J. Math. Anal. Appl. 435 (2016) 1688–1700

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Approaching the solving of constrained variational inequalities via penalty term-based dynamical systems

Radu Ioan Boţ [*,1], Ernö Robert Csetnek [2]

University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

ARTICLE INFO

Article history:
Received 6 March 2015
Available online 29 November 2015
Submitted by A. Daniilidis

Keywords:
Dynamical systems
Lyapunov analysis
Monotone inclusions
Forward–backward algorithm
Penalty schemes

ABSTRACT

We investigate the existence and uniqueness of (locally) absolutely continuous trajectories of a penalty term-based dynamical system associated to a constrained variational inequality expressed as a monotone inclusion problem. Relying on Lyapunov analysis and on the ergodic continuous version of the celebrated Opial Lemma we prove weak ergodic convergence of the orbits to a solution of the constrained variational inequality under investigation. If one of the operators involved satisfies stronger monotonicity properties, then strong convergence of the trajectories can be shown.

1. Introduction and preliminaries

This paper is motivated by the increasing interest in solving constrained variational inequalities expressed as monotone inclusion problems of the form

$$0 \in Ax + N_C(x), \tag{1}$$

where $\mathcal{H}$ is a real Hilbert space, $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $C = \operatorname{argmin} \Psi$ is the set of global minima of the proper, convex and lower semicontinuous function $\Psi : \mathcal{H} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ fulfilling $\min \Psi = 0$ and $N_C : \mathcal{H} \rightrightarrows \mathcal{H}$ is the normal cone of the set $C \subseteq \mathcal{H}$ (see [4–6,8,17,18,26,27]). One can find in the literature iterative schemes based on the forward–backward paradigm for solving (1) (see [5,6,26,27]), that perform in each iteration a proximal step with respect to A and a subgradient step with respect to the penalization function Ψ.

Recently, even more complex structures have been analyzed, like monotone inclusion problems of the form

* Corresponding author.
E-mail addresses: radu.bot@univie.ac.at (R.I. Boţ), ernoe.robert.csetnek@univie.ac.at (E.R. Csetnek).
[1] Research partially supported by DFG (German Research Foundation), project BO 2516/4-1.
[2] Research supported by FWF (Austrian Science Fund), Lise Meitner Programme, project M 1682-N25.

http://dx.doi.org/10.1016/j.jmaa.2015.11.032
0022-247X/

$$0 \in Ax + Dx + N_C(x), \tag{2}$$

where $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $D : \mathcal{H} \to \mathcal{H}$ is a (single-valued) cocoercive operator and $C \subseteq \mathcal{H}$ is the (nonempty) set of zeros of another cocoercive operator $B : \mathcal{H} \to \mathcal{H}$, see [8,17,18].

In this paper we are concerned with addressing monotone inclusion problem (2) from the perspective of dynamical systems. More precisely, we associate to this constrained variational inequality a first-order dynamical system formulated in terms of the resolvent of the maximal monotone operator A, which has as discrete counterparts penalty-type numerical schemes already considered in the literature in the context of solving (2). Let us mention that dynamical systems of similar implicit type have been investigated in [1,3,9,12,19–21].

In the first part of the manuscript we study the existence and uniqueness of (locally) absolutely continuous trajectories generated by the dynamical system, by appealing to arguments based on the Cauchy–Lipschitz–Picard Theorem (see [25,29]). In the second part of the paper we investigate the convergence of the trajectories to a solution of the constrained variational inequality (2). We use as tools Lyapunov analysis combined with the continuous version of the Opial Lemma. Under the fulfillment of a condition expressed in terms of the Fitzpatrick function of the cocoercive operator B we are able to show ergodic weak convergence of the orbits. Moreover, if the operator A is strongly monotone, we can prove even strong (non-ergodic) convergence for the generated trajectories.

For the reader's convenience we present in the following some notations which are used throughout the paper (see [10,14,28]).

Let $\mathcal{H}$ be a real Hilbert space with *inner product* $\langle \cdot, \cdot \rangle$ and associated *norm* $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. The *normal cone* of $S \subseteq \mathcal{H}$ is defined by $N_S(x) = \{u \in \mathcal{H} : \langle y - x, u \rangle \leq 0 \ \forall y \in S\}$, if $x \in S$ and $N_S(x) = \emptyset$ for $x \notin S$. Notice that for $x \in S$, $u \in N_S(x)$ if and only if $\sigma_S(u) = \langle x, u \rangle$, where σ_S is the support function of S, defined by $\sigma_S(u) = \sup_{y \in S} \langle y, u \rangle$.

For an arbitrary set-valued operator $M : \mathcal{H} \rightrightarrows \mathcal{H}$ we denote by $\operatorname{Gr} M = \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Mx\}$ its *graph*, by $\operatorname{dom} M = \{x \in \mathcal{H} : Mx \neq \emptyset\}$ its *domain*, by $\operatorname{ran} M = \{u \in \mathcal{H} : \exists x \in \mathcal{H} \text{ s.t. } u \in Mx\}$ its *range* and $M^{-1} : \mathcal{H} \rightrightarrows \mathcal{H}$ its *inverse operator*, defined by $(u, x) \in \operatorname{Gr} M^{-1}$ if and only if $(x, u) \in \operatorname{Gr} M$. We use also the notation $\operatorname{zer} M = \{x \in \mathcal{H} : 0 \in Mx\}$ for the *set of zeros* of the operator M. We say that M is *monotone* if $\langle x - y, u - v \rangle \geq 0$ for all $(x, u), (y, v) \in \operatorname{Gr} M$. A monotone operator M is said to be *maximally monotone*, if there exists no proper monotone extension of the graph of M on $\mathcal{H} \times \mathcal{H}$. Let us mention that in case M is maximally monotone, $\operatorname{zer} M$ is a convex and closed set [10, Proposition 23.39]. We refer to [10, Section 23.4] for conditions ensuring that $\operatorname{zer} M$ is nonempty. If M is maximally monotone, then one has the following characterization for the set of its zeros:

$$z \in \operatorname{zer} M \text{ if and only if } \langle u - z, w \rangle \geq 0 \text{ for all } (u, w) \in \operatorname{Gr} M. \tag{3}$$

The operator M is said to be γ-*strongly monotone* with $\gamma > 0$, if $\langle x - y, u - v \rangle \geq \gamma \|x - y\|^2$ for all $(x, u), (y, v) \in \operatorname{Gr} M$. Notice that if M is maximally monotone and strongly monotone, then $\operatorname{zer} M$ is a singleton, thus nonempty (see [10, Corollary 23.37]).

