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Four small limit cycles around a Hopf singular point in 3-dimensional competitive Lotka–Volterra systems

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A B S T R A C T

The 3-dimensional competitive Lotka–Volterra (LV) systems have been studied for more than two decades, and particular attention has been focused on bifurcation of limit cycles. For such a system, Zeeman (1993) identified 33 stable equivalence classes on a carrying simplex, among which only classes 26–31 may have limit cycles. It has been shown that all these 6 classes may possess two limit cycles, and the existence of three limit cycles was claimed in some of these classes. Recently, Gyllenberg and Yan (2009) studied the existence of four limit cycles, three of them are small-amplitude limit cycles due to Hopf bifurcation and one additional limit cycle, enclosing all the three small-amplitude limit cycles, is due to the existence of a heteroclinic cycle, and proposed a new conjecture including: (i) There exists a 3-d competitive LV system with at least 5 limit cycles. (ii) In the case of a heteroclinic cycle on the boundary of the carrying simplex of a 3-d competitive LV system, the vanishing of the first four focus values (the vanishing of the zero-order focus value means that there is a pair of purely imaginary eigenvalues at the positive equilibrium) does not imply that the heteroclinic cycle is neutrally stable, and hence it does not imply that the positive equilibrium is a center. (iii) In the case of a heteroclinic cycle on the boundary of the carrying simplex of a 3-d competitive LV system, the vanishing of the first three focus values and that the heteroclinic cycle is neutrally stable do not imply the vanishing of the third-order focus value, and hence they do not imply that the positive equilibrium is a center. In this paper, we will present two examples belonging to class 27 and another two examples belonging to class 26, which exhibit at least four small-amplitude limit cycles in the vicinity of the positive equilibrium due to Hopf bifurcations, and prove that the items (ii) and (iii) in the conjecture are true. Moreover, showing the existence of four small-amplitude limit cycles is a necessary step towards proving item (i) of the conjecture.

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1. Introduction

Limit cycle is an important phenomenon in dynamical systems, which can occur in almost all fields of science and engineering such as physics, mechanics, electronics, ecology, economy, biology, finance etc. Analysis of limit cycles plays an important role in the study of nonlinear systems since bifurcation of limit cycles can cause complex dynamics in such systems. In recent years, bifurcation of limit cycles has been investigated in many biological systems (e.g. see [2,15]). One well-known example is the 3-dimensional Lotka–Volterra (LV) system. It is known that 2-dimensional LV systems cannot have limit cycles [1]. While for general 3-dimensional LV systems, complicated dynamical behavior such as period doubling route to chaos has been found (e.g., see [6]). On the other hand, for 3-dimensional competitive LV systems, described by

$$\dot{z}_i = z_i\Big(r_i - \sum_{j=1}^{3} a_{ij} z_j\Big), \quad i = 1, 2, 3, \quad r_i > 0, \quad a_{ij} > 0, \tag{1.1}$$

the dynamical possibilities are more restricted. Here, r_i and a_{ij} are constant parameters. In general, the system can have one positive equilibrium and seven boundary equilibria. Hirsch [14] first showed that all non-trivial orbits of system (1.1) approach an invariant two-dimensional manifold, called “carrying simplex”, leading to a Poincaré–Bendixson theorem to be held for system (1.1). Thus, the long-term behavior of system (1.1) is determined by the dynamics on this simplex. Later, Zeeman [28] identified 33 stable equivalence classes in system (1.1), and showed that in 27 of these classes, all the compact limit sets are fixed points, so the dynamical behavior for these cases have been fully described. Further, Hopf bifurcation theory was applied to show that the remaining 6 classes (26–31) can possess isolated periodic orbits or limit cycles, and only one class (class 27) may have heteroclinic cycle. Since then, the open question of how many limit cycles can surround the positive equilibrium has attracted many researchers to investigate. Some results were obtained and a couple of conjectures were posed on the maximal number of limit cycles.

Twenty years ago, Hofbauer and So [16] showed two limit cycles for class 27, one of which is generated by a Hopf bifurcation and the other is guaranteed by the Poincaré–Bendixson theorem due to the existence of a heteroclinic cycle. They proposed a conjecture as described below.

Conjecture 1.1. *(See [16].) For system (1.1), in the case of heteroclinic cycle on the boundary of the carrying simplex, the following conditions are equivalent to having a center:*

(a) There is a pair of purely imaginary eigenvalues at the positive equilibrium.
(b) The first focus value vanishes.
(c) The heteroclinic cycle is neutrally stable.
And further, condition (c) might be replaced by the condition,
(c’) The second focus value vanishes.

Thus, according to this conjecture, the maximal number of the limit cycles that system (1.1) can have might be two.

Later, Xiao and Li [24] proved that the number of limit cycles bifurcating in system (1.1) is finite if the system does not have heteroclinic cycles. Moreover, in this paper, an example is also given to show the existence of two limit cycles for class 27. Similar to the example given in [16] these two limit cycles contain one limit cycle due to a Hopf bifurcation and the other due to the existence of a heteroclinic cycle. Further, two limit cycles were found in [19] for classes 26, 28, 29, and in [9] for classes 30, 31. In 2003, Lu and Luo claimed that they obtained three limit cycles for class 27 [20]. Three years later, three limit cycles were also found for class 29 [11]. Recently, Gyllenberg and Yan constructed examples for class 27 and claimed that they obtained four limit cycles, three of which are due to a Hopf bifurcation and one due to the existence of a heteroclinic cycle [10]. The existence of three and four limit cycles disproved Conjecture 1.1. In [10] the authors proposed a new conjecture as follows.

Conjecture 1.2. *(See [10].)*

(i) *System (1.1) can have at least five limit cycles.*
(ii) *For system (1.1), in the case of heteroclinic cycle existing on the boundary of the carrying simplex, the conditions (a), (b), (c'), and (d) the third focus value vanishes, do not imply (c). In particular, the conditions (a), (b), (c') and (d) do not imply that the positive equilibrium is a center.*
(iii) *For system (1.1), in the case of heteroclinic cycle existing on the boundary of the carrying simplex, the conditions (a), (b), (c), and (c') do not imply (d). In particular, the conditions (a), (b), (c) and (c') do not imply that the positive equilibrium is a center.*

In this paper, we will consider bifurcation of limit cycles around the positive equilibrium of system (1.1), and show that there exist at least four small-amplitude limit cycles near this equilibrium due to a Hopf bifurcation for classes 27 and 26. Although the examples we constructed for class 27 have heteroclinic cycles, we cannot claim one more limit cycle. However, based on the results of four small-amplitude limit cycles obtained for class 27, we can prove that the items (ii) and (iii) in Conjecture 1.2 are true. Moreover, proving the existence of four small-amplitude limit cycles is a necessary step towards proving item (i) of Conjecture 1.2.

The rest of the paper is organized as follows. In the next section, we discuss the methods for computing the limit cycles, and then provide a review on the existing literature in Section 3. In Section 4, we prove our main result by constructing two examples for class 27 and another two examples for class 26, each of which exhibits at least four small-amplitude limit cycles in the vicinity of the positive equilibrium. We also prove that the items (ii) and (iii) in Conjecture 1.2 are true. Finally, conclusions are drawn in Section 5.

2. Methodology

There are many approaches developed for computing the focus values of planar vector fields, such as Poincaré Takens method [8], the perturbation method [25], the singular point value method [18], etc. However, for higher dimensional dynamical systems, besides the normal form computation, there is only one exception which is applicable for 3-dimensional systems by using the method of Lyapunov constants approach [9–11,16,19,20]. In the following, we briefly describe the two methods for computing focus values.

2.1. Computation of normal forms for n-dimensional systems

In this subsection, we briefly introduce the method of normal forms for computing the focus values of general n-dimensional dynamical systems, which is applicable not only for Hopf bifurcation, but also for other singularities. The general normal form theory can be found in [4,5,7,8,17] and computations using computer algebra systems can be found in [13,21,22,25,27].

Consider the following general n-dimensional differential system:

$$\dot{\boldsymbol{z}} = A\,\boldsymbol{z} + \boldsymbol{f}(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathbf{R}^n, \quad \boldsymbol{f}:\ \mathbf{R}^n \to \mathbf{R}^n, \tag{2.1}$$

where $A\,\boldsymbol{z}$ and $\boldsymbol{f}(\boldsymbol{z})$ represent the linear and nonlinear terms of the system, respectively. We suppose that $\boldsymbol{z} = 0$ is a fixed point (or an equilibrium) of the system, implying that $\boldsymbol{f}(\mathbf{0}) = D\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$. (Otherwise, a simple shift can make the fixed point be at the origin.) It is assumed that $\boldsymbol{f}(\boldsymbol{z})$ is analytic and can be expanded into Taylor series in $\boldsymbol{z}$. In general, matrix A may contain the eigenvalues with negative, positive and zero real parts, and thus system (2.1) may consist of stable, unstable and center manifolds. However, in real applications, we may assume that the unstable manifold is empty since a real system with unstable manifold is usually unstable and the first task will be stabilizing the system. Therefore, we assume that system (2.1) only contain stable and center manifolds.

In normal form computation, the first step is usually to introduce a linear transformation into (2.1) such that its linear part becomes the Jordan canonical form. (For linear systems, the Jordan canonical form is the normal form.) Without loss of generality, suppose under the linear transformation $\boldsymbol{z} = T\binom{\boldsymbol{x}}{\boldsymbol{y}}$, system (2.1) becomes

$$\begin{aligned} \dot{\boldsymbol{x}} &= J_1\, \boldsymbol{x} + \boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{x} \in \mathbf{R}^k, \qquad \boldsymbol{f}_1 :\ \mathbf{R}^n \to \mathbf{R}^k, \\ \dot{\boldsymbol{y}} &= J_2\, \boldsymbol{y} + \boldsymbol{f}_2(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{y} \in \mathbf{R}^{n-k}, \quad \boldsymbol{f}_2 :\ \mathbf{R}^n \to \mathbf{R}^{n-k}, \end{aligned} \tag{2.2}$$

where $J_1 = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k)$, and $J_2 = \operatorname{diag}(\lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_n)$, with $\operatorname{Re}(\lambda_j) = 0$, $j = 1, 2, \cdots, k$ and $\operatorname{Re}(\lambda_j) < 0$, $j = k+1, \cdots, n$.

The second step is to apply center manifold theory [3] to system (2.2) so that $\boldsymbol{y}$ can be expressed on the center manifold as $\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{x})$, satisfying $\boldsymbol{H}(\mathbf{0}) = D\boldsymbol{H}(\mathbf{0}) = \mathbf{0}$. Therefore, the first equation of (2.2) becomes

$$\dot{\boldsymbol{x}} = J_1\, \boldsymbol{x} + \boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{H}(\boldsymbol{x})) = J_1\, \boldsymbol{x} + \boldsymbol{f}_1^2(\boldsymbol{x}) + \boldsymbol{f}_1^2(\boldsymbol{x}) + \cdots + \boldsymbol{f}_1^s(\boldsymbol{x}) + \cdots, \tag{2.3}$$

where $\boldsymbol{f}_1^j \in \mathrm{M}_j,\ j = 2, 3, \ldots,$ M_j defining a linear space of vector fields whose elements are homogeneous polynomials of degree j. Equation (2.3) describes the dynamics on the center manifold of system (2.2), and $\boldsymbol{H}(\boldsymbol{x})$ can be found from the following equation:

$$D\boldsymbol{H}(\boldsymbol{x}) \left[J_1\, \boldsymbol{x} + \boldsymbol{f}_1(\boldsymbol{x},\, \boldsymbol{H}(\boldsymbol{x})) \right] - J_2\, \boldsymbol{H}(\boldsymbol{x}) - \boldsymbol{f}_2(\boldsymbol{x},\, \boldsymbol{H}(\boldsymbol{x})) = \mathbf{0}, \tag{2.4}$$

with the boundary conditions $\boldsymbol{H}(\mathbf{0}) = D\boldsymbol{H}(\mathbf{0}) = \mathbf{0}$.

Next, using normal form theory [5,7,8,17] we introduce the near-identity transformation:

$$\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{Q}(\boldsymbol{u}) = \boldsymbol{u} + \boldsymbol{q}^2(\boldsymbol{u}) + \boldsymbol{q}^3(\boldsymbol{u}) + \cdots + \boldsymbol{q}^s(\boldsymbol{u}) + \cdots, \tag{2.5}$$

where $\boldsymbol{q}^j \in \mathrm{M}_j,\ j = 2, 3, \ldots$ into (2.3) to obtain the normal form,

$$\dot{\boldsymbol{u}} = J_1 \boldsymbol{u} + \boldsymbol{C}(\boldsymbol{u}) = J_1 \boldsymbol{u} + \boldsymbol{c}^2(\boldsymbol{u}) + \boldsymbol{c}^3(\boldsymbol{u}) + \cdots + \boldsymbol{c}^s(\boldsymbol{u}) + \cdots, \tag{2.6}$$

where $\boldsymbol{c}^j \in \mathrm{M}_j,\ j = 2, 3, \ldots$. The basic idea of normal form theory is to use the transformation $\boldsymbol{q}^j$ to simplify $\boldsymbol{c}^j$ "as simple as possible", order by order for $j = 2, 3, \ldots$. Thus, assuming the normal form reduction up to order $s-1$ has been preformed, we apply the transformation $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{q}^s(\boldsymbol{u})$ for the s-order process, and differentiating it gives

$$\dot{\boldsymbol{x}} = \left[I + D\boldsymbol{q}^s(\boldsymbol{u}) \right] \dot{\boldsymbol{u}}. \tag{2.7}$$

Now, we introduce the following map, called homological operator,

$$\mathrm{L}\!:\ \mathrm{M}_j \to \mathrm{M}_j, \quad \text{with} \quad \mathrm{L}(\boldsymbol{\xi}) = [\boldsymbol{\xi}, \mathrm{L}] = D\mathrm{L}\boldsymbol{\xi} - D\boldsymbol{\xi}\mathrm{L}, \quad \forall\, \boldsymbol{\xi} \in \mathrm{M}_j, \tag{2.8}$$

where $[\bullet,\, \bullet]$ is called *Lie bracket* operation, with $\mathrm{L} = J_1 \boldsymbol{x}$, and define the subspace induced by the map as $\mathrm{L}(\mathrm{M}_j)$. Thus $\mathrm{M}_j = \mathrm{L}(\mathrm{M}_j) + \mathrm{G}_j$, where G_j is the complement for $\mathrm{L}(\mathrm{M}_j)$ in M_j.

Then, it follows from (2.7) that

$$\begin{aligned}\dot{\boldsymbol{u}} &= [\,I + D\boldsymbol{q}^s(\boldsymbol{u})\,]^{-1}\,\boldsymbol{f}_1\,[\,\boldsymbol{u} + \boldsymbol{q}^s(\boldsymbol{u})\,] \\ &= \left[I - D\boldsymbol{q}^s(\boldsymbol{u}) + O(\|\boldsymbol{u}\|^{2(s-1)})\right]\left[J_1\,(\boldsymbol{u} + \boldsymbol{q}^s(\boldsymbol{u})) + \sum_{k=2}^{s}\boldsymbol{f}_1\,(\boldsymbol{u} + \boldsymbol{q}^s(\boldsymbol{u})) + O(\|\boldsymbol{u}\|^{s+1})\right] \\ &= J_1\boldsymbol{u} + \boldsymbol{c}^2(\boldsymbol{u}) + \cdots + \boldsymbol{c}^{s-1}(\boldsymbol{u}) + \boldsymbol{f}_1^s(\boldsymbol{u}) + \mathrm{L}\boldsymbol{q}^s(\boldsymbol{u}) + O(\|\boldsymbol{u}\|^{s+1}).\end{aligned} \tag{2.9}$$

Here it should be noted that $\boldsymbol{c}^j = \boldsymbol{f}_1^j$ for $j = 2, 3, \ldots, s-1$ since it is assumed that the normal form reduction has been carried out up to order $s-1$, and $\mathrm{L}\boldsymbol{q}^s(\boldsymbol{u}) = D\mathrm{L}\boldsymbol{q}^s(\boldsymbol{u}) - D\boldsymbol{q}^s(\boldsymbol{u})\,\mathrm{L}$. Hence, to simplify the s-order term, $\forall\, \boldsymbol{f}_1^s \in \mathrm{M}_s$, we need to find suitable $\boldsymbol{q}^s \in \mathrm{M}_s$ such that $\boldsymbol{f}_1^s(\boldsymbol{u}) + \mathrm{L}\boldsymbol{q}^s(\boldsymbol{u}) = \boldsymbol{c}^s(\boldsymbol{u})$ becomes as simple as possible, where $\mathrm{L}\boldsymbol{q}^s(\boldsymbol{u}) \in \mathrm{L}(\mathrm{M}_s)$ and $\boldsymbol{c}^s(\boldsymbol{u}) \in \mathrm{G}_s$. Therefore, once the basis of $\mathrm{L}(\mathrm{M}_s)$ is found, one can determine the basis of the complementary space G_s and thus the "form" of the normal form. It is well known that the normal form is not unique since the basis of G_j is not unique.

