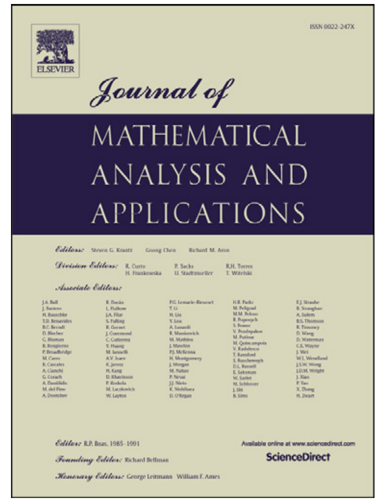

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Revisit to sign-changing solutions for the nonlinear Schrödinger-Poisson system in $\mathbb{R}^3$ *

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Abstract

In this paper, we investigate the existence of solutions for the nonlinear Schrödinger-Poisson system with zero mass. By introducing some new ideas, we prove, via the constraint variational method and the quantitative deformation lemma, that the system has a sign-changing solution.

Keywords: Schrödinger-Poisson system; Sign-changing solution; Zero mass

1 Introduction

In recent years, the following Schrödinger-Poisson system,

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

has been investigated extensively, specially on the existence of positive solutions, ground state solutions and multiple solutions, see for examples [4, 8, 9, 15]. For the mathematical and physical background of the system (1.1), we refer the reader to the papers [5, 6] and the references therein. To the authors' knowledge, there are very few results on the existence of sign-changing solutions for system (1.1) except [12, 16].

In [16], by introducing the constraint variational method and the Brouwer degree theory, Wang and Zhou successfully show the existence of sign-changing solutions for the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \mu\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $p \in (3, 5)$, μ is a positive parameter. In [16], the authors define

$$H = \begin{cases} H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}, & \text{if } V(x) \text{ is a constant,} \\ \left\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 < \infty\right\}, & \text{if } V(x) \text{ is not a constant} \end{cases}$$

and assume that

(V) $V \in C(\mathbb{R}^3, \mathbb{R}_+)$ such that $H \subset H^1(\mathbb{R}^3)$ and the embedding $H \hookrightarrow L^q(\mathbb{R}^3)(2 < q < 6)$ is compact, where $\mathbb{R}_+ = [0, \infty)$.

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They get a sign-changing solution of (1.2) by seeking first a minimizer of the energy functional I_μ over the constraint

$$\mathcal{M}_\mu = \{u \in H : u^\pm \neq 0, \langle I'_\mu(u), u^+\rangle = \langle I'_\mu(u), u^-\rangle = 0\}$$

and then showing that the minimizer is a sign-changing solution of (1.2) via degree theory, where

$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2) + \frac{\mu}{4}\int_{\mathbb{R}^3}\phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}$$

and $\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}\mathrm{d}y, u \in H$. The difficulty and novelty of [16] is to explain $\mathcal{M}_\mu \neq \emptyset$ and the minimizer is a critical point of I_μ.

Inspired by above references, more precisely by [16], we are concerned in this paper with the existence of sign-changing solutions for a class of nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where more general conditions involving the function V are assumed. Throughout this paper, we say that $(V, K) \in \mathcal{K}$ if the following conditions hold:

(H_0) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$;

(H_1) if $\{A_n\} \subset \mathbb{R}^3$ is a sequence of Borel sets such that the Lebesgue measure of A_n is less than R, for all n and some $R > 0$, then

$$\lim_{r\to\infty}\int_{A_n \cap B_r^c(0)} K = 0, \quad \text{uniformly in } n = 1, 2, \ldots;$$

(H_2) $K/V \in L^\infty(\mathbb{R}^3)$; or

(H_3) there exists $p \in (2, 6)$ such that

$$\frac{K(x)}{V(x)^{(6-p)/4}} \to 0, \quad |x| \to \infty.$$

The hypotheses (H_0)-(H_3) on functions V and K were firstly introduced in [1] and characterize system (1.3) as zero mass. Nonlinear elliptic equations with zero mass have been studied by many authors, for instance, [1, 2, 3, 7, 10, 11] and the references therein. As for the function f, we assume $f \in C^1(\mathbb{R}, \mathbb{R})$ and satisfies the following conditions:

(f_1) $\lim_{t\to 0} f(t)/t = 0$, if (H_2) hold;

(f_2) $\lim_{t\to 0} f(t)/|t|^{p-1} = A \in \mathbb{R}$, if ($\mathrm{H}_3$) hold;

(f_3) f has a "quasicritical growth", namely, $\lim_{|t|\to\infty} f(t)/t^5 = 0$;

(f_4) $\lim_{|t|\to\infty} F(t)/t^4 = \infty$, where $F(t) = \int_0^t f(s)\mathrm{d}s$;

(f_5) the map $t \mapsto f(t)/|t|^3$ is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$ respectively.

Since (1.3) is a zero mass problem, it seems that the approprite working space should be

$$X = \left\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y < \infty\right\}$$

with the norm

$$\|u\|_X^2 = \int_{\mathbb{R}^3}|\nabla u|^2 + \left(\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

By [13], we know that $(X, \|\cdot\|_X)$ is a uniformly convex Banach space. Let us define $\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\mathrm{d}y, u \in X$. Then, $u \in X$ if and only if both u and ϕ_u belong to $D^{1,2}(\mathbb{R}^3)$. In such a case, $-\Delta\phi_u = u^2$ in a weak sense, and

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 = \int_{\mathbb{R}^3} \phi_u u^2 = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{1.4}$$

Using the expression of (1.4), we obtain that the system (1.3) is merely a single equation on u.

The condition $(V, K) \in \mathcal{K}$ is fascinating. It can be used to certify that the space E given by

$$E = \left\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V|u|^2 < \infty\right\}$$

endowed with the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V|u|^2)$$

is compactly embedded into the weighted Lebesgue space

$$L_K^q(\mathbb{R}^3) = \left\{u : u \text{ is measurable on } \mathbb{R}^3 \text{ and } \int_{\mathbb{R}^3} K|u|^q < \infty\right\},$$

for some $q \in (2, 6)$, see Proposition 2.1 below. However, because of zero mass situation, we need to consider a new space

$$B = \left\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V|u|^2 < \infty,\ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y < \infty\right\}$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V|u|^2) + \left(\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

Since $(X, \|\cdot\|_X)$ is a Banach space and $(E, \|\cdot\|_E)$ is a Banach space because of (H_0), $(B, \|\cdot\|)$ is also a Banach space. Denote the usual norm of $L^p(\mathbb{R}^3)$ by $|\cdot|_p$. By Sobolev embedding theorem, the embedding $E \hookrightarrow D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous. Let $S' > 0$ be the embedding constant, i.e,

$$|u|_6^2 \leqslant S'^{-1}\|u\|_E^2, \quad u \in E. \tag{1.5}$$

Define the energy functional $J : B \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} KF(u), \quad u \in B. \tag{1.6}$$

The functional J is well-defined on B and belongs to $C^1(B, \mathbb{R})$. In addition, we have

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + Vuv) + \int_{\mathbb{R}^3} \phi_u uv - \int_{\mathbb{R}^3} Kf(u)v, \quad u, v \in B. \tag{1.7}$$

As is well known, a critical point of J is a weak solution of (1.3). If $u \in B$ is a weak solution of (1.3) and $u^{\pm} \neq 0$, we say that u is a sign-changing solution of (1.3), where

$$u^+(x) = \max\{u(x), 0\}, \quad u^-(x) = \min\{u(x), 0\}.$$

Motivated by [16], for the purpose of getting a sign-changing solution of (1.3), we first try to seek a minimizer of the energy functional J over the following constraint:

$$\mathcal{M} = \left\{u \in B : u^{\pm} \neq 0, \langle J'(u), u^+\rangle = \langle J'(u), u^-\rangle = 0\right\}, \tag{1.8}$$

and then show that the minimizer is a sign-changing solution of (1.3).

The main result of this paper is stated below.

Theorem 1.1. *Suppose that* $(V, K) \in \mathcal{K}$ *and* f *satisfies (f$_1$)-(f$_5$). Then the system* (1.3) *possesses at least one sign-changing solution.*

In this paper, the main tools used are the minimization argument and the quantitative deformation lemma. We must point out that the difficulty in the proof of Theorem 1.1 is still to show that $\mathcal{M} \neq \emptyset$ and the minimizer is a critical point of J. To show $\mathcal{M} \neq \emptyset$, we use the Brouwer fixed point theorem, which is entirely different from the method presented in [16]; to prove the minimizer is a critical point of J, we make use of the quantitative deformation lemma which is introduced in [14]. There are some new ingredients in the proof process of using the quantitative deformation lemma.

The paper is organized as follows. In Section 2 we give some propositions and lemmas for convenience. In Section 3, we establish some technical lemmas which play a critical role in showing $\mathcal{M} \neq \emptyset$. In Section 4 we accomplish the proof of Theorem 1.1 by the quantitative deformation lemma.

2 Preliminaries

In this section we give some propositions and lemmas for convenience. In order to recover compactness, for $q \in [1, \infty)$, we define the weighted Lebesgue space $L^q_K(\mathbb{R}^3)$ with the norm

$$\|u\|_K = \left(\int_{\mathbb{R}^3} K|u|^q\right)^{1/q}, \quad u \in L^q_K(\mathbb{R}^3).$$

By (H$_0$), $L^q_K(\mathbb{R}^3)$ is a Banach space. We firstly state two important consequences owing to Alves and Souto [1, Proposition 2.1 and Lemma 2.2].

Proposition 2.1. *Assume* $(V, K) \in \mathcal{K}$. *If (H$_2$) holds, then* E *is compactly embedded in* $L^q_K(\mathbb{R}^3)$ *for every* $q \in (2, 6)$; *if (H$_3$) holds, then* E *is compactly embedded in* $L^p_K(\mathbb{R}^3)$.

Proposition 2.2. *Suppose that* f *satisfies (f$_1$)-(f$_5$) and* $(V, K) \in \mathcal{K}$. *Let* $\{v_n\}$ *be such that* $v_n \rightharpoonup v$ *in* E. *Then*

$$\int_{\mathbb{R}^3} KF(v_n) \to \int_{\mathbb{R}^3} KF(v), \quad \int_{\mathbb{R}^3} Kf(v_n)v_n \to \int_{\mathbb{R}^3} Kf(v)v.$$

Secondly, we consult a proposition for the sake of proving Lemma 3.3 below.

Proposition 2.3. *[13] Given a sequence* $\{u_n\}$ *in* X, $u_n \rightharpoonup u$ *in* X *if and only if* $u_n \rightharpoonup u$ *in* $D^{1,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y$ *is bounded. In such a case,* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$.

Finally, we need to introduce some lemmas which will play a crucial role in the proof of Lemma 3.1 below.

Lemma 2.4. *Assume that* $(V, K) \in \mathcal{K}$ *and* f *satisfies (f$_1$)-(f$_5$). Then, for any* $u \in E\backslash\{0\}$,

$$\lim_{|t|\to\infty} \int_{\mathbb{R}^3} \frac{Kf(tu)u}{t^3} = \infty. \tag{2.1}$$

Proof. First of all, by the conditions (f$_4$) and (f$_5$), we have that

$$\lim_{|t|\to\infty} \frac{f(t)}{t^3} = \infty. \tag{2.2}$$

Suppose that (H$_2$) is true. It follows from (2.2) that for any given $M > 0$, there exists $R > 0$ such that

$$f(t)t \geqslant Mt^4, \quad |t| > R.$$

By the condition (f$_1$), we get

$$\lim_{t\to 0} \frac{f(t)t - Mt^4}{t^2} = 0. \tag{2.3}$$

It follows from (2.3) that there exists $C > 0$ such that

$$\frac{f(t)t - Mt^4}{t^2} \geqslant -C, \quad |t| \in (0, R].$$

Therefore we can conclude that

$$f(t)t \geqslant Mt^4 - Ct^2, \quad t \in \mathbb{R}. \tag{2.4}$$

From (2.4), we get

$$\int_{\mathbb{R}^3} Kf(tu)tu \geqslant Mt^4 \int_{\mathbb{R}^3} Ku^4 - Ct^2 \int_{\mathbb{R}^3} Ku^2. \tag{2.5}$$

Dividing by t^4 and letting $|t| \to \infty$, we deduce (2.1). Here, we used the integrability about Ku^4 and Ku^2. In order to show its rationality, we need to prove that $\int_{\mathbb{R}^3} Ku^4 < \infty$ and $\int_{\mathbb{R}^3} Ku^2 < \infty$. In fact, since (H_2) holds, $\int_{\mathbb{R}^3} Ku^2 \leqslant |K/V|_\infty \|u\|_E^2 < \infty$. By Proposition 2.1, we know that $E \hookrightarrow L_K^q(\mathbb{R}^3)$ is compactly for $q \in (2, 6)$. Hence $\int_{\mathbb{R}^3} Ku^4 \leqslant C\|u\|_E^4 < \infty$.