The *resolvent* of M, $J_M : \mathcal{H} \rightrightarrows \mathcal{H}$, is defined by $J_M = (\operatorname{Id} + M)^{-1}$, where $\operatorname{Id} : \mathcal{H} \to \mathcal{H}$, $\operatorname{Id}(x) = x$ for all $x \in \mathcal{H}$, is the *identity operator* on $\mathcal{H}$. Moreover, if M is maximally monotone, then $J_M : \mathcal{H} \to \mathcal{H}$ is single-valued and maximally monotone (cf. [10, Proposition 23.7 and Corollary 23.10]). We will also use the *Yosida approximation* of the operator M, which is defined by $M_\alpha = \frac{1}{\alpha}(\operatorname{Id} - J_{\alpha M})$, for $\alpha > 0$.

The *Fitzpatrick function* associated to a monotone operator M, defined as

$$\varphi_M : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}}, \ \varphi_M(x, u) = \sup_{(y,v) \in \operatorname{Gr} M} \{\langle x, v \rangle + \langle y, u \rangle - \langle y, v \rangle\},$$

is a convex and lower semicontinuous function and it will play an important role throughout the paper. Introduced by Fitzpatrick in [24], this notion opened the gate towards the employment of convex analysis

specific tools when investigating the maximality of monotone operators (see [10,11,13–16,23,28] and the references therein). In case M is maximally monotone, φ_M is proper and it fulfills

$$\varphi_M(x,u) \geq \langle x,u\rangle \ \forall (x,u) \in \mathcal{H}\times\mathcal{H},$$

with equality if and only if $(x,u) \in \operatorname{Gr} M$. We refer the reader to [11], for formulae of the corresponding Fitzpatrick functions computed for particular classes of monotone operators.

Let $\gamma > 0$ be arbitrary. A single-valued operator $M : \mathcal{H} \to \mathcal{H}$ is said to be γ-*cocoercive*, if $\langle x-y, Mx-My\rangle \geq \gamma\|Mx-My\|^2$ for all $(x,y) \in \mathcal{H}\times\mathcal{H}$, and γ-*Lipschitz continuous*, if $\|Mx-My\| \leq \gamma\|x-y\|$ for all $(x,y) \in \mathcal{H}\times\mathcal{H}$.

In this paper we are concerned with the solving of the following constrained variational inequality expressed as monotone inclusion problem (see also [17]).

Problem 1. Let $\mathcal{H}$ be a real Hilbert space, $A : \mathcal{H} \rightrightarrows \mathcal{H}$ a maximally monotone operator, $D : \mathcal{H} \to \mathcal{H}$ an η-cocoercive operator with $\eta > 0$, $B : \mathcal{H} \to \mathcal{H}$ a μ-cocoercive operator with $\mu > 0$ and suppose that $C = \operatorname{zer} B \neq \emptyset$. The monotone inclusion problem to solve is

$$0 \in Ax + Dx + N_C(x).$$

Let us mention that a (discrete) iterative scheme for solving this problem has been proposed and investigated in [6] for D taken as zero operator and B as the gradient of a convex and differentiable function with Lipschitz continuous gradient.

2. A penalty term-based dynamical system

We associate to Problem 1 the following dynamical system:

$$\begin{cases} \dot{x}(t) + x(t) = J_{\lambda(t)A}\Big(x(t) - \lambda(t)Dx(t) - \lambda(t)\beta(t)Bx(t)\Big) \\ x(0) = x_0, \end{cases} \tag{4}$$

where $x_0 \in \mathcal{H}$ is fixed and $\lambda, \beta : [0,+\infty) \to (0,+\infty)$.

Remark 2. The dynamical system (4) can be seen as an extension of similar implicit first-order constructions considered in the last years in the literature. For instance, the resulting dynamical system when B is the zero operator and λ is a constant function has been investigated in [19] in connection with approaching the set of zeros of $A+D$. Moreover, the situation when A is the convex subdifferential of a proper, convex and lower semicontinuous function has been addressed in [1], while the even more particular case when this function is the indicator function of a nonempty, convex and closed subset of $\mathcal{H}$ has been considered in [12].

Remark 3. The explicit discretization of (4) with respect to the time variable t, with step size $h_n > 0$, yields for an initial point $x_0 \in \mathcal{H}$ the following iterative scheme

$$\frac{x_{n+1}-x_n}{h_n} + x_n = J_{\lambda_n A}\Big(x_n - \lambda_n Dx_n - \lambda_n\beta_n Bx_n\Big) \ \forall n \geq 0,$$

which for $h_n = 1$ becomes

$$x_{n+1} = J_{\lambda_n A}\Big(x_n - \lambda_n Dx_n - \lambda_n\beta_n Bx_n\Big) \ \forall n \geq 0, \tag{5}$$

where $(\lambda_n)_{n\geq 0}$, $(\beta_n)_{n\geq 0}$ are sequences of positive real numbers.

Let us mention that a convergence analysis for (5) has been carried out in [17]. There we have shown that, if

$$\begin{cases} (i)\ A + N_C \text{ is maximally monotone and } \operatorname{zer}(A + D + N_C) \neq \emptyset; \\ (ii)\ \text{For every } p \in \operatorname{ran} N_C, \sum_{n\in\mathbb{N}} \lambda_n\beta_n \left[\sup_{u\in C} \varphi_B\left(u, \frac{p}{\beta_n}\right) - \sigma_C\left(\frac{p}{\beta_n}\right)\right] < +\infty; \\ (iii)\ (\lambda_n)_{n\geq 0} \in \ell^2 \setminus \ell^1 \end{cases}$$

and $\limsup_{n\to+\infty} \lambda_n\beta_n < 2\mu$, then $(z_n)_{n\geq 1}$ converges weakly to an element in $\operatorname{zer}(A+D+N_C)$ as $n \to +\infty$, where

$$z_n = \frac{1}{\tau_n}\sum_{k=1}^{n} \lambda_k x_k, \text{ and } \tau_n = \sum_{k=1}^{n} \lambda_k \ \forall n \geq 1.$$

If, additionally, A is strongly monotone, then $(x_n)_{n\geq 0}$ converges strongly to the unique element in $\operatorname{zer}(A + D + N_C)$ as $n \to +\infty$.

The situation when D is the zero operator has been addressed in [6] under the supplementary assumption that B is the gradient of a convex and differentiable function with Lipschitz continuous gradient. Other penalty-type iterative schemes have been considered in the context of solving monotone inclusion problems and convex optimization problems in [5,8,18,26,27].

As in [2,7], we consider the following definition of an absolutely continuous function.