Many methodologies and efficient algorithms with the aid of computer algebra systems have been developed for computing normal forms associated with different singularities (e.g., see [7,13,27]). Recently, an explicit recursive formula is given for computing the normal form together with center manifold for general n-dimensional differential systems associated with semisimple singularities. To achieve this, rewrite (2.5) and (2.6) as

$$\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{Q}(\boldsymbol{u}) = \boldsymbol{u} + \sum_{m\ge 2}\sum_{m(k)} \boldsymbol{q}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k} \equiv \boldsymbol{q}(\boldsymbol{u}), \tag{2.10}$$

and

$$\dot{\boldsymbol{u}} = J_1\boldsymbol{u} + \boldsymbol{C}(\boldsymbol{u}) = J_1\boldsymbol{u} + \sum_{m\ge 2}\sum_{m(k)} \boldsymbol{c}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k}, \tag{2.11}$$

respectively, where $m(k)$ represents a vector $(m_1, m_2, \cdots, m_k)$ of k nonnegative integers, satisfying $\sum_{j=1}^{k} m_j = m$, and the index $m(k)$ in the summation denotes that the summation goes over all the sets for $m \ge 2$. Then, the center manifold can be expressed in the new variable $\boldsymbol{u}$ as

$$\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{q}(\boldsymbol{u})) = \sum_{m\ge 2}\sum_{m(k)} \boldsymbol{h}_{m(k)} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k} \equiv \boldsymbol{h}(\boldsymbol{u}). \tag{2.12}$$

Combining the center manifold and normal form computations yields the following compact equation,

$$D_{\boldsymbol{u}}\,\tilde{\boldsymbol{h}}(\boldsymbol{u}) J_1 \boldsymbol{u} - J\,\tilde{\boldsymbol{h}}(\boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u}) - D_{\boldsymbol{u}}\,\tilde{\boldsymbol{h}}(\boldsymbol{u}) - \tilde{\boldsymbol{C}}(\boldsymbol{u}), \tag{2.13}$$

where

$$\tilde{\boldsymbol{h}}(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{Q}(\boldsymbol{u}) \\ \boldsymbol{h}(\boldsymbol{u}) \end{pmatrix}, \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \boldsymbol{F}(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{u}, \boldsymbol{h}(\boldsymbol{u})) \\ \boldsymbol{f}_2(\boldsymbol{u}, \boldsymbol{h}(\boldsymbol{u})) \end{pmatrix}, \quad \tilde{\boldsymbol{C}}(\boldsymbol{u}) = \begin{pmatrix} \boldsymbol{C}(\boldsymbol{u}) \\ 0 \end{pmatrix}.$$

Finally, comparing the coefficients on both sides of (2.13) we obtain the recursive formulas for computing the coefficients of $\tilde{\boldsymbol{h}}(\boldsymbol{u})$ and $\boldsymbol{C}(\boldsymbol{u})$. We omit the detailed formulas and algorithms, as well as the Maple programs here, which can be found in [22].

Writing the normal form associated with Hopf bifurcation in polar coordinates we obtain

$$\begin{aligned}\dot{r} &= r\left(v_0 + r_1\, r^2 + r_2\, r^4 + \cdots + r_k\, r^{2k} + \cdots\right), \\ \dot{\theta} &= \omega_c + \tau_0 + \tau_1\, r^2 + \tau_2\, r^4 + \cdots + \tau_k\, r^{2k} + \cdots\,,\end{aligned} \tag{2.14}$$

where r and θ represent the amplitude and phase of motion, respectively. v_k $(k = 0, 1, 2, \cdots)$ is called the kth-order focus value. v_0 and τ_0 are obtained from linear analysis. The first equation of (2.14) can be used for studying bifurcation of limit cycles and their stability, while the second equation can be applied to find the frequency of bifurcating periodic motion. Moreover, the coefficients τ_j can be used to determine the order (or critical periods) of a center (i.e., when $v_j = 0$, $j \geq 0$).

The Maple programs developed in [21,22,25] for computing the normal form of Hopf bifurcation have been cross verified for many mathematical and practical systems. They produce the normal forms which are either identical or different by only a positive constant.

2.2. Computation of Lyapunov constants for 3-dimensional systems

In this subsection, we introduce the Lyapunov constant method for computing the focus values of 3-dimensional systems associated with Hopf bifurcation, which has been used in [9–11,16,19,20]. This method does not need to transform the linear part of the system to the Jordan canonical form. Consider system (2.1) with $n = 3$. Let $A = (a_{ij})$ and $\mathrm{Tr}(A) = a_{11} + a_{22} + a_{33}$. Then, suppose at the critical point, defined by

$$\det(A) = (a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32})\mathrm{Tr}(A) \equiv \omega_c^2 \mathrm{Tr}(A), \tag{2.15}$$

A contains a negative eigenvalue, $d = \mathrm{Tr}(A) < 0$, and a purely imaginary pair, $\pm i\omega_c$, with $\omega_c^2 = \det(A)/\mathrm{Tr}(A) > 0$, implying that $\det(A) < 0$. Now, introducing the affine transformation $\boldsymbol{z} = I + T^{-1}\boldsymbol{x}$ into (2.1) yields

$$\dot{\boldsymbol{x}} = J\,\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \tag{2.16}$$

where the same notation $\boldsymbol{f}$ used in (2.1) is still used for simplicity, and

$$J = \begin{bmatrix} J_{11} & J_{12} & 0 \\ J_{21} & -J_{11} & 0 \\ 0 & 0 & d \end{bmatrix}, \quad \text{satisfying} \quad J_{11} > 0, \quad -(J_{11}^2 + J_{12}J_{21}) = \omega_c^2. \tag{2.17}$$

Here, the linear transformation $T = T_2 T_1$ with

$$T_1 = \begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ B_{11} & B_{21} & B_{31} \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = B = A - d\,I_{3\times 3},$$

and

$$B_{11} = \begin{bmatrix} b_{22} & b_{32} \\ b_{23} & b_{33} \end{bmatrix}, \quad B_{21} = -\begin{bmatrix} b_{12} & b_{32} \\ b_{13} & b_{33} \end{bmatrix}, \quad B_{31} = \begin{bmatrix} b_{12} & b_{22} \\ b_{13} & b_{23} \end{bmatrix}.$$

Then,

$$T_1 A T_1^{-1} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & -J_{11} & J_{23} \\ 0 & 0 & d \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ 0 & d \end{bmatrix},$$

which in turn yields T_2 in the form of

$$T_2 = \begin{bmatrix} I_{2\times 2} & (C_1 - d\, I_{2\times 2})^{-1} C_2 \\ 0 & 1 \end{bmatrix}.$$

It should be noted that now the origin, $\boldsymbol{x} = \mathbf{0}$, of system (2.16) is an equilibrium, corresponding to the positive equilibrium of the original system (1.1).

Next, we apply center manifold theory and let

$$x_3 = h(x_1, x_2) = \sum_{m\geq 2} \sum_{j=0}^{m} h_{(m-j)j} x_1^{m-j} x_2^{j}. \tag{2.18}$$

Then differentiating the above equation with respect to time and using equation (2.16) to balance the coefficients of like-powers in the resulting equations for solving h_{kj}. Having obtained x_3, the first two equations become

$$\begin{aligned} \dot{x}_1 &= J_{11}\, x_1 + J_{12}\, x_2 + \sum_{m=2}^{\infty} f_{1m}(x_1, x_2), \\ \dot{x}_2 &= J_{21}\, x_1 - J_{11}\, x_2 + \sum_{m=2}^{\infty} f_{2m}(x_1, x_2), \end{aligned} \tag{2.19}$$

where f_{1m} and f_{2m} are homogeneous polynomials of degree m in x_1 and x_2.

To compute the Lyapunov constants, we take a Lyapunov function,

$$F(x_1, x_2) = -J_{21}\, x_1^2 + 2\, J_{11}\, x_1 x_2 + J_{12}\, x_2^2 + \sum_{m\geq 3} F_m(x_1, x_2), \tag{2.20}$$

where $F_m(x_1, x_2) = \sum_{j=0}^{m} a_k x_1^{m-k} x_2^k$, satisfying

$$\left.\frac{dF}{dt}\right|_{(2.19)} = \mathrm{LV}_1 x_2^4 + \mathrm{LV}_2 x_2^6 + \cdots \mathrm{LV}_n x_2^{2n+2} + \cdots, \tag{2.21}$$

where LV_n denotes the nth-order Lyapunov constant, which is equivalent to the nth-order focus value in the sense of that

$$v_1 = v_2 = \cdots = v_n = 0 \quad \Longleftrightarrow \quad \mathrm{LV}_1 = \mathrm{LV}_2 = \cdots = \mathrm{LV}_n = 0.$$

Now, differentiating (2.20) with respect to time and then using (2.19) and (2.21) we obtain the equations:

$$\begin{aligned} &-\Big[(J_{11}x_1 + J_{12}x_2)\,\frac{\partial F_n}{\partial x_1} + (J_{21}x_1 - J_{11}x_2)\,\frac{\partial F_n}{\partial x_2} \\ &= -2\big[(J_{21}x_1 - J_{11}x_2) f_{1(n-1)} - (J_{11}x_1 + J_{12}x_2) f_{2(n-1)}\big] \\ &\quad + \sum_{k=3}^{n-1} \Big[f_{1(n-k-1)}\frac{\partial F_k}{\partial x_1} + f_{2(n-k-1)}\frac{\partial F_k}{\partial x_2}\Big], \quad \text{for} \ \ n = \text{odd}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} &\mathrm{LV}_{\frac{n}{2}-1}\, x_2^n - \Big[(J_{11}x_1 + J_{12}x_2)\,\frac{\partial F_n}{\partial x_1} + (J_{21}x_1 + J_{11}x_2)\,\frac{\partial F_n}{\partial x_2} \\ &= -2\,\big[(J_{21}x_1 - J_{11}x_2) f_{1(n-1)} - (J_{11}x_1 + J_{12}x_2) f_{2(n-1)}\big] \\ &\quad + \sum_{k=3}^{n-1} \Big[f_{1(n-k-1)}\frac{\partial F_k}{\partial x_1} + f_{2(n-k-1)}\frac{\partial F_k}{\partial x_2}\Big], \quad \text{for} \ \ n = \text{even}. \end{aligned} \tag{2.23}$$

Denoting the right-hand sides of the above two equations by $\sum_{k=0}^{n} b_k x_1^{n-k} x_2^k$, we then balance the coefficients of the like powers in (2.22) and (2.23) to obtain the recursive formulas, with $a_{-1} = a_{n+1} = 0$, as follows:

$$\begin{aligned} &(n-2k)\, J_{11}\, a_k + (k+1)\, J_{21}\, a_{k+1} + (n-k+1)\, J_{12}\, a_{k-1} = -b_k, \\ &\qquad \text{for } k = 0, 1, \ldots, n-1, \text{ and } \text{ for } k = n \text{ when } n = \text{odd}; \\ &\mathrm{LV}_{\frac{n}{2}-1} = -n\, J_{11}\, a_n + J_{12}\, a_{n-1} + b_n, \quad \text{for } k = n \text{ when } n = \text{even}. \end{aligned} \tag{2.24}$$

More details can be found in [19]. With the above explicit formulas, we have coded a Maple program for computing the Lyapunov constants LV_n, $n = 1, 2, \cdots$.

Remark 2.1.

1. The linear transformation used for system (2.1) does not change the signs of the focus values, v_k. It may change the absolute values of the focus values, but does not change the number of limit cycles.
2. It is usually to consider the second method being simpler than the first one since it does not need the Jordan canonical form. However, from the view point of computation, it has been noted that the second method is not simpler than the first one. In other words, computation complexity for these two methods is more or less the same, but the second method needs a separate transformation for the center manifold while the first method unifies the transformation into the one procedure. In fact, once the matrix J is obtained in (2.16) by using the second method, one can apply one more simple linear transformation:

$$T^* = \begin{bmatrix} 1 & 0 & 0 \\ \frac{J_{11}}{J_{12}} & \frac{\omega_c}{J_{12}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   into (2.16) and then apply the first method to directly obtain the focus values, without applying the formula (2.18).
3. When applying the second method, the linear transformation used for system (2.1) is important since it may change the signs of the Lyapunov constants. The correct transformed linear part must be in the form given by (2.17). If the matrix J uses the following form,

$$J = \begin{bmatrix} -J_{11} & J_{12} & 0 \\ J_{21} & J_{11} & 0 \\ 0 & 0 & d \end{bmatrix} \quad (J_{11} > 0), \tag{2.25}$$

   then all the signs of the Lyapunov constants would be reversed. This can cause problem when a heteroclinic cycle exists in system (1.1) and one wants to use the Poincaré–Bendixson theory to determine bifurcation of a limit cycle.

Once we obtain the focus values or the Lyapunov constants for a given system, we want to use these quantities to determine bifurcation of limit cycles. The basic idea of finding k small-amplitude limit cycles in system (2.1) associated with a Hopf bifurcation around the origin is as follows: First, find the conditions such that $v_0 = v_1 = \cdots = v_{k-1} = 0$ (note that $v_0 = 0$ is automatically satisfied at the critical point), but $v_k \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. This indicates that the procedure for finding multiple limit cycles involves two steps: Computing the focus values (i.e., computing the normal form) or the Lyapunov constants, and solving multivariate coupled nonlinear polynomial equations: $v_0 = v_1 = \cdots = v_{k-1} = 0$. In the following lemma, we give sufficient conditions for the existence of k small-amplitude limit cycles. (The proofs can be found in [26].)

Lemma 2.1. *Suppose that the focus values depend on k parameters, expressed as*

$$v_j = v_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_k), \quad j = 0, 1, \ldots, k, \tag{2.26}$$

satisfying

$$\begin{aligned} & v_j(\epsilon_{1c}, \cdots, \epsilon_{kc}) = 0, \quad j = 0, 1, \ldots, k-1, \quad v_k(\epsilon_{1c}, \cdots, \epsilon_{kc}) \neq 0, \\ \text{and} \quad & \det\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)}(\epsilon_{1c}, \cdots, \epsilon_{kc})\right] \neq 0. \end{aligned} \tag{2.27}$$

Then, for any given $\epsilon_0 > 0$, there exist $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ and $\delta > 0$ with $|\epsilon_j - \epsilon_{jc}| < \epsilon_0$, $j = 1, 2, \ldots, k$ such that the equation $\dot{r} = 0$ has exactly k real positive roots r (i.e., system (1.1) has exactly k limit cycles) in a δ-ball with the center at the origin.

3. Literature review

In this section, we will give a review on the existing literature related to the problem we discuss in this paper. In particular, we shall discuss the systems considered in [9–11,16,19,20,24]. We have applied the algorithms and Maple programs developed in [21,22,25], and that for the second method presented above, as well as one algorithm with Maple program for 3-dimensional systems associated with Hopf bifurcation [23]. We used these five different programs to verify each other for each case. Note that the second method is used to compute the Lyapunov constants in all the articles [9–11,16,19,20].

It is noted that proving the existence of the two limit cycles in the classes 30 and 31 [9] applies Hopf bifurcation theory and the Poincaré–Bendixson theorem. However, due to the sign problem in the computation of the Lyapunov constants, the application of the Poincaré–Bendixson theorem in [9] is invalid and thus the conclusion does not hold. But for these two cases, we can still use the same examples to show that two small-amplitude limit cycles can exist merely from Hopf bifurcations, and so the general conclusion on the existence of two limit cycles in classes 30 and 31 are still true. For three limit cycles, we will show that the conclusion on the first example given in [20] for class 27 is not valid since a sign problem is identified. The proof for the second example of three limit cycles given in [11] for class 29 also applies both Hopf bifurcation theory and the Poincaré–Bendixson theorem. But for this case one cannot prove the existence of three small-amplitude limit cycles. Moreover, we will show that the first example presented in [10] for the existence of four limit cycles in class 27 also has a sign problem and thus the conclusion does not hold for this example. However, we will prove that the second example in [10], which was not used to prove the existence of four limit cycles, actually can have four limit cycles, and thus the general conclusion is still valid.

Now consider system (1.1). Without loss of generality, we may assume that the positive equilibrium, E, be located at $(x_1, x_2, x_3) = (1, 1, 1)$, implying that

$$r_i = \sum_{j=1}^{3} a_{ij}, \quad i = 1, 2, 3. \tag{3.1}$$

Further, introducing $z_i = 1 + x_i$ into (1.1) we shift the positive equilibrium E(1, 1, 1) to the origin, together with the condition (3.1), to obtain the system,

$$\dot{x}_i = (1 + x_i) \sum_{j=1}^{3} \big(-a_{ij}\big)\, x_j, \quad i = 1, 2, 3, \tag{3.2}$$

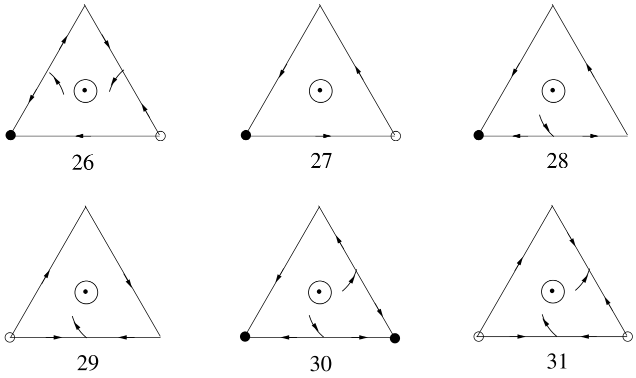


Fig. 1. The six classes in Zeeman's classification which can have limit cycles.

which has a fixed point at $\boldsymbol{x} = \mathbf{0}$, and a linear part $-A\,\boldsymbol{x}$. Noticing $a_{ij} > 0$, we know that under condition (2.15), this system has eigenvalues, $d = \mathrm{Tr}(-A) < 0$ and $\pm i\,\omega_c$ with $\omega_c^2 = \det(-A)/\mathrm{Tr}(-A) > 0$ and hence $\det(A) > 0$. Note that the nonlinear terms of this system only contain quadratic terms.

For system (1.1), there are six cases in Zeeman's classification [28] which can exhibit limit cycles, as shown in Fig. 1, among them only case 27 can have heteroclinic cycle. To determine the existence of the heteroclinic cycle and its stability, we define

$$\lambda_{ij} = r_j - \frac{a_{ji} r_i}{a_{ii}}, \quad \text{with} \quad R_{ij} = \mathrm{sign}(\lambda_{ij}), \tag{3.3}$$

and

$$P = \lambda_{12}\lambda_{23}\lambda_{31} + \lambda_{21}\lambda_{13}\lambda_{32}. \tag{3.4}$$

Then we have the following result.

Lemma 3.1. *If $R_{12} = R_{23} = R_{31}$, $R_{21} = R_{13} = R_{32}$, and $R_{ij}R_{ji} < 0$, then the system has a heteroclinic cycle, which is an attractor (a repellor) if $P < 0$ (> 0).*

In the following we will give a brief review for each of the articles [9–11,16,19,20,24] and point out some errors in some papers.

3.1. The first example giving two limit cycles in class 27 [16]

The first system considered by Hofbauer and So [16] is with the matrix

$$A = \begin{bmatrix} 2 & 5 & \frac{1}{2} \\ \frac{1}{2} & 1 & \mu \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \quad (\mu > 0).$$

A linear analysis yields the zero-order focus value as $v_0 = \frac{288}{1657}(\mu - \frac{71}{48})$. Thus, at the critical point, defined by $\mu_c = \frac{71}{48}$, we obtain the eigenvalues: $d = -4$ and $\pm i\,\omega_c$ with $\omega_c = (2 - \frac{1}{2}\mu_c)^{1/2} = \frac{11}{4\sqrt{6}}$. The positive equilibrium $\mathrm{E}(1,1,1)$ is stable (unstable) for $\mu < \frac{71}{48}$ ($> \frac{71}{48}$). Then, under the condition $\mu = \frac{71}{48}$, the first focus value obtained by using the first method with the Maple programs in [22,25] is $v_1 = \frac{220\,473}{58\,574\,950}$, while with the Maple programs in [21,23] is $v_1' = \frac{440\,946}{29\,287\,475}$, and the first Lyapunov constant obtained by using the second method with the normalized system (2.16) having J given in the form of (2.17) is $\mathrm{LV}_1 = \frac{690\,227\,472}{630\,881\,498\,975}$. All

v_1, v_1' and LV_1 are positive, though not equal, as expected. Note that if we use the form of J given in (2.25) we will get a negative $\mathrm{LV}_1 = -\frac{1\,849\,621\,488}{1\,698\,527\,112\,625}$, which is certainly not correct. Also note that $v_1'/v_1 = 4$ due to a constant multiplier, while $v_1/\mathrm{LV}_1 = \frac{64\,623}{18\,784}$ due to the different linear transformations used in the two methods. This indicates that the Hopf bifurcation is subcritical, and thus when $\mu < \frac{71}{48}$ and near this critical point, the positive equilibrium $\mathrm{E}(1,1,1)$ is stable and there exists an unstable limit cycle enclosing the equilibrium.

To find whether one more limit cycle can exist or not, we need to determine under what conditions a heteroclinic cycle can exist in this system. To achieve this, we calculate the quantities defined in (3.3) to obtain

$$\lambda_{12} = -\frac{3}{8} + \mu, \quad \lambda_{23} = \frac{7}{4} - \frac{\mu}{2}, \quad \lambda_{31} = \frac{25}{4}, \quad \lambda_{21} = -5\mu, \quad \lambda_{13} = -\frac{5}{4}, \quad \lambda_{32} = \frac{3}{2}(1-\mu).$$

It is easy to see that for $1 < \mu < \frac{7}{2}$, $\lambda_{ij}\lambda_{ji} < 0$, and $R_{12} = R_{23} = R_{31} = 1$, $R_{21} = R_{13} = R_{32} = -1$. Hence, for $1 < \mu < \frac{7}{2}$, a heteroclinic cycle exists. Further, a simple calculation using formula (3.4) results in $P = \frac{25}{128}(32\mu - 7)(3 - 2\mu)$, implying that $P > 0$ when $1 < \mu < \frac{3}{2}$, for which the heteroclinic cycle is unstable. So by applying Poincaré–Bendixson theorem, we conclude that there exists at least one stable limit cycle between the unstable limit cycle and the unstable heteroclinic cycle.

Summarizing above results show that for $\mu < \frac{71}{48}$ and near this point, there exist at least two limit cycles: one is due to Hopf bifurcation and the other due to the existence of a heteroclinic cycle. It is interesting to note that the two limit cycles can be also obtained just from the Hopf bifurcation, since $v_1 = \frac{220\,473}{58\,574\,950} \approx 0.003764$, and the second focus value $v_2 = -\frac{53\,186\,380\,685\,062\,360\,594\,081}{5\,305\,663\,551\,017\,822\,846\,700\,000} \approx -0.010024$. If we further change μ from $\mu = \frac{71}{48}$ to $\mu = \frac{71}{48} - 0.001$, yielding $v_0 \approx -0.000174$, then we obtain the approximations for the amplitudes of the two limit cycles from the truncated equation: $v_0 + v_1 r^2 + v_2 r^4 = 0$ as $r_1 \approx 0.232207$ and $r_2 \approx 0.567059$. It should be noted that here r_j's represent the average values of the amplitudes of the limit cycles, different from the quantity x_1 shown in Fig. 1(a) in [16].

In the following examples taken from the review articles, we will mainly apply the focus values obtained using the Maple program in [25], for which we have used all other four approaches and Maple programs to verify.

3.2. Another example giving two limit cycles in class 27 [24]

The example given in [24] has the matrix A given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ \frac{13}{5} + \varepsilon_1 & \frac{8}{5} + \varepsilon_2 & 3 \end{bmatrix} \quad (|\varepsilon_1| \ll 1, \; |\varepsilon_2| \ll 1).$$

Note that for this system when $\varepsilon_1 = \varepsilon_2 = 0$, on the center manifold, all the focus values evaluated at the positive equilibrium vanish, and thus we may apply the approach and Maple programs developed in [12] to perturb the 2-dimensional differential system on the center manifold. However, we here still apply the methods described in this paper to the original 3-dimensional differential system to obtain the focus values. First, it is easy to show that the zero-order focus value is equal to $v_0 = \frac{5}{84}(2\varepsilon_1 + 3\varepsilon_2)$, which defines a critical line,

$$\mathrm{L}_1: \quad 2\,\varepsilon_1 + 3\,\varepsilon_2 = 0, \tag{3.5}$$

in the ε_1–ε_2 parameter plane, as shown in Fig. 2. On the critical line L_1, we have $\varepsilon_1 = -\frac{3}{2}\varepsilon_2$. Then, the eigenvalues of the linearized system evaluated at the positive equilibrium are -5 and $\pm i\sqrt{1/5 - \varepsilon_2/2}$.

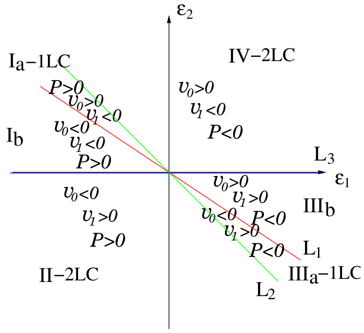


Fig. 2. Distribution of limit cycles for the example given in [24].

Further, with a linear transformation and application of the Maple program we obtain the first focus value, given by

$$v_1 = -\frac{6875}{303\,408}\,\varepsilon_2,$$

which is negative (positive) in the upper-half (lower-half) parameter plane.

Similarly, using the formulas in (3.3) and (3.4) we obtain

$$\begin{aligned}
&\lambda_{12} = 1, \quad \lambda_{23} = \tfrac{4}{5} + \varepsilon_1 - 3\varepsilon_2, \quad \lambda_{31} = \tfrac{5}{3} - \tfrac{1}{3}(\varepsilon_1 + \varepsilon_2), \quad \lambda_{21} = -1, \\
&\lambda_{13} = -\tfrac{3}{5} - 2\varepsilon_1 + \varepsilon_2, \quad \lambda_{32} = -\tfrac{4}{5} - \tfrac{2}{3}(\varepsilon_1 + \varepsilon_2), \quad P = -\tfrac{5}{3}(1 + \varepsilon_1 - \varepsilon_2)(\varepsilon_1 + \varepsilon_2).
\end{aligned}$$

It is obvious that for $|\varepsilon_1| \ll 1$, $|\varepsilon_2| \ll 1$, $\lambda_{ij}\lambda_{ji} < 0$ and $R_{12} = R_{23} = R_{31} = 1$, $R_{21} = R_{13} = R_{32} = -1$, which implies the existence of a heteroclinic cycle. Moreover, this heteroclinic cycle is stable (unstable) for $\varepsilon_1 + \varepsilon_2 > 0$ (< 0). This defines another critical line,

$$\mathrm{L}_2: \quad \varepsilon_1 + \varepsilon_2 = 0. \tag{3.6}$$

Note that $v_1 = 0$, i.e., $\varepsilon_2 = 0$ also defines a critical line,

$$\mathrm{L}_3: \quad \varepsilon_2 = 0. \tag{3.7}$$

The three critical lines L_k, $k = 1, 2, 3$, divides the ε_1–ε_2 parameter plane into six regions, as shown in Fig. 2. We obtain the results as follows: There are at least two limit cycles bifurcating in regions II and IV since $v_0 v_1 < 0$ and $v_1 P > 0$, but they have opposite stability in the two regions. The smaller limit cycle is due to Hopf bifurcation while the larger one due to the existence of heteroclinic cycle. In regions I_a and III_a there exists at least one limit cycle due to Hopf bifurcation since $v_0 v_1 < 0$ but $v_1 P < 0$, though the heteroclinic cycle exists. We cannot determine the number of limit cycles bifurcating in regions I_b and III_b because $v_0 v_1 > 0$ and $v_1 P < 0$, which may need a global bifurcation analysis. We conjecture that there exists at least one limit cycle in regions I_b and III_b. The details are shown in Fig. 2.

3.3. More examples giving two limit cycles in classes 26–29 [19]

In [19], the authors present the detailed calculation of the Lyapunov constants and consider four classes 26–29, each of them can exhibit two limit cycles. In addition, one non-competitive case is shown to have two limit cycles. For all the five cases, the correct J in the form of (2.17) is used. All these limit cycles are small-amplitude limit cycles due to Hopf bifurcations. We used all the five methods to verify these results.

The system used in this paper is given in the form

$$\dot{x}_i = x_i \sum_{j=1}^{3} a_{ij}(x_j - 1), \quad i = 1, 2, 3, \quad \text{with} \;\; a_{ij} < 0. \tag{3.8}$$

Note that system (3.8) is exactly the same as system (1.1) if the condition (3.1) is considered, since here $a_{ij} < 0$. The matrices A for the four competitive cases are given respectively by [19]

$$A = \begin{bmatrix} -2 & -5 & \lambda \\ -\frac{1}{10} & -1 & \mu \\ -1 & -13 & -5 \end{bmatrix}, \; \begin{bmatrix} -2 & -5 & \lambda \\ -\frac{13}{10} & -2 & \mu \\ -1 & -13 & -5 \end{bmatrix}, \; \begin{bmatrix} -\frac{1}{5} & -1 & \mu \\ -2 & -5 & \lambda \\ -1 & -13 & -5 \end{bmatrix}, \; \begin{bmatrix} -2 & -5 & \lambda \\ -\frac{1}{5} & -1 & \mu \\ -1 & -13 & -5 \end{bmatrix}.$$

In our formulas and programs, we take $-A$ for computation since we use system (1.1). All the results of two limit cycles for the above four cases and a non-competitive case are correct. In the following, we only give the results for the second case (Class 27) as a comparison of the first two methods. For the first method, we only show the result using the Maple program given in [25]. Using the second method, we obtain

$$\mathrm{LV}_1 = \frac{239(2147\lambda+5300)(47\,853\,681\,073\lambda^3+394\,605\,348\,396\lambda^2+603\,987\,570\,800\lambda+909\,087\,240\,000)}{9\,219\,218(2147\lambda-14\,140)(2147\lambda-72\,460)^2},$$

which seems different from that given in [19], $\mathrm{LV}_1^* = f(\lambda) = \frac{f_1(\lambda)}{f_2(\lambda)}$ (see page 59 in [19]). But a careful checking on the f_1 and f_2 given in [19] shows that they can actually be factorized to yield

$$\mathrm{LV}_1^* = \frac{239(2147\lambda+5300)(47\,853\,681\,073\lambda^3+394\,605\,348\,396\lambda^2+603\,987\,570\,800\lambda+909\,087\,240\,000)}{18\,438\,436(2147\lambda-14\,140)(2147\lambda-72\,460)^2},$$

showing that the result given in [19] for Class 27 is correct. Note that the denominator cannot equal zero since $\lambda < 0$, and the factor on the numerator, $2147\lambda + 5300$, is negative since $\omega_c = [-(2147\lambda + 5300)/960]^{1/2} > 0$.

The focus value v_1, on the other hand, obtained using the Maple program [25] is

$$v_1 = \frac{47\,853\,681\,073\lambda^3+394\,605\,348\,396\lambda^2+603\,987\,570\,800\lambda+909\,087\,240\,000}{68\,704(2147\lambda-14\,140)(2147\lambda-72\,460)},$$

which is certainly equivalent to LV_1 in the sense of having the same sign and giving the same roots for $v_1 = 0$ and $\mathrm{LV}_1 = 0$. In fact, the cubic polynomial on the numerator has one negative root, $\lambda = -6.800956\cdots$, which is located in the interval given in [19],

$$\left[-\frac{501\,822\,010\,581\,998\,678\,817}{73\,786\,976\,294\,838\,206\,464}, \; -\frac{1\,003\,644\,021\,163\,997\,357\,633}{147\,573\,952\,589\,676\,412\,928}\right].$$

At this critical point, we have $\mathrm{LV}_1 = v_1 = 0$, and $\mathrm{LV}_2 \approx -0.453257 \times 10^{-7}$, $v_2 \approx -0.540033 \times 10^{-4}$, as well as $\omega_c \approx 3.112752$. This clearly indicates that the second focus value (or Lyapunov constant) is negative, agreeing with what shown in [19].

Further, using the formulas (3.3) and (3.4) yields $R_{12} = R_{23} = R_{31} = -1$, $R_{21} = R_{13} = R_{32} = 1$, and $P \approx 0.764793$, implying that this example belongs to Class 27 and the heteroclinic cycle is unstable. However, we cannot apply Poincaré–Bendixson theorem here since $v_2 P < 0$. Hence, there are at least two small-amplitude limit cycles around the positive equilibrium due to Hopf bifurcation.

3.4. Two limit cycles in classes 30 and 31

In 2009, Gyllenberg and Yan [9] tried to find two limit cycles in the remaining two classes 30 and 31. Note that they used system (1.1) (with $a_{ij} > 0$) in the introduction for general discussion, but used system

(3.8) (with $a_{ij} < 0$) in their two examples. We will show that their first Lyapunov constants do not have correct signs and thus their conclusion on the second limit cycles by using the Poincaré–Bendixson theorem is not valid.

The two matrices in their two examples are given as follows:

$$A = \begin{bmatrix} -2 & -\frac{20\,051}{10\,000} & \lambda \\ -\frac{399}{200} & -2 & \mu \\ -\frac{211\,399}{123\,421\,530} & -\frac{21\,152\,242\,153}{12\,342\,153\,000\,000} & -\frac{1}{30} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -3 & -\frac{31}{10} & \lambda \\ -\frac{29}{10} & -3 & \mu \\ -\frac{2201}{100\,000} & -\frac{1001}{100\,000} & -\frac{1}{500} \end{bmatrix},$$

for classes 31 and 30, respectively, where $\lambda < 0$, $\mu < 0$. Note that in our calculations we again take $-A$ for consistence. It has been noted that these two systems are different from all the cases discussed above. In above cases, at the critical values of the parameters for which the first focus value (Lyapunov constant) vanishes, the critical frequency ω_c is still positive, as usually required to yield a fine focus at the critical point. Thus, for those cases, one does not need to worry about the sign change of ω_c^2 when small perturbation is applied to obtain two limit cycles. While for the two cases considered in [9] the critical frequency ω_c becomes imaginary (i.e., $\omega_c^2 < 0$, implying that the equilibrium point is not an elementary center at the critical point, but a saddle). Hence, the authors carefully choose perturbations near the critical point such that the frequency becomes positive and $v_0 v_1 < 0$ simultaneously, leading to one small-amplitude limit cycle due to Hopf bifurcation. To prove the existence of the second limit cycle, the authors first identified the class that the system belongs to and then applied Poincaré–Bendixson theorem. We will show that Poincaré–Bendixson theorem is not applicable for the two examples, but fortunately we can obtain two small-amplitude limit cycles due to Hopf bifurcation. This way, we need to find the second focus value.

