsilon_0,\epsilon_1,\epsilon_2,r) \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} |u|^{r-2}|\nabla|u||^2 \mathrm{d}x \\
&\quad + \int\limits_{\mathbb{R}^3\cap\{|u|>0\}} \mu r(1-\epsilon_0)\epsilon_2|u|^r \left|\nabla\frac{u}{|u|}\right|^2 \mathrm{d}x \leq C(C_0,\mu,r,\epsilon_0),
\end{aligned} \tag{3.12}$$

where

$$f(\epsilon_0,\epsilon_1,\epsilon_2,r) = \mu(1-\epsilon_0)(1-\epsilon_2)\phi(\epsilon_0,\epsilon_1,r) + \mu(r-1-\epsilon_0) - \frac{(r-2)^2(\mu+\lambda)}{4}. \tag{3.13}$$

Subcase 1: $4 \in \left\{r\middle| -\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\mu+\lambda)}{4(r-1)} > 0\right\}$, i.e., $\lambda + \epsilon_0\mu > 0$. It is easy to get

$$[4,+\infty) \subset \left\{r\middle| -\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\mu+\lambda)}{4(r-1)} > 0\right\}.$$

Therefore, we have

$$\phi(\epsilon_0,\epsilon_1,r) = \frac{\mu\epsilon_1(r-1)}{3\left(-\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\lambda+\mu)}{4(r-1)}\right)}, \quad \text{for } r \in [4,+\infty). \tag{3.14}$$

Substituting (3.14) into (3.13), for $r \in [4,\infty)$, we have

$$\begin{aligned}
&f(\epsilon_0,\epsilon_1,\epsilon_2,r) \\
&\quad = \frac{\mu^2\epsilon_1(1-\epsilon_0)(1-\epsilon_2)(r-1)}{3\left(-\frac{(4-\epsilon_0)\mu}{3} - \lambda + \frac{r^2(\lambda+\mu)}{4(r-1)}\right)} + \mu(r-1-\epsilon_0) - \frac{(r-2)^2(\mu+\lambda)}{4}.
\end{aligned} \tag{3.15}$$

Taking $(\epsilon_1,\epsilon_2,r) = (1,0,4)$, we have

$$f(\epsilon_0,1,0,4) = \frac{3(1-\epsilon_0)\mu^2}{\lambda+\epsilon_0\mu} + 2\mu - \lambda - \epsilon_0\mu = -\frac{(\lambda-a_1\mu)(\lambda-a_2\mu)}{\lambda+\epsilon_0\mu}, \tag{3.16}$$

where

$$\begin{aligned} a_1(\epsilon_0) &= 1 - \epsilon_0 + \sqrt{4 - 3\epsilon_0}, \\ a_2(\epsilon_0) &= 1 - \epsilon_0 - \sqrt{4 - 3\epsilon_0}. \end{aligned} \tag{3.17}$$

Since $\lambda - a_2(\epsilon_0)\mu = \lambda + \epsilon_0\mu + \left(\sqrt{4-3\epsilon_0} - 1\right)\mu > 0$, so to make sure that $f(\epsilon_0, 1, 0, 4) > 0$, it suffices to assume that

$$-\epsilon_0\mu < \lambda < a_1(\epsilon_0)\mu. \tag{3.18}$$

Due to the fact that $a_1(0) = 3$ and $a_1'(\epsilon_0) < 0$ for $\epsilon_0 \in [0, 1/4]$, we can choose some $\epsilon_0 \in (0, 1/4)$ small enough such that $a_1(\epsilon_0) \leq 3 - \alpha_{\lambda\mu}$.

Since $f(\epsilon_0, \epsilon_1, \epsilon_2, 4)$ is continuous with respect to (ϵ_1, ϵ_2) over $[0, 1] \times [0, 1]$, there exists $(\epsilon_1, \epsilon_2) \in (0, 1) \times (0, 1)$ such that

$$f(\epsilon_0, \epsilon_1, \epsilon_2, 4) > 0,$$

which, together with (3.12) (letting $r = 4$), implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \rho|u|^4 \mathrm{d}x + 4\mu(1 - \epsilon_0)\epsilon_2 \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 |\nabla u|^2 \mathrm{d}x \leq C(C_0, \mu, \epsilon_0). \tag{3.19}$$

Subcase 2: $4 \notin \{r | \frac{r^2(\mu+\lambda)}{4(r-1)} - \frac{(4-\epsilon_0)\mu}{3} - \lambda > 0\}$, i.e., $\lambda \leq -\epsilon_0\mu$. In this case, for $r = 4$, it is easy to get from (3.13) that

$$\begin{aligned} 4f(\epsilon_0, \epsilon_1, \epsilon_2, 4) &= 4\left(\mu(3 - \epsilon_0) - (\mu + \lambda)\right) \\ &> 4\Big(\frac{11}{4}\mu - (\mu + \lambda)\Big) = 4\Big(\frac{7\mu}{4} - \lambda\Big) \geq 4\Big(\frac{7\mu}{4} + \epsilon_0\mu\Big) > 7\mu, \end{aligned} \tag{3.20}$$

which, together with (3.12) (letting $r = 4$), implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \rho|u|^4 \mathrm{d}x + 7\mu \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^2 |\nabla u|^2 \mathrm{d}x \leq C(C_0, \mu, \epsilon_0). \tag{3.21}$$