Definition 1. A function $f : [0, b] \to \mathcal{H}$ (where $b > 0$) is said to be absolutely continuous if one of the following equivalent properties holds:

(i) there exists an integrable function $g : [0, b] \to \mathcal{H}$ such that

$$f(t) = f(0) + \int_0^t g(s)ds \quad \forall t \in [0, b];$$

(ii) f is continuous and its distributional derivative is Lebesgue integrable on $[0, b]$;

(iii) for every $\varepsilon > 0$, there exists $\eta > 0$ such that for any finite family of intervals $I_k = (a_k, b_k)$ we have the implication:

$$\left(I_k \cap I_j = \emptyset \text{ and } \sum_k |b_k - a_k| < \eta\right) \Longrightarrow \sum_k \|f(b_k) - f(a_k)\| < \varepsilon.$$

Remark 4.

(a) It follows from the above definition that an absolutely continuous function is differentiable almost everywhere, its derivative coincides with its distributional derivative almost everywhere and one can recover the function from its derivative $\dot{f} = g$ by the integration formula (i).

(b) If $f : [0, b] \to \mathcal{H}$ (where $b > 0$) is absolutely continuous and $B : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous (where $L \geq 0$), then the function $h = B \circ f$ is absolutely continuous, too. This can be easily verified by considering the characterization in Definition 1(iii). Moreover, h is almost everywhere differentiable and the inequality $\|\dot{h}(\cdot)\| \leq L\|\dot{f}(\cdot)\|$ holds almost everywhere.

Definition 2. We say that $x : [0, +\infty) \to \mathcal{H}$ is a strong global solution of (4) if the following properties are satisfied:

(i) $x : [0, +\infty) \to \mathcal{H}$ is absolutely continuous on each interval $[0, b]$, $0 < b < +\infty$;
(ii) $\dot{x}(t) + x(t) = J_{\lambda(t)A}(x(t) - \lambda(t)Dx(t) - \lambda(t)\beta(t)Bx(t))$ for almost every $t \in [0, +\infty)$;
(iii) $x(0) = x_0$.

In what follows we discuss the existence and uniqueness of strong global solutions of (4). To this end we use the Cauchy–Lipschitz theorem for absolutely continuous trajectories (see for example [25, Proposition 6.2.1], [29, Theorem 54]). To this end we will make use of the following Lipschitz property of the resolvent operator as a function of the step size, which actually is a consequence of the classical results [22, Proposition 2.6] and [10, Proposition 23.28]; see also [2, Proposition 3.1].

Proposition 5. *Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator, $x \in \mathcal{H}$ and $0 < \delta < +\infty$. Then the mapping $\tau \mapsto J_{\tau A}x$ is Lipschitz continuous on $[\delta, +\infty)$. More precisely, for any $\lambda, \mu \in [\delta, +\infty)$ the following inequality holds:*

$$\|J_{\lambda A}x - J_{\mu A}x\| \leq |\lambda - \mu| \|A_\delta x\|. \tag{6}$$

For proving the existence of strong global solutions of (4), we need the following natural assumption:

(H1) $\lambda, \beta : [0, +\infty) \to (0, +\infty)$ are continuous on each interval $[0, b]$, for $0 < b < +\infty$.

Notice that the dynamical system (4) can be written as

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0, \end{cases} \tag{7}$$

where $f : [0, +\infty) \times \mathcal{H} \to \mathcal{H}$ is defined by

$$f(t, x) = \left[J_{\lambda(t)A} \circ \left(\mathrm{Id} - \lambda(t)D - \lambda(t)\beta(t)B\right) - \mathrm{Id}\right] x. \tag{8}$$

(a) We claim that for every $t \geq 0$ and every $x, y \in \mathcal{H}$ we have

$$\|f(t, x) - f(t, y)\| \leq \left(2 + \frac{\lambda(t)}{\eta} + \frac{\lambda(t)\beta(t)}{\mu}\right) \|x - y\|. \tag{9}$$

Indeed, since the resolvent operator is nonexpansive (see [10, Corollary 23.10 and Definition 4.1]), D is $(1/\eta)$-Lipschitz continuous and B is $(1/\mu)$-Lipschitz continuous, it holds

$$\begin{aligned} \|f(t, x) - f(t, y)\| \leq & \left\|J_{\lambda(t)A}\left(x - \lambda(t)Dx - \lambda(t)\beta(t)Bx\right) - J_{\lambda(t)A}\left(y - \lambda(t)Dy - \lambda(t)\beta(t)By\right)\right\| \\ & + \|x - y\| \\ \leq & \lambda(t)\|Dx - Dy\| + \lambda(t)\beta(t)\|Bx - By\| + 2\|x - y\| \\ \leq & \left(2 + \frac{\lambda(t)}{\eta} + \frac{\lambda(t)\beta(t)}{\mu}\right) \|x - y\|, \end{aligned}$$

hence (9) holds. Further, notice that due to (H1),

$$L_f : [0, +\infty) \to \mathbb{R}, L_f(t) = 2 + \frac{\lambda(t)}{\eta} + \frac{\lambda(t)\beta(t)}{\mu},$$

which is for every $t \geq 0$ equal to the Lipschitz-constant of $f(t,\cdot)$, satisfies

$$L_f(\cdot) \in L^1([0,b]) \text{ for any } 0 < b < +\infty.$$

(b) We show now that

$$\forall x \in \mathcal{H},\ \forall b > 0, \quad f(\cdot, x) \in L^1([0,b], \mathcal{H}). \tag{10}$$

Let us fix $x \in \mathcal{H}$ and $b > 0$. Due to (H1), there exist $\lambda_{min}, \beta_{min} > 0$ such that

$$0 < \lambda_{min} \leq \lambda(t) \text{ and } 0 < \beta_{min} \leq \beta(t)\ \forall t \in [0,b].$$

We obtain for all $t \in [0,b]$ the following chain of inequalities:

$$\begin{aligned}
\|f(t,x)\| &\leq \|x\| + \left\|J_{\lambda(t)A}\big(x - \lambda(t)Dx - \lambda(t)\beta(t)Bx\big)\right\| \\
&\leq \|x\| + \|x - \lambda(t)Dx - \lambda(t)\beta(t)Bx - x + \lambda_{min}Dx + \lambda_{min}\beta_{min}Bx\| \\
&\quad + \left\|J_{\lambda(t)A}\big(x - \lambda_{min}Dx - \lambda_{min}\beta_{min}Bx\big)\right\| \\
&\leq \|x\| + (\lambda(t) - \lambda_{min})\|Dx\| + (\lambda(t)\beta(t) - \lambda_{min}\beta_{min})\|Bx\| \\
&\quad + (\lambda(t) - \lambda_{min})\|A_{\lambda_{min}}(x - \lambda_{min}Dx - \lambda_{min}\beta_{min}Bx)\| \\
&\quad + \left\|J_{\lambda_{min}A}\big(x - \lambda_{min}Dx - \lambda_{min}\beta_{min}Bx\big)\right\|,
\end{aligned}$$

where in the second inequality we used the nonexpansiveness of the resolvent operator and in the third one the statement of Proposition 5. The claim (10) follows now easily by integrating and by taking into account (H1).