Next, supposing that (H_3) is true, we divide the proof process into three cases. Consider the case $p \in (2, 4)$. Similar to the argument about (2.4), it follows from (f_2) and (2.2) that

$$f(t)t \geqslant Mt^4 - C|t|^p, \quad t \in \mathbb{R}. \tag{2.6}$$

From (2.6), we get

$$\int_{\mathbb{R}^3} Kf(tu)(tu) \geqslant Mt^4 \int_{\mathbb{R}^3} Ku^4 - C|t|^p \int_{\mathbb{R}^3} K|u|^p. \tag{2.7}$$

Dividing by t^4 and letting $|t| \to \infty$, we can also deduce (2.1). In the following, we still need to illustrate that $\int_{\mathbb{R}^3} Ku^4 < \infty$ and $\int_{\mathbb{R}^3} K|u|^p < \infty$. Indeed, by (H_3), we know that $E \hookrightarrow L_K^p(\mathbb{R}^3)$ is compact. Hence there is $C' > 0$ such that $\int_{\mathbb{R}^3} K|u|^p \leqslant C'\|u\|_E^p < \infty$. By the interpolation inequality and (1.5), there exists some $0 < \lambda < 1$ such that

$$\begin{aligned}
\int_{\mathbb{R}^3} Ku^4 &= \int_{\mathbb{R}^3} (K|u|^p)^{1-\lambda}(Ku^6)^\lambda \\
&\leqslant \left(\int_{\mathbb{R}^3} K|u|^p\right)^{1-\lambda} \left(\int_{\mathbb{R}^3} Ku^6\right)^\lambda \\
&\leqslant (C'\|u\|_E^p)^{1-\lambda}(|K|_\infty S'^{-3}\|u\|_E^6)^\lambda < \infty.
\end{aligned}$$

We now consider the case $p = 4$. By the condition (f_2), we get

$$\lim_{t\to 0^\pm} \frac{f(t)}{t^3} = \lim_{t\to 0^\pm} \frac{f(t)}{|t|^{p-1}} \frac{|t|^{p-1}}{t^3} = \pm A. \tag{2.8}$$

Thus, from (f_5) and (2.8), there exists $C > 0$ such that

$$\frac{f(t)}{t^3} \geqslant -C, \quad t \in \mathbb{R}, \tag{2.9}$$

namely, $f(t)/t^3$ is bounded from below. According to Proposition 2.1, $\int_{\mathbb{R}^3} Ku^4 < \infty$. Thus, by (2.2), (2.9) and Fatou's Lemma, it follows that

$$\lim_{|t|\to\infty} \int_{\mathbb{R}^3} \frac{Kf(tu)u}{t^3} = \lim_{|t|\to\infty} \int_{\mathbb{R}^3} \frac{Kf(tu)}{(tu)^3} u^4 = \infty. \tag{2.10}$$

We consider the case $p \in (4, 6)$. By the condition (f_2), we get

$$\lim_{t\to 0^\pm} \frac{f(t)}{t^3} = \lim_{t\to 0^\pm} \frac{f(t)}{|t|^{p-1}} \frac{|t|^{p-1}}{t^3} = 0. \tag{2.11}$$

Thus, it follows from (f_5) and (2.11) that

$$\frac{f(t)}{t^3} \geqslant 0, \quad t \in \mathbb{R}. \tag{2.12}$$

From (2.2), (2.12) and Fatou's Lemma, (2.10) also holds. The proof is completed. □

Lemma 2.5. *Assume that* $(V, K) \in \mathcal{K}$ *and* f *satisfies* (f_1)-(f_5). *Then, for any* $u \in E\backslash\{0\}$,

$$\lim_{|t|\to\infty} \int_{\mathbb{R}^3} \frac{KF(tu)}{t^4} = \infty.$$

Because the proof is analogous to that of Lemma 2.4, we omit it here. In fact, it is sufficient to transform $f(t)t$ into $F(t)$ in the proof process of Lemma 2.4.

Lemma 2.6. *Assume that* $(V, K) \in \mathcal{K}$ *and* f *satisfies* (f_1)-(f_5). *Then, for any* $u \in E\backslash\{0\}$,

$$\lim_{t\to 0} \int_{\mathbb{R}^3} \frac{Kf(tu)u}{t} = 0.$$

Proof. First of all, suppose that (H_2) is true. It follows from (f_1) and (f_3) that for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leqslant \varepsilon|t| + C_\varepsilon|t|^5, \quad t \in \mathbb{R}.$$

It follows from (1.5) that

$$\begin{aligned} \left|\int_{\mathbb{R}^3} Kf(u)u\right| &\leqslant \varepsilon \int_{\mathbb{R}^3} Ku^2 + C_\varepsilon \int_{\mathbb{R}^3} Ku^6 \\ &\leqslant \varepsilon|K/V|_\infty \|u\|_E^2 + C_\varepsilon|K|_\infty S'^{-3}\|u\|_E^6. \end{aligned} \tag{2.13}$$

Next, by assuming that (H_3) is true. From [1], there is a constant $C_p > 0$. For every given $\varepsilon \in (0, C_p)$, there exists $R > 0$ large enough leading to

$$\int_{|x|\geqslant R} K|u|^p \leqslant \varepsilon \int_{|x|\geqslant R} (Vu^2 + u^6), \quad u \in E. \tag{2.14}$$

From (f_2) and (f_3), there are $C_1, C_2 > 0$ such that

$$|f(t)| \leqslant C_1|t|^{p-1} + C_2|t|^5, \quad t \in \mathbb{R}.$$

By (1.5), (2.14) and Hölder's inequality, we obtain

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} Kf(u)u\right| \\ \leqslant\ & C_1 \int_{\mathbb{R}^3} K|u|^p + C_2 \int_{\mathbb{R}^3} Ku^6 \\ \leqslant\ & C_1\varepsilon \int_{|x|\geqslant R} (Vu^2 + u^6) + C_1 \left(\int_{|x|<R} K^{6/(6-p)}\right)^{(6-p)/6} \left(\int_{|x|<R} u^6\right)^{p/6} + C_2|K|_\infty \int_{\mathbb{R}^3} u^6 \\ \leqslant\ & C_1 \left(\varepsilon\|u\|_E^2 + S'^{-3}\varepsilon\|u\|_E^6 + |K|_{L^{6/(6-p)}B_R(0)} S'^{-p/2}\|u\|_E^p\right) + C_2|K|_\infty S'^{-3}\|u\|_E^6. \end{aligned} \tag{2.15}$$

Consequently, either (H_2) or (H_3) holds, there exist $C_3, C_4, C_5 > 0$ such that

$$\left|\int_{\mathbb{R}^3} Kf(tu)tu\right| \leqslant C_3\varepsilon t^2\|u\|_E^2 + C_4 t^6\|u\|_E^6 + C_5|t|^p\|u\|_E^p.$$

Dividing by t^2 and letting $t \to 0$, we deduce that

$$\lim_{t\to 0} \int_{\mathbb{R}^3} \frac{Kf(tu)u}{t} = 0.$$

The proof is completed. $\square$

3 Technical Lemmas

In this section, we prove some technical lemmas related to the existence of sign-changing solutions of (1.3). To this end, for $u \in B$ with $u^{\pm} \neq 0$, we define $G_u : \mathbb{R}_+^2 \to \mathbb{R}$ by $G_u(t,s) = J(tu^+ + su^-)$.