First, we consider the second example. At the critical point, defined as $\mu = -\frac{6\,601\,200+2\,952\,450\lambda}{3\,052\,501}$, the eigenvalues of the linearized system, evaluated at the positive equilibrium, are $-\frac{3001}{500}$ and $\pm i\,\omega_c$, where $\omega_c = \left(\frac{1\,941\,971\lambda+2\,154\,020}{6\,105\,002\,000}\right)^{1/2}$. After using a linear transformation to make its linear part in Jordan canonical form, we apply the Maple program [25] to the resulting system to obtain $v_1 = \frac{v_{1n}}{v_{1d}}$, where

$$\begin{aligned} v_{1n} &= -(31\,981\,184\,327\,405\,191\,475\,060\,680\,500\,000\lambda^3 \\ &\quad + 73\,199\,457\,380\,504\,291\,265\,748\,720\,095\,500\lambda^2 \\ &\quad - 29\,578\,679\,415\,951\,235\,492\,386\,007\,815\,992\,041\lambda \\ &\quad - 33\,115\,498\,750\,883\,661\,600\,551\,100\,029\,064\,000), \\ v_{1d} &= 62\,143\,072(970\,985\,500\lambda + 27\,491\,904\,068\,501)(242\,746\,375\lambda + 27\,491\,096\,311\,001), \end{aligned}$$

where $v_{1d} > 0$ since $\omega_c > 0$ implying that $1\,941\,971\lambda + 2\,154\,020 > 0$. This first focus value seems different from the Lyapunov constant $\mathrm{LV}_1^* = \frac{f_1(\lambda)}{f_2(\lambda)}$, given in [9]. But actually, the f_1 and f_2 given in [9] can be further factorized as

$$\begin{aligned} f_1 &= \tfrac{123\,018\,750\,000}{3\,771\,251\,364\,841}(1\,941\,971\lambda + 2\,154\,020) \\ &\quad \times (31\,981\,184\,327\,405\,191\,475\,060\,680\,500\,000\lambda^3 \\ &\quad + 73\,199\,457\,380\,504\,291\,265\,748\,720\,095\,500\lambda^2 \\ &\quad - 29\,578\,679\,415\,951\,235\,492\,386\,007\,815\,992\,041\lambda \\ &\quad - 33\,115\,498\,750\,883\,661\,600\,551\,100\,029\,064\,000), \\ f_2 &= (970\,985\,500\lambda + 27\,491\,904\,068\,501)(242\,746\,375\lambda + 27\,491\,096\,311\,001)^2. \end{aligned}$$

It is obvious that the first focus value v_1 and the first Lyapunov constant LV_1^* have opposite signs though they produce the same roots. This sign difference is due to that the matrix J used in [9] (see Example 2 on page 351) is in the form (2.25) rather than the correct form (2.17). This causes a problem in proving the existence of the second limit cycle by applying Poincaré–Bendixson theorem.

Letting $v_{1n} = 0$ results in three real roots for λ, two of them are negative: $-1.117990\cdots$ and $-31.024361\cdots$. But, unfortunately, for these two values, ω_c^2 becomes negative, $-0.279869\cdots$ and $-0.009515\cdots$. Thus, we have to choose a perturbation on λ such that $\omega_c^2 > 0$ and simultaneously $v_0\, v_1 < 0$, leading to bifurcation of a small-amplitude limit cycle. To achieve this, note that the value of λ yielding $\omega_c = 0$ is $-1.109192\cdots$, which is close to $\lambda = -1.117990\cdots$. Therefore, we can take this value of λ as the critical value and choose a proper perturbation. In fact, in [9] the authors chose $\lambda = -\frac{16\,503}{15\,005} = 1.099833\cdots$, at which $\omega_c = \sqrt{2\,498\,291\,310\,645\,351\,187}/916\,055\,550\,100 > 0$. For this value of λ, we find

$$\lambda_{12} = \frac{106\,989\,382\,327}{2\,748\,166\,650\,300}, \qquad \lambda_{23} = -\frac{64\,377\,534\,852\,829}{27\,481\,666\,503\,000\,000}, \qquad \lambda_{31} = -\frac{13\,066\,103}{3\,001\,000},$$
$$\lambda_{21} = -\frac{88\,570\,870\,619}{2\,748\,166\,650\,300}, \qquad \lambda_{13} = -\frac{2\,691\,397}{900\,300\,000}, \qquad \lambda_{32} = -\frac{166\,495\,296\,811}{36\,642\,222\,004},$$

and so $R_{23} = R_{31} = R_{21} = R_{13} = R_{32} = -R_{12} = 1$, indicating that this is class 30 (see Fig. 1), which agrees with that shown in [9]. Since their calculation gives $\mathrm{LV}_1^* \approx -0.153495 \times 10^{-4} < 0$, indicating that the small-amplitude limit cycle is stable, they can apply Poincaré–Bendixson theorem to get a second limit cycle (see class 30 in Fig. 1). But we know that LV_1 for this example should be positive, indicating that the small-amplitude limit cycle is unstable and thus Poincaré–Bendixson theorem is not applicable here, as shown in Fig. 1 (see class 30).

However, fortunately we can show that for this example two small-amplitude limit cycles can be obtained from Hopf bifurcation. To achieve this, we use our Maple program [25] to obtain that for $\lambda = -\frac{16\,503}{15\,005}$,

$$v_1 \approx 0.011453, \qquad v_2 \approx -0.156462.$$

Further, we can perturb μ from its critical value as $\mu = -\frac{6\,601\,200+2\,952\,450\lambda}{3\,052\,501} + \varepsilon$, where $\varepsilon = 0.01$ to get $v_0 \approx -0.00000847$. Thus, the roots of the truncated equation, $v_0 + v_1\, r^2 + v_2\, r^4 = 0$, generate the approximation of the amplitudes of two small-amplitude limit cycles as $r_1 \approx 0.027340$ and $r_2 \approx 0.269170$.

Next, we consider the first example for which we have the eigenvalues $-\frac{121}{30}$ and $\pm\, i\, \omega_c$, where

$$\omega_c = \Big(\frac{27\,020\,304\,349\,285\,775\,602\,730+694\,460\,191\,889\,319\,911\,809\lambda}{210\,796\,993\,837\,474\,793\,660\,000\,000}\Big)^{1/2}$$

at the critical point $\mu = -\frac{198\,861\,623\,716\,180}{2\,561\,915\,176\,033} - \frac{51\,109\,911\,857\,141}{51\,238\,303\,520\,660}\lambda$. Similarly, we obtain the first focus value, given by $v_1 = \frac{v_{1n}}{v_{1d}}$, where

$$\begin{aligned}
v_{1n} = {} & -12\,342\,153 \\
& \times (2\,896\,744\,181\,458\,462\,236\,920\,535\,979\,772\,285\,792\,936\,317\,677\,325\,983\,615\,057\,706\,053\,800\,000\lambda^3 \\
& + 333\,913\,695\,853\,530\,775\,330\,889\,316\,047\,452\,150\,648\,929\,877\,598\,119\,093\,293\,136\,874\,198\,455\,580\lambda^2 \\
& + 13\,524\,585\,317\,482\,810\,284\,763\,966\,304\,816\,040\,419\,150\,059\,997\,463\,121\,264\,098\,249\,898\,635\,179\,933\lambda \\
& + 191\,344\,855\,663\,934\,493\,200\,078\,251\,392\,330\,684\,681\,438\,514\,510\,807\,640\,824\,449\,995\,451\,239\,188\,400), \\
v_{1d} = {} & 288\,895\,439\,825\,957\,083\,312\,544 \\
& \times (593\,533\,857\,667\,331\,900\,532\,351\,890 + 480\,780\,132\,846\,452\,246\,637\,\lambda) \\
& \times (30\,863\,031\,050\,483\,828\,111\,741\,024\,570 + 6\,250\,141\,727\,003\,879\,206\,281\,\lambda).
\end{aligned}$$

The first Lyapunov constant given in [9] is $\mathrm{LV}_1^* = \frac{f_1(\lambda)}{f_2(\lambda)}$, where f_1 and f_2 are lengthy expressions and can be actually further factorized as

$$
\begin{aligned}
f_1 = {} & 3\,892\,754\,254\,614\,721\,494\,929\,872\,834\,500\,000 \\
& \times (27\,020\,304\,349\,285\,775\,602\,730 + 694\,460\,191\,889\,319\,911\,809\lambda) \\
& \times (2\,896\,744\,181\,458\,462\,236\,920\,535\,979\,772\,285\,792\,936\,317\,677\,325\,983\,615\,057\,706\,053\,800\,000\lambda^3 \\
& + 333\,913\,695\,853\,530\,775\,330\,889\,316\,047\,452\,150\,648\,929\,877\,598\,119\,093\,293\,136\,874\,198\,455\,580\lambda^2 \\
& + 13\,524\,585\,317\,482\,810\,284\,763\,966\,304\,816\,040\,419\,150\,059\,997\,463\,121\,264\,098\,249\,898\,635\,179\,933\lambda \\
& + 191\,344\,855\,663\,934\,493\,200\,078\,251\,392\,330\,684\,681\,438\,514\,510\,807\,640\,824\,449\,995\,451\,239\,188\,400), \\
f_2 = {} & 6\,269\,574\,455\,546\,363\,434\,471\,082\,455\,513\,577\,989\,482\,253 \\
& \times (593\,533\,857\,667\,331\,900\,532\,351\,890 + 480\,780\,132\,846\,452\,246\,637\,\lambda) \\
& \times (30\,863\,031\,050\,483\,828\,111\,741\,024\,570 + 6\,250\,141\,727\,003\,879\,206\,281\,\lambda).
\end{aligned}
$$

Noticing that the first factor in f_1 is positive due to $\omega_c > 0$. Again, we see that v_1 and LV_1^* have opposite signs though they give the same unique real root: $\lambda = -38.909278\cdots$. This indicates that LV_1^* has a wrong sign even it seems they used the correct J in the form of (2.17) (see Example 1 in [9] on page 349). At this unique root, $\omega_c^2 \approx -0.303929 \times 10^{-8} < 0$. Hence, the authors of [9] slightly change this value to $\lambda = -\frac{41\,086\,603}{1\,056\,995} = -38.871142\cdots$. Then, at this value of λ, we can show that $R_{12} = R_{23} = R_{31} = R_{13} = R_{32} = -R_{21} = 1$, implying that this example belongs to class 31 (see Fig. 1), which agrees with that shown in [9]. Therefore, based on their calculation with $\mathrm{LV}_1^* \approx 0.000754 > 0$, they concluded that a second limit cycle exists by applying Poincaré–Bendixson theorem. Since their LV_1^* has a wrong sign, this claim is invalid.

However, we can show that two small-amplitude limit cycles exist for this example. In fact, for $\lambda = -\frac{41\,086\,603}{1\,056\,995}$, we obtain

$$v_1 \approx -0.061921, \quad v_2 \approx 177.129645.$$

Note that since v_2 is not small, we have to restrict the limit cycles to be very small and thus higher order terms in the normal form can be neglected. Next, we perturb μ from its critical value to $\mu = -\frac{198\,861\,623\,716\,180}{2\,561\,915\,176\,033} - \frac{51\,109\,911\,857\,141}{51\,238\,303\,520\,660}\lambda - \varepsilon$, where $\varepsilon = -0.0001$ to obtain $v_0 \approx 0.212664 \times 10^{-7}$. Then, solving the truncated equation, $v_0 + v_1 r^2 + v_2 r^4 = 0$, yields two positive roots: $r_1 \approx 0.000586$ and $r_2 \approx 0.018687$, giving the approximations for the amplitudes of bifurcating small-amplitude limit cycles.

Summarizing the results in this subsection we have the following result.

Theorem 3.1. *The 3-dimensional LV competitive system (1.1) has at least two small-amplitude limit cycles in classes 30 and 31.*

3.5. First attempt for finding three limit cycles in class 27 [20]

One year later, the same authors of [19] tried to give an example in [20] to show three limit cycles. They also used system (3.8) in their analysis. But unfortunately, due to an error in their calculation for computing the Lyapunov constants, their claim does not hold. In [19] they described the procedure to compute the Lyapunov constants using the correct formulas (2.16) and (2.19). But in [20] they used the wrong formula (2.25) in calculating the Lyapunov constants, which results in an opposite sign. Moreover, there should have two solutions. However, even with the additional solution, one cannot conclude the existence of three limit cycles. The matrix A in (3.8) is given by

$$A = \begin{bmatrix} -1 & -2 & \lambda \\ -\frac{3}{5} & -3 & \mu \\ -2 & -\frac{11}{10} & -5 \end{bmatrix} \quad (\lambda < 0,\ \mu < 0), \tag{3.9}$$

which, at the critical point $\mu = -\frac{3(3120+211\lambda)}{640}$, has eigenvalues, -9 and $\pm i\,\omega_c$, where $\omega_c = \frac{1}{80}(5837\lambda + 36\,560)^{1/2}$. After a linear transformation, they obtained the system (2.16) with J in the form of

$$J = \begin{bmatrix} -\frac{505}{211} & -\frac{20\,943}{1688} - \frac{633}{640}\lambda & 0 \\ \frac{5837}{6330} & \frac{505}{211} & 0 \\ 0 & 0 & -9 \end{bmatrix}, \tag{3.10}$$

which is in the wrong form of (2.25). As a matter of fact, using the correct form of J:

$$J = \begin{bmatrix} \frac{505}{211} & \frac{20\,943}{1688} + \frac{633}{640}\lambda & 0 \\ -\frac{5837}{6330} & -\frac{505}{211} & 0 \\ 0 & 0 & -9 \end{bmatrix}, \tag{3.11}$$

we obtain

$$\mathrm{LV}_1 = -\frac{600(44\,521\lambda+558\,480)(194\,341\,522\,781\lambda^3+4\,397\,246\,049\,014\lambda^2+28\,169\,289\,409\,536\lambda+43\,772\,534\,807\,040)}{1\,231\,607(5837\lambda+166\,160)(5837\lambda+554\,960)^2},$$

and the focus value obtained using the Maple program [25] is

$$v_1 = -\frac{15(194\,341\,522\,781\lambda^3+4\,397\,246\,049\,014\lambda^2+28\,169\,289\,409\,536\lambda+43\,772\,534\,807\,040)}{23\,348(5837\lambda+166\,160)(5837\lambda+554\,960)},$$

showing that LV_1 and v_1 have the same sign, and $\mathrm{LV}_1 = 0$ and $v_1 = 0$ give the same roots since except the factor of cubic polynomial, all other factors are positive due to $\omega_c > 0$. Note that the first Lyapunov constant given in [20] is

$$\mathrm{LV}_1^* = -\frac{63\,300(5837\lambda+36\,560)(194\,341\,522\,781\lambda^3+4\,397\,246\,049\,014\lambda^2+28\,169\,289\,409\,536\lambda+43\,772\,534\,807\,040)}{34\,070\,569(5837\lambda+166\,160)(5837\lambda+554\,960)^2},$$

which shows that all LV_1, v_1 and LV_1^* have negative sign, indicating that the correct format (3.11) might have been used in their calculations [20]. Note that the first factors in the numerators of LV_1 and LV_1^* are different, and in fact, executing our Maple program for the second method with the wrong form of J given in (3.10) yields the exact expression $-2\,\mathrm{LV}_1^*$, indicating a sign problem. The cubic polynomial factor has three negative roots: $\lambda_1 = -12.406568\cdots$, $\lambda_2 = -7.930662\cdots$, $\lambda_3 = -2.289153\cdots$, but only the last one is located in the interval given in [20],

$$\left[-\frac{337\,819\,463\,133\,526\,323\,681}{147\,573\,952\,589\,676\,412\,928},\ -\frac{84\,454\,865\,783\,381\,581\,545}{36\,893\,488\,147\,419\,103\,232}\right],$$

which yields a positive critical frequency, $\omega_c \approx 1.903869$, and $\mathrm{LV}_2 \approx 0.000126$, $v_2 \approx 0.005392$. These two values indicate that at the critical value of λ_3, solved from $\mathrm{LV}_1 = v_1 = \mathrm{LV}_1^* = 0$, the second focus value (Lyapunov constant) is positive. However, it is shown in [20] that $\mathrm{LV}_2^* < 0$ at this critical value. To investigate this difference, which raises the question about the existence of three limit cycles, we have applied our five approaches and Maple programs (four of them using the method of normal forms [21–23,25] and one using the method of Lyapunov constants, as presented in this paper), we obtain the second focus value and the second Lyapunov constant evaluated at $\lambda = \lambda_3$ as

$$\begin{aligned} v_2^a &= 0.0053921796\cdots \ [25], \quad & v_2^b &= 0.0862748740\cdots \ [21], \\ v_2^c &= 0.0028322207\ \cdots [22], \quad & v_2^d &= 0.0862748740\cdots \ [23], \\ \mathrm{LV}_2 &= 0.0001261198\cdots \ [\text{this paper}], \end{aligned}$$

all of them are positive, and $v_2^b = v_2^d = 16v_2^c$, as expected.

With the formula given in [20], $\mathrm{LV}_2^* = \frac{g_1}{g_2}$, we substitute $\lambda = \lambda_3$ into LV_2^* to obtain

$$\mathrm{LV}_2^* \approx -0.613499 \times 10^{-3} < 0,$$

implying that the wrong form J given in (3.10) might be used in computation. This inconsistency with the first Lyapunov constant computation may cause some errors in their computation.

Next, with the formulas (3.3) and (3.4) we can show that at the critical value of $\lambda = \lambda_3$, $R_{12} = R_{23} = R_{31} = 1$, $R_{21} = R_{13} = R_{32} = -1$, and $P \approx -8.131719$. Thus, due to $v_2 P < 0$, one cannot apply Poincaré–Bendixson theorem to claim the existence of one more limit cycle. Hence, the example given in [20] can have at least 2 limit cycles due to Hopf bifurcation, similar to the cases considered in [19].

3.6. Three limit cycles in class 29 [11]

Later, another try for finding three limit cycles in class 29 was a success [11]. System (1.1) with the matrix A, given by

$$A = \begin{bmatrix} 2 & \frac{129}{26} & \lambda \\ \frac{27}{136} & 1 & \mu \\ \frac{99}{100} & \frac{79}{6} & \frac{181}{36} \end{bmatrix} \qquad (\lambda > 0,\ \mu > 0),$$

is used in this paper to discuss bifurcation of limit cycles. The Jacobian of the system evaluated at the positive equilibrium, at the critical point $\mu = \frac{148\,137\,475 - 11\,422\,593\lambda}{100\,576\,964}$, has eigenvalues $-\frac{289}{36}$ and $\pm i\,\omega_c$ with

$$\omega_c = \left(\tfrac{66\,074\,164\,182\lambda - 430\,763\,319\,725}{130\,750\,053\,200}\right)^{1/2}.$$

Using these values, we obtain $v_1 = \frac{v_{1n}}{v_{1d}}$, where

$$\begin{aligned} v_{1n} = {} & -75(8\,910\,617\,491\,309\,507\,914\,601\,934\,593\,541\,676\lambda^3 \\ & - 103\,023\,993\,494\,982\,666\,728\,650\,319\,100\,938\,388\lambda^2 \\ & + 218\,258\,608\,138\,994\,756\,699\,580\,568\,887\,833\,521\lambda \\ & - 243\,985\,609\,490\,033\,480\,902\,512\,448\,327\,852\,625), \\ v_{1d} = {} & 2\,007\,951\,808\,(21\,408\,029\,194\,968\lambda + 542\,956\,133\,991\,425) \\ & \times (2\,676\,003\,649\,371\lambda + 323\,815\,810\,342\,300). \end{aligned}$$

The first Lyapunov constant given in [11] is $\mathrm{LV}_1^* = \frac{f_1(\lambda)}{f_2(\lambda)}$, and a factorization on f_1 and f_2 yields

$$\begin{aligned} f_1 = {} & -7\,139\,120\,625(66\,074\,164\,182\lambda - 430\,763\,319\,725) \\ & \times (8\,910\,617\,491\,309\,507\,914\,601\,934\,593\,541\,676\lambda^3 \\ & - 103\,023\,993\,494\,982\,666\,728\,650\,319\,100\,938\,388\lambda^2 \\ & + 218\,258\,608\,138\,994\,756\,699\,580\,568\,887\,833\,521\lambda \\ & - 243\,985\,609\,490\,033\,480\,902\,512\,448\,327\,852\,625), \\ f_2 = {} & 3\,937\,373\,499\,268\,036(21\,408\,029\,194\,968\lambda + 542\,956\,133\,991\,425) \\ & \times (2\,676\,003\,649\,371\lambda + 323\,815\,810\,342\,300)^2. \end{aligned}$$

Since the first factor in f_1 is positive due to $\omega_c > 0$, v_1 and LV_1^* have the same sign, and yield a same positive root: $\lambda = 9.229463\cdots$, at which $\omega_c \approx 1.170268 > 0$ and $v_2 \approx 0.011432 > 0$. Therefore, we can make perturbations on λ and μ such that $0 < v_0 \ll -v_1 \ll v_2$ to get two small-amplitude limit cycles, with the inner one stable and outer unstable, enclosing an unstable equilibrium. To obtain the third limit cycle, we use (3.3) to obtain

$$R_{12} = R_{21} = R_{13} = R_{32} = 1, \quad R_{23} = R_{31} = -1,$$

implying that this example belongs to class 29 (see Fig. 1). Since the outer small-amplitude limit cycle is unstable, one can apply Poincaré–Bendixson theorem to this case to obtain one more limit cycle.

It is noted however that for this example, without the application of Poincaré–Bendixson theorem, one cannot obtain three (small-amplitude) limit cycles by using the third focus value. Actually, at $\lambda = 9.229463\cdots$, $v_0 = v_1 = 0$, $v_2 \approx 0.011432$ and $v_3 \approx -0.001876$. Since $v_2 \gg -v_3$ though $v_2 v_3 < 0$, it is not able to find proper perturbations to yield three small-amplitude limit cycles. For example, we may perturb $\lambda = 9.229463\cdots + \varepsilon_2$ and then $\mu = \frac{148\,137\,475 - 11\,422\,593\lambda}{100\,576\,964} + \varepsilon_1$, where $\varepsilon_1 = 0.0000001$ and $\varepsilon_2 = 0.001$ to obtain

$$v_0 \approx 0.640259 \times 10^{-7}, \quad v_1 \approx -0.859320 \times 10^{-4}, \quad v_2 \approx 0.011551, \quad v_3 \approx -0.001790.$$

Then, the truncated equation, $v_0 + v_1 r^2 + v_2 r^4 + v_3 r^6 = 0$, yields three positive roots: $r_1 \approx 0.028971$, $r_2 \approx 0.081832$, $r_3 \approx 6.451191$, but obviously the third root does not lead to a small-amplitude limit cycle. Thus, one of the three limit cycles obtained for the example must be obtained by applying Poincaré–Bendixson theorem.

3.7. First attempt for finding four limit cycles in class 27 *[10]*

Finally, we come to the examples given in [10] used to prove the existence of four limit cycles. Similarly, they used system (1.1) (with $a_{ij} > 0$) for general discussion as well as deriving the formulas in (3.3) and (3.4). However, they also used system (3.8) (with $a_{ij} < 0$) in their two examples. Both examples belong to class 27, with the matrices given below:

$$A = \begin{bmatrix} -1 & -2 & \lambda \\ -\frac{3}{5} & -3 & \mu \\ n & n & -5 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 & \lambda \\ -\frac{1}{2} & -3 & \mu \\ n & n & -5 \end{bmatrix} \quad (\lambda < 0,\ \mu < 0,\ n < 0).$$

The first example was used to prove the existence of four limit cycles, while the second one used to disprove Conjecture 1.1 [16], and then a new Conjecture 1.2 is proposed.

First, consider example 1, which has a critical point defined by $\mu = \frac{3(312-11n)\lambda}{50n}$ at which A has eigenvalues -9 and $\pm i\,(\frac{154-17n\lambda}{50})^{1/2}$. A linear transformation is applied to obtain a J in the form of (2.25) (see [16] on page 651), which results in opposite signs in the Lyapunov constants. In fact, the first Lyapunov constant given in [16] is $\mathrm{LV}_1^* = \frac{f_1(n,\lambda)}{f_2(n,\lambda)}$, where

$$\begin{aligned} f_1 = {} & -33(154 - 17n\lambda)(619\,927n^3\lambda^3 - 67\,592n^3\lambda^2 - 22\,963\,596n^2\lambda^2 - 225\,420n^3\lambda \\ & - 4\,307\,432n^2\lambda + 244\,214\,240n\lambda - 8\,698\,560n^2 + 51\,148\,032n - 669\,981\,312), \\ f_2 = {} & -289n^3(2333 - 34n\lambda)(4204 - 17n\lambda)^2. \end{aligned}$$

Using the focus value computation, we obtain $v_1 = \frac{v_{1n}}{v_{1d}}$, where

$$\begin{aligned} v_{1n} = &\ 3\,(619\,927n^3\lambda^3 - 67\,592n^3\lambda^2 - 22\,963\,596n^2\lambda^2 - 225\,420n^3 - 4\,307\,432n^2\lambda\lambda \\ &+ 244\,214\,240n\lambda - 8\,698\,560n^2 + 51\,148\,032n - 669\,981\,312), \\ v_{1d} = &\ 272n^2(2333 - 34n\lambda)(4204 - 17n\lambda). \end{aligned}$$

Since $n < 0$ and $154 - 17n\lambda > 0$ (due to $\omega_c > 0$), we have $f_2 > 0$, $v_{1d} > 0$, and that f_1 and v_{1n} have opposite signs, leading to that LV_1^* and v_1 have opposite signs, as expected. In fact, all our focus values and all of their Lyapunov constants have opposite signs.

In the following, we will use our computed focus values to find the conditions on the parameters n and λ such that $v_1 = v_2 = 0$, but $v_3 \neq 0$. In [10] the authors found one solution $(n_0, \lambda_0) = (-2.662554649, -2.537333171)$ satisfying $\mathrm{LV}_1^* = \mathrm{LV}_2^* = 0$ (or $v_1 = v_2 = 0$). But actually, there exist two solutions, as shown below. The second focus value $v_2 = \frac{v_{2n}}{v_{2d}}$, where

$$v_{2d} = 2\,774\,400n^4(154 - 17n\lambda)(604 - 17n\lambda)(2333 - 34n\lambda)^3(4204 - 17n\lambda)^3 > 0,$$

and the lengthy expression of v_{2n} is omitted here for brevity. Eliminating λ from the two equations $v_{1n} = v_{2n} = 0$ yields a solution for $\lambda = \lambda(n)$ and a resultant equation:

$$\begin{aligned} F_1(n) = &\ (83n - 234)(11\,271n - 41\,083)(45\,084n - 2\,537\,857)(n+2)(867n + 2629) \\ &\times (31\,212n + 2\,483\,689)(29\,315\,871n^2 + 2\,341\,682\,034n - 3\,653\,192\,079\,416) \\ &\times (88\,049\,569\,592\,213\,722\,972\,439\,491\,781\,250\,n^{12} \\ &\quad - 41\,710\,228\,970\,436\,880\,399\,350\,763\,202\,619\,375\,n^{11} \\ &\quad + 5\,237\,299\,319\,334\,114\,473\,187\,618\,527\,061\,249\,400\,n^{10} \\ &\quad - 102\,269\,800\,789\,270\,732\,027\,540\,953\,340\,621\,497\,880\,n^{9} \\ &\quad + 465\,320\,020\,069\,729\,018\,872\,125\,899\,033\,932\,657\,648\,n^{8} \\ &\quad + 252\,002\,009\,127\,862\,741\,539\,827\,584\,565\,214\,881\,103\,n^{7} \\ &\quad + 1\,667\,998\,020\,294\,418\,710\,629\,537\,611\,678\,737\,654\,606\,n^{6} \\ &\quad - 680\,081\,559\,118\,121\,613\,815\,617\,231\,021\,758\,841\,896\,n^{5} \\ &\quad - 76\,814\,451\,844\,219\,144\,429\,959\,470\,876\,106\,124\,635\,400\,n^{4} \\ &\quad - 23\,344\,849\,575\,051\,062\,410\,831\,292\,297\,284\,678\,295\,984\,n^{3} \\ &\quad + 1\,011\,923\,454\,279\,486\,301\,995\,116\,814\,577\,210\,693\,693\,536\,n^{2} \\ &\quad + 3\,420\,855\,811\,193\,893\,263\,603\,513\,782\,545\,935\,636\,410\,112\,n \\ &\quad + 2\,949\,258\,308\,419\,850\,969\,880\,028\,406\,854\,418\,090\,652\,288), \end{aligned}$$

which has six negative solutions for n:

$$\begin{aligned} &n_1 = -2.6625546487\cdots, \quad n_2 = -1.3463316619\cdots, \quad n_3 = -3.0322952710\cdots, \\ &n_4 = -79.5748109701\cdots, \quad n_5 = -395.1991822306\cdots, \quad n_6 = -2, \end{aligned}$$

among which only the solutions n_1 and n_2 yield $\omega_c > 0$ and $\lambda < 0$, and n_1 gives the solution n_0 obtained in [10]. We list the two solutions as follows:

$$(n_1, \lambda_1) = (-2.6625546487\cdots, -2.5373331705\cdots),$$
$$(n_2, \lambda_2) = (-1.3463316619\cdots, -3.7238575401\cdots).$$

For these two sets of parameter values, we have

$$\text{for } (n_1, \lambda_1): \quad v_0 = v_1 = v_2 = 0, \quad v_3 = 0.0000979074\cdots > 0,$$
$$\text{for } (n_2, \lambda_2): \quad v_0 = v_1 = v_2 = 0, \quad v_3 = 0.0238636659\cdots > 0.$$

Thus, after proper perturbations, we can have three small-amplitude limit cycles bifurcating from the Hopf critical point near the positive equilibrium, since

$$\left.\frac{\partial(v_1, v_2)}{\partial(n, \lambda)}\right|_{(n_1,\lambda_1)} \approx 0.00002694 \neq 0, \quad \left.\frac{\partial(v_1, v_2)}{\partial(n, \lambda)}\right|_{(n_2,\lambda_2)} \approx -0.00152843 \neq 0,$$

and the outer small-amplitude limit cycle is unstable due to $v_3 > 0$. In [10], the authors obtained $\mathrm{LV}_3^* \approx -0.002850 < 0$, as expected to have an opposite sign.

Next, we apply the formulas in (3.3) and (3.4) to obtain

$$R_{12} = R_{23} = R_{31} = 1, \quad R_{21} = R_{13} = R_{32} = -1,$$

for both the two solutions, indicating that this example belongs to class 27, and

$$\begin{aligned} P &= -\tfrac{9}{12\,500n}(154 - 17n\lambda)(22n^2\lambda^2 - 24n^2\lambda - 429n\lambda + 60n^2 + 198n + 1560) \\ &= \begin{cases} -0.0451924039\cdots & \text{for } (n_1, \lambda_1), \\ -1.2374371624\cdots & \text{for } (n_2, \lambda_2). \end{cases} \end{aligned}$$

$P < 0$ implies that the heteroclinic cycle is stable for both the two solutions. Therefore, one cannot apply Poincaré–Bendixson theorem to claim the existence of one more limit cycle, and so the conclusion given in [10] on the existence of four limit cycles based on example 1 does not hold.

Next, we consider example 2 which was only used in [10] to disprove Conjecture 1.1, and thus the authors computed the Lyapunov constants only up to second order. Fortunately, we found that this example can have four limit cycles, and detailed analysis is given below.

First, for this example the eigenvalues of A are -9 and $\pm i\,(\frac{64-n\lambda}{20})^{1/2}$ at the critical point $\mu = \frac{376-13n\lambda}{20n}$. Then, after a linear transformation is applied to the system such that its linear part is in Jordan canonical form, we apply Maple program [25] to obtain $v_i = \frac{v_{in}}{v_{id}}$, $i = 1, 2$, where

$$\begin{aligned} v_{1n} &= 115\,401n^3\lambda^3 - 2716n^3\lambda^2 - 46\,060n^3\lambda - 4\,388\,728n^2\lambda^2 - 1\,075\,136n^2\lambda \\ &\quad - 1\,947\,680n^2 + 47\,500\,640n\lambda + 12\,083\,136n - 129\,861\,376, \\ v_{1d} &= 1568n^2(67 - n\lambda)(1684 - 7n\lambda), \\ v_{2n} &= -262\,471\,073\,054\,190n^{10}\lambda^{10} - (7\,998\,710\,800\,080n^2 - 172\,018\,160\,740\,216n \\ &\quad - 342\,297\,096\,252\,795\,397)n^9\lambda^9 + (9\,892\,398\,516\,000n^3 + 7\,247\,212\,318\,206\,700n^2 \\ &\quad - 121\,822\,545\,116\,543\,068n - 74\,200\,186\,206\,874\,349\,321)n^8\lambda^8 \\ &\quad - (6\,457\,306\,779\,098\,000n^3 + 2\,379\,894\,887\,154\,529\,080n^2 \\ &\quad - 24\,745\,631\,712\,017\,878\,284n - 6\,414\,837\,288\,594\,228\,884\,068)n^7\lambda^7 \\ &\quad - (1\,316\,253\,953\,126\,000n^4 - 1\,468\,171\,957\,849\,366\,000n^3 \end{aligned}$$

$$
\begin{aligned}
&- 321\,173\,225\,124\,284\,337\,980n^2 + 2\,229\,942\,129\,925\,443\,270\,512n \\
&+ 281\,192\,411\,460\,640\,110\,028\,384)n^6\lambda^6 + (713\,359\,711\,182\,202\,000n^4 \\
&- 136\,599\,554\,247\,456\,468\,000n^3 - 20\,142\,293\,231\,832\,256\,733\,360n^2 \\
&+ 111\,659\,696\,134\,043\,851\,999\,296n + 6\,897\,145\,672\,968\,670\,616\,672\,512)n^5\lambda^5 \\
&- (122\,406\,992\,322\,048\,560\,000n^4 - 5\,209\,844\,948\,778\,976\,240\,000n^3 \\
&- 637\,865\,716\,306\,259\,025\,365\,760n^2 + 3\,345\,467\,072\,093\,461\,387\,296\,512n \\
&+ 98\,033\,745\,270\,833\,041\,155\,048\,704)n^4\lambda^4 + (8\,687\,604\,856\,427\,238\,560\,000n^4 \\
&- 61\,437\,606\,163\,833\,647\,552\,000n^3 - 10\,983\,978\,259\,735\,404\,266\,560\,000n^2 \\
&+ 60\,215\,735\,289\,602\,477\,180\,619\,776n + 797\,330\,432\,325\,486\,507\,532\,239\,872)n^3\lambda^3 \\
&- (273\,783\,196\,619\,554\,531\,840\,000n^4 + 232\,818\,075\,145\,210\,001\,664\,000n^3 \\
&- 108\,766\,943\,691\,675\,900\,097\,039\,360n^2 + 624\,732\,495\,673\,404\,558\,453\,567\,488n \\
&+ 3\,459\,474\,928\,185\,108\,515\,962\,290\,176)n^2\lambda^2 + (3\,440\,002\,405\,086\,520\,942\,592\,000n^4 \\
&+ 10\,105\,372\,238\,152\,222\,732\,288\,000n^3 - 614\,115\,070\,536\,121\,720\,186\,613\,760n^2 \\
&+ 3\,394\,850\,061\,788\,245\,143\,273\,488\,384n + 6\,766\,756\,123\,585\,322\,428\,205\,989\,888)n\lambda \\
&- 14\,521\,440\,223\,175\,938\,719\,744\,000n^4 - 52\,423\,044\,664\,001\,829\,945\,344\,000n^3 \\
&+ 1\,564\,142\,951\,599\,349\,992\,477\,982\,720n^2 - 7\,432\,390\,001\,532\,099\,134\,627\,971\,072n \\
&- 4\,132\,258\,253\,599\,200\,903\,619\,149\,824, \\
v_{2d} &= 387\,233\,280n^4(64 - 7n\lambda)(244 - 7n\lambda)(67 - n\lambda)^3(1684 - 7n\lambda)^3.
\end{aligned}
$$

Similarly, eliminating λ from the equations, $v_{1n} = v_{2n} = 0$, we obtain a solution for $\lambda = \lambda(n)$, and a resultant equation:

$$
\begin{aligned}
F_2 = {} & (33n - 94)(94n - 4997)(2303n - 8644)(n + 2)(49n + 142)(238n + 16\,063) \\
& \times (1\,241\,317n^2 + 38\,468\,528n - 129\,961\,471\,412) \\
& \times (13\,577\,406\,839\,627\,262\,063\,432\,000n^{12} \\
& \quad - 5\,918\,165\,943\,152\,122\,352\,145\,243\,700n^{11} \\
& \quad + 700\,585\,215\,767\,706\,068\,407\,119\,498\,840n^{10} \\
& \quad - 13\,191\,031\,948\,276\,720\,898\,989\,353\,898\,691n^9 \\
& \quad + 55\,604\,513\,671\,148\,633\,317\,214\,637\,262\,028n^8 \\
& \quad + 104\,811\,087\,966\,334\,990\,229\,300\,928\,941\,433n^7 \\
& \quad + 22\,225\,813\,497\,904\,059\,134\,789\,783\,518\,870n^6 \\
& \quad - 955\,759\,975\,030\,448\,322\,004\,857\,617\,880\,092n^5 \\
& \quad - 7\,958\,566\,929\,093\,036\,242\,950\,224\,009\,421\,452n^4 \\
& \quad + 1\,297\,887\,459\,599\,762\,222\,903\,065\,098\,535\,676n^3 \\
& \quad + 143\,664\,771\,734\,509\,018\,013\,888\,334\,944\,526\,616n^2 \\
& \quad + 465\,528\,628\,784\,736\,031\,135\,032\,578\,026\,490\,640n \\
& \quad + 391\,343\,764\,618\,894\,630\,997\,428\,356\,110\,445\,536).
\end{aligned}
$$

The equation $F_2 = 0$ has six negative solutions for n:

$$\begin{aligned} &n_1 = -2.6007711762\cdots, \qquad n_2 = -1.3463714575\cdots, \qquad n_3 = -2.8979591836\cdots, \\ &n_4 = -67.4915966386\cdots, \qquad n_5 = -339.4341404696\cdots, \qquad n_6 = -2, \end{aligned}$$

and only the first two solutions satisfy $\omega_c > 0$ and $\lambda(n) < 0$, as given below:

$$\begin{aligned} (n_1, \lambda_1) &= (-2.6007711762\cdots, -2.6701865551\cdots), \\ (n_2, \lambda_2) &= (-1.3463714575\cdots, -3.7431794372\cdots), \end{aligned}$$

for which we obtain

$$\begin{aligned} &\text{for } \ (n_1, \lambda_1): \quad v_0 = v_1 = v_2 = 0, \quad v_3 = 0.0000885049\cdots > 0, \\ &\text{for } \ (n_2, \lambda_2): \quad v_0 = v_1 = v_2 = 0, \quad v_3 = 0.0271341458\cdots > 0. \end{aligned}$$

Then, by applying appropriate perturbations, we can have three small-amplitude limit cycles bifurcating from the Hopf critical point near the positive equilibrium because

$$\left.\frac{\partial(v_1, v_2)}{\partial(n, \lambda)}\right|_{(n_1,\lambda_1)} \approx 0.000029644 \neq 0, \quad \left.\frac{\partial(v_1, v_2)}{\partial(n, \lambda)}\right|_{(n_2,\lambda_2)} \approx -0.00168198 \neq 0,$$

and the outer small-amplitude limit cycles are unstable since $v_3 > 0$.

Next, similarly applying the formulas in (3.3) and (3.4) for this example yields

$$R_{12} = R_{23} = R_{31} = 1, \quad R_{21} = R_{13} = R_{32} = -1,$$

for both the two solutions, indicating that this example belongs to class 27, and further, we have

$$\begin{aligned} P &= -\tfrac{3}{2000n}(64 - 7n\lambda)(26n^2\lambda^2 - 32n^2\lambda - 517n\lambda + 80n^2 + 254n + 1880) \\ &= \begin{cases} 0.0181685809\cdots & \text{for } (n_1, \lambda_1), \\ -1.4398192224\cdots & \text{for } (n_2, \lambda_2). \end{cases} \end{aligned}$$

Hence, the first solution yields $v_3\,P > 0$ while the second solution gives $v_3\,P < 0$, implying that we can apply Poincaré–Bendixson theorem to this example on the first solution to conclude the existence of one more limit cycle. Therefore, Theorem 2.1 in [10] about the existence of four limit cycles in system (1.1) for class 27 with a heteroclinic cycle on the boundary of the carrying simplex still holds, but the proof must use a different solution from their second example.

4. Four small-amplitude limit cycles in classes 27 and 26

Now, in this section we prove our main result: there exist 3-dimensional competitive LV systems which can exhibit at least four small-amplitude limit cycles near the positive equilibrium due to Hopf bifurcation in classes 27 and 26. Also we will show that the items (ii) and (iii) in Conjecture 1.2 are true.

4.1. Four small-amplitude limit cycles in class 27

For this class, we have two examples with the matrices, given by

$$A = \begin{bmatrix} 1 & 2 & p_2 \\ p_3 & 3 & p_4 \\ p_1 & p_1 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} p_1 & 1 & 1 \\ p_2 & 1 & p_4 \\ p_3 & p_1 & 3 \end{bmatrix}, \tag{4.1}$$

where $p_i > 0$, $i = 1, 2, 3, 4$. We have the following theorem.

Theorem 4.1. *The 3-dimensional LV competitive system (1.1) has at least four small-amplitude limit cycles in class 27.*

Proof. We will only give a detailed proof for system (1.1) with the first matrix A, and summarize the results for the second A. The critical point, defined by

$$p_4 = \frac{1}{10p_1}\big[192 - 6p_1p_2 - (8 + p_1p_2)p_3\big], \tag{4.2}$$

at which the Jacobian of system (1.1), evaluated at the positive equilibrium E(1, 1, 1) ($J = -A$), has a negative eigenvalue -9 and a purely imaginary pair, $\pm i\,\omega_c$, where

$$\omega_c = \Big[\frac{1}{10}(38 + p_1p_2p_3 - 4p_1p_2 - 12p_3)\Big]^{1/2}. \tag{4.3}$$

Using the following linear transformation,

$$T = \begin{bmatrix} \frac{2(p_3+1)}{p_1(p_3-4)} & \frac{10\omega_c}{p_1(p_3-4)} & \frac{p_1p_2+8}{10p_1} \\ \frac{18-7p_3}{p_1(p_3-4)} & \frac{(p_3+6)\omega_c}{p_1(4-p_3)} & \frac{32-p_1p_2}{10p_1} \\ 1 & 0 & 1 \end{bmatrix}, \tag{4.4}$$

we can transform system (1.1) into a new system such that its linear part is in Jordan canonical form. Then, we apply the Maple program (e.g. [25]) to obtain the focus values, $v_i = \frac{v_{in}}{v_{id}}$, $i = 1, 2, 3$, where

$$\begin{aligned}
v_{1d} &= 16p_1^2(2(p_1p_2 - 12)^2p_3^3 - (p_1p_2 - 12)(24p_1p_2 - 2273)p_3^2 \\
&\quad + 8(12p_1^2p_2^2 - 2273p_1p_2 + 64\,048)p_3 - 16(8p_1p_2 - 481)(p_1p_2 - 212), \\
v_{2d} &= 1\,866\,240p_1^4(p_3 - 4)^2(p_1p_2p_3 - 4p_1p_2 - 12p_3 + 38)(p_1p_2p_3 - 4p_1p_2 - 12p_3 + 128) \\
&\quad \times (p_1p_2p_3 - 4p_1p_2 - 12p_3 + 848)^3(2p_1p_2p_3 - 8p_1p_2 - 24p_3 + 481)^3, \\
v_{3d} &= 278\,628\,139\,008\,000\,000\,000\,000p_1^6(p_3 - 4)^3(8p_1p_2p_3 - 32p_1p_2 - 96p_3 + 709) \\
&\quad \times (p_1p_2p_3 - 4p_1p_2 - 12p_3 + 38)^2(p_1p_2p_3 - 4p_1p_2 - 12p_3 + 128)^2 \\
&\quad \times (p_1p_2p_3 - 4p_1p_2 - 12p_3 + 848)^5(2p_1p_2p_3 - 8p_1p_2 - 24p_3 + 481)^5, \\
v_{1n} &= -(13p_1^3p_2^3 - 252p_1^3p_2^2 + 20p_1^3p_2 + 576p_1^2p_2^2 + 208p_1^2p_2 + 3360p_1^2 - 7840p_1p_2 \\
&\quad + 19\,392p_1 + 3072)p_3^3 - 2(207p_1^3p_2^3 - 1258p_1^3p_2^2 - 320p_1^3p_2 - 3766p_1^2p_2^2 \\
&\quad + 11\,532p_1^2p_2 - 49\,360p_1^2 + 95\,360p_1p_2 - 302\,432p_1 - 56\,512)p_3^2 \\
&\quad - 8(508p_1^3p_2^3 + 1978p_1^3p_2^2 + 520p_1^3p_2 - 2199p_1^2p_2^2 - 41\,962p_1^2p_2 + 56\,560p_1^2 \\
&\quad - 120\,880p_1p_2 + 434\,512p_1 + 206\,592)p_3 - 12\,288p_1^3p_2^3 + 1152(6p_1 + 467)p_1^2p_2^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + 1536(5p_1^2 - 193p_1 - 4150)p_1p_2 + 445\,440p_1^2 + 3\,099\,648p_1 + 17\,031\,168,\\
v_{2n} &= \cdots,\\
v_{3n} &= \cdots.
\end{aligned}
$$

Here, the lengthy expressions of v_{2n} (having 546 terms) and v_{3n} (having 2422 terms) are not listed for brevity. Now we need to solve $v_{1n} = v_{2n} = v_{3n} = 0$ for p_1, p_2, p_3. Since none of them is a linear equation with respect to one of the parameters, we have to solve them using a different approach. Also, due to the large and complicated expressions, the built-in Maple command *eliminate* does not work. We shall use another built-in Maple command *resultant* to find solutions, but it is more involved. The commands "resultant (v_{1n}, v_{2n}, p_2)" and "resultant (v_{1n}, v_{3n}, p_2)" yield two resultant equations: $F_{12} = F_0\,R_{12}$ and $F_{13} = F_0\,R_{13}$, where F_0, R_{12} and R_{13} are polynomials in p_1 and p_3. Then, again using the command "resultant (R_{12}, R_{13}, p_1)" we obtain $F_{123} = F_1\,R_{123}$, where

$$
\begin{aligned}
F_1 &= C_1\,p_3^3(p_3+6)^{78}(p_3+16)^3(2p_3+27)^2(13p_3+128)^2,\\
R_{123} &= R_{123a}R_{123b}R_{123c}R_{123d}R_{123e}R_{123f}R_{123g}R_{123h}R_{123i},\\
R_{123a} &= 3p_3^2 - 62p_3 + 360,\\
R_{123b} &= 1431p_3^5 - 43\,248p_3^4 + 387\,988p_3^3 - 519\,488p_3^2 - 5\,003\,712p_3 + 2\,861\,568,\\
R_{123c} &= 34\,992p_3^6 - 2\,550\,240p_3^5 + 66\,441\,610p_3^4 - 670\,030\,595p_3^3 + 706\,773\,150p_3^2\\
&\quad + 16\,315\,647\,552p_3 + 42\,699\,363\,840,\\
R_{123d} &= 4860p_3^7 - 248\,373p_3^6 + 4\,615\,605p_3^5 - 39\,391\,410p_3^4 + 152\,439\,000p_3^3\\
&\quad + 14\,871\,040p_3^2 - 2\,634\,835\,968p_3 - 832\,757\,760,\\
R_{123e} &= 16\,092p_3^7 - 528\,807p_3^6 + 8\,230\,671p_3^5 - 66\,406\,545p_3^4 + 305\,962\,625p_3^3\\
&\quad - 821\,758\,608p_3^2 + 1\,207\,592\,388p_3 - 761\,379\,264,\\
R_{123f} &= 18\,576p_3^9 - 412\,320p_3^8 + 3\,837\,048p_3^7 - 19\,177\,776p_3^6 + 53\,179\,885p_3^5\\
&\quad - 66\,565\,142p_3^4 - 37\,703\,547p_3^3 + 248\,579\,280p_3^2 - 324\,282\,528p_3 + 151\,375\,392,\\
R_{123g} &= 25\,981\,560p_3^{18} - 6\,783\,087\,663p_3^{17} + 183\,805\,902\,756p_3^{16} + 74\,149\,123\,298\,382p_3^{15}\\
&\quad - 8\,646\,644\,928\,599\,760p_3^{14} + 432\,134\,707\,392\,093\,120p_3^{13}\\
&\quad - 11\,193\,107\,743\,022\,469\,024p_3^{12} + 79\,441\,720\,395\,640\,995\,168p_3^{11}\\
&\quad + 4\,117\,244\,407\,715\,938\,345\,856p_3^{10} - 80\,237\,699\,508\,671\,744\,952\,320p_3^9\\
&\quad - 815\,037\,765\,917\,899\,962\,736\,640p_3^8 + 10\,376\,512\,356\,893\,516\,058\,607\,616p_3^7\\
&\quad + 47\,080\,782\,729\,938\,511\,076\,589\,568p_3^6 - 636\,488\,168\,241\,770\,497\,071\,120\,384p_3^5\\
&\quad - 1\,811\,106\,555\,190\,360\,094\,602\,690\,560p_3^4\\
&\quad + 15\,368\,204\,827\,836\,961\,010\,535\,628\,800p_3^3\\
&\quad + 68\,182\,592\,605\,320\,391\,833\,893\,732\,352p_3^2\\
&\quad + 83\,710\,159\,285\,007\,934\,689\,856\,454\,656p_3\\
&\quad + 19\,433\,465\,641\,293\,486\,859\,738\,939\,392,\\
R_{123h} &= \cdots,\\
R_{123i} &= \cdots,
\end{aligned}
$$

in which C_1 is a big integer, and the lengthy expressions of R_{123h} and R_{123i} are omitted.