In the light of the statements proven in (a) and (b), the existence and uniqueness of a strong global solution of the dynamical system (4) follow from [25, Proposition 6.2.1] (see also [29, Theorem 54]).

3. Convergence of the generated trajectories

In this section we investigate the convergence properties of the trajectories generated by the dynamical system (4). Our analysis relies on Lyapunov analysis combined with the continuous ergodic version of the Opial Lemma.

We split the proof of the convergence into several lemmas.

Lemma 6. *Consider the setting of Problem 1 and the associated dynamical system (4) under the assumption that* (H1) *holds. Take* $(z,w) \in \mathrm{Gr}(A + D + N_C)$ *such that* $w = v + p + Dz$, *where* $v \in Az$ *and* $p \in N_C(z)$. *Then the following inequality holds for almost every* $t \geq 0$

$$\begin{aligned}
&\frac{d}{dt}\|x(t) - z\|^2 + \lambda(t)(2\eta - 3\lambda(t))\|Dx(t) - Dz\|^2 \leq \\
&2\lambda(t)\beta(t)\left[\sup_{u \in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] + 3\lambda^2(t)\beta^2(t)\|Bx(t)\|^2 + 3\lambda^2(t)\|Dz + v\|^2 + \\
&2\lambda(t)\langle z - x(t), w\rangle.
\end{aligned} \tag{11}$$

Proof. From the definition of the resolvent we have for almost every $t \geq 0$

$$-\frac{1}{\lambda(t)}\dot{x}(t) - Dx(t) - \beta(t)Bx(t) \in A(\dot{x}(t) + x(t)),$$

which combined with $v \in Az$ and the monotonicity of A gives

$$\langle \dot{x}(t) + x(t) - z, \dot{x}(t)\rangle \leq \lambda(t)\langle z - \dot{x}(t) - x(t), \beta(t)Bx(t) + Dx(t) + v\rangle.$$

From here it follows that for almost every $t \geq 0$

$$\begin{aligned}
\frac{d}{dt}\|x(t) - z\|^2 &= 2\langle \dot{x}(t) + x(t) - z, \dot{x}(t)\rangle - 2\|\dot{x}(t)\|^2 \\
&\leq 2\lambda(t)\langle z - \dot{x}(t) - x(t), \beta(t)Bx(t) + Dx(t) + v\rangle - 2\|\dot{x}(t)\|^2 \\
&= 2\lambda(t)\langle z - x(t), \beta(t)Bx(t) + Dx(t) + v\rangle \\
&\quad + 2\lambda(t)\langle -\dot{x}(t), \beta(t)Bx(t) + Dx(t) + v\rangle - 2\|\dot{x}(t)\|^2 \\
&\leq 2\lambda(t)\langle z - x(t), \beta(t)Bx(t) + Dx(t) + v\rangle + \lambda^2(t)\|\beta(t)Bx(t) + Dx(t) + v\|^2 \\
&\leq 2\lambda(t)\langle z - x(t), \beta(t)Bx(t) + Dx(t) + v\rangle + 3\lambda^2(t)\beta^2(t)\|Bx(t)\|^2 \\
&\quad + 3\lambda^2(t)\|Dz + v\|^2 + 3\lambda^2(t)\|Dx(t) - Dz\|^2.
\end{aligned}$$

It remains to evaluate the first term on the right-hand side of the last of the above inequalities. By noticing that $v = w - Dz - p$, from the cocoercivity of D, the definition of the Fitzpatrick function and using that $\sigma_C\left(\frac{p}{\beta(t)}\right) = \left\langle z, \frac{p}{\beta(t)}\right\rangle$, we obtain for almost every $t \geq 0$

$$\begin{aligned}
&2\lambda(t)\langle z - x(t), \beta(t)Bx(t) + Dx(t) + w - Dz - p\rangle = \\
&2\lambda(t)\langle z - x(t), Dx(t) - Dz\rangle + 2\lambda(t)\langle z - x(t), \beta(t)Bx(t) - p\rangle + 2\lambda(t)\langle z - x(t), w\rangle = \\
&2\lambda(t)\langle z - x(t), Dx(t) - Dz\rangle + \\
&2\lambda(t)\beta(t)\left[\langle z, Bx(t)\rangle + \left\langle x(t), \frac{p}{\beta(t)}\right\rangle - \langle x(t), Bx(t)\rangle - \left\langle z, \frac{p}{\beta(t)}\right\rangle\right] + 2\lambda(t)\langle z - x(t), w\rangle \leq \\
&-2\eta\lambda(t)\|Dx(t) - Dz\|^2 + 2\lambda(t)\beta(t)\left[\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] + 2\lambda(t)\langle z - x(t), w\rangle
\end{aligned}$$

and the desired conclusion follows. □

Lemma 7. *Consider the setting of Problem 1 and the associated dynamical system (4) under the assumption that* (H1) *holds. Let $z \in C \cap \operatorname{dom} A$ and $v \in Az$. Then for every $\varepsilon \geq 0$ and almost every $t \geq 0$ we have*

$$\begin{aligned}
&\frac{d}{dt}\|x(t) - z\|^2 + \frac{1+2\varepsilon}{1+\varepsilon}\|\dot{x}(t)\|^2 + \frac{2\varepsilon}{1+\varepsilon}\lambda(t)\beta(t)\langle x(t) - z, Bx(t)\rangle \leq \\
&\lambda(t)\beta(t)\left((1+\varepsilon)\lambda(t)\beta(t) - \frac{2\mu}{1+\varepsilon}\right)\|Bx(t)\|^2 + 2\lambda(t)\langle z - \dot{x}(t) - x(t), Dx(t) + v\rangle. \qquad (12)
\end{aligned}$$

Proof. As it follows from the first inequality obtained in the proof of Lemma 6, we have for almost every $t \geq 0$

$$\begin{aligned}
\frac{d}{dt}\|x(t) - z\|^2 + 2\|\dot{x}(t)\|^2 &\leq 2\lambda(t)\beta(t)\langle z - x(t), Bx(t)\rangle + 2\lambda(t)\beta(t)\langle -\dot{x}(t), Bx(t)\rangle \\
&\quad + 2\lambda(t)\langle z - \dot{x}(t) - x(t), Dx(t) + v\rangle.
\end{aligned}$$

Since B is μ-cocoercive and $Bz = 0$, we have for almost every $t \geq 0$

$$\langle z - x(t), Bx(t)\rangle \leq -\mu\|Bx(t)\|^2,$$

hence

$$2\lambda(t)\beta(t)\langle z-x(t),Bx(t)\rangle \leq -\frac{2\mu}{1+\varepsilon}\lambda(t)\beta(t)\|Bx(t)\|^2+\frac{2\varepsilon}{1+\varepsilon}\lambda(t)\beta(t)\langle z-x(t),Bx(t)\rangle.$$

The conclusion follows by using that for almost every $t\geq 0$

$$2\lambda(t)\beta(t)\langle -\dot{x}(t),Bx(t)\rangle \leq \frac{1}{1+\varepsilon}\|\dot{x}(t)\|^2+(1+\varepsilon)\lambda^2(t)\beta^2(t)\|Bx(t)\|^2. \qquad \square$$

Lemma 8. *Consider the setting of Problem 1 and the associated dynamical system (4) under the assumption that* (H1) *holds. Moreover, suppose that* $\limsup_{t\to+\infty}\lambda(t)\beta(t)<2\mu$ *and let be* $z\in C\cap \operatorname{dom}A$ *and* $v\in Az$. *Then there exist* $a,b>0$ *and* $t_0>0$ *such that for almost every* $t\geq t_0$ *it holds*

$$\begin{aligned}&\frac{d}{dt}\|x(t)-z\|^2+a\left(\|\dot{x}(t)\|^2+\lambda(t)\beta(t)\langle x(t)-z,Bx(t)\rangle+\lambda(t)\beta(t)\|Bx(t)\|^2\right)\leq\\&\left(b\lambda^2(t)-2\eta\lambda(t)\right)\|Dx(t)-Dz\|^2+2\lambda(t)\langle z-x(t),v+Dz\rangle+b\lambda^2(t)\|Dz+v\|^2.\end{aligned}\tag{13}$$

Proof. Let us evaluate the second term of right-hand side of the inequality (12). The cocoercivity of D yields for almost every $t\geq 0$

$$\begin{aligned}&2\lambda(t)\langle z-\dot{x}(t)-x(t),Dx(t)+v\rangle=\\&2\lambda(t)\langle -\dot{x}(t),Dx(t)+v\rangle+2\lambda(t)\langle z-x(t),Dx(t)-Dz\rangle+2\lambda(t)\langle z-x(t),v+Dz\rangle\leq\\&\frac{\varepsilon}{2(1+\varepsilon)}\|\dot{x}(t)\|^2+\frac{2(1+\varepsilon)}{\varepsilon}\lambda^2(t)\|Dx(t)+v\|^2-\\&2\eta\lambda(t)\|Dx(t)-Dz\|^2+2\lambda(t)\langle z-x(t),v+Dz\rangle\leq\\&\frac{\varepsilon}{2(1+\varepsilon)}\|\dot{x}(t)\|^2+\frac{4(1+\varepsilon)}{\varepsilon}\lambda^2(t)\|Dx(t)-Dz\|^2+\frac{4(1+\varepsilon)}{\varepsilon}\lambda^2(t)\|Dz+v\|^2-\\&2\eta\lambda(t)\|Dx(t)-Dz\|^2+2\lambda(t)\langle z-x(t),v+Dz\rangle.\end{aligned}$$

Combining this inequality with (12) we obtain for almost every $t\geq 0$

$$\begin{aligned}&\frac{d}{dt}\|x(t)-z\|^2+\frac{2+3\varepsilon}{2(1+\varepsilon)}\|\dot{x}(t)\|^2+\frac{2\varepsilon}{1+\varepsilon}\lambda(t)\beta(t)\langle x(t)-z,Bx(t)\rangle+\frac{\varepsilon}{1+\varepsilon}\lambda(t)\beta(t)\|Bx(t)\|^2\leq\\&\lambda(t)\beta(t)\left((1+\varepsilon)\lambda(t)\beta(t)-\frac{2\mu}{1+\varepsilon}+\frac{\varepsilon}{1+\varepsilon}\right)\|Bx(t)\|^2+\\&\left(\frac{4(1+\varepsilon)}{\varepsilon}\lambda^2(t)-2\eta\lambda(t)\right)\|Dx(t)-Dz\|^2+\\&2\lambda(t)\langle z-x(t),Dz+v\rangle+\frac{4(1+\varepsilon)}{\varepsilon}\lambda^2(t)\|Dz+v\|^2.\end{aligned}$$

Further, there exist α and $\varepsilon_0>0$ such that

$$\limsup_{t\to+\infty}\lambda(t)\beta(t)<\alpha<2\mu$$

and

$$(1+\varepsilon_0)\alpha-\frac{2\mu}{1+\varepsilon_0}+\frac{\varepsilon_0}{1+\varepsilon_0}<0.$$

By taking $a:=\frac{\varepsilon_0}{2(1+\varepsilon_0)}$ and $b:=\frac{4(1+\varepsilon_0)}{\varepsilon_0}$ the desired conclusion follows. $\square$

Lemma 9. *Consider the setting of Problem 1 and the associated dynamical system (4) under the assumption that (H1) holds. Moreover, suppose that* $\limsup_{t\to+\infty}\lambda(t)\beta(t) < 2\mu$ *and* $\liminf_{t\to+\infty}\lambda(t) = 0$ *and let be* $(z,w)\in \mathrm{Gr}(A+D+N_C)$ *such that* $w = v+p+Dz$*, where* $v\in Az$ *and* $p\in N_C(z)$*. Then there exist* $a,b>0$ *and* $t_1>0$ *such that for almost every* $t\geq t_1$ *it holds*

$$\frac{d}{dt}\|x(t)-z\|^2 + a\left(\|\dot{x}(t)\|^2 + \frac{\lambda(t)\beta(t)}{2}\langle x(t)-z, Bx(t)\rangle + \lambda(t)\beta(t)\|Bx(t)\|^2\right)\leq$$
$$\frac{a\lambda(t)\beta(t)}{2}\left[\sup_{u\in C}\varphi_B\left(u,\frac{4p}{a\beta(t)}\right) - \sigma_C\left(\frac{4p}{a\beta(t)}\right)\right] + 2\lambda(t)\langle z-x(t), w\rangle + b\lambda^2(t)\|Dz+v\|^2. \tag{14}$$

Proof. According to Lemma 8, there exist $a,b>0$ and $t_0>0$ such that for almost every $t\geq t_0$ the inequality (13) holds. Since $\liminf_{t\to\infty}\lambda(t)=0$, there exists $t_1\geq t_0$ such that $b\lambda^2(t)-2\eta\lambda(t)\leq 0$ for every $t\geq t_1$, hence for almost every $t\geq t_1$

$$\frac{d}{dt}\|x(t)-z\|^2 + a\left(\|\dot{x}(t)\|^2 + \lambda(t)\beta(t)\langle x(t)-z, Bx(t)\rangle + \lambda(t)\beta(t)\|Bx(t)\|^2\right)\leq$$
$$2\lambda(t)\langle z-x(t), v+Dz\rangle + b\lambda^2(t)\|Du+v\|^2.$$

The conclusion follows by combining this inequality with the following one, which holds for almost every $t\geq 0$

$$2\lambda(t)\langle z-x(t), v+Dz\rangle + \frac{a\lambda(t)\beta(t)}{2}\langle z-x(t), Bx(t)\rangle =$$
$$2\lambda(t)\langle z-x(t), -p\rangle + \frac{a\lambda(t)\beta(t)}{2}\langle z-x(t), Bx(t)\rangle + 2\lambda(t)\langle z-x(t), w\rangle =$$
$$\frac{a\lambda(t)\beta(t)}{2}\left(\langle z, Bx(t)\rangle + \left\langle x(t), \frac{4p}{a\beta(t)}\right\rangle - \langle x(t), Bx(t)\rangle - \left\langle z, \frac{4p}{a\beta(t)}\right\rangle\right) + 2\lambda(t)\langle z-x(t), w\rangle \leq$$
$$\frac{a\lambda(t)\beta(t)}{2}\left[\sup_{u\in C}\varphi_B\left(u,\frac{4p}{a\beta(t)}\right) - \sigma_C\left(\frac{4p}{a\beta(t)}\right)\right] + 2\lambda(t)\langle z-x(t), w\rangle. \qquad \square$$

For proving the convergence statement for the trajectories generated by the dynamical system (4) we will make use of the following ergodic version of the continuous Opial Lemma. The proof of this result follows similarly to the one of [4, Lemma 2.3] and therefore we omit it.

Lemma 10. *Let* $S\subseteq\mathcal{H}$ *be a nonempty set,* $x:[0,+\infty)\to\mathcal{H}$ *a given map and* $\lambda:[0,+\infty)\to(0,+\infty)$ *such that* $\int_0^{+\infty}\lambda(t) = +\infty$*. Define* $\tilde{x}:[0,+\infty)\to\mathcal{H}$ *by*

$$\tilde{x}(t) = \frac{1}{\int_0^t\lambda(s)ds}\int_0^t\lambda(s)x(s)ds.$$

Assume that

(i) *for every* $z\in S$, $\lim_{t\to+\infty}\|x(t)-z\|$ *exists;*
(ii) *every weak sequential cluster point of the map* $\tilde{x}$ *belongs to* S*.*

Then there exists $x_\infty\in S$ *such that* $w-\lim_{t\to+\infty}\tilde{x}(t) = x_\infty$*.*

We will prove the convergence results under the following hypotheses, which can be seen as continuous counterparts of the conditions considered in [17] in the discrete case (see also [4,6]):

(H2) $A + N_C$ is maximally monotone and $\operatorname{zer}(A + D + N_C) \neq \emptyset$;
(H3) $\lambda(\cdot) \in L^2([0,+\infty)) \setminus L^1([0,+\infty))$;
(H_{fitz}) For every $p \in \operatorname{ran} N_C$, $\int_0^{+\infty} \lambda(t)\beta(t) \left[\sup_{u\in C} \varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] dt < +\infty$.

Remark 11.

(a) Since A is maximally monotone and C is a nonempty, convex and closed set, $A+N_C$ is maximally monotone, provided that a so-called regularity condition is fulfilled. We refer the reader to [10,13–16,28,30] for conditions guaranteeing the maximal monotonicity of the sum of two maximally monotone operators. Further, as D is maximally monotone (see [10, Example 20.28]) and $\operatorname{dom} D = \mathcal{H}$, (H2) guarantees that $A + D + N_C$ is maximally monotone, too (see [10, Corollary 24.4]).
(b) The condition (H3) is obviously satisfied for the function $\lambda(t) = \frac{1}{t+1}$.
(c) Let us turn now our attention to (H_{fitz}). The discrete version of this condition has been considered for the first time in [17]. We notice that for each $p \in \operatorname{ran} N_C$ we have

$$\sup_{u\in C} \varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right) \geq 0 \ \forall t \geq 0.$$

Indeed, if $p \in \operatorname{ran} N_C$, then there exists $\overline{u} \in C$ such that $p \in N_C(\overline{u})$. This implies that

$$\sup_{u\in C} \varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right) \geq \left\langle \overline{u}, \frac{p}{\beta_n}\right\rangle - \sigma_C\left(\frac{p}{\beta_n}\right) = 0 \ \forall t \geq 0.$$

Let us consider the particular case $B = \nabla\Psi$, where $\Psi : \mathcal{H} \to \mathbb{R}$ is a convex and differentiable function with Lipschitz continuous gradient and satisfies $\min \Psi = 0$. In this case $C = \operatorname{argmin} \Psi$, $\Psi(x) = 0$ for $x \in C$ and it holds (see [11])

$$\varphi_{\nabla\Psi}(x,u) \leq \Psi(x) + \Psi^*(u) \ \forall (x,u) \in \mathcal{H} \times \mathcal{H}, \tag{15}$$

where $\Psi^* : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, $\Psi^*(u) = \sup_{x\in\mathcal{H}}\{\langle u, x\rangle - \Psi(x)\}$, is the Fenchel conjugate of Ψ.
This means that (H_{fitz}) is in this particular case fulfilled, if one has:
(H) For every $p \in \operatorname{ran} N_C$, $\int_0^{+\infty} \lambda(t)\beta(t) \left[\Psi^*\left(\frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] dt < +\infty$.
Let us mention that (H) is the continuous counterpart of a condition used in [6] in the context of proving convergence for penalty-type iterative schemes. It has its origins in the work [4], where a similar condition has been used in the convergence analysis of coupled dynamical systems with multiscale aspects.
Let us present a particular setting in which (H) and, consequently, (H_{fitz}) are fulfilled. This example is inspired by [4, Section 1.3(b)]. Take $\Psi : \mathcal{H} \to \mathbb{R}$, $\Psi(x) = \frac{1}{2}d^2(x,C) = \frac{1}{2}\inf_{y\in C}\|x-y\|^2$. For its conjugate function one gets $\Psi^*(x) = \frac{1}{2}\|x\|^2 + \sigma_C(x)$ $\forall x \in \mathcal{H}$, hence ($H$) reduces to

$$\int_0^{+\infty} \frac{\lambda(t)}{\beta(t)} dt < +\infty,$$

which is obviously fulfilled for $\lambda(t) = \frac{1}{t+1}$ and $\beta(t) = 1 + t$. For other particular instances where (H_{fitz}) (in its continuous or discrete version) holds we refer the reader to [4–6,8,26,27].

Let us state now the main result concerning the convergence of the trajectories generated by the dynamical system (4).

Theorem 12. *Consider the setting of Problem 1 and the associated dynamical system (4). Assume that* $\limsup_{t\to+\infty}\lambda(t)\beta(t)<2\mu$ *and that* (H1)–(H3) *and* (H_{fitz}) *hold. Let* $\tilde{x}:[0,+\infty)\to\mathcal{H}$ *be defined by*

$$\tilde{x}(t)=\frac{1}{\int_0^t\lambda(s)ds}\int_0^t\lambda(s)x(s)ds.$$

Then the following statements are true:

(i) *for every* $z\in\operatorname{zer}(A+D+N_C)$, $\|x(t)-z\|$ *converges as* $t\to+\infty$*; moreover,* $\int_0^{+\infty}\|\dot{x}(t)\|^2dt<+\infty$, $\int_0^{+\infty}\lambda(t)\beta(t)\langle Bx(t),x(t)-z\rangle dt<+\infty$ *and* $\int_0^{+\infty}\lambda(t)\beta(t)\|Bx(t)\|^2dt<+\infty$*;*
(ii) $\tilde{x}(t)$ *converges weakly to an element in* $\operatorname{zer}(A+D+N_C)$ *as* $t\to+\infty$*;*
(iii) *if, additionally, A is strongly monotone, then* $x(t)$ *converges strongly to the unique element in* $\operatorname{zer}(A+D+N_C)$ *as* $t\to+\infty$*.*

Proof. (i) According to (H3), the function $\lambda(\cdot)$ satisfies the relation $\liminf_{t\to\infty}\lambda(t)=0$. Take $z\in\operatorname{zer}(A+D+N_C)$ and $v\in Az$ and $p\in N_C(z)$ fulfilling $0=v+p+Dz$. Applying Lemma 9 for $w=0$, it follows that there exist $a,b>0$ and $t_1>0$ such that for almost every $t\geq t_1$ it holds

$$\frac{d}{dt}\|x(t)-z\|^2+a\left(\|\dot{x}(t)\|^2+\frac{\lambda(t)\beta(t)}{2}\langle x(t)-z,Bx(t)\rangle+\lambda(t)\beta(t)\|Bx(t)\|^2\right)\leq$$
$$\frac{a\lambda(t)\beta(t)}{2}\left[\sup_{u\in C}\varphi_B\left(u,\frac{4p}{a\beta(t)}\right)-\sigma_C\left(\frac{4p}{a\beta(t)}\right)\right]+b\lambda^2(t)\|Dz+v\|^2.$$

Since the function (having as argument t) on the right-hand side of the above inequality belongs to $L^1([0,+\infty))$, by using also [2, Lemma 5.1], the statements follow.

(ii) According to Lemma 10, it is enough to show that every weak sequential cluster limit of $\tilde{x}$ belongs to $\operatorname{zer}(A+D+N_C)$. Let $\overline{x}$ be such a weak sequential cluster limit, that is, there exists a sequence $(t_n)_{n\geq0}\to+\infty$ such that $\tilde{x}(t_n)$ weakly converges to $\overline{x}$ as $n\to+\infty$.

Take an arbitrary $(z,w)\in\operatorname{Gr}(A+D+N_C)$ such that $w=v+p+Dz$, where $v\in Az$ and $p\in N_C(z)$. From Lemma 6 and by using that $\liminf_{t\to+\infty}\lambda(t)=0$ it follows that there exists $t_2>0$ such that for almost every $t\geq t_2$ we have

$$\frac{d}{dt}\|x(t)-z\|^2\leq2\lambda(t)\beta(t)\left[\sup_{u\in C}\varphi_B\left(u,\frac{p}{\beta(t)}\right)-\sigma_C\left(\frac{p}{\beta(t)}\right)\right]$$
$$+3\lambda^2(t)\beta^2(t)\|Bx(t)\|^2+3\lambda^2(t)\|Dz+v\|^2+2\lambda(t)\langle z-x(t),w\rangle.$$

By integrating from t_2 to T, where $T\geq t_2$, we obtain

$$\|x(T)-z\|^2-\|x(t_2)-z\|^2\leq L+2\left\langle\left(\int_{t_2}^T\lambda(t)dt\right)z-\int_{t_2}^T\lambda(t)x(t)dt,w\right\rangle,\tag{16}$$

where, according to the hypotheses and statement (i),

$$L:=2\int_0^{+\infty}\lambda(t)\beta(t)\left[\sup_{u\in C}\varphi_B\left(u,\frac{p}{\beta(t)}\right)-\sigma_C\left(\frac{p}{\beta(t)}\right)\right]dt$$
$$+3\int_0^{+\infty}\lambda^2(t)\beta^2(t)\|Bx(t)\|^2dt+3\|Dz+v\|^2\int_0^{+\infty}\lambda^2(t)dt<+\infty.$$

Now dividing (16) by $\int_0^T \lambda(t)dt$ and discarding the nonnegative term $\|x(T)-z\|^2$, we obtain

$$\frac{-\|x(t_2)-z\|^2}{\int_0^T \lambda(t)dt} \leq \frac{L'}{\int_0^T \lambda(t)dt} + 2\left\langle z - \frac{1}{\int_0^T \lambda(t)dt}\int_0^T \lambda(t)x(t)dt, w\right\rangle, \tag{17}$$

where

$$L' := L + 2\left\langle \left(-\int_0^{t_2}\lambda(t)dt\right)z + \int_0^{t_2}\lambda(t)x(t)dt, w\right\rangle < +\infty.$$

Letting $T := t_n$ in (17) for any $n \geq 0$, passing to $n \to +\infty$ and using (H3) and the definition of $\tilde{x}$, it follows

$$0 \leq 2\langle z - \overline{x}, w\rangle.$$

Since (z,w) was taken arbitrary in $\mathrm{Gr}(A+D+N_C)$, we obtain from (3) that $\overline{x} \in \mathrm{zer}(A+D+N_C)$ and from here the conclusion follows.

(iii) Suppose that A is γ-strongly monotone, where $\gamma > 0$. Let z be the unique element in $\mathrm{zer}(A+D+N_C)$ and $v \in Az$ and $p \in N_C(z)$ such that $0 = v + p + Dz$. Following the lines of the proof of Lemma 6, one can prove that for almost every $t \geq 0$

$$2\gamma\lambda(t)\|\dot{x}(t)+x(t)-z\|^2 + \frac{d}{dt}\|x(t)-z\|^2 + \lambda(t)(2\eta - 3\lambda(t))\|Dx(t)-Dz\|^2 \leq$$
$$2\lambda(t)\beta(t)\left[\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] + 3\lambda^2(t)\beta^2(t)\|Bx(t)\|^2 + 3\lambda^2(t)\|Dz+v\|^2.$$

Since $\liminf_{t\to+\infty}\lambda(t) = 0$, it follows that there exists $t_2 > 0$ such that for almost every $t \geq t_2$

$$2\gamma\lambda(t)\|\dot{x}(t)+x(t)-z\|^2 + \frac{d}{dt}\|x(t)-z\|^2 \leq$$
$$2\lambda(t)\beta(t)\left[\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] + 3\lambda^2(t)\beta^2(t)\|Bx(t)\|^2 + 3\lambda^2(t)\|Dz+v\|^2,$$

thus

$$\gamma\lambda(t)\|x(t)-z\|^2 + \frac{d}{dt}\|x(t)-z\|^2 \leq 2\gamma\lambda(t)\|\dot{x}(t)\|^2 +$$
$$2\lambda(t)\beta(t)\left[\sup_{u\in C}\varphi_B\left(u, \frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] + 3\lambda^2(t)\beta^2(t)\|Bx(t)\|^2 + 3\lambda^2(t)\|Dz+v\|^2.$$

By using the hypotheses and statement (i), after integration of the last inequality one obtains

$$\int_0^{+\infty}\lambda(t)\|x(t)-z\|^2dt < +\infty.$$

Using the convergence of $\|x(t)-z\|$ as $t \to +\infty$ and (H3) it follows that $x(t)$ must converge to z as $t \to +\infty$. □

References

[1] B. Abbas, H. Attouch, Dynamical systems and forward–backward algorithms associated with the sum of a convex subdifferential and a monotone cocoercive operator, Optimization 64 (10) (2015) 2223–2252.
[2] B. Abbas, H. Attouch, B.F. Svaiter, Newton-like dynamics and forward–backward methods for structured monotone inclusions in Hilbert spaces, J. Optim. Theory Appl. 161 (2) (2014) 331–360.
[3] A.S. Antipin, Minimization of convex functions on convex sets by means of differential equations, Differ. Uravn. 30 (9) (1994) 1475–1486 (in Russian); translation in: Differ. Equ. 30 (9) (1994) 1365–1375.
[4] H. Attouch, M.-O. Czarnecki, Asymptotic behavior of coupled dynamical systems with multiscale aspects, J. Differential Equations 248 (6) (2010) 1315–1344.
[5] H. Attouch, M.-O. Czarnecki, J. Peypouquet, Prox-penalization and splitting methods for constrained variational problems, SIAM J. Optim. 21 (1) (2011) 149–173.
[6] H. Attouch, M.-O. Czarnecki, J. Peypouquet, Coupling forward–backward with penalty schemes and parallel splitting for constrained variational inequalities, SIAM J. Optim. 21 (4) (2011) 1251–1274.
[7] H. Attouch, B.F. Svaiter, A continuous dynamical Newton-like approach to solving monotone inclusions, SIAM J. Control Optim. 49 (2) (2011) 574–598.
[8] S. Banert, R.I. Boţ, Backward penalty schemes for monotone inclusion problems, J. Optim. Theory Appl. 166 (3) (2015) 930–948.
[9] S. Banert, R.I. Boţ, A forward–backward–forward differential equation and its asymptotic properties, arXiv:1503.07728, 2015.
[10] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, CMS Books in Mathematics, Springer, New York, 2011.
[11] H.H. Bauschke, D.A. McLaren, H.S. Sendov, Fitzpatrick functions: inequalities, examples and remarks on a problem by S. Fitzpatrick, J. Convex Anal. 13 (3–4) (2006) 499–523.
[12] J. Bolte, Continuous gradient projection method in Hilbert spaces, J. Optim. Theory Appl. 119 (2) (2003) 235–259.
[13] J.M. Borwein, Maximal monotonicity via convex analysis, J. Convex Anal. 13 (3–4) (2006) 561–586.
[14] J.M. Borwein, J.D. Vanderwerff, Convex Functions: Constructions, Characterizations and Counterexamples, Cambridge University Press, Cambridge, 2010.
[15] R.I. Boţ, Conjugate Duality in Convex Optimization, Lecture Notes in Economics and Mathematical Systems, vol. 637, Springer, Berlin, Heidelberg, 2010.
[16] R.I. Boţ, E.R. Csetnek, An application of the bivariate inf-convolution formula to enlargements of monotone operators, Set-Valued Anal. 16 (7–8) (2008) 983–997.
[17] R.I. Boţ, E.R. Csetnek, Forward–backward and Tseng's type penalty schemes for monotone inclusion problems, Set-Valued Var. Anal. 22 (2014) 313–331.
[18] R.I. Boţ, E.R. Csetnek, A Tseng's type penalty scheme for solving inclusion problems involving linearly composed and parallel-sum type monotone operators, Vietnam J. Math. 42 (4) (2014) 451–465.
[19] R.I. Boţ, E.R. Csetnek, A dynamical system associated with the fixed points set of a nonexpansive operator, J. Dynam. Differential Equations (2015), http://dx.doi.org/10.1007/s10884-015-9438-x, in press.
[20] R.I. Boţ, E.R. Csetnek, Second order forward–backward dynamical systems for monotone inclusion problems, arXiv: 1503.04652, 2015.
[21] R.I. Boţ, E.R. Csetnek, Convergence rates for forward–backward dynamical systems associated with strongly monotone inclusions, arXiv:1504.01863, 2015.
[22] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Notas de Matemática (50), North-Holland Mathematics Studies, vol. 5, North-Holland/Elsevier, New York, 1973.
[23] R.S. Burachik, B.F. Svaiter, Maximal monotone operators, convex functions and a special family of enlargements, Set-Valued Anal. 10 (4) (2002) 297–316.
[24] S. Fitzpatrick, Representing monotone operators by convex functions, in: Workshop/Miniconference on Functional Analysis and Optimization, Canberra, 1988, in: Proceedings of the Centre for Mathematical Analysis, vol. 20, Australian National University, Canberra, 1988, pp. 59–65.
[25] A. Haraux, Systèmes Dynamiques Dissipatifs et Applications, Rech. Math. Appl., vol. 17, Masson, Paris, 1991.
[26] N. Noun, J. Peypouquet, Forward–backward penalty scheme for constrained convex minimization without inf-compactness, J. Optim. Theory Appl. 158 (3) (2013) 787–795.
[27] J. Peypouquet, Coupling the gradient method with a general exterior penalization scheme for convex minimization, J. Optim. Theory Appl. 153 (1) (2012) 123–138.
[28] S. Simons, From Hahn–Banach to Monotonicity, Springer, Berlin, 2008.
[29] E.D. Sontag, Mathematical Control Theory. Deterministic Finite-Dimensional Systems, second edition, Texts in Applied Mathematics, vol. 6, Springer-Verlag, New York, 1998.
[30] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.