Lemma 3.1. *Assume that $(V,K) \in \mathcal{K}$ and f satisfies (f$_1$)-(f$_5$). Then,*

(i) The pair (t,s) is a critical point of G_u with $t,s>0$ if and only if $tu^+ + su^- \in \mathcal{M}$.

(ii) The map G_u has a unique critical point (t_+, s_-), with $t_+ = t_+(u) > 0$ and $s_- = s_-(u) > 0$, which is the unique maximum point of G_u.

Proof. Since

$$\begin{aligned}\nabla G_u(t,s) &= (\langle J'(tu^+ + su^-), u^+\rangle, \langle J'(tu^+ + su^-), u^-\rangle) \\ &= \left(\frac{1}{t}\langle J'(tu^+ + su^-), tu^+\rangle, \frac{1}{s}\langle J'(tu^+ + su^-), su^-\rangle\right) \\ &:= \left(\frac{1}{t} g_u(t,s), \frac{1}{s} h_u(t,s)\right),\end{aligned}$$

where

$$g_u(t,s) = t^2\|u^+\|_E^2 + t^4 \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + t^2 s^2 \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 - \int_{\mathbb{R}^3} Kf(tu^+)(tu^+), \tag{3.1}$$

$$h_u(t,s) = s^2\|u^-\|_E^2 + s^4 \int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2 + t^2 s^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 - \int_{\mathbb{R}^3} Kf(su^-)(su^-), \tag{3.2}$$

we can prove the item (i).

By (1.4) and Fubini theorem, we see that

$$\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 = \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2.$$

Now we prove (ii). First we prove the existence of a critical point of G_u, namely, $\mathcal{M} \neq \emptyset$. For $u \in B$ with $u^{\pm} \neq 0$ and $s_0 \geqslant 0$ fixed. By (3.1) and Lemma 2.6, we obtain

$$g_u(t,s_0) = t^2\left(\|u^+\|_E^2 + t^2\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + s_0^2 \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 - \int_{\mathbb{R}^3} \frac{Kf(tu^+)u^+}{t}\right)$$

and that $g_u(t,s_0) > 0$ for t small enough. At the same time, by (3.1) and Lemma 2.4, we get

$$g_u(t,s_0) = t^4\left(\frac{1}{t^2}\|u^+\|_E^2 + \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + \frac{s_0^2}{t^2} \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 - \int_{\mathbb{R}^3} \frac{Kf(tu^+)u^+}{t^3}\right)$$

and that $g_u(t,s_0) < 0$ for t large enough. From continuity of $g_u(t,s_0)$, there is a $t_0 > 0$ such that $g_u(t_0,s_0) = 0$. We claim t_0 is unique. In fact, supposing by contradiction there are $0 < t_1 < t_2$ such that $g_u(t_1,s_0) = g_u(t_2,s_0) = 0$. Then

$$\frac{\|u^+\|_E^2}{t_1^2} + \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + \frac{s_0^2}{t_1^2}\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 = \int_{\mathbb{R}^3} \frac{Kf(t_1u^+)}{(t_1u^+)^3}(u^+)^4$$

and this identity is also true if t_2 is substituted for t_1. Hence,

$$\left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right)\left(\|u^+\|_E^2 + s_0^2\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2\right) = \int_{\mathbb{R}^3} K\left[\frac{f(t_1u^+)}{(t_1u^+)^3} - \frac{f(t_2u^+)}{(t_2u^+)^3}\right](u^+)^4,$$

which is absurd in view of (f$_5$) and $0 < t_1 < t_2$. Therefore, there exists a unique $t_0 > 0$ such that $g_u(t_0,s_0) = 0$. Let $\varphi_1(s) := t(s)$, where $t(s)$ satisfies the properties as it mentioned before with s in the place of s_0. Then, the map $\varphi_1 : \mathbb{R}_+ \to (0,\infty)$ is well defined. According to the definition of $\partial G_u/\partial t$, we have

$$\frac{\partial G_u}{\partial t}(\varphi_1(s), s) = 0, \quad s \geqslant 0,$$

that is,

$$\varphi_1(s)\|u^+\|_E^2 + \varphi_1^3(s)\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2 + \varphi_1(s)s^2\int_{\mathbb{R}^3}\phi_{u^-}(u^+)^2 = \int_{\mathbb{R}^3} Kf(\varphi_1(s)u^+)u^+, \quad s \geqslant 0. \tag{3.3}$$

In the following, we prove some properties of function φ_1.

(a) φ_1 is continuous. In fact, let $s_n \to s_0$ as $n \to \infty$. We firstly prove that $\{\varphi_1(s_n)\}$ is bounded. Suppose, by contradiction, that there is a subsequence, still denoted by $\{s_n\}$, such that $\varphi_1(s_n) \to \infty$ as $n \to \infty$. So, for n large, we have $\varphi_1(s_n) \geqslant s_n$. From (3.3), we get,

$$\frac{\|u^+\|_E^2}{\varphi_1^2(s_n)} + \int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2 + \frac{s_n^2}{\varphi_1^2(s_n)}\int_{\mathbb{R}^3}\phi_{u^-}(u^+)^2 = \int_{\mathbb{R}^3} K\frac{f(\varphi_1(s_n)u^+)}{(\varphi_1(s_n)u^+)^3}(u^+)^4. \tag{3.4}$$

From Lemma 2.4, passing to the limit as $n \to \infty$, we have a contradiction. So $\{\varphi_1(s_n)\}$ is bounded. Therefore, there exists a $t_0 \geqslant 0$ such that, passing to a subsequence,

$$\varphi_1(s_n) \to t_0. \tag{3.5}$$

Passing to the limit as $n \to \infty$ in (3.3) with $s = s_n$ and using (3.5), we obtain

$$t_0\|u^+\|_E^2 + t_0^3\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2 + t_0 s_0^2\int_{\mathbb{R}^3}\phi_{u^-}(u^+)^2 = \int_{\mathbb{R}^3} Kf(t_0u^+)u^+.$$

The last equality shows

$$\frac{\partial G_u}{\partial t}(t_0, s_0) = 0.$$

As a result, $t_0 = \varphi_1(s_0)$ implies that φ_1 is continuous.