Case 2: we assume that

$$\int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^r \left|\nabla\left(\frac{u}{|u|}\right)\right|^2 \mathrm{d}x \leq \phi(\epsilon_0, \epsilon_1, r) \int_{\mathbb{R}^3 \cap \{|u|>0\}} |u|^{r-2} \big|\nabla|u|\big|^2 \mathrm{d}x. \tag{3.22}$$

A direct calculation yields for $|u| > 0$ that

$$\operatorname{div} u = |u| \operatorname{div} \frac{u}{|u|} + \frac{u \cdot \nabla|u|}{|u|}. \tag{3.23}$$

Combining (3.23) and (3.9)–(3.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|u|^r\mathrm{d}x+\int_{\mathbb{R}^3\cap\{|u|>0\}}\mu r(1-\epsilon_0)|u|^{r-2}|\nabla u|^2\mathrm{d}x$$
$$+\int_{\mathbb{R}^3\cap\{|u|>0\}}r(\mu+\lambda)|u|^{r-2}|\operatorname{div}u|^2\mathrm{d}x+\int_{\mathbb{R}^3\cap\{|u|>0\}}\mu r(r-2)|u|^{r-2}\big|\nabla|u|\big|^2\mathrm{d}x$$
$$=-r(r-2)(\mu+\lambda)\int_{\mathbb{R}^3\cap\{|u|>0\}}\left(|u|^{r-2}u\cdot\nabla|u|\operatorname{div}\frac{u}{|u|}+|u|^{r-4}|u\cdot\nabla|u||^2\right)\mathrm{d}x. \tag{3.24}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|u|^r\mathrm{d}x+\int_{\mathbb{R}^3\cap\{|u|>0\}}r|u|^{r-4}G_r\mathrm{d}x\le C(C_0,\mu,r,\epsilon_0), \tag{3.25}$$

where

$$G_r=\mu(1-\epsilon_0)|u|^2|\nabla u|^2+(\mu+\lambda)|u|^2|\operatorname{div}u|^2+\mu(r-2)|u|^2\big|\nabla|u|\big|^2$$
$$+(r-2)(\mu+\lambda)|u|^2u\cdot\nabla|u|\operatorname{div}\frac{u}{|u|}+(r-2)(\mu+\lambda)|u\cdot\nabla|u||^2. \tag{3.26}$$

Now we analyze how to get the positiveness of the term with G_r in (3.25). Substituting (3.8) and (3.23) into (3.27), we get

$$G_r=\mu(1-\epsilon_0)|u|^4\left|\nabla\frac{u}{|u|}\right|^2+\mu(r-1-\epsilon_0)|u|^2\big|\nabla|u|\big|^2+(r-1)(\mu+\lambda)|u\cdot\nabla|u||^2$$
$$+r(\mu+\lambda)|u|^2u\cdot\nabla|u|\operatorname{div}\frac{u}{|u|}+(\mu+\lambda)|u|^4\left(\operatorname{div}\frac{u}{|u|}\right)^2$$
$$=\mu(1-\epsilon_0)|u|^4\left|\nabla\frac{u}{|u|}\right|^2+\mu(r-1-\epsilon_0)|u|^2\big|\nabla|u|\big|^2$$
$$+(r-1)(\mu+\lambda)\left(u\cdot\nabla|u|+\frac{r}{2(r-1)}|u|^2\operatorname{div}\frac{u}{|u|}\right)^2$$
$$+(\mu+\lambda)|u|^4\left(\operatorname{div}\frac{u}{|u|}\right)^2-\frac{r^2(\mu+\lambda)}{4(r-1)}|u|^4\left(\operatorname{div}\frac{u}{|u|}\right)^2, \tag{3.27}$$

which, combining with the fact $\left(\operatorname{div}\frac{u}{|u|}\right)^2\le 3\left|\nabla\frac{u}{|u|}\right|^2$, implies that

$$G_r\ge\frac{\mu(1-\epsilon_0)}{3}|u|^4\left(\operatorname{div}\frac{u}{|u|}\right)^2+\mu(r-1-\epsilon_0)|u|^2\big|\nabla|u|\big|^2$$
$$+\left(\mu+\lambda-\frac{r^2(\mu+\lambda)}{4(r-1)}\right)|u|^4\left(\operatorname{div}\frac{u}{|u|}\right)^2$$
$$=\mu(r-1-\epsilon_0)|u|^2\big|\nabla|u|\big|^2+\left(\frac{(4-\epsilon_0)\mu}{3}+\lambda-\frac{r^2(\mu+\lambda)}{4(r-1)}\right)|u|^4\left(\operatorname{div}\frac{u}{|u|}\right)^2. \tag{3.28}$$

Thus, from the definition of $\phi(\epsilon_0,\epsilon_1,r)$ and (3.22), we get

$$\begin{aligned}\int\limits_{\mathbb{R}^3\cap\{|u|>0\}} r|u|^{r-4}G_r\mathrm{d}x &\geq \mu r(r-1-\epsilon_0)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}}|u|^{r-2}\big|\nabla|u|\big|^2\mathrm{d}x\\
&\quad + r\left(\frac{(4-\epsilon_0)\mu}{3}+\lambda-\frac{r^2(\mu+\lambda)}{4(r-1)}\right)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}}|u|^r\left(\operatorname{div}\frac{u}{|u|}\right)^2\mathrm{d}x\\
&\geq \mu r(r-1-\epsilon_0)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}}|u|^{r-2}\big|\nabla|u|\big|^2\mathrm{d}x\\
&\quad + 3r\left(\frac{(4-\epsilon_0)\mu}{3}+\lambda-\frac{r^2(\mu+\lambda)}{4(r-1)}\right)\phi(\epsilon_0,\epsilon_1,r)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}}|u|^{r-2}\big|\nabla|u|\big|^2\mathrm{d}x\\
&= g(\epsilon_0,\epsilon_1,r)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}}|u|^{r-2}\big|\nabla|u|\big|^2\mathrm{d}x,\end{aligned}\tag{3.29}$$

where

$$g(\epsilon_0,\epsilon_1,r)=3r\left(\frac{(4-\epsilon_0)\mu}{3}+\lambda-\frac{r^2(\mu+\lambda)}{4(r-1)}\right)\phi(\epsilon_0,\epsilon_1,r)+\mu r(r-1-\epsilon_0)>0.\tag{3.30}$$

Here we need that ϵ_0 be sufficiently small such that $\epsilon_0<(r-1)(1-\epsilon_1)$. Then from (3.8), (3.25) and (3.29)–(3.30), when $r=4$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}\rho|u|^4\mathrm{d}x+g(\epsilon_0,\epsilon_1,4)\int\limits_{\mathbb{R}^3\cap\{|u|>0\}}|u|^2|\nabla u|^2\mathrm{d}x\leq C(C_0,\mu,\epsilon_0).\tag{3.31}$$

Combining (3.19), (3.21), and (3.31), and using Gronwall's inequality, we conclude that if $\lambda<(3-\alpha_{\lambda\mu})\mu$, there exist positive constants C_1 and C_2 depending only on C_0, μ, λ, and γ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}\rho|u|^4\mathrm{d}x+C_1\int\limits_{\mathbb{R}^3\cap\{|u|>0\}}|u|^2|\nabla u|^2\mathrm{d}x\leq C_2.\tag{3.32}$$

Integrating (3.32), it is shown that there exists a positive constant C depending only on (ρ_0,u_0,P_0), C_0, μ, λ, γ and T such that (ρ_0,u_0,P_0), C_0, μ, λ, γ and T such that (3.4) holds. □

The next lemma will give a key estimate on ∇u.

Lemma 3.3. *Let (1.14) hold. Then we have*

$$|\nabla u(t)|_2^2+\int\limits_0^T|\sqrt{\rho}u_t|_2^2\mathrm{d}t\leq C,\quad \textit{for } 0\leq t<T,$$

where C is a positive constant depending only on (ρ_0,u_0,P_0), C_0, μ, λ, γ and T.

Proof. It follows from the momentum equations $(1.6)_2$ that

$$\triangle G=\operatorname{div}(\rho\dot{u}),\qquad \mu\triangle\omega=\nabla\times(\rho\dot{u}),\tag{3.33}$$

where $\dot{f}=f_t+u\cdot\nabla f$ stands for the material derivative of f, and

$$G = (2\mu + \lambda)\operatorname{div} u - P, \quad \text{and} \quad \omega = \operatorname{rot} u$$

are the effective viscous flux, and the vorticity, respectively.

Applying the standard L^p-estimates in Lemma 2.2 to the elliptic system (3.33), we have

$$|\nabla G|_2 + |\nabla \omega|_2 \le C|\rho \dot{u}|_2 \le C\left(|\sqrt{\rho} u_t|_2 + \left|\sqrt{\rho}|u||\nabla u|\right|_2\right). \tag{3.34}$$

Multiplying the momentum equations $(1.6)_2$ by u_t and integrating over $\mathbb{R}^3$ lead to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2)\mathrm{d}x + \int_{\mathbb{R}^3} \rho|u_t|^2 \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \Big(P \operatorname{div} u_t - \rho(u\cdot\nabla)u\cdot u_t\Big)\mathrm{d}x := A + B.
\end{aligned} \tag{3.35}$$

From the definition of G we have

$$\begin{aligned}
A &= \int_{\mathbb{R}^3} P \operatorname{div} u_t \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} P \operatorname{div} u \mathrm{d}x - \int_{\mathbb{R}^3} P_t \operatorname{div} u \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} P \operatorname{div} u \mathrm{d}x - \frac{1}{2\mu+\lambda}\int_{\mathbb{R}^3} P_t G \mathrm{d}x - \frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} P^2 \mathrm{d}x \\
&:= A_1 + A_2 + A_3.
\end{aligned} \tag{3.36}$$

Denote

$$E = \frac{1}{2}\rho|u|^2 + \frac{P}{\gamma - 1},$$

and it is easy to show from (1.6) that

$$E_t + \operatorname{div}(Eu + Pu) = \operatorname{div}(u\mathbb{T}). \tag{3.37}$$

Then we have

$$\begin{aligned}
A_2 &= -\frac{1}{2\mu+\lambda}\int_{\mathbb{R}^3} P_t G \mathrm{d}x = -\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3} \Big(E_t - \frac{1}{2}(\rho|u|^2)_t\Big) G \mathrm{d}x \\
&:= A_{21} + A_{22}.
\end{aligned} \tag{3.38}$$

From (3.37) we get

$$\begin{aligned}
A_{21} &= -\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3} E_t G \mathrm{d}x \\
&= -\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3} \left(\frac{1}{2}\rho|u|^2 u + \frac{\gamma}{\gamma-1} P u - u\mathbb{T}\right)\cdot \nabla G \mathrm{d}x \\
&\le -\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3} \frac{1}{2}\rho|u|^2 u \cdot \nabla G \mathrm{d}x + C|\nabla G|_2 \left(|u|_6 |P|_3 + \left||u||\nabla u|\right|_2\right).
\end{aligned} \tag{3.39}$$

From (3.3), Lemma 3.2 and (3.34), we have

$$
\begin{aligned}
|\sqrt{\rho}|u||\nabla u||_2 &\le C|\rho^{\frac{1}{4}}u|_4|\nabla u|_4 \le C\big(|G|_4+|\omega|_4+1\big)\\
&\le C(|G|_2^{\frac{1}{4}}|\nabla G|_2^{\frac{3}{4}}+|\omega|_2^{\frac{1}{4}}|\nabla\omega|_2^{\frac{3}{4}}+1)\\
&\le \epsilon(|\nabla G|_2+|\nabla\omega|_2)+C(\epsilon)(|G|_2+|\omega|_2)+C\\
&\le \epsilon(|\sqrt{\rho}u_t|_2+|\sqrt{\rho}|u||\nabla u||_2)+C(\epsilon)(|\nabla u|_2+1),
\end{aligned}
\tag{3.40}
$$

where we used Hölder's inequality, Gagliardo–Nirenberg's inequality (Lemma 2.1) and Young's inequality. Therefore,

$$
\big|\sqrt{\rho}|u||\nabla u|\big|_2 \le \epsilon|\sqrt{\rho}u_t|_2+C(\epsilon)(|\nabla u|_2+1),
\tag{3.41}
$$

where $\epsilon>0$ is a sufficiently small constant. Substituting (3.41) into (3.34), we have

$$
|\nabla G|_2+|\nabla\omega|_2 \le C(|\sqrt{\rho}u_t|_2+|\nabla u|_2+1),
\tag{3.42}
$$

which, together with (3.39), implies that

$$
A_{21} \le -\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\frac{1}{2}\rho|u|^2u\cdot\nabla G\mathrm{d}x+\epsilon|\sqrt{\rho}u_t|_2^2+C(\epsilon)\left(|\nabla u|_2^2+\big||u||\nabla u|\big|_2^2+1\right).
\tag{3.43}
$$

Now we consider A_{22}. From the continuity equation $(1.6)_1$ and Lemma 3.1, we have

$$
\begin{aligned}
A_{22} &= \frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\frac{1}{2}\left(\rho|u|^2\right)_t G\mathrm{d}x\\
&= \frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\frac{1}{2}\rho_t|u|^2G\mathrm{d}x+\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\rho u\cdot u_tG\mathrm{d}x\\
&\le -\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\frac{1}{2}\operatorname{div}(\rho u)|u|^2G\mathrm{d}x+\epsilon|\sqrt{\rho}u_t|_2^2+C(\epsilon)\int_{\mathbb{R}^3}\rho|u|^2|G|^2\mathrm{d}x\\
&= \frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\rho u\cdot\nabla u\cdot uG\mathrm{d}x+\frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\frac{1}{2}\rho|u|^2u\cdot\nabla G\mathrm{d}x\\
&\quad+\epsilon|\sqrt{\rho}u_t|_2^2+C(\epsilon)\int_{\mathbb{R}^3}\rho|u|^2|G|^2\mathrm{d}x\\
&\le \frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\frac{1}{2}\rho|u|^2u\cdot\nabla G\mathrm{d}x+\epsilon|\sqrt{\rho}u_t|_2^2+C(\epsilon)\left(\big|\sqrt{\rho}|u||\nabla u|\big|_2^2+1\right),
\end{aligned}
\tag{3.44}
$$

which implies that

$$
A_{22} \le \frac{\gamma-1}{2\mu+\lambda}\int_{\mathbb{R}^3}\frac{1}{2}\rho|u|^2u\cdot\nabla G\mathrm{d}x+\epsilon|\sqrt{\rho}u_t|_2^2+C(\epsilon)\left(\big||u||\nabla u|\big|_2^2+1\right).
\tag{3.45}
$$

Then substituting (3.43) and (3.45) into (3.38), we deduce that

$$
A_2 \le \epsilon|\sqrt{\rho}u_t|_2^2+C(\epsilon)\left(|\nabla u|_2^2+\big||u||\nabla u|\big|_2^2+1\right).
\tag{3.46}
$$

For the term B, we have

$$B = \int_{\mathbb{R}^3} -\rho \cdot \nabla u \cdot u_t \mathrm{d}x \le \epsilon|\sqrt{\rho}u_t|_2^2 + C(\epsilon)\big|\sqrt{\rho}|u||\nabla u|\big|_2^2$$
$$\le \epsilon|\sqrt{\rho}u_t|_2^2 + C(\epsilon)\big||u||\nabla u|\big|_2^2. \tag{3.47}$$

Then from (3.35)–(3.36) and (3.46)–(3.47), by letting $\epsilon > 0$ be sufficiently small, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2)\mathrm{d}x + \int_{\mathbb{R}^3} \rho|u_t|^2\mathrm{d}x$$
$$\le C\left(|\nabla u|_2^2 + \big||u||\nabla u|\big|_2^2 + 1\right). \tag{3.48}$$

From Gronwall's inequality and Lemma 3.2, we obtain the desired conclusions. □

In the following two lemmas, we give the estimates on $|\nabla u|_6$ and $|u|_{L^\infty([0,T]\times\mathbb{R}^3)}$.

Lemma 3.4. *Let (1.14) hold. Then we have*

$$|\sqrt{\rho}\dot{u}(t)|_2^2 + \int_0^T |\dot{u}|_{D^1}^2 \mathrm{d}t \le C, \quad \textit{for } 0 \le t < T,$$

where C only depends on (ρ_0, u_0, P_0), C_0, μ, λ, γ and T.

Proof. We will follow an idea due to Hoff [5]. Applying $\dot{u}^j[\partial/\partial t + \operatorname{div}(u\cdot)]$ to the j-th component of $(1.6)_2$ and integrating by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \rho|\dot{u}|^2\mathrm{d}x = -\int_{\mathbb{R}^3} \Big(\dot{u}^j\big(\partial_j P_t + \operatorname{div}(\partial_j P u)\big) + \mu\dot{u}^j\big(\triangle u_t^j + \operatorname{div}(u\triangle u^j)\big)\Big)\mathrm{d}x$$
$$+ (\lambda+\mu)\int_{\mathbb{R}^3} \dot{u}^j\big(\partial_j \operatorname{div} u_t + \operatorname{div}(u\partial_j \operatorname{div} u)\big)\mathrm{d}x \equiv: \sum_{i=1}^3 L_i. \tag{3.49}$$

According to Lemmas 3.1–3.3, Hölder's inequality, Lemma 2.1 and Young's inequality, we have

$$L_1 = -\int_{\mathbb{R}^3} \dot{u}^j\big(\partial_j P_t + \operatorname{div}(\partial_j P u)\big)\mathrm{d}x = \int_{\mathbb{R}^3} \big(\partial_j\dot{u}^j P_t + \partial_k\dot{u}^j\partial_j P u^k\big)\mathrm{d}x$$
$$= \int_{\mathbb{R}^3} \big(-\partial_j\dot{u}^j u^k\partial_k P - \gamma P \operatorname{div} u\partial_j\dot{u}^j + (\gamma-1)Q(u)\partial_j\dot{u}^j + \partial_k\dot{u}^j\partial_j P u^k\big)\mathrm{d}x$$
$$= \int_{\mathbb{R}^3} \big(-\gamma P \operatorname{div} u\partial_j\dot{u}^j + (\gamma-1)Q(u)\partial_j\dot{u}^j + P\partial_k(\partial_j\dot{u}^j u^k) - P\partial_j(\partial_k\dot{u}^j u^k)\big)\mathrm{d}x$$
$$\le C\left(|\nabla\dot{u}|_2|\nabla u|_2 + |\nabla\dot{u}|_2|\nabla u|_4^2\right) \le \epsilon|\nabla\dot{u}|_2^2 + C(\epsilon)(|\nabla u|_4^4 + 1),$$

$$
\begin{aligned}
L_2 &= \int_{\mathbb{R}^3} \mu \dot{u}^j \big(\triangle u_t^j + \operatorname{div}(u \triangle u^j)\big)\mathrm{d}x = -\int_{\mathbb{R}^3} \mu\big(\partial_i \dot{u}^j \partial_i u_t^j + \triangle u^j u \cdot \nabla \dot{u}^j\big)\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \mu\big(|\nabla \dot{u}|^2 - \partial_i \dot{u}^j u^k \partial_k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j + \triangle u^j u \cdot \nabla \dot{u}^j\big)\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \mu\big(|\nabla \dot{u}|^2 + \partial_i \dot{u}^j \partial_k u^k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j - \partial_i u^j \partial_i u^k \partial_k \dot{u}^j\big)\mathrm{d}x \\
&\le -\frac{\mu}{2}|\nabla \dot{u}|_2^2 + C|\nabla u|_4^4,
\end{aligned}
\tag{3.50}
$$

and similarly, we have

$$
L_3 = (\lambda+\mu)\int_{\mathbb{R}^3} \dot{u}^j \big(\partial_j \operatorname{div} u_t + \operatorname{div}(u \partial_j \operatorname{div} u)\big)\mathrm{d}x \le -\frac{\mu+\lambda}{2}|\nabla \dot{u}|_2^2 + C|\nabla u|_4^4. \tag{3.51}
$$

Letting ϵ be sufficiently small, from (3.49)–(3.51), Lemma 2.2 and (3.33) we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \rho|\dot{u}|^2\mathrm{d}x + |\dot{u}|_{D^1}^2 &\le C\left(|\nabla u|_4^4 + 1\right) \le C\left(|G|_4^4 + |\omega|_4^4 + 1\right) \\
&\le C\left(|G|_2^{\frac{5}{2}}|\nabla G|_6^{\frac{3}{2}} + |\omega|_2^{\frac{5}{2}}|\nabla \omega|_6^{\frac{3}{2}} + 1\right) \\
&\le C\left(|\nabla \dot{u}|_2^{\frac{3}{2}} + 1\right) \le \epsilon |\nabla \dot{u}|_2^2 + C,
\end{aligned}
\tag{3.52}
$$

which implies that

$$
\int_{\mathbb{R}^3} \rho|\dot{u}|^2(t)\mathrm{d}x + \int_0^t |\dot{u}|_{D^1}^2 \le C, \quad \text{for } 0 \le t \le T. \qquad \square \tag{3.53}
$$

Based on Lemma 3.4, we have the following estimate.

Lemma 3.5. *Let (1.14) hold. Then we have*

$$
|(\nabla G, \nabla \omega)(t)|_2 + |\nabla u(t)|_6 + |u(t)|_\infty + \int_0^T \left(|\operatorname{div} u|_\infty^2 + |\omega|_\infty^2\right)\mathrm{d}t \le C, \tag{3.54}
$$

for $0 \le t < T$, where C only depends on (ρ_0, u_0, P_0), C_0, μ, λ, γ and T.

Proof. From Lemmas 2.2, 3.1–3.4 and noting (3.33), we have

$$
\begin{aligned}
&|\nabla G|_2 + |\nabla \omega|_2 \le C|\rho \dot{u}|_2 \le C|\sqrt{\rho}\dot{u}|_2 \le C, \\
&|\nabla G|_6 + |\nabla \omega|_6 \le C|\rho \dot{u}|_6 \le C|\dot{u}|_6 \le C|\nabla \dot{u}|_2,
\end{aligned}
\tag{3.55}
$$

which imply that

$$
|\nabla u|_6 \le C(|G|_6 + |P|_6 + |\omega|_6) \le C\left(|\nabla G|_2 + |\nabla \omega|_2 + 1\right) \le C. \tag{3.56}
$$

From Lemma 2.1, we have

$$\begin{aligned} |(\operatorname{div} u, \omega)|_\infty &\le C\left(|G|_\infty + |\omega|_\infty + 1\right) \\ &\le C\left(|\nabla G|_6 + |\nabla \omega|_6 + |G|_6 + |\omega|_6\right) \\ &\le C(1 + |\nabla \dot{u}|_2), \end{aligned} \tag{3.57}$$

which, together with Lemma 3.4, implies that

$$\int_0^T \left(|\operatorname{div} u|_\infty^2 + |\omega|_\infty^2\right) \mathrm{d}t \le C. \qquad \square$$

Next we will show the estimates on $|(\rho, P)|_{D^{1,q}}$.

Lemma 3.6. *Let (1.14) hold. Then we have*

$$|(\rho, P)(t)|_{D^1 \cap D^{1,q}} + |\rho_t(t)|_{L^2 \cap L^q} + |P_t(t)|_2 + \int_0^T |\nabla u|_\infty \mathrm{d}t \le C, \tag{3.58}$$

for $0 \le t < T$*, where* C *only depends on* (ρ_0, u_0, P_0)*,* C_0*,* μ*,* λ*,* γ *and* T*.*

Proof. Noting that $Lu = -\rho u_t - \rho u \cdot \nabla u - \nabla P$, for any $2 \le r \le q$, from Lemma 2.2 we have

$$|\nabla^2 u|_r \le C\left(|\rho \dot{u}|_r + |\nabla P|_r\right) \le C\left(1 + |\nabla \dot{u}|_2 + |\nabla P|_r\right). \tag{3.59}$$

Applying ∇ to $(1.6)_1$ and multiplying by $r|\nabla \rho|^{r-2}\nabla \rho$, we have

$$\begin{aligned} &(|\nabla \rho|^r)_t + \operatorname{div}(|\nabla \rho|^r u) + (r-1)|\nabla \rho|^r \operatorname{div} u \\ &\quad = -r|\nabla \rho|^{r-2}(\nabla \rho)^\top \nabla u(\nabla \rho) - r\rho|\nabla \rho|^{r-2}\nabla \rho \cdot \nabla \operatorname{div} u. \end{aligned} \tag{3.60}$$

Integrating (3.60) over $\mathbb{R}^3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \rho|_r \le C\left(|\nabla u|_\infty |\nabla \rho|_r + |\nabla^2 u|_r\right), \tag{3.61}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \rho|_r \le C\left((1 + |\nabla u|_\infty)(|\nabla \rho|_r + |\nabla P|_r) + |\nabla \dot{u}|_2^2 + 1\right). \tag{3.62}$$

Next, applying ∇ to $(1.6)_2$ and multiplying by $r|\nabla P|^{r-2}\nabla P$, we have

$$\begin{aligned} &(|\nabla P|^r)_t + \operatorname{div}(|\nabla P|^r u) + (r\gamma - 1)|\nabla P|^r \operatorname{div} u \\ &\quad = -r|\nabla P|^{r-2}(\nabla P)^\top \nabla u(\nabla P) - r\gamma P|\nabla P|^{r-2}\nabla P \cdot \nabla \operatorname{div} u \\ &\qquad + r(\gamma - 1)\nabla Q(u) \cdot \nabla P|\nabla P|^{r-2}. \end{aligned} \tag{3.63}$$

Integrating (3.63) over $\mathbb{R}^3$, we then get

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla P|_r^r \le C\left(|\nabla u|_\infty |\nabla P|_r^r + |\nabla^2 u|_r |\nabla P|_r^{r-1} + |\nabla u|_\infty |u|_{D^{2,r}} |\nabla P|_r^{r-1}\right), \tag{3.64}$$

which means that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla P|_r &\leq C\left((1+|\nabla u|_\infty)|\nabla P|_r + (1+|\nabla u|_\infty)|u|_{D^{2,r}}\right)\\ &\leq C\left((1+|\nabla u|_\infty)(1+|\nabla P|_r) + |\nabla \dot{u}|_2 + |\nabla u|_\infty|\nabla\dot{u}|_2\right).\end{aligned}\tag{3.65}$$

According to Lemma 2.4 and (3.59), we obtain

$$\begin{aligned}|\nabla u|_{L^\infty(\mathbb{R}^3)} &\leq C\left((|\operatorname{div} u|_\infty + |\omega|_\infty)\ln\left(e+|\nabla^2 u|_q\right) + |\nabla u|_2 + 1\right)\\ &\leq C\left((|\operatorname{div} u|_\infty + |\omega|_\infty)\ln\left(e+|\nabla\dot{u}|_2 + |\nabla P|_q\right) + 1\right)\\ &\leq C\left((|\operatorname{div} u|_\infty + |\omega|_\infty)\left(\ln(e+|\nabla\dot{u}|_2) + \ln(e+|\nabla P|_q)\right) + 1\right).\end{aligned}\tag{3.66}$$

From Lemma 2.1 we have

$$|\nabla u|_\infty \leq C|\nabla u|_q^{1-\frac{3}{q}}|\nabla^2 u|_q^{\frac{3}{q}} \leq C\left(1+|\nabla\dot{u}|_2 + |\nabla P|_q\right)^{\frac{3}{q}},\tag{3.67}$$

which leads to

$$\begin{aligned}|\nabla u|_\infty|\nabla\dot{u}|_2 &\leq C\left(1+|\nabla\dot{u}|_2+|\nabla P|_q\right)^{\frac{3}{q}}|\nabla\dot{u}|_2\\ &\leq C\left((1+|\nabla\dot{u}|_2)(1+|\nabla P|_q) + |\nabla\dot{u}|_2^2\right).\end{aligned}\tag{3.68}$$

Combining (3.62), (3.65)–(3.66) and (3.68), we easily have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla\rho|_q + |\nabla P|_q\right) &\leq C\left((1+|\nabla u|_\infty)(1+|\nabla P|_q + |\nabla\rho|_q) + |\nabla\dot{u}|_2^2 + |\nabla u|_\infty|\nabla\dot{u}|_2\right)\\ &\leq C\Big(\left(1+|\operatorname{div} u|_\infty + |\omega|_\infty\right)\ln(e+|\nabla\dot{u}|_2)(1+|\nabla P|_q+|\nabla\rho|_q)\\ &\quad + \left(1+|\operatorname{div} u|_\infty + |\omega|_\infty\right)\ln(e+|\nabla P|_q)(1+|\nabla P|_q+|\nabla\rho|_q)\\ &\quad + (1+|\nabla\dot{u}|_2)(1+|\nabla P|_q) + |\nabla\dot{u}|_2^2\Big).\end{aligned}\tag{3.69}$$

Using the notations

$$f = e + |\nabla\rho|_q + |\nabla P|_q, \qquad g = \left(1+|\operatorname{div} u|_\infty + |\omega|_\infty\right)\ln(e+|\nabla\dot{u}|_2),$$

(3.69) yields

$$f_t \leq C\left(gf + gf\ln f + |\nabla\dot{u}|_2^2 + (1+|\nabla\dot{u}|_2)f\right),$$

which, together with Lemma 3.4 and Gronwall's inequality, yields

$$\ln f(t) \leq C, \quad \text{for } 0\leq t\leq T.$$

Thus we have

$$|\nabla\rho|_q + |\nabla P|_q \leq C, \quad \text{for } 0 \leq t \leq T,$$

which, along with (3.66) and Lemma 3.4, implies that

$$\int_0^t |\nabla u|_\infty \mathrm{d}s \le C, \quad \text{for } 0 \le t \le T. \tag{3.70}$$

Combining (3.62), (3.65) and (3.70), it is not hard to have

$$|\nabla \rho|_r + |\nabla P|_r \le C, \quad \text{for } 0 \le t \le T.$$

Finally, the estimates for ρ_t and P_t can be obtained easily by noting the following relations:

$$\rho_t = -u \cdot \nabla \rho - \rho \operatorname{div} u, \qquad P_t = -u \cdot \nabla P - \gamma P \operatorname{div} u + (\gamma - 1) Q(u), \tag{3.71}$$

and the estimates obtained in Lemmas 3.1–3.6. $\square$

According to the estimates obtained in Lemmas 3.1–3.6, we deduce that

Lemma 3.7. *Let (1.14) hold. Then we have*

$$|u(t)|_{D^2} + |\sqrt{\rho} u_t(t)|_2 + |\nabla P(t)|_q + \int_0^T \left(|u_t|^2_{D^1} + |u|^2_{D^{2,q}}\right) \mathrm{d}t \le C,$$

for $0 \le t \le T$, where C only depends on (ρ_0, u_0, P_0), C_0, μ, λ, γ and T.

Proof. From the momentum equations $(1.6)_2$, Lemmas 2.2, 3.1–3.6 and estimate (3.59), we have

$$|u|_{D^2} \le (1 + |\rho \dot{u}|_2 + |\nabla P|_2) \le C, \qquad |u|_{D^{2,q}} \le C(1 + |\nabla \dot{u}|_2),$$

which, together with (3.71) imply that

$$\int_0^T |u|^2_{D^{2,q}} \mathrm{d}t \le C, \qquad |P_t|_q \le C(1 + |Q(u)|_q) \le C(1 + \|\nabla u\|_1) \le C.$$

According to Lemmas 3.2–3.3 and 3.6, we quickly have

$$|\sqrt{\rho} u_t|_2 \le C\left(|\sqrt{\rho}\dot{u}|_2 + |\sqrt{\rho} u \cdot \nabla u|_2\right) \le C\left(1 + |\rho^{\frac{1}{4}} u|_4 |\nabla u|_2\right) \le C.$$

Similarly, from Lemma 3.4, we have

$$\int_0^T |u_t|^2_{D^1} \mathrm{d}t \le C \int_0^T \left(|\dot{u}|^2_{D^1} + |u \cdot \nabla u|^2_{D^1}\right) \mathrm{d}t \le C. \qquad \square$$

At last, in view of the estimates obtained in Lemmas 3.1–3.7, we know that the functions $(\rho, u, P)|_{t=\overline{T}} = \lim_{t \to \overline{T}} (\rho, u, P)$ satisfy the conditions imposed on the initial data (1.9)–(1.10). Then, we take $(\rho, u, P)|_{t=\overline{T}}$ as the initial data and apply the local existence Theorem 1.1 to extend our local strong solution beyond $t \ge \overline{T}$. This contradicts the maximum assumption on $\overline{T}$. Therefore, the blowup criterion showed by Theorem 1.2 is proved.

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References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math. 12 (1959) 623–727.
[2] T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3D Euler equations, Comm. Math. Phys. 94 (1984) 61–66.
[3] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, J. Differential Equations 228 (2006) 377–411.
[4] G. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Springer, New York, 1994.
[5] D. Hoff, Global solutions of the Navier–Stokes equations for multidimensional compressible flow with discontinuous initial data, J. Differential Equations 120 (1995) 215–254.
[6] X. Huang, J. Li, Z. Xin, Blow-up criterion for viscous barotropic flows with vacuum states, Comm. Math. Phys. 301 (2011) 23–35.
[7] X. Huang, J. Li, Z. Xin, Serrin-type criterion for the three-dimensional viscous compressible flows, SIAM J. Math. Anal. 43 (2011) 1872–1886.
[8] X. Huang, J. Li, Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, Comm. Pure Appl. Math. 65 (2012) 549–585.
[9] X. Huang, J. Li, Z. Xin, Global well-posedness for classical solutions to the multi-dimensional isentropic compressible Navier–Stokes system with vacuum on bounded domains, preprint, 2012.
[10] O. Ladyzenskaja, N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, American Mathematical Society, Providence, RI, 1968.
[11] G. Ponce, Remarks on a paper: "Remarks on the breakdown of smooth solutions for the 3D Euler equations", Comm. Math. Phys. 98 (1985) 349–353.
[12] Y. Sun, C. Wang, Z. Zhang, A Beale–Kato–Majda blow-up criterion to the compressible Navier–Stokes equation, J. Math. Pures Appl. 95 (2011) 36–47.
[13] H. Wen, C. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier–Stokes equations with vacuum, Adv. Math. 248 (2013) 534–572.
[14] Z. Xin, Blow-up of smooth solutions to the compressible Navier–Stokes equations with compact density, Comm. Pure Appl. Math. 51 (1998) 0229–0240.
[15] Z. Xin, W. Yan, On blow-up of classical solutions to the compressible Navier–Stokes equations, Comm. Math. Phys. 321 (2013) 529–541.