Now, solving these univariate polynomials in p_3, we obtain 93 positive solutions for p_3. Then, among these 93 solutions, we use the equations $R_{12} = R_{13} = 0$ to find 38 positive solutions for (p_3, p_1). Next, among the 38 solutions, we use $v_{1n} = v_{2n} = v_{3n} = 0$ to obtain 10 positive $(p_3,\, p_1,\, p_2)$ solutions. In the last step, we need to check if these 10 solutions satisfy $\omega_c > 0$ and $p_4 > 0$, and finally find only one solution satisfying these two conditions, given by

$$p_1 = 1.460400\cdots, \quad p_2 = 3.151879\cdots, \quad p_3 = 2.128078\cdots, \quad p_4 = 9.419456\cdots, \tag{4.5}$$

for which $\omega_c = 0.620209\cdots$. For this solution, the focus values become

$$v_0 = v_1 = v_2 = v_3 = 0, \quad \text{and} \quad v_4 \approx -0.452369 < 0.$$

Moreover, at the solution given in (4.5) we have

$$\frac{\partial(v_1,\, v_2,\, v_3)}{\partial(p_1,\, p_2,\, p_3)} \approx 0.363684 \neq 0.$$

Thus, according to Lemma 3.1, we can perturb v_1, v_2 and v_3 by using p_1, p_2 and p_3 to obtain three small-amplitude limit cycles. Further, we change p_4 to perturb v_0 such that $v_0\, v_1 < 0$ and $|v_0| \ll |v_1|$ to get one more small limit cycle, and thus obtain four small-amplitude limit cycles.

Finally, we want to check if we are luck to be able to apply Poincaré–Bendixson theorem to get a 5th limit cycle. For the above unique solution given in (4.5), it is easy to use the formulas given in (3.3) and (3.4) to obtain

$$R_{12} = R_{23} = R_{31} = 1, \quad R_{21} = R_{13} = R_{32} = -1, \quad P \approx 0.003560 > 0,$$

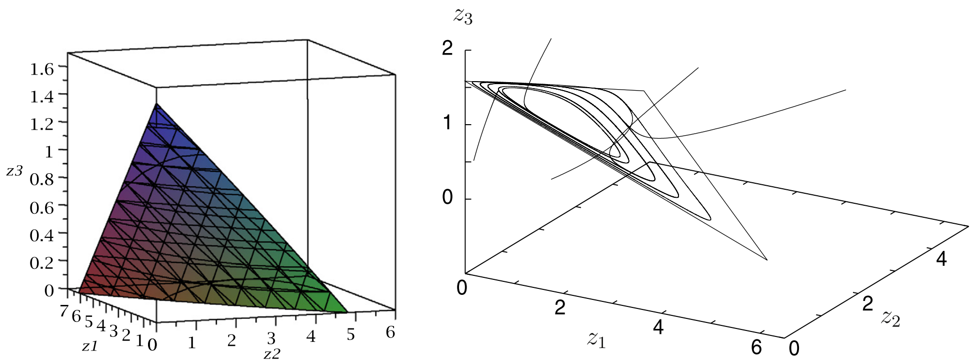
indicating that unfortunately Poincaré–Bendixson theorem is not applicable for this case due to $v_4\, P < 0$.

It should be noted that it is hard to use simulation to show the four small-amplitude limit cycles for the critical values of the parameters given in (4.5) with proper perturbations because they are around a fourth-order fine focus, causing extremely slow convergence. However, what we can demonstrate using numerical simulation is to show that for these critical parameter values all trajectories converge to the carrying simplex which contains the positive equilibrium. Also we can show the center manifold for the critical values, which actually contains the whole carrying simplex. The center manifold is obtained by using our Maple program for the second method after the affine transformation $\boldsymbol{z} = I + T^{-1}\boldsymbol{x}$ and then using the formula (2.18) yields the approximated center manifold up to third-order terms:

$$\begin{aligned} x_3 \approx{} & -0.00026852\, x_1^2 - 0.03506069\, x_1 x_2 - 0.03854293\, x_2^2 \\ & + 0.00007096\, x_1^3 + 0.00067345\, x_1^2 x_2 + 0.00129539\, x_1 x_2^2 + 0.00063851\, x_2^3. \end{aligned}$$

It is clear that the quadratic approximation of the center manifold near the positive equilibrium represents an elliptic paraboloid. However, when we transform the center manifold back to the original coordinates, we obtain

$$\begin{aligned} F(z_1, z_2, z_3) \approx{} & 10.24382231(z_1 - 1) + 12.60300630(z_2 - 1) + 37.75019310(z_3 - 1) \\ & - 0.10795713(z_1 - 1)^2 + 0.00625568(z_2 - 1)^2 + 0.41448552(z_3 - 1)^2 \\ & + 0.29208424(z_1 - 1)(z_2 - 1) - 0.36161739(z_1 - 1)(z_3 - 1) \\ & - 0.27553495(z_2 - 1)(z_3 - 1), \end{aligned}$$



(a) (b)

Fig. 3. (a) System behavior for the first case of class 27 with the critical values given in (4.5): (a) the center manifold near the positive equilibrium; and (b) numerical simulation showing convergence to the positive equilibrium.

which shows that the center manifold looks like almost a plane near the positive equilibrium, as shown in Fig. 3(a). Hence, the neighborhood of the positive equilibrium is very large and the quadratic elliptic paraboloid covers a very large area. In fact, the whole carrying complex is within the local area of the center manifold. This is reasonable since the nonlinearity is very weak in this case due to the positive equilibrium being a fourth-order fine focus.

To check if there are any other possible limit cycles on the carrying simplex, we will use simulation to verify this. Note that the carrying simplex has three vertexes on the z_1-axis, z_2-axis and z_3-axis with the coordinates

$$z_1^* = 3 + p_2 \approx 6.1519, \quad z_2^* = 1 + \tfrac{p_3+p_4}{3} \approx 4.8492 \;\; \text{and} \;\; z_3^* = 1 + \tfrac{2p_1}{5} \approx 1.5842,$$

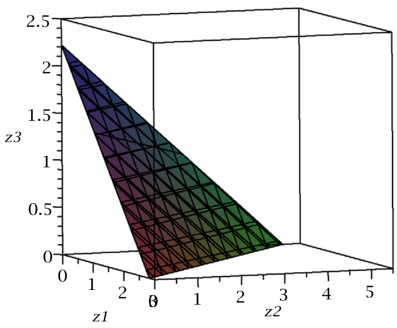
respectively. We can see from the simulation result given in Fig. 3(b), which has the exactly same carrying simplex as that shown in Fig. 3(a), that all simulated trajectories with different initial points quickly converge to the carrying simplex and almost stay there as a closed orbit. This is not surprising since the convergence on this carrying simplex (with an interior equilibrium being a fourth-order fine focus) is extremely slow once the trajectories reach the carrying simplex. Theoretically, these trajectories on the carrying simplex should converge to the equilibrium since $v_4 < 0$, but the convergence is too slow to be demonstrated by simulation. The above discussions show an agreement between the simulation and the theoretical prediction, implying that besides the four limit cycles, no other limit cycles can exist on the carrying simplex.

Next, we summarize the results obtained for the example with the second A given in (4.1). With a similar procedure used in the first example, we obtain four solutions, given as follows:

$$\begin{array}{llll}
p_1^a = 1.093819\cdots, & p_2^a = 1.092774\cdots, & p_3^a = 2.665493\cdots, & p_4^a = 2.999624\cdots, \\
p_1^b = 2.589229\cdots, & p_2^b = 4.435498\cdots, & p_3^b = 5.387046\cdots, & p_4^b = 3.172367\cdots, \\
p_1^c = 9.210496\cdots, & p_2^c = 9.351798\cdots, & p_3^c = 24.537382\cdots, & p_4^c = 3.302650\cdots, \\
p_1^d = 27.848014\cdots, & p_2^d = 33.074642\cdots, & p_3^d = 66.433114\cdots, & p_4^d = 7.426532\cdots,
\end{array} \tag{4.6}$$

and the corresponding focus values evaluated at these solutions are

$$v_0 = v_1 = v_2 = v_3 = 0, \quad \text{and} \quad \begin{cases} v_4^a = -0.1508181513\cdots \times 10^{-8}, \\ v_4^b = 0.5095370267\cdots \times 10^{-6}, \\ v_4^c = 0.1198056699\cdots \times 10^{-7}, \\ v_4^d = 0.3653554211\cdots \times 10^{-6}. \end{cases}$$



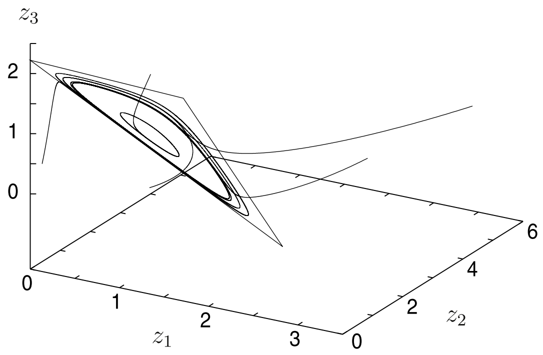


Fig. 4. (a) System behavior for the second case of class 27 with the first set of critical values given in (4.6): (a) the center manifold near the positive equilibrium; and (b) numerical simulation showing convergence to the positive equilibrium.

All these four solutions yield

$$R_{12} = R_{23} = R_{31} = 1, \quad R_{21} = R_{13} = R_{32} = -1,$$

and thus this example belongs to class 27 for all the four solutions. Moreover, we obtain

$$P^a \approx 0.00001531, \quad P^b \approx -0.00464832, \quad P^c \approx -0.23665111, \quad P^d \approx -78.58702849,$$

and so for all the four solutions we have $v_4 P < 0$, implying that Poincaré–Bendixson theorem cannot be applied for any of the four solutions. To obtain four small-amplitude limit cycles, we only need to verify the following determinant for the four solutions:

$$\frac{\partial(v_1,\, v_2,\, v_3)}{\partial(p_1,\, p_2,\, p_3)} = \left\{ \begin{array}{r} 0.7494729616\cdots \times 10^{-14} \\ -0.2763365707\cdots \times 10^{-10} \\ -0.3655856013\cdots \times 10^{-13} \\ 0.1982620287\cdots \times 10^{-12} \end{array} \right\} \neq 0.$$

Thus, three small-amplitude limit cycles are obtained by perturbing p_1, p_2, p_3 and one more small-amplitude limit cycle is given by perturbing p_4.

Similarly, we can obtain the expressions of the center manifolds for this case and simulate trajectories of system (1.1) for the critical values given in (4.6). For example, as shown in Fig. 4, we present the results for the first set of the critical values given in (4.6), which yields the center manifold, described in the original coordinates, as

$$\begin{aligned} F(z_1, z_2, z_3) \approx\ & 5.57209437(z_1 - 1) + 3.09381944(z_2 - 1) + 7.09344382(z_3 - 1) \\ & - 0.00002820(z_1 - 1)^2 + 0.00007664(z_2 - 1)^2 - 0.00016095(z_3 - 1)^2 \\ & - 0.00053096(z_1 - 1)(z_2 - 1) + 0.00048101(z_1 - 1)(z_3 - 1) \\ & + 0.00011515(z_2 - 1)(z_3 - 1), \end{aligned}$$

which again shows that the linear part dominate the expression and so the center manifold including the whole carrying simplex is almost a plane (see Fig. 3(a)). The numerical simulation shown in Fig. 3(b) also indicates the similar situation: all trajectories with different initial conditions are firstly all converging to the carrying simplex, which has the three vertexes at

$$z_1^* = 1 + \tfrac{2}{p_1} \approx 2.8285, \quad z_2^* = 1 + p_2 + p_3 \approx 5.0924 \quad \text{and} \quad z_3^* = \tfrac{p_3+4}{3} \approx 2.2218,$$

and then almost stay there, implying the very slow convergence on the carrying simplex because the positive equilibrium is a fourth-order fine focus. The agreement between the simulation and the analytical prediction implies that except the four small-amplitude limit cycles, there are no other limit cycles on the carrying simplex.

The proof for Theorem 4.1 is complete. □

4.2. Conclusions on items (ii) and (iii) of Conjecture 1.2

Now, we are ready to consider Conjecture 1.2. Although we did not get five limit cycles, we did compute the focus values up to v_4 as well as the index P for the heteroclinic cycle. Thus, we are able to make decision on Items (ii) and (iii) of Conjecture 1.2. In fact, we can prove that these two items are true. That is, we have the following result.

Theorem 4.2. *For system (1.1), in the case of heteroclinic cycle existing on the boundary of the carrying simplex, suppose that the first three focus values vanish ($v_0 = v_1 = v_2 = 0$). Then, neither the vanish of the third focus value ($v_3 = 0$) implies that the heteroclinic cycle is neutrally stable ($P = 0$), nor conversely that the heteroclinic cycle is neutrally stable ($P = 0$) implies the vanish of the third focus value ($v_3 = 0$). In particular, neither the conditions $v_0 = v_1 = v_2 = v_3 = 0$, nor $v_0 = v_1 = v_2 = P = 0$ imply that the positive equilibrium of system (1.1) is a center.*

Proof. We need to consider the two cases in class 27 since these two systems have heteroclinic cycles. The proof for the first part of the theorem (i.e., the item (ii) in Conjecture 1.2) is straightforward since we have already shown in the proof of Theorem 4.2 that $v_0 = v_1 = v_2 = v_3 = 0$ does not yield $P = 0$, and so the first part is true.

To show that the second part of the theorem (i.e., the item (iii) in Conjecture 1.2) is also true, we need to prove $v_3 \neq 0$ under the conditions: $v_0 = v_1 = v_2 = 0$ and $P = 0$. We first consider the first example in class 27, in which P is given by

$$\begin{aligned} P = \frac{3}{500p_1}\,(38 + p_1p_2p_3 - 4p_1p_2 - 12p_3)\big[&p_3(p_1p_2 - 10p_1 + 8)(2p_1p_2 + 4p_1 - 5) \\ &+ 12p_1^2(p_2^2 + 2p_2 + 5) - 264p_1p_2 - 168p_1 + 960\big]. \end{aligned}$$

Since the first factor is positive (due to $\omega_c > 0$), we may solve p_3 from the second factor to obtain

$$p_{3c} = \frac{12p_1^2(p_2^2 + 2p_2 + 5) - 264p_1p_2 - 168p_1 + 960}{(p_1p_2 - 10p_1 + 8)(2p_1p_2 + 4p_1 - 5)},$$

at which v_{1n} and v_{2n} are simplified to $v_{in} = \frac{v_{inn}}{v_{ind}}$, $i = 1, 2$, where v_{inn} and v_{ind} are polynomials of p_1 and p_2. Solving the two equations: $v_{1nn} = 0$ and $v_{2nn} = 0$ we obtain three positive solutions for (p_1, p_2) such that $p_3 > 0$ and $\omega_c > 0$ but none of them satisfies $p_4 > 0$. Moreover, among these three solutions, only one satisfies $v_0 = v_1 = v_2 = 0$. Thus, we have one solution satisfying $v_0 = v_1 = v_2 = P = 0$, but with one of the parameters, p_4 taking negative value (outside the physical limitation). Even for this solution, $v_3 \approx -6777.688726 \neq 0$. This shows that the conclusion of Theorem 4.2 is true for the first example.

Next, considering the second example. we have the critical value of p_2, solved from $P = 0$, as

$$p_{2c} = \frac{(p_1 + 2)(12p_1^2 - 11p_1p_3 + 3p_3^2 + 16p_1 - 6p_3)}{(3p_1 - 2p_3 - 2)(p_1 - p_3 - 2)},$$

for which v_{1n} and v_{2n} become rational functions of p_1 and p_3. Letting the numerators of these two functions equal zero and eliminating p_3 from these two polynomials we obtain a solution $p_3 = p_3(p_1)$ and a resultant equation:

$$\begin{aligned} R_{12} = {} & (p_1+1)(3p_1+4)(p_1+3)(p_1+4)(p_1+2)(3p_1^3+44p_1^2+192p_1+192) \\ & \times (12p_1^5+311p_1^4+2896p_1^3+12\,304p_1^2+23\,888p_1+17\,088)(p_1-4) \\ & \times (2p_1^2-p_1-12)(4368p_1^{21}-2\,208\,348p_1^{20}-62\,412\,886p_1^{19}-720\,547\,221p_1^{18} \\ & -3\,516\,987\,941p_1^{17}+9\,230\,386\,223p_1^{16}+276\,352\,849\,087p_1^{15}+2\,351\,832\,392\,006p_1^{14} \\ & +13\,503\,291\,495\,696p_1^{13}+60\,571\,893\,748\,128p_1^{12}+210\,845\,598\,341\,664p_1^{11} \\ & +490\,525\,009\,852\,448p_1^{10}+313\,072\,362\,303\,168p_1^{9}-2\,733\,450\,253\,244\,416p_1^{8} \\ & -12\,509\,314\,819\,438\,080p_1^{7}-27\,975\,482\,128\,885\,760p_1^{6}-34\,991\,231\,653\,621\,760p_1^{5} \\ & -15\,757\,485\,834\,600\,448p_1^{4}+21\,929\,674\,230\,202\,368p_1^{3}+42\,449\,333\,699\,215\,360p_1^{2} \\ & +29\,479\,620\,110\,712\,832p_1+7\,940\,884\,779\,761\,664), \end{aligned}$$

which has 6 positive solutions for p_1:

$$1.025382\cdots, \quad 2.621433\cdots, \quad 8.222485\cdots, \quad 532.969720\cdots, \quad 2.712214\cdots, \quad 4.$$

For these 6 solutions of p_1, we use the formula $p_3(p_1)$ to find solution for p_3, and then using the formula p_{2c} to find solution for p_{2c}, and finally verifying p_4 indicates that all the 6 solutions yield $p_1 > 0$, $p_3 > 0$, $p_{2c} > 0$ and $p_4 > 0$. One more thing needs to check is ω_c and we find that only the first four solutions yield $\omega_c > 0$. Therefore, we find four solutions:

$$\begin{aligned} & p_1^A = 1.025382\cdots, && p_2^A = 0.980948\cdots, && p_3^A = 2.526887\cdots, && p_4^A = 2.983402\cdots, \\ & p_1^B = 2.621433\cdots, && p_2^B = 4.562640\cdots, && p_3^B = 5.429149\cdots, && p_4^B = 3.163067\cdots, \\ & p_1^C = 8.222485\cdots, && p_2^C = 8.194314\cdots, && p_3^C = 22.52972\cdots, && p_4^C = 2.917147\cdots, \\ & p_1^C = 532.9697\cdots, && p_2^D = 636.1192\cdots, && p_3^D = 1247.183\cdots, && p_4^D = 108.7045\cdots, \end{aligned}$$

for which $\omega_c > 0$, and

$$v_0 = v_1 = v_2 = 0, \quad \text{and} \quad \begin{cases} v_3^A = 0.6089032187\cdots\times 10^{-6}, \\ v_3^B = -0.1540246260\cdots\times 10^{-5}, \\ v_3^C = 0.2411287902\cdots\times 10^{-6}, \\ v_3^D = -0.0002396875\cdots, \end{cases}$$

showing that the third focus value is nonzero for all the four solutions, and so the conclusion of Theorem 4.2 is also true for the second example.

This finishes the proof. □

Remark 4.1. It should be noted from the proof of Theorem 4.2 that under the conditions $v_0 = v_1 = v_2 = 0$ and $P = 0$, but $v_3 \neq 0$, we may get three small-amplitude limit cycles from perturbing v_0, v_1 and v_2, and one more limit cycle by applying Poincaré–Bendixson theorem if $v_3 P > 0$. We only consider the second example since the first example does not have feasible parameter solutions for $v_0 = v_1 = v_2 = P = 0$. For simplicity, we take $p_2 = p_{2c} + 1$ and then solve $v_{1n} = 0$ and $v_{2n} = 0$ to get two feasible solutions:

$$p_1^I = 29.68240\cdots, \quad p_2^I = 35.38205\cdots, \quad p_3^I = 70.65660\cdots, \quad p_4^I = 7.78072\cdots,$$
$$p_1^{II} = 340.9426\cdots, \quad p_2^{II} = 407.9100\cdots, \quad p_3^{II} = 797.8659\cdots, \quad p_4^{II} = 69.9927\cdots,$$

which yield $\omega_c > 0$ and

$$v_0 = v_1 = v_2 = 0, \quad \begin{cases} v_3^I = -0.0000021092\cdots, & P = -93.4628545\cdots \\ v_3^{II} = -0.0001818308\cdots, & P = -10097.8387\cdots, \end{cases}$$

showing that $v_3 P > 0$ and thus Poincaré–Bendixson theorem can be applied. Further,

$$\frac{\partial(v_1, v_2)}{\partial(p_1, p_3)} = \begin{Bmatrix} 0.2092511160\cdots \times 10^{-6} \\ 0.2589286844\cdots \times 10^{-7} \end{Bmatrix} \neq 0,$$

implying that two small-amplitude limit cycles can be obtained by perturbing p_1 and p_3. Finally, perturbing p_4 such that $v_0 v_1 < 0$ to get one more small-amplitude limit cycle.

Item (i) in Conjecture 1.2, i.e., the existence of five limit cycles in the 3-dimensional LV system (1.1), remains open. However, the results of four small-amplitude limit cycles obtained in this paper for class 27 (which needs the computations on v_i, $i = 1, 2, 3, 4$ and P) give us hope that a positive answer may be confirmed in near future for this conjecture.

4.3. Four small-amplitude limit cycles in class 26

Finally, in this section, we will show that system (1.1) can also have four small-amplitude limit cycles in class 26. This may promote studies on other classes 28–31 to see if there exist four limit cycles in those classes. For class 26, we also have two examples, with the matrices given by

$$A = \begin{bmatrix} p_3 & 2 & p_2 \\ \frac{1}{2} & 3 & p_4 \\ p_1 & p_1 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} p_3 & 2 & p_2 \\ \frac{3}{5} & 3 & p_4 \\ p_1 & p_1 & 5 \end{bmatrix}, \quad p_i > 0 \quad (i = 1, 2, 3, 4). \tag{4.7}$$

We have the following result.

Theorem 4.3. *The 3-dimensional LV competitive system (1.1) has at least four small-amplitude limit cycles in class* 26*.*

Proof. Since these two matrices given in (4.7) are almost identical (except one constant entry), we expect that the results for the two systems should be similar. Thus, we only prove the system with the first matrix A and then summarize the results for the second one. The Jacobian matrix of system (1.1) with the first matrix A, evaluated on the positive equilibrium $\mathrm{E}(1, 1, 1)$, at the critical point, defined by

$$p_4 = \frac{1}{20 p_1}(234 - 2p_1p_2p_3 + 126p_3 - 11p_1p_2 + 16p_3^2),$$

has a negative eigenvalue $-(p_3 + 8)$ and an imaginary pair, $\pm i\,\omega_c$, where

$$\omega_c = \left[\frac{1}{20}(46 + 2p_1p_2p_3 + 34p_3 - 9p_1p_2 - 16p_3^2)\right]^{1/2}.$$

Then, using a linear transformation T, where

$$T = \begin{bmatrix} \frac{2(11-p_3)}{p_1(2p_3-9)} & \frac{20\omega_c}{p_1(2p_3-9)} & \frac{p_1p_2+2p_3+8}{10p_1} \\ \frac{6p_3+23}{p_1(2p_3-9)} & \frac{(2p_3+11)\omega_c}{p_1(9-2p_3)} & \frac{24-8p_3-p_1p_2}{10p_1} \\ 1 & 0 & 1 \end{bmatrix},$$

we have a system with its linear part in Jordan canonical form, and apply our Maple program to this system to obtain the focus values, $v_i = \frac{v_{in}}{v_{id}}$, $i = 1, 2, 3$ (where v_{in} and v_{id} are not listed). Then, eliminating p_2 from the equations $v_{1n} = v_{2n} = 0$ and $v_{1n} = v_{3n} = 0$ yields two solutions $p_{2a}(p_1, p_3)$ and $p_{2b}(p_1, p_3)$ for p_2, and two resultant equations: $R_{12}(p_1, p_3) = 0$ and $R_{13}(p_1, p_3) = 0$. Further, using the command "resultant (R_{12}, R_{13}, p_1)" we obtain the resultant equation $R_{123}(p_3) = 0$. Solving this univariate polynomial equation results in 95 positive solutions for p_3. Then, among these 95 solutions, we use two resultant equations, $R_{12}(p_1, p_3) = 0$ and $R_{13}(p_1, p_3) = 0$, to find 26 positive solutions (p_3, p_1), among which 12 solutions satisfy $p_{2a} = p_{2b} > 0$. Finally, from the 12 solutions we only find one solution satisfying $p_4 > 0$ and $\omega_c > 0$, given by

$$p_1 = 2.472968\cdots, \quad p_2 = 3.401336\cdots, \quad p_3 = 1.202017\cdots, \quad p_4 = 5.981176\cdots, \tag{4.8}$$

for which $\omega_c = 0.643031\cdots$, and

$$v_0 = v_1 = v_2 = v_3 = 0, \quad v_4 \approx -0.393059 \times 10^{-5}.$$

Moreover, at the solution given in (4.8), we obtain

$$\frac{\partial(v_1, v_2, v_3)}{\partial(p_1, p_2, p_3)} \approx -0.591553 \times 10^{-8} \neq 0,$$

implying that three small-amplitude limit cycles can be produced by perturbing v_1, v_2, v_3 using p_1, p_2, p_3 and the fourth (the smallest) small limit cycle can be obtained by perturbing v_0 using p_4.

Finally, we need to check whether this example belongs to class 26 or not. Evaluating the formulas (3.3) and (3.4) at the solution (4.8) yields

$$\mathrm{R}_{12} = \mathrm{R}_{21} = \mathrm{R}_{23} = 1, \quad \mathrm{R}_{13} = \mathrm{R}_{31} = \mathrm{R}_{32} = -1, \tag{4.9}$$

showing that the first example indeed belongs to class 26.

The simulation result for this example is shown in Fig. 5(b), which is different from the previous two examples in class 27. In the two examples for class 27, the carrying simplex is almost on a plane (see Figs. 3 and 4). But here the carrying simplex is not a plane, where a part of the boundary is shown in Fig. 5(b), and it is seen that the converging trajectories located on the carrying simplex are obviously not on the plane based on the triangle (see the triangle in Fig. 5(b)). In fact, the center manifold expressed in the original coordinates can be approximated as

$$\begin{aligned} F(z_1, z_2, z_3) \approx\ & 11.26973329(z_1 - 1) + 16.81543285(z_2 - 1) + 33.05750294(z_3 - 1) \\ & + 0.45803447(z_1 - 1)^2 + 0.48822428(z_2 - 1)^2 - 0.91752650(z_3 - 1)^2 \\ & - 2.36131219(z_1 - 1)(z_2 - 1) + 1.78377462(z_1 - 1)(z_3 - 1) \\ & + 0.61041412(z_2 - 1)(z_3 - 1), \end{aligned}$$

which clearly shows the nonlinearity, and the quadratic center manifold is shown in Fig. 5(a), which is indeed not a plane.

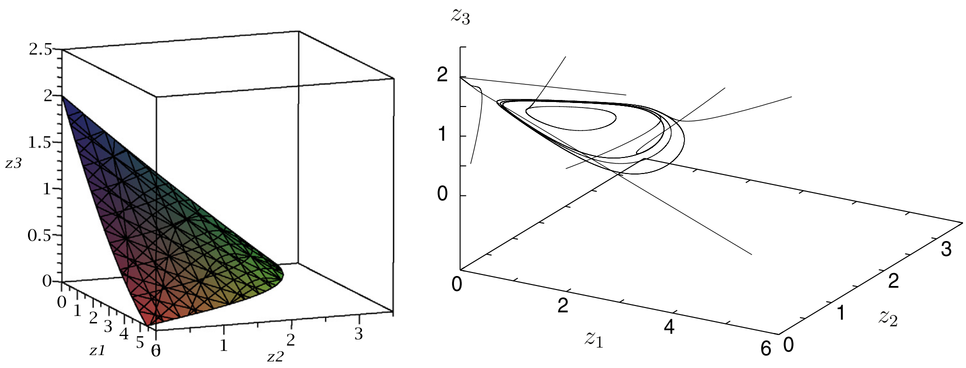


Fig. 5. (a) System behavior for the first case of class 26 with the critical values given in (4.8): (a) the center manifold near the positive equilibrium; and (b) numerical simulation showing convergence to the positive equilibrium.

It should be noted that since this is class 26, there are additional two equilibrium points on the boundary of the carrying simplex, one of them is stable and the other one is unstable. The coordinates of the five equilibrium points located on the boundary of the carrying simplex are approximated as

$$(5.4936, 0, 0), \quad (0, 3.1604, 0), \quad (0, 0, 1.9892), \quad (0.3253, 3.1062, 0), \quad (0.3384, 0, 1.8218),$$

among which the last second one is unstable, and the last one is stable. This can be seen from Fig. 5(b) that one trajectory converges to the above last equilibrium point, while other trajectories quickly converge to the carrying simplex and then almost stay there as a closed orbit because the positive equilibrium is a fourth-order fine focus. The central area of the carrying simplex is obviously above the triangle plane. The simulation indicates that except for the four small-amplitude limit cycles, there are no other limit cycles which may occur on the carrying simplex.

For the second example, following a similar procedure, we also obtain only one solution (which is slightly different from that given in (4.8), as expected):

$$p_1 = 2.494009\cdots, \quad p_2 = 3.390952\cdots, \quad p_3 = 1.230410\cdots, \quad p_4 = 5.934852\cdots, \tag{4.10}$$

at which $\omega_c \approx 0.620188$. Further, at the solution we obtain that

$$v_0 = v_1 = v_2 = v_3 = 0, \quad v_4 \approx -0.363907 \times 10^{-5},$$

and

$$\frac{\partial(v_1, v_2, v_3)}{\partial(p_1, p_2, p_3)} \approx -0.496536 \times 10^{-8} \neq 0,$$

as well as the conditions given in (4.9) hold for this example. Hence, the second example also belongs to class 26, and we perturb the first four focus values, v_i, $i = 0, 1, 2, 3$ using p_i, $i = 1, 2, 3, 4$ to obtain four small-amplitude limit cycles.

For this example, since the matrix A is almost the same as that for the first example (see Eq. (4.7)), and the unique solution given in (4.10) does not have much difference from that for the first example (see Eq. (4.8)). So it is expected that the center manifold and simulation result should be very similar to that shown in Fig. 5. Indeed, as seen from Fig. 6(a), the center manifold is not a plane, and it can also be seen from Fig. 6(b) that all trajectories with different initial points quickly converge to the carrying simplex, with one of them particularly converging to the stable equilibrium on the boundary of the carrying simplex.

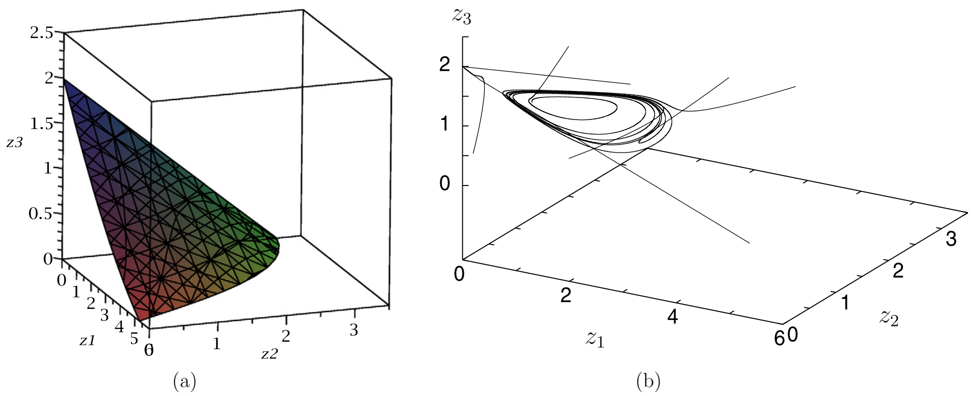


Fig. 6. (a) System behavior for the second case of class 26 with the critical values given in (4.10): (a) the center manifold near the positive equilibrium; and (b) numerical simulation showing convergence to the positive equilibrium.

This indicates that except for the four small-amplitude limit cycles, there are no other limit cycles which may occur on the carrying simplex. The approximate equations for the center manifold is given by

$$\begin{aligned} F(z_1, z_2, z_3) \approx\ & 11.55561330(z_1 - 1) + 16.91789018(z_2 - 1) + 32.99673290(z_3 - 1) \\ & + 0.41892209(z_1 - 1)^2 + 0.47012405(z_2 - 1)^2 - 0.87731188(z_3 - 1)^2 \\ & - 2.25521701(z_1 - 1)(z_2 - 1) + 1.72721322(z_1 - 1)(z_3 - 1) \\ & + 0.57273933(z_2 - 1)(z_3 - 1), \end{aligned}$$

and the coordinates of the five equilibrium points located on the boundary of the carrying simplex are approximated as

$$(5.3814, 0, 0), \quad (0, 3.1783, 0), \quad (0, 0, 1.9976), \quad (0.3189, 3.1145, 0), \quad (0.3306, 0, 1.8327),$$

among which the last second one is unstable, and the last one is stable.

The proof for Theorem 4.3 is complete. □

5. Conclusion

By properly choosing parameter values, we have constructed four 3-dimensional competitive Lotka–Volterra systems, showing the existence of at least four small-amplitude limit cycles around the positive equilibrium due to Hopf bifurcation in classes 27 and 26. Poincaré–Bendixson theorem is not applicable for the constructed examples in class 27 though heteroclinic cycles exist. We also give positive answers to the Items (ii) and (iii) of a conjecture proposed by Gyllenberg and Yan [10]. Although Item (i) in this conjecture remains open, our new results may provide clues to find possible combination of parameter values such that besides the four small-amplitude limit cycles, there may exist an additional limit cycle due to the existence of heteroclinic cycle. Also, whether there are four limit cycles in other classes 28–31 is still open and worth for future studies.

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