(b) $\varphi_1(s)$ is bounded below form 0. Suppose that there exists a sequence $\{s_n\}$ such that $\varphi_1(s_n) \to 0^+$ as $n \to \infty$. Then by (3.3) and Lemma 2.6, we have

$$\|u^+\|_E^2 \leqslant \lim_{n\to\infty}\int_{\mathbb{R}^3} K\frac{f(\varphi_1(s_n)u^+)u^+}{\varphi_1(s_n)} = 0.$$

This is absurd. Hence, there exists $\tilde{C} > 0$ such that $\varphi_1(s) \geqslant \tilde{C}$.

(c) $\varphi_1(s) < s$ for s large. In fact, if there exists a sequence $\{s_n\}$ with $s_n \to \infty$ such that $\varphi_1(s_n) \geqslant s_n$ for all $n \in \mathbb{N}$, then it follows from (3.3) that (3.4) holds. This is a contradiction to Lemma 2.4. Thus, $\varphi_1(s) < s$ for s large.

Analogously, for $h_u(t, s)$, we can define a map $\varphi_2 : \mathbb{R}_+ \to (0, \infty)$ by $\varphi_2(t) = s(t)$ satisfying (a), (b) and (c).

By (c), there exists $C_1 > 0$ such that $\varphi_1(s) \leqslant s$ and $\varphi_2(t) \leqslant t$ respectively when $t, s > C_1$. Let

$$C_2 = \max\left\{\max_{s\in[0,C_1]}\varphi_1(s), \max_{t\in[0,C_1]}\varphi_2(t)\right\}.$$

Let $C = \max\{C_1, C_2\}$. We define $T : [0, C] \times [0, C] \to \mathbb{R}_+^2$ by $T(t, s) = (\varphi_1(s), \varphi_2(t))$. Now we show $T(t, s) \in [0, C] \times [0, C]$ for all $(t, s) \in [0, C] \times [0, C]$. In fact,

$$\begin{cases} \varphi_2(t) \leqslant t \leqslant C, & t > C_1, \\ \varphi_2(t) \leqslant \max_{t\in[0,C_1]}\varphi_2(t) \leqslant C_2, & t \leqslant C_1, \end{cases}$$

that is to say $\varphi_2(t) \leqslant C$. Similarly, we have $\varphi_1(s) \leqslant C$. Note that T is continuous. Then, by Brouwer fixed point theorem, there exists $(t_+, s_-) \in [0, C] \times [0, C]$ such that

$$(\varphi_1(s_-), \varphi_2(t_+)) = (t_+, s_-). \tag{3.6}$$

Since $\varphi_i > 0$, (3.6) implies $t_+, s_- > 0$. By the definition, we have

$$\frac{\partial G_u}{\partial t}(t_+, s_-) = \frac{\partial G_u}{\partial s}(t_+, s_-) = 0.$$

Now we need to show the uniqueness of (t_+, s_-). Assuming that $w \in \mathcal{M}$, we have,

$$\begin{aligned} \nabla G_w(1,1) &= \left(\frac{\partial G_w}{\partial t}(1,1), \frac{\partial G_w}{\partial s}(1,1) \right) \\ &= (\langle J'(w^+ + w^-), w^+ \rangle, \langle J'(w^+ + w^-), w^- \rangle) = (0,0), \end{aligned}$$

which indicates that $(1,1)$ is a critical point of G_w. In the following, we prove that $(1,1)$ is the unique critical point of G_w with positive coordinates. Assume that (t_0, s_0) is a critical point of G_w. Without loss of generality, we assume that $0 < t_0 \leqslant s_0$. Then,

$$t_0^2 \|w^+\|_E^2 + t_0^4 \int_{\mathbb{R}^3} \phi_{w^+}(w^+)^2 + t_0^2 s_0^2 \int_{\mathbb{R}^3} \phi_{w^-}(w^+)^2 = \int_{\mathbb{R}^3} K f(t_0 w^+)(t_0 w^+), \tag{3.7}$$

and

$$s_0^2 \|w^-\|_E^2 + s_0^4 \int_{\mathbb{R}^3} \phi_{w^-}(w^-)^2 + t_0^2 s_0^2 \int_{\mathbb{R}^3} \phi_{w^+}(w^-)^2 = \int_{\mathbb{R}^3} K f(s_0 w^-)(s_0 w^-). \tag{3.8}$$

By (3.8) we get

$$s_0^2 \|w^-\|_E^2 + s_0^4 \int_{\mathbb{R}^3} \phi_w(w^-)^2 \geqslant \int_{\mathbb{R}^3} K f(s_0 w^-)(s_0 w^-).$$

Hence,

$$\frac{\|w^-\|_E^2}{s_0^2} + \int_{\mathbb{R}^3} \phi_w(w^-)^2 \geqslant \int_{\mathbb{R}^3} K \frac{f(s_0 w^-)}{(s_0 w^-)^3}(w^-)^4. \tag{3.9}$$

On the other hand, since $w \in \mathcal{M}$, we have

$$\|w^-\|_E^2 + \int_{\mathbb{R}^3} \phi_w(w^-)^2 = \int_{\mathbb{R}^3} K \frac{f(w^-)}{(w^-)^3}(w^-)^4. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\left(\frac{1}{s_0^2} - 1 \right) \|w^-\|_E^2 \geqslant \int_{\mathbb{R}^3} K \left[\frac{f(s_0 w^-)}{(s_0 w^-)^3} - \frac{f(w^-)}{(w^-)^3} \right] (w^-)^4.$$

From the last inequality and (f_5) we conclude that $0 < t_0 \leqslant s_0 \leqslant 1$. Now we prove that $t_0 \geqslant 1$. In fact, from (3.7) and $0 < t_0 \leqslant s_0$, we have

$$\frac{\|w^+\|_E^2}{t_0^2} + \int_{\mathbb{R}^3} \phi_w(w^+)^2 \leqslant \int_{\mathbb{R}^3} K \frac{f(t_0 w^+)}{(t_0 w^+)^3}(w^+)^4. \tag{3.11}$$

For another hand, since $w \in \mathcal{M}$, we have

$$\|w^+\|_E^2 + \int_{\mathbb{R}^3} \phi_w(w^+)^2 = \int_{\mathbb{R}^3} K \frac{f(w^+)}{(w^+)^3}(w^+)^4. \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\left(\frac{1}{t_0^2} - 1 \right) \|w^+\|_E^2 \leqslant \int_{\mathbb{R}^3} K \left[\frac{f(t_0 w^+)}{(t_0 w^+)^3} - \frac{f(w^+)}{(w^+)^3} \right] (w^+)^4.$$

If $t_0 < 1$, the left side of this inequality is positive. But from (f_5), the right side is negative. Thus we must have $t_0 \geqslant 1$. Consequently, $t_0 = s_0 = 1$, which implies that $(1,1)$ is the unique critical point of G_w with positive coordinates. Now, assume that $u \in B, u^\pm \neq 0$ and $(t_1, s_1), (t_2, s_2)$ are the critical points of G_u with positive coordinates. From item (i), we conclude that

$$w_1 = t_1 u^+ + s_1 u^- \in \mathcal{M}, \quad w_2 = t_2 u^+ + s_2 u^- \in \mathcal{M}.$$

Therefore,

$$w_2 = \left(\frac{t_2}{t_1}\right) t_1 u^+ + \left(\frac{s_2}{s_1}\right) s_1 u^- = \left(\frac{t_2}{t_1}\right) w_1^+ + \left(\frac{s_2}{s_1}\right) w_1^- \in \mathcal{M}.$$

Since that $w_1 \in B$ with $w_1^\pm \neq 0$, we have that $\left(\frac{t_2}{t_1}, \frac{s_2}{s_1}\right)$ is a critical point of the map G_{w_1} with positive coordinates. Since $w_1 \in \mathcal{M}$, we have that

$$\frac{t_2}{t_1} = \frac{s_2}{s_1} = 1,$$

which implies that $t_1 = t_2, s_1 = s_2$.

Finally, we prove that the unique critical point is the unique maximum point of G_u. By Lemma 2.5, we deduce that $G_u(t,s) \to -\infty$ as $|(t,s)| \to \infty$. So it is sufficient to check that the maximum point cannot be achieved on the boundary of $\mathbb{R}^2_+$. Without loss of generality, we only prove that $(0,\bar{s})$ is not a maximum point of G_u. In fact, since

$$\begin{aligned} G_u(t,\bar{s}) \quad = \quad & \frac{t^2}{2}\|u^+\|_E^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 - \int_{\mathbb{R}^3} KF(tu^+) + \frac{t^2\bar{s}^2}{2}\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 \\ & + \frac{\bar{s}^2}{2}\|u^-\|_E^2 + \frac{\bar{s}^4}{4}\int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2 - \int_{\mathbb{R}^3} KF(\bar{s}u^-) \end{aligned}$$

is an increasing function with respect to t if t is small enough by Lemma 2.6, the pair $(0,\bar{s})$ is not a maximum point of G_u in $\mathbb{R}^2_+$. □

Lemma 3.2. *Assume that $(V,K) \in \mathcal{K}$ and f satisfies (f_1)-(f_5). If $u \in B$ with $u^\pm \neq 0$ such that $g_u(1,1) \leqslant 0$ and $h_u(1,1) \leqslant 0$, where $g_u(t,s)$, $h_u(t,s)$ are given by* (3.1) *and* (3.2)*, then the unique pair (t_+, s_-) obtained in Lemma 3.1 satisfies $0 < t_+, s_- \leqslant 1$.*

Proof. Without loss of generality, we suppose that $t_+ \geqslant s_- > 0$. Since $t_+u^+ + s_-u^- \in \mathcal{M}$, we have

$$\begin{aligned} & t_+^2\|u^+\|_E^2 + t_+^4\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + t_+^4\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 \\ \geqslant \quad & t_+^2\|u^+\|_E^2 + t_+^4\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + t_+^2 s_-^2\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 \\ = \quad & \int_{\mathbb{R}^3} Kf(t_+u^+)(t_+u^+). \end{aligned} \tag{3.13}$$

The assumption $g_u(1,1) \leqslant 0$ gives that

$$\|u^+\|_E^2 + \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 \leqslant \int_{\mathbb{R}^3} Kf(u^+)u^+. \tag{3.14}$$

Together (3.13) with (3.14), we then get

$$\left(\frac{1}{t_+^2} - 1\right)\|u^+\|_E^2 \geqslant \int_{\mathbb{R}^3} K\left[\frac{f(t_+u^+)}{(t_+u^+)^3} - \frac{f(u^+)}{(u^+)^3}\right](u^+)^4.$$

If $t_+ > 1$, the left side of this inequality is negative. But from (f_5), the right side is positive. Thus we must have $t_+ \leqslant 1$. Then the proof is completed. □

By Lemma 3.1, we may define

$$m = \inf\{J(u) : u \in \mathcal{M}\}. \tag{3.15}$$

Lemma 3.3. *Suppose that $(V,K) \in \mathcal{K}$ and f satisfies (f_1)-(f_5). Then $m > 0$ can be achieved.*

Proof. For every $u \in \mathcal{M}$, we have $\langle J'(u), u\rangle = 0$. Then by (2.13), we get

$$\|u\|_E^2 \leqslant \int_{\mathbb{R}^3}(|\nabla u|^2 + Vu^2) + \int_{\mathbb{R}^3} \phi_u u^2 = \int_{\mathbb{R}^3} Kf(u)u \leqslant \varepsilon|K/V|_\infty\|u\|_E^2 + C_\varepsilon|K|_\infty S'^{-3}\|u\|_E^6. \tag{3.16}$$

Choosing $\varepsilon < 1/|K/V|_\infty$, there exists a constant $\alpha_1 > 0$ such that $\|u\|_E^2 \geqslant \alpha_1$. Similarly, by (2.15), we have

$$\begin{aligned} \|u\|_E^2 &\leqslant \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) + \int_{\mathbb{R}^3} \phi_u u^2 \\ &= \int_{\mathbb{R}^3} Kf(u)u \leqslant C_1\varepsilon\|u\|_E^2 + C_1 S'^{-3}(\varepsilon + C_2|K|_\infty)\|u\|_E^6 + |K|_{L^{6/(6-p)}B_R(0)} S'^{-p/2}\|u\|_E^p. \quad (3.17) \end{aligned}$$

Choosing $\varepsilon < 1/C_1$, there exists a constant $\alpha_2 > 0$ such that $\|u\|_E^2 \geqslant \alpha_2$. In a word, there exists a constant $\alpha = \max\{\alpha_1, \alpha_2\} > 0$ such that $\|u\|_E^2 \geqslant \alpha$. By the condition ($f_5$), we have

$$f'(t)t^2 - 3f(t)t \geqslant 0, \quad t \in \mathbb{R}. \tag{3.18}$$

Therefore

$$H(t) := f(t)t - 4F(t) \geqslant 0, \quad t \in \mathbb{R}, \tag{3.19}$$

and H is increasing when $t > 0$ and decreasing when $t < 0$. Hence,

$$\begin{aligned} J(u) &= J(u) - \frac{1}{4}\langle J'(u), u\rangle \\ &= \frac{1}{4}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(f(u)u - 4F(u)) \\ &\geqslant \frac{1}{4}\alpha. \end{aligned}$$

This implies that $m \geqslant \alpha/4 > 0$.

Let $\{u_n\} \subset \mathcal{M}$ such that $J(u_n) \to m$. Then $\|u_n\|_E \leqslant C$. Hence, we may assume that there exists $u \in E$ such that $u_n \rightharpoonup u$, $u_n^\pm \rightharpoonup u^\pm$ weakly in E. By Proposition 2.2, we know that $\int_{\mathbb{R}^3} KF(u_n) \to \int_{\mathbb{R}^3} KF(u)$. Hence $\{\int_{\mathbb{R}^3} KF(u_n)\}$ is bounded. From (1.6), we get $\frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \frac{1}{2}\|u_n\|_E^2 = J(u_n) + \int_{\mathbb{R}^3} KF(u_n)$, which implies that $\{u_n\}$ is bounded in B. Choosing a subsequence necessarily, by the uniqueness of the convergence, we deduce $u_n^\pm \rightharpoonup u^\pm$ weakly in B. Since $u_n \in \mathcal{M}$, we have $\langle J'(u_n), u_n^\pm\rangle = 0$, that is,

$$\|u_n^\pm\|_E^2 + \int_{\mathbb{R}^3} \phi_{u_n}(u_n^\pm)^2 = \int_{\mathbb{R}^3} Kf(u_n^\pm)u_n^\pm. \tag{3.20}$$

By (3.20) and Proposition 2.2, we get

$$0 < \alpha \leqslant \|u_n^\pm\|_E^2 + \int_{\mathbb{R}^3} \phi_{u_n}(u_n^\pm)^2 = \int_{\mathbb{R}^3} Kf(u_n^\pm)u_n^\pm = \int_{\mathbb{R}^3} Kf(u^\pm)u^\pm + o(1),$$

where $o(1)$ denotes the quantity tending to zero as $n \to \infty$. Thus, $u^\pm \neq 0$. Since $u_n \rightharpoonup u$ weakly in B, from Proposition 2.3, we have $\phi_{u_n} \rightharpoonup \phi_u$ weakly in $D^{1,2}(\mathbb{R}^3)$. By the weakly lower semicontinuity of norm and Fatou's lemma, we have

$$\|u^\pm\|_E^2 + \int_{\mathbb{R}^3} \phi_u(u^\pm)^2 \leqslant \liminf_{n\to\infty} \left[\|u_n^\pm\|_E^2 + \int_{\mathbb{R}^3} \phi_{u_n}(u_n^\pm)^2\right]. \tag{3.21}$$

Then from Proposition 2.2 we get

$$\|u^\pm\|_E^2 + \int_{\mathbb{R}^3} \phi_u(u^\pm)^2 \leqslant \int_{\mathbb{R}^3} Kf(u^\pm)u^\pm. \tag{3.22}$$

From (3.22), Lemmas 3.1 and 3.2, there exists $(\bar{t}, \bar{s}) \in (0,1] \times (0,1]$ such that

$$\bar{u} := \bar{t}u^+ + \bar{s}u^- \in \mathcal{M}.$$

By (3.19), we then have

$$\begin{aligned} m &\leqslant J(\bar{u}) - \frac{1}{4}\langle J'(\bar{u}), \bar{u}\rangle \\ &= \frac{1}{4}\|\bar{u}\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} K[f(\bar{u})\bar{u} - 4F(\bar{u})] \\ &= \frac{1}{4}(\|\bar{t}u^+\|_E^2 + \|\bar{s}u^-\|_E^2) + \frac{1}{4}\int_{\mathbb{R}^3} K[f(\bar{t}u^+)(\bar{t}u^+) - 4F(\bar{t}u^+)] + \frac{1}{4}\int_{\mathbb{R}^3} K[f(\bar{s}u^-)(\bar{s}u^-) - 4F(\bar{s}u^-)] \\ &\leqslant \frac{1}{4}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} K[f(u)u - 4F(u)] \\ &\leqslant \liminf_{n\to\infty}\left[J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle\right] = m. \end{aligned}$$

By the above inequality we deduce that $\bar{t} = \bar{s} = 1$. Thus $\bar{u} = u$ and $J(u) = m$. $\square$

4 Proof of the main result

The main aim of this section is to prove that the minimizer u for (3.15) is indeed a sign-changing solution of (1.3) using the quantitative deformation lemma from [17].

Proof of Theorem 1.1. It is clear that $\langle J'(u), u^+\rangle = 0 = \langle J'(u), u^-\rangle$. It follows from Lemma 3.1 that, for $(t,s) \in \mathbb{R}_+^2$ and $(t,s) \neq (1,1)$,

$$J(tu^+ + su^-) < J(u^+ + u^-) = m. \tag{4.1}$$

Set $\xi_1 = |u^+|_6, \xi_2 = |u^-|_6$ and $\xi = \min\{\xi_1, \xi_2\}$. We denote $\tilde{S}$ the imbedding constant of $B \hookrightarrow L^6(\mathbb{R}^3)$, that is, $|u|_6 \leqslant \tilde{S}\|u\|, u \in B$.

By contradiction, we assume that $J'(u) \neq 0$. Then there exist $r, \lambda > 0$ such that

$$\|J'(v)\| \geqslant \lambda, \quad \|v - u\| \leqslant r. \tag{4.2}$$

Choose $\delta \in (0, \min\{\xi/(2\tilde{S}), r/3\})$ and $\sigma \in (0, \min\{1/2, \delta/(\sqrt{2}\|u\|)\})$. Let $D = (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and $\psi(t,s) := tu^+ + su^-, (t,s) \in D$. It follows from (4.1) that

$$\bar{m} := \max_{\partial D} J \circ \psi < m. \tag{4.3}$$

Let $0 < \varepsilon < \min\{(m - \bar{m})/2, \lambda\delta/8\}$ and $S := \{v \in B, \|v - u\| \leqslant \delta\}$. Then it follows from (4.2) that

$$\|J'(v)\| \geqslant 8\varepsilon/\delta, \quad v \in J^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_{2\delta}. \tag{4.4}$$

Applying (4.4) and [17, Lemma 2.3, p.38], there exists a deformation $\eta \in C([0,1] \times B, B)$ such that

(a) $\eta(1,u) = u$ if $u \notin J^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}$;

(b) $\eta(1, J^{m+\varepsilon} \cap S) \subset J^{m-\varepsilon}$;

(c) $\|\eta(1,u) - u\| \leqslant \delta$ for all $u \in B$.

In the following, we firstly show that

$$\max_{(t,s)\in\bar{D}} J(\eta(1, \psi(t,s))) < m. \tag{4.5}$$

Indeed, by Lemma 3.1, we know $J(\psi(t,s)) \leqslant m < m + \varepsilon$, that is, $\psi(t,s) \in J^{m+\varepsilon}$. Moreover,

$$\begin{aligned} \|\psi(t,s) - u\|^2 &\leqslant 2\left((t-1)^2\|u^+\|^2 + (s-1)^2\|u^-\|^2\right) \\ &\leqslant 2\sigma^2\|u\|^2 \\ &< \delta^2, \end{aligned}$$

which yields that $\psi(t,s) \in S$ for all $(s,t) \in \bar{D}$. Therefore, by (b), we have, $J(\eta(1,\psi(t,s))) < m - \varepsilon$. Hence (4.5) holds.

Next, we claim that

$$\eta(1,\psi(D)) \cap \mathcal{M} \neq \emptyset. \tag{4.6}$$

In fact, let us define $\gamma(t,s) := \eta(1,\psi(t,s))$ and

$$\Phi_0(t,s) = (\langle J'(\psi(t,s)), tu^+\rangle, \langle J'(\psi(t,s)), su^-\rangle) = (\langle J'(tu^+ + su^-), tu^+\rangle, \langle J'(tu^+ + su^-), su^-\rangle),$$

$$\Phi_1(t,s) = (\langle J'(\gamma(t,s)), (\gamma(t,s))^+\rangle, \langle J'(\gamma(t,s)), (\gamma(t,s))^-\rangle).$$

We will utilize the degree theory to prove the result (4.6). From (3.1) and (3.2), by a direct calculation, we have

$$\begin{aligned}
\frac{\partial g_u}{\partial t}(1,1) &= \|u^+\|_E^2 + 3\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 - \int_{\mathbb{R}^3} Kf'(u^+)(u^+)^2, \\
\frac{\partial g_u}{\partial s}(1,1) &= 2\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2, \\
\frac{\partial h_u}{\partial t}(1,1) &= 2\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2, \\
\frac{\partial h_u}{\partial s}(1,1) &= \|u^-\|_E^2 + 3\int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2 + \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 - \int_{\mathbb{R}^3} Kf'(u^-)(u^-)^2.
\end{aligned}$$

Set the matrix

$$M = \begin{bmatrix} \frac{\partial g_u}{\partial t}(1,1) & \frac{\partial g_u}{\partial s}(1,1) \\ \frac{\partial h_u}{\partial t}(1,1) & \frac{\partial h_u}{\partial s}(1,1) \end{bmatrix}.$$

From (3.18), we have

$$\begin{aligned}
\frac{\partial g_u}{\partial t}(1,1) &= \|u^+\|_E^2 + 3\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 - \int_{\mathbb{R}^3} Kf'(u^+)(u^+)^2 \\
&\leqslant \|u^+\|_E^2 + 3\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 + \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 - 3\int_{\mathbb{R}^3} Kf(u^+)u^+ \\
&= -2\|u^+\|_E^2 - 2\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2.
\end{aligned}$$

Similarly,

$$\frac{\partial h_u}{\partial s}(1,1) \leqslant -2\|u^-\|_E^2 - 2\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2.$$

Thus, we conclude that

$$\begin{aligned}
\det M &\geqslant 4\left[\|u^+\|_E^2 + \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2\right]\left[\|u^-\|_E^2 + \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2\right] \\
&\quad -4\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 > 0.
\end{aligned}$$

Since Φ_0 belongs to class C^1 and $(1,1)$ is the unique isolated zero point of Φ_0, we know that

$$\deg(\Phi_0, D, 0) = \mathrm{ind}(\Phi_0,(1,1)) = \mathrm{sgn} J_{\Phi_0}(1,1) = 1.$$

It follows from $\bar{m} < m - 2\varepsilon$, (4.3) and (a) that $\psi = \gamma$ on ∂D. Thus, $\deg(\Phi_0, D, 0) = \deg(\Phi_1, D, 0) = 1$. Hence, there exists a pair $(t_0,s_0) \in D$ such that $\Phi_1(t_0,s_0) = 0$. Since $|u^{\pm}|_6 \geqslant \xi$ and $(t_0,s_0) \in D$, we have $|(\psi(t_0,s_0))^+|_6 = t_0|u^+|_6 \geqslant \xi/2$ and $|(\psi(t_0,s_0))^-|_6 = s_0|u^-|_6 \geqslant \xi/2$. By (c), we get $|\gamma(t_0,s_0) - \psi(t_0,s_0)|_6 \leqslant \tilde{S}\|\gamma(t_0,s_0) - \psi(t_0,s_0)\| \leqslant \tilde{S}\delta$. This implies that $|(\gamma(t_0,s_0))^{\pm} - (\psi(t_0,s_0))^{\pm}|_6 \leqslant |\gamma(t_0,s_0) - \psi(t_0,s_0)|_6 \leqslant \tilde{S}\delta$. Thus we have $|(\gamma(t_0,s_0))^{\pm}|_6 \geqslant |(\psi(t_0,s_0))^{\pm}|_6 - \tilde{S}\delta \geqslant \xi/2 - \tilde{S}\delta > 0$. That is to say $(\gamma(t_0,s_0))^{\pm} \neq 0$. Hence, $\eta(1,\psi(t_0,s_0)) = \gamma(t_0,s_0) \in \mathcal{M}$, which is a contradiction with (4.5). Hence, u is a critical point of J, that is, a sign-changing solution for the problem (1.3).

□

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