



On a class of hyperbolic equations in Weyl–Hörmander calculus



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ABSTRACT

In this paper we study the Cauchy problem for hyperbolic equations in the setting of Hörmander $S(m, g)$ classes. We provide regularity estimates, existence and uniqueness in the scale of Sobolev spaces $H(m, g)$ adapted to the Weyl–Hörmander calculus. We also obtain estimates for some parabolic evolution equations.

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1. Introduction

In this work we study regularity estimates, existence and uniqueness for the Cauchy problem corresponding to an equation of the form

$$\frac{\partial^\ell u}{\partial t^\ell} = P_\ell(t, x, D_x)u,$$

where for every $t \in \mathbb{R}$, $P_\ell(t, x, D_x)$ is a suitable pseudodifferential operator of order ℓ ($\ell = 1, 2$) in the sense of Hörmander $S(m, g)$ classes (cf. [15–17]). The Cauchy problem for hyperbolic partial differential and pseudo-differential operators has been intensively studied for a long time. Here we consider the situation for a very general class of pseudodifferential operators allowing degeneracies in space that cannot be treated with more classical classes.

An adaptation to the Weyl–Hörmander calculus of the general method of order reduction for hyperbolic equations (cf. [24], Chap. 4; [25], Chap. 7) will naturally lead us to consider more general equations. In particular, we will also obtain regularity estimates for hyperbolic systems and parabolic evolution equations.

In order to give a better illustration of this work we briefly describe the problematic. The simplest case of order 2 corresponds to the wave equation, which is associated to an operator of the form $\frac{\partial^2}{\partial t^2} - P$, where

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P is an elliptic second order differential operator. The notion of ellipticity can be extended to the setting of Hörmander $S(m, g)$ classes by means of the Planck function corresponding to the Hörmander's metric g on $\mathbb{R}^n \times \mathbb{R}^n$. The *Planck function* or the *uncertainty parameter* $h(x, \xi)$ associated to g is defined by $h(x, \xi) = \sup_{(y, \eta) \neq 0} \left(\frac{g(x, \xi)(y, \eta)}{g^\sigma(x, \xi)(y, \eta)} \right)^{1/2}$, where g^σ is the dual metric of g with respect to the canonical symplectic form $[(y, \eta), (z, \zeta)] = z \cdot \eta - y \cdot \zeta$. For a Hörmander metric g and a g -weight m , the class of pseudodifferential operators corresponding to a class of symbols $S(m, g)$ is denoted by $OpS(m, g)$. The Planck function h can be used in particular to extend the notion of operators of order μ for a real number μ (cf. Section 2). Indeed, the weight $h(x, \xi)^{-\mu}$ will correspond to the order μ . In particular, we will denote by $\Delta_{(g)}$ an operator of the form

$$\Delta_{(g)} := h^{-2}(x, D) + R(x, D), \quad (1.1)$$

where $R \in OpS(h^{-1}, g)$. The notation $\Delta_{(g)}$ is inspired from the notation for the Laplace–Beltrami operators associated to a Riemannian metric. In our case the metric g is defined on the phase space.

If g is a Hörmander metric, we consider the following Cauchy problem with initial data in the scale of Sobolev spaces $H(m, g)$ adapted to the Weyl–Hörmander calculus:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = -\gamma(t)\Delta_{(g)}u + w, \\ u(0) = f_0, \\ \frac{\partial u}{\partial t}(0) = f_1, \end{cases} \quad (1.2)$$

where $0 < \gamma \in C^\infty(\mathbb{R})$, $f_0 \in H(h^{-s}, g)$, $f_1 \in H(h^{-(s-1)}, g)$ and $w \in L^2([0, T], H(h^{-(s-1)}, g))$ for some $s \in \mathbb{R}$. Under such assumptions we will prove the existence and uniqueness of a solution $u \in C([0, T], H(h^{-s}, g))$. In particular, if $g = g^{1,0} = dx^2 + \frac{d\xi^2}{\langle \xi \rangle^2}$, one has $h^{-2}(x, D) = -\Delta + I$, where Δ is the Laplacian on \mathbb{R}^n and I the identity operator. By choosing $\gamma = 1$, $R = -1$, the problem (1.2) becomes the classical Cauchy problem for the wave equation. In general, the operator $\Delta_{(g)}$ is not elliptic, a parametrix of $\Delta_{(g)}$ does not even exist as a classical pseudodifferential operator and the setting of Weyl–Hörmander calculus is appropriate to deal with such situations.

Analogous regularity estimates are well known when $\Delta_{(g)}$ is replaced by a second order partial differential operator $P(t, x, D_x)$ such that $\frac{\partial^\ell u}{\partial t^\ell} - P$ is strictly hyperbolic.

We recall (cf. [18,19]) that a pseudodifferential operator with symbol $0 \leq L \in S(h^{-2}, g)$ is called *g-subelliptic* with index of subellipticity $0 < \tau \leq 2$ if

$$\|f\|_{H(h^{-\tau}, g)} \leq C(\|L(x, D)f\|_{L^2} + \|f\|_{L^2}),$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. In particular, if $g = g^{1,0}$ the definition above absorbs the classical notion of subellipticity. If the order of subellipticity is $\tau = 2$, we say that L is *g-elliptic*. When the metric is $g = g^{1,0}$, we will also simply say subelliptic instead of $g^{1,0}$ -subelliptic.

We now turn on to a more specific situation. Let L be a second order differential operator, formally self-adjoint,

$$Lf = - \sum a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f + \text{lower order terms}, \quad f \in C_0^\infty(\mathbb{R}^n), \quad (1.3)$$

where $(a_{ij}(x))$ is a positive semi-definite matrix of real functions and $a_{ij} \in C_b^\infty(\mathbb{R}^n)$ (C^∞ functions which are bounded as well as all their derivatives). If the operator L is not elliptic, $(L + C)^{-1}$ with $C > 0$ is

not a classical pseudodifferential operator. This is the kind of fact that has led to the introduction of more general classes as the $\varphi - \Phi$ Beals–Fefferman classes (cf. [2,1]) and the $S(m, g)$ -classes of Hörmander. If L is subelliptic with order of subellipticity $1 \leq \tau \leq 2$, we associate to L the Hörmander metric g defined by

$$g_X(dx, d\xi) = m(X)^{-2}(\langle \xi \rangle^2 dx^2 + d\xi^2), \quad (X = (x, \xi) \in \mathbb{R}^{2n}) \quad (1.4)$$

where

$$m(X) = m(x, \xi) = (a(x, \xi) + \langle \xi \rangle^\tau)^{\frac{1}{2}}, \quad (1.5)$$

$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ and $a(x, \xi)$ is the principal symbol of L . Metrics of this type with $\tau = 1$ have been associated to the operator L with loss of one derivative by Beals (cf. [1]). The metric g with $\tau = 1$ has also been applied in [8] for the study of parametrices for sum of squares satisfying the Hörmander condition of order 2. The invertibility of $L + C$ in the setting of $S(m, g)$ classes has been studied in [12]. The case $0 < \tau \leq 1$ has been investigated for special highly degenerate Grushin operators in [26] and [13]. The study of invertibility requires of lower bounds estimates in a crucial way [21,20]. Fundamental solutions for highly degenerate elliptic operators (or in other terms small τ) have been obtained in [3,4]. Formulas for the propagators of degenerate hyperbolic equations were established in [5]. The L^p boundedness in the setting of $S(m, g)$ calculus for fractional powers of subelliptic operators has been studied in [11] and [10]. Recent developments on hyperbolic equations with rough time dependent coefficients can be found in [14] and the references therein.

For L as in (1.3), the corresponding Planck function for the metric (1.4) is given by $h(x, \xi) = \frac{\langle \xi \rangle}{a(x, \xi) + \langle \xi \rangle^\tau}$, and an example of a g -elliptic operator is $P = (L + \Lambda^\tau)^2 \Lambda^{-2}$ where Λ is the Bessel potential of order 1 on \mathbb{R}^n , i.e. corresponding to the symbol $\langle \xi \rangle$.

2. Basics of Weyl–Hörmander calculus

In this section we recall the basic elements of the Weyl–Hörmander calculus which will be essential for us as well as some notations. For the details of this theory we refer the reader to [16] and [17].

For a function $u \in C^\infty(\mathbb{R}^n)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will employ the following useful notations:

$$|\alpha| = \alpha_1 + \dots + \alpha_n; D_i^l u = (-i)^l \frac{\partial^l u}{\partial x_i^l}; D^\alpha u = D_n^{\alpha_n} \dots D_2^{\alpha_2} D_1^{\alpha_1} u; x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

The Kohn–Nirenberg and Weyl quantizations are recalled below:

Definition 2.1. For $a(x, \xi) \in S'(\mathbb{R}^n \times \mathbb{R}^n)$ ($x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$), we define the *Kohn–Nirenberg or classic quantization* as the operator $a(x, D) : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$ given by

$$a(x, D)u(x) = (2\pi)^{-n} \int \int e^{i(x-y)\xi} a(x, \xi) u(y) dy d\xi.$$

The *Weyl quantization* has a fundamental relationship with the symplectic structure of $\mathbb{R}^n \times \mathbb{R}^n = T^*\mathbb{R}^n$ as we will see below. The Weyl quantisation of $a(x, \xi)$, is given by the operator $a^w : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$ where

$$a^w u(x) = (2\pi)^{-n} \int \int e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

Definition 2.2. Let $a(x, \xi), b(x, \xi) \in S(\mathbb{R}^n \times \mathbb{R}^n)$ we define

$$(a \# b)(X) = \pi^{-2n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} e^{-2i\sigma(X-Y_1, X-Y_2)} a(Y_1) b(Y_2) dY_1 dY_2,$$

where $\sigma(X, Y) = y \cdot \xi - x \cdot \eta$ for $X = (x, \xi)$ and $Y = (y, \eta)$.

The operation $\#$ is useful in order to describe the composition $a^w \circ b^w$, indeed one has $a^w \circ b^w = (a \# b)^w$. Moreover, from the formula for the composition we notice how the symplectic form naturally appears in the integral expression above.

We shall now recall the definition of Hörmander metrics on the phase-space. A Hörmander metric is a special Riemannian metric which carries basic information of the differential operator target into the phase space.

Definition 2.3. For $X \in \mathbb{R}^n \times \mathbb{R}^n$ let $g_X(\cdot)$ be a positive definite quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$, we say that $g(\cdot)$ is a Hörmander's metric if the following three conditions are satisfied:

1. **Continuity.** There exist constants $C, c, c' \in \mathbb{R}$ such that $g_X(Y) \leq C$, for $X, Y \in \mathbb{R}^n \times \mathbb{R}^n$, implies $c' \cdot g_{X+Y}(T) \leq g_X(T) \leq c \cdot g_{X+Y}(T)$ for every $T \in \mathbb{R}^n \times \mathbb{R}^n$.
2. **Uncertainty principle.** For $Y = (y, \eta)$ and $Z = (z, \zeta)$ we define $\sigma(Y, Z) = z \cdot \eta - y \cdot \zeta$, and

$$g_X^\sigma(Y) = \sup_{Z \neq 0} \frac{\sigma(Y, Z)^2}{g_X(Z)}.$$

We say that g satisfies the uncertainty principle if

$$\lambda_g(X) = \inf_{T \neq 0} \left(\frac{g_X^\sigma(T)}{g_X(T)} \right)^{1/2} \geq 1,$$

for all $X \in \mathbb{R}^n \times \mathbb{R}^n$.

3. **Temperateness.** We say that g is temperate if there exist $C > 0$ and $N_0 \in \mathbb{N}$ such that

$$\left(\frac{g_X(\cdot)}{g_Y(\cdot)} \right)^{\pm 1} \leq C(1 + g_Y^\sigma(X - Y))^{N_0}.$$

Let g be a Hörmander's metric, the *uncertainty parameter* or the *Planck function* associated to g is defined by

$$h(X)^2 = \sup_{T \neq 0} \frac{g_X(T)}{g_X^\sigma(T)}$$

and it is clear that $h(X) = (\lambda_g(X))^{-1}$.

The uncertainty principle can be translated then by the condition

$$h(X) \leq 1.$$

Remark 2.4. (i) For a *split metric* g , i.e. of type

$$g_X(dx, d\xi) = \sum_{i=1}^n \frac{dx_i^2}{a_i(X)} + \frac{d\xi_i^2}{b_i(X)},$$

where $a_i(X)$ and $b_i(X)$ are positive functions, one can prove

$$g_X^\sigma(dx, d\xi) = \sum_{i=1}^n b_i(X) dx_i^2 + a_i(X) d\xi_i^2.$$

(ii) A special case of (i) is the one of *symmetrically split metric* i.e. of type

$$g_X(dx, d\xi) = \frac{dx^2}{a(X)} + \frac{d\xi^2}{b(X)}.$$

The metric (1.4) is an example of a such type.

(iii) If g is a split metric one can prove the following formula for λ_g

$$\lambda_g(X) = \min_j \sqrt{a_j(X)b_j(X)}. \quad (2.1)$$

(iv) In particular, if g is symmetrically split like in (ii) then

$$\lambda_g(X) = \sqrt{a(X)b(X)}. \quad (2.2)$$

(v) If g is given by (1.4) then

$$\lambda_g(X) = \frac{a(x, \xi) + \langle \xi \rangle^\tau}{\langle \xi \rangle} = \frac{m(x, \xi)^2}{\langle \xi \rangle}. \quad (2.3)$$

(vi) The metric

$$g = \frac{dx^2}{1 + |x|^2 + |\xi|^2} + \frac{d\xi^2}{1 + |x|^2 + |\xi|^2} \quad (2.4)$$

is a symmetrically split metric and

$$\lambda_g(X) = 1 + |x|^2 + |\xi|^2. \quad (2.5)$$

The metric (2.4) is useful in spectral theory and with it one recovers the Shubin classes (cf. [23], Chap. IV).

The classical weight $\langle \xi \rangle^m$ is generalized in the following way for a corresponding Hörmander metric.

Definition 2.5. We say that a strictly positive function M is a g -weight or g -continuous if there exist $\tilde{C} > 0$ and $N \in \mathbb{N}$ such that

$$\begin{aligned} \left(\frac{M(X+Y)}{M(X)} \right)^{\pm 1} &\leq \tilde{C}, \quad \text{if } g_X(Y) \leq \frac{1}{\tilde{C}}, \\ \left(\frac{M(Y)}{M(X)} \right)^{\pm 1} &\leq \tilde{C}(1 + g_Y^\sigma(X-Y))^N. \end{aligned}$$

Definition 2.6. For a Hörmander metric g and a g -weight M , we denote by $S(M, g)$ the set of all smooth functions a on $\mathbb{R}^n \times \mathbb{R}^n$ such that for any integer k there exists $C_k \geq 0$, such that for all $X, T_1, \dots, T_k \in \mathbb{R}^n \times \mathbb{R}^n$

$$|a^{(k)}(X; T_1, \dots, T_k)| \leq C_k M(X) \prod_{i=1}^k g_X^{1/2}(T_i).$$

For $a \in S(M, g)$ we denote by $\|a\|_{k, S(M, g)}$ the minimum C_k satisfying the above inequality. The class $S(M, g)$ becomes a Frechet space endowed with the family of seminorms $\|\cdot\|_{k, S(M, g)}$.

Remark 2.7. The function λ_g is a g -weight for the metric g (cf. [16]). Given a g -weight M , it is possible to construct an equivalent smooth weight \widetilde{M} such that $\widetilde{M} \in S(M, g)$ (cf. [16, 17]). In particular, for λ_g there exists an equivalent smooth weight $\widetilde{\lambda}_g$ such that $\widetilde{\lambda}_g \in S(\lambda_g, g)$. Hence, $\widetilde{\lambda}_g \in S(\lambda_g, g)$. A weight M such that $M \in S(M, g)$ is called *regular*. Thus the weight λ_g and consequently the Planck function h can be assumed to be regulars.

If $g = g^{\rho, \delta}$ is a (ρ, δ) -metric, we have

$$\lambda_g(X) = \langle \xi \rangle^{(\rho - \delta)}.$$

In the special case $\rho = 1, \delta = 0$ then one has $\lambda_g(X) = \langle \xi \rangle$. The weight λ_g can be seen as an extension of the basic one $\langle \xi \rangle$, for the (ρ, δ) classes. The symbols in $S(\lambda_g^\mu, g)$ for $\mu \in \mathbb{R}$ can be seen as the symbols of order μ with respect to the metric g . In particular, λ_g^μ is a symbol of order μ with respect to g .

We recall the following theorem which gives the asymptotic expansion for the composition of two symbols. The proof can also be found in [7] or [17].

Theorem 2.8. Let g be a Hörmander metric on $\mathbb{R}^n \times \mathbb{R}^n$, M_1, M_2 two g -weights. If $a \in S(M_1, g)$, $b \in S(M_2, g)$, then for all $N \in \mathbb{N}$,

$$a \# b - \sum_{0 \leq j < N} \frac{1}{j!} (i[D_{X_1}, D_{X_2}])^j (a \otimes b)|_{\text{diagonal}} \in S(M_1 M_2 \lambda_g^{-N}, g). \quad (2.6)$$

We shall now define the Sobolev spaces adapted to the Weyl–Hörmander calculus. Here we adopt the Beals’s definition for simplicity in the presentation of the basic theory. For a comprehensive development of the Sobolev spaces in this setting the reader should be addressed to the works [6, 17].

Definition 2.9. Let g be a Hörmander metric and M a g -weight. We will call Sobolev space relative to M and it will be denoted by $H(M, g)$, the set of tempered distributions u on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$a^w u \in L^2, \quad \forall a \in S(M, g). \quad (2.7)$$

Remark 2.10. It has been shown that the metric (1.4) is a Hörmander’s metric and the weight (1.5) is g -continuous (cf. [8]). Moreover, the metric (1.4) is dominated by the strongly temperate metric $g^{\frac{1}{2}, \frac{1}{2}}$ on $\mathbb{R}^n \times \mathbb{R}^n$.

The action of the Weyl quantization on the Sobolev spaces is determined by the following theorem (cf. [6]).

Theorem 2.11. Let g be a Hörmander metric, M and M_1 be two g -weights. For every $a \in S(M, g)$, we have

$$a^w : H(M_1, g) \rightarrow H(M_1/M, g).$$

It is customary to identify $H(1, g)$ ($M = 1$) with L^2 (cf. [6]):

Theorem 2.12. For a Hörmander’s metric g we have $H(1, g) = L^2$.

Remark 2.13. The Beals's [Definition 2.9](#) of the Sobolev space $H(M, g)$ requires a test over all the symbols in $S(M, g)$. In contrast, we note that the classical Sobolev spaces $H^m = H(\langle \xi \rangle^m, g^{1,0})$ are defined by the condition on the tempered distribution u :

$$a(x, D)u \in L^2, \text{ with } a(x, \xi) = \langle \xi \rangle^m.$$

This means that, in the classical case the Sobolev space is defined by a fixed symbol. This property can be generalized to the Sobolev spaces $H(M, g)$. That is a consequence of the existence (cf. [\[6\]](#)), for every g -weight M of a one parameter group (for the operation $\#$) of symbols $b_t \in S(M^t, g)$. Thus, $b_s \# b_t = b_{s+t}$ for all $s, t \in \mathbb{R}$. In particular:

Given a g -weight M , there exist $b \in S(M, g)$ and $b' \in S(M^{-1}, g)$, such that $b \# b' = b' \# b = 1$. Moreover, for every g -weight M_1 , the operator b^w is an isomorphism from $H(M_1, g)$ onto $H(M_1/M, g)$.

For a such b , one has that a tempered distribution u belongs to $H(M, g)$ if and only if $b^w u \in L^2(\mathbb{R}^n)$.

3. Existence and regularity estimates

In this section we prove the main results of this work. First, we consider the problem of existence and regularity for a corresponding extension to the $S(M, g)$ calculus of the notion of hyperbolic equations of order 1. Second, we obtain existence and regularity estimates for the problem [\(1.2\)](#).

The reduction of hyperbolic equations of higher order yields a system of equations of lower order. This kind of situation motivates the introduction of a more general setting of classes including matrix-valued symbols. In order to study the equation [\(1.2\)](#) we shall require to deal with systems and specifically for the finite dimensional case. With suitable adaptations it is well known that most of the results for the scalar-valued symbols can be extended to the matrix-valued case. In particular, the basic theorems relative to the L^2 -Sobolev boundedness of pseudo-differential operators in the $S(m, g)$ calculus are still valid for systems. However, the Fefferman–Phong inequality cannot in general be extended to the case of systems. Some counterexamples for systems and the Fefferman–Phong inequality are given in [\[22\]](#). In the last part of this section we consider a parabolic type equation, in the scalar case and we will be in a position to apply the Fefferman–Phong inequality.

An $\ell \times \ell$ -matrix symbol belongs to $S(M, g)_{\ell \times \ell}$ if each one of its entries is a symbol in $S(M, g)$. In the following theorems we will write $K(t, x, \xi) \in S(\lambda_g, g)_{\ell \times \ell}$, which should be understood in the sense that for each $t \in \mathbb{R}$ fixed, $K(t, \cdot, \cdot) \in S(\lambda_g, g)_{\ell \times \ell}$. We will also define the spaces $H_g^{+\infty} = \bigcap_{s \in \mathbb{R}} H(\lambda_g^s, g)$ and $H_g^{-\infty} = \bigcup_{s \in \mathbb{R}} H(\lambda_g^s, g)$. A classical result for energy estimates on hyperbolic equations of order 1 can be stated in $S(M, g)$ -calculus as follows:

Theorem 3.1. *Let g be a Hörmander metric. Let $K(t, x, \xi) \in S(\lambda_g, g)_{\ell \times \ell}$ depending smoothly on t . Assume that $K^*(t, x, D_x) + K(t, x, D_x) \in OpS(1, g)_{\ell \times \ell}$, where for t fixed, $K(t, x, D_x) = K(t)$ denotes the pseudodifferential operator corresponding to $K(t, x, \xi)$. Let $s \in \mathbb{R}$, $T > 0$. If $v \in C([0, T], H(\lambda_g^{s+1}, g)) \cap C^1([0, T], H(\lambda_g^s, g))$ and $Q = \partial_t - K(t)$. Then v satisfies*

$$\|v(t)\|_{H(\lambda_g^s, g)}^2 \leq e^{Ct} \left(\|v(0)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|Qv(\tau)\|_{H(\lambda_g^s, g)}^2 d\tau \right) \quad (3.1)$$

for all $t \in [0, T]$. Moreover, we can replace $v(0)$ by $v(T)$ on the right-hand side of [\(3.1\)](#). The same conclusion holds for the operator Q^* .

Proof. We consider the one parameter group b_s as in [Remark 2.13](#), and we will quantize b_s with respect to the Kohn–Nirenberg quantization and write $b_s(x, D) = \Lambda_g^s$. Thus, a tempered distribution u belongs to

$H(\lambda_g^s, g)$ if and only if $\Lambda_g^s u \in L^2(\mathbb{R}^n)$. We now assume that $v \in C([0, T], H(\lambda_g^{s+1}, g)) \cap C^1([0, T], H(\lambda_g^s, g))$ and write $\omega = Qv$. Since $\partial_t v = (\partial_t - K(t))v + K(t)v = Qv + K(t)v = \omega + K(t)v$, we observe that

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{H(\lambda_g^s, g)}^2 &= \frac{d}{dt} \langle \Lambda_g^s v, \Lambda_g^s v \rangle \\ &= 2\operatorname{Re} \langle \Lambda_g^s v_t, \Lambda_g^s v \rangle \\ &= 2\operatorname{Re} \langle \Lambda_g^s (K(t)v + \omega), \Lambda_g^s v \rangle \\ &= 2\operatorname{Re} \langle \Lambda_g^s K(t)v, \Lambda_g^s v \rangle \\ &\quad - 2\operatorname{Re} \langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle \\ &\quad + 2\operatorname{Re} \langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle \\ &\quad + 2\operatorname{Re} \langle \Lambda_g^s \omega, \Lambda_g^s v \rangle \\ &= 2\operatorname{Re} \langle [\Lambda_g^s, K(t)]v, \Lambda_g^s v \rangle \\ &\quad + 2\operatorname{Re} \langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle \\ &\quad + 2\operatorname{Re} \langle \Lambda_g^s \omega, \Lambda_g^s v \rangle. \end{aligned} \tag{3.2}$$

The term (3.2) can be written in the following way

$$\begin{aligned} 2\operatorname{Re} \langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle &= \langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle + \overline{\langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle} \\ &= \langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle + \langle \Lambda_g^s v, K(t) \Lambda_g^s v \rangle \\ &= \langle K(t) \Lambda_g^s v, \Lambda_g^s v \rangle + \langle K(t)^* \Lambda_g^s v, \Lambda_g^s v \rangle \\ &= \langle (K(t) + K(t)^*) \Lambda_g^s v, \Lambda_g^s v \rangle. \end{aligned}$$

Now, applying Theorem 2.8 and the invariance of the symbol class after switching between Weyl and Kohn–Nirenberg quantizations (cf. Theorem 2.3.19 of [17]) we get $A(t) = [\Lambda_g^s, K(t)] \in \operatorname{Op}S(\lambda_g^s, g)$. Since we also have $K(t) + K(t)^* \in \operatorname{Op}S(1, g)_{\ell \times \ell}$, it follows that

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{H(\lambda_g^s, g)}^2 &\leq \\ &\leq \|A(t)v\|_{L^2} \|v\|_{H(\lambda_g^s, g)} + C_1 \|v\|_{H(\lambda_g^s, g)}^2 + C_2 \|\omega\|_{H(\lambda_g^s, g)} \|v\|_{H(\lambda_g^s, g)} \\ &\leq C \|v\|_{H(\lambda_g^s, g)} \|v\|_{H(\lambda_g^s, g)} + C_1 \|v\|_{H(\lambda_g^s, g)}^2 + C_2 \|\omega\|_{H(\lambda_g^s, g)} \|v\|_{H(\lambda_g^s, g)} \\ &\leq C \|v\|_{H(\lambda_g^s, g)}^2 + C \|\omega\|_{H(\lambda_g^s, g)}^2. \end{aligned}$$

An application of the Gronwall inequality gives us the energy inequality

$$\|v(t)\|_{H(\lambda_g^s, g)}^2 \leq e^{Ct} \left(\|v(0)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|\omega(\tau)\|_{H(\lambda_g^s, g)}^2 d\tau \right). \tag{3.3}$$

We can also prove an analogous estimate with $v(T)$ instead of $v(0)$ on the right-hand side of the inequality (3.3). The conclusion for Q^* follows analogously. \square

We now establish a theorem concerning existence, uniqueness and regularity as an application of the above estimates.

Theorem 3.2. Let g be a Hörmander metric. Let $K(t, x, \xi) \in S(\lambda_g, g)_{\ell \times \ell}$ depending smoothly on t . Assume that $K^*(t, x, D_x) + K(t, x, D_x) \in OpS(1, g)_{\ell \times \ell}$. Let $s \in \mathbb{R}$, $T > 0$, $f \in H(\lambda_g^s, g)$, $\omega \in L^2([0, T], H(\lambda_g^s, g))$. Then, there exists a unique $v \in C([0, T], H(\lambda_g^s, g))$ such that

$$\begin{cases} \frac{\partial v}{\partial t} = K(t)v + \omega, & (\text{in the sense of } \mathcal{D}'([0, T] \times \mathbb{R}^n)) \\ v(0) = f. \end{cases} \quad (3.4)$$

Moreover, the solution v satisfies the energy estimate (3.1). If $\omega \in C^\infty([0, T], H_g^{+\infty})$ and $f \in H_g^{+\infty}$ then $v \in C^\infty([0, T], H_g^{+\infty})$.

Proof. We will now prove the existence of a solution v of (3.4) in $C([0, T], H(\lambda_g^s, g))$. The proof is an adaptation of the corresponding part in the proof of Theorem 4.5 in [9]. We write $Q = \frac{\partial}{\partial t} - K$ and we introduce the space $E = \{\varphi \in C^\infty([0, T], H_g^{-\infty}) | \varphi(T) = 0\}$. We will see that we can define a linear form β on Q^*E by

$$Q^*\varphi \rightarrow \beta(Q^*\varphi) = \int_0^T (\omega(t, \cdot), \varphi(t, \cdot)) dt + \frac{1}{i} (f, \varphi(0, \cdot)).$$

We note that the energy estimate (3.1) holds for $-s$ for the Cauchy problem (3.4) corresponding to the operator Q^* with $\|v(T, \cdot)\|_{H(\lambda_g^s, g)}$ on the right hand side of (3.1). Thus, for $\varphi \in E$ we have

$$\|\varphi(t, \cdot)\|_{H(\lambda_g^{-s}, g)}^2 \leq C \int_0^T \|Q^*\varphi(t', \cdot)\|_{H(\lambda_g^{-s}, g)}^2 dt', \quad t \in [0, T],$$

so that

$$|\beta(Q^*\varphi)|^2 \leq C' \int_0^T \|Q^*\varphi(t', \cdot)\|_{H(\lambda_g^{-s}, g)}^2 dt'.$$

We deduce that β is well defined and continuous with respect to the topology induced on Q^*E by $L^2([0, T], H(\lambda_g^{-s}, g))$. An application of the Hahn–Banach theorem implies the existence of an element $v \in (L^2([0, T], H(\lambda_g^{-s}, g)))' = L^2([0, T], H(\lambda_g^s, g))$ such that

$$(v, Q^*\varphi) = \int_0^T (\omega(t, \cdot), \varphi(t, \cdot)) dt + \frac{1}{i} (f, \varphi(0, \cdot)) \quad (3.5)$$

for all $\varphi \in E$. In particular, if $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^n)$, (3.5) implies that $Qv = \omega$ in $\mathcal{D}'([0, T] \times \mathbb{R}^n)$. Thus $\frac{\partial}{\partial t}v = Kv + \omega \in L^2([0, T], H(\lambda_g^{s-1}, g))$. An integration by parts with respect to t in (3.5) implies that $(v(0, \cdot), \varphi(0, \cdot)) = (f, \varphi(0, \cdot))$ for all $\varphi \in E$ and consequently $v(0) = v(0, \cdot) = f$.

Now, if $\omega \in C^\infty([0, T], H_g^{+\infty})$ and $f \in H_g^{+\infty}$, the above argument shows that $v \in C([0, T], H_g^{+\infty})$. Moreover, since $\frac{\partial}{\partial t}v = Kv + \omega$, one can deduce step by step that $v \in C^k([0, T], H_g^{+\infty})$ for all $k \geq 0$. Consequently $v \in C^\infty([0, T], H_g^{+\infty})$.

We will now prove that $v \in C([0, T], H(\lambda_g^s, g))$ and that it satisfies the energy estimate (3.1). Suppose we have sequences $(\omega)_j$ in $C_0^\infty([0, T] \times \mathbb{R}^n)$ and $(f)_j$ in $C_0^\infty(\mathbb{R}^n)$ such that $\omega_j \rightarrow \omega$ in $L^2([0, T], H(\lambda_g^s, g))$ and $f_j \rightarrow f$ in $H(\lambda_g^s, g)$. Let $v_j \in C^\infty([0, T], H_g^{+\infty})$ be the solution of $Qv_j = \omega_j$, $v_j(0, \cdot) = f_j$. The inequality (3.1) applied to the $v_j - v_k$ shows that v_j is a Cauchy sequence in $C([0, T], H(\lambda_g^s, g))$ so that $v_j \rightarrow \tilde{v}$

in $C([0, T], H(\lambda_g^s, g))$. In the limit, we have $Q\tilde{v} = \omega$, $\tilde{v}(0, \cdot) = f$; consequently, the uniqueness shows that $\tilde{v} = v$.

The corresponding inequality (3.1) for v is obtained passing to the limit in this inequality applied to v_j . In this way we conclude the proof of the Theorem. The uniqueness of the solution v follows from the energy inequality (3.1). \square

Now, let us come back to the problem (1.2). We first note that, for a given Hörmander metric g we can choose $\widetilde{\lambda}_g$, as in Remark 2.7, equivalent to λ_g such that $\widetilde{\lambda}_g \in S(\widetilde{\lambda}_g, g)$. In particular, $\widetilde{\lambda}_g$ is a smooth symbol and we can justify the use of quantization for the Planck function in (1.1) and reformulate (1.2), by writing

$$\Delta_{(g)} := \widetilde{\lambda}_g^2(x, D) + R(x, D),$$

where $R \in OpS(\lambda_g, g)$.

As it will become clear below, we will assume the following property on the Hörmander metric g .

Definition 3.3. Let g be a Hörmander metric, we will say that g is *regular* if there exists a regular weight $\widetilde{\lambda}_g$ equivalent to λ_g (see Remark 2.7) which yields an invertible pseudodifferential operator $\widetilde{\lambda}_g(x, D)$ from $H(\lambda_g, g)$ into L^2 .

We note that if two g -weights M_1 and M_2 are equivalent, then $S(M_1, g) = S(M_2, g)$ and $H(M_1, g) = H(M_2, g)$. In particular, with the notation of the Definition 3.3, we have $S(\lambda_g^s, g) = S(\widetilde{\lambda}_g^s, g)$ and $H(\lambda_g^s, g) = H(\widetilde{\lambda}_g^s, g)$ for all $s \in \mathbb{R}$.

Example. (i) Given a Hörmander metric g , we recall that $\lambda_g = h^{-1}$, where h is the Planck or the uncertainty parameter associated to g . The metrics $g^{\rho, \delta}$ are regular since $\lambda_g(x, \xi) = \langle \xi \rangle^{\rho - \delta}$.

(ii) The metric g defined by (1.4) associated to the operator L is regular. In this case, $\lambda_g(x, \xi) = \frac{a(x, \xi) + \langle \xi \rangle^\tau}{\langle \xi \rangle}$, and the invertibility of $\lambda_g(x, D)$ will follow from the invertibility of $L + C$ (see Lemma 3.6).

(iii) The metric g defined by (2.4) is regular since

$$\lambda_g(x, D) = -\Delta + |x|^2 + I.$$

Under the assumption that g is regular and in order to apply Theorem 3.1 to the study of the problem (1.2), we will decompose

$$\frac{\partial^2 u}{\partial t^2} = -\gamma(t)\Delta_{(g)}u + w,$$

into the form

$$\frac{\partial}{\partial t} \frac{\partial u}{\partial t} = -\gamma(t)\Delta_{(g)}A^{-1}Au + w,$$

by taking $A = \gamma(t)^{\frac{1}{2}}\widetilde{\lambda}_g(x, D)$.

We will also apply the following mild lemma which requires of an additional condition on the metric holding in most of important cases in applications.

Lemma 3.4. Let g be Hörmander metric such that $g_X(y, \eta) = g_X(y, -\eta)$ for all $X = (x, \xi)$, $Y = (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$. Let m be a g -weight and $a \in S(m, g)$. Then $(a(x, D))^* - \bar{a}(x, D) \in OpS(m\lambda_g^{-1}, g)$.

Proof. We consider the isomorphism $J^t : S(m, g) \rightarrow S(m, g)$ (see Theorem 2.3.18 of [17]) given by $J^t a = \exp itD_x \cdot D_\xi a$. In particular, for $t = \frac{1}{2}$ we have

$$\begin{aligned} (a(x, D))^* &= ((J^{\frac{1}{2}} a)^w)^* \\ &= \overline{(J^{\frac{1}{2}} a)}^*. \end{aligned} \quad (3.6)$$

On the other hand, if $b \in S(m, g)$ one has $J^{\frac{1}{2}} b - b \in S(m\lambda_g^{-1}, g)$ (see identity (2.3.29) of [17]) with $\nu = 1$). By applying this to $b = a$ in the identity (3.6) we conclude the proof. \square

We can now state a theorem for existence and regularity estimates for a solution of (1.2).

Theorem 3.5. *Let g be a regular Hörmander metric such that $g_X(y, \eta) = g_X(y, -\eta)$ for all $X = (x, \xi)$, $Y = (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $s \in \mathbb{R}$, $T > 0$, $0 < \gamma \in C^\infty(\mathbb{R})$, $f_0 \in H(\lambda_g^s, g)$, $f_1 \in H(\lambda_g^{s-1}, g)$ and $w \in L^2([0, T], H(\lambda_g^{s-1}, g))$. Then, there exists a unique solution $u \in C([0, T], H(\lambda_g^s, g))$ of the Cauchy problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = -\gamma(t)\Delta_{(g)}u + w, & (\text{in the sense of } \mathcal{D}'([0, T] \times \mathbb{R}^n)) \\ u(0) = f_0, \\ \frac{\partial u}{\partial t}(0) = f_1. \end{cases} \quad (3.7)$$

Moreover, the solution u satisfies the following energy estimate

$$\|u(t)\|_{H(\lambda_g^s, g)}^2 \leq Ce^{Ct} \left(\|f_0\|_{H(\lambda_g^s, g)}^2 + \|f_1\|_{H(\lambda_g^{s-1}, g)}^2 + \int_0^t \|w(\tau)\|_{H(\lambda_g^{s-1}, g)}^2 d\tau \right). \quad (3.8)$$

If $w \in C^\infty([0, T], H_g^{+\infty})$ and $f \in H_g^{+\infty}$ then $u \in C^\infty([0, T], H_g^{+\infty})$.

Proof. We put $A(t) = \gamma(t)^{\frac{1}{2}} \widetilde{\lambda}_g(x, D)$ with $\widetilde{\lambda}_g$ as in Definition 3.3. Since $A(t) \in OpS(\lambda_g, g)$ for every t , $A(t)$ is invertible and $A(t)^{-1} \in OpS(\lambda_g^{-1}, g)$, we can now write

$$\begin{aligned} \underbrace{\frac{\partial}{\partial t}}_{v_2} \underbrace{\frac{\partial u}{\partial t}}_{v_1} &= -\gamma(t)\Delta_{(g)}A(t)^{-1} \underbrace{A(t)u}_{v_1} \\ \frac{\partial}{\partial t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 & A \\ -\gamma(t)\Delta_{(g)}A^{-1} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ w \end{bmatrix}. \end{aligned}$$

It is clear that $K(t) \in OpS(\lambda_g, g)_{2 \times 2}$, where $K(t)$ is the 2×2 matrix-valued operator:

$$K(t) = \begin{bmatrix} 0 & A \\ -\gamma(t)\Delta_{(g)}A^{-1} & 0 \end{bmatrix}.$$

We are now going to prove that $K + K^* \in OpS(1, g)$. We have

$$\begin{aligned} -\gamma(t)\Delta_{(g)}A^{-1} &= (-\gamma(t)\widetilde{\lambda}_g^2(x, D) - \gamma(t)R)\gamma(t)^{-\frac{1}{2}}\widetilde{\lambda}_g(x, D)^{-1}, \text{ where } R \in OpS(\lambda_g, g), \\ &= -\gamma(t)^{\frac{1}{2}}\widetilde{\lambda}_g(x, D) - \gamma(t)^{\frac{1}{2}}R\widetilde{\lambda}_g(x, D)^{-1}, \\ &= -\gamma(t)^{\frac{1}{2}}\widetilde{\lambda}_g(x, D) - \gamma(t)^{\frac{1}{2}}T_1, \text{ where } T_1 \in OpS(1, g). \end{aligned}$$

We now observe by applying [Lemma 3.4](#) that

$$A^* = (\gamma(t)^{\frac{1}{2}} \widetilde{\lambda}_g(x, D))^* = \gamma(t)^{\frac{1}{2}} \widetilde{\lambda}_g(x, D) + \gamma(t)^{\frac{1}{2}} T_2, \quad \text{where } T_2 \in OpS(1, g).$$

Hence

$$-\gamma(t) \Delta_{(g)} A^{-1} + A^* = \gamma(t)^{\frac{1}{2}} (-T_1 + T_2) \in OpS(1, g). \quad (3.9)$$

Consequently, by taking adjoints in [\(3.9\)](#) we also have

$$A + (-\gamma(t) \Delta_{(g)} A^{-1})^* = \gamma(t)^{\frac{1}{2}} (-T_1^* + T_2^*) \in OpS(1, g).$$

Then $K + K^* \in OpS(1, g)_{2 \times 2}$. Now, since $Af_0 \in H(\lambda_g^{s-1}, g)$ and applying [Theorem 3.2](#) to

$$v(0) = f = \begin{bmatrix} Af_0 \\ f_1 \end{bmatrix} \in H(\lambda_g^{s-1}, g), \quad \omega = \begin{bmatrix} 0 \\ w \end{bmatrix} \in L^2([0, T], H(\lambda_g^{s-1}, g)), \quad (3.10)$$

we obtain $v \in C([0, T], H(\lambda_g^{s-1}, g))$. Since $u = A^{-1}v_1 \in H(\lambda_g^s, g)$ we deduce that $u \in C([0, T], H(\lambda_g^s, g))$. The uniqueness of u follows from the uniqueness of v and the invertibility of A since $u = A^{-1}v_1$. The inequality [\(3.8\)](#) is an immediate consequence of the inequality [\(3.1\)](#) applied to the data [\(3.10\)](#). The last conclusion of the theorem also follows from the analogous part of [Theorem 3.2](#). \square

We will now consider the special case of the metric g defined by [\(1.4\)](#) and the corresponding λ_g .

Lemma 3.6. *Let g be the Hörmander metric given by [\(1.4\)](#). Then g is regular.*

Proof. The function λ_g is in this case given by $\frac{a(x, \xi) + \langle \xi \rangle^\tau}{\langle \xi \rangle}$. The classical weight $\langle \xi \rangle$ is regular for the metric g and $a(x, \xi) + \langle \xi \rangle^\tau \in S(m^2, g)$ (cf. [\[12\]](#)). Hence λ_g is regular. On the other hand, as a consequence of the main theorem in [\[12\]](#), the operator $L + \Lambda^\tau$ is invertible in $OpS(m^2, g)$, for L is as in [\(1.3\)](#). We recall that here, Λ^τ denotes the Bessel potential of order τ on \mathbb{R}^n , i.e. with symbol $\langle \xi \rangle^\tau$. Now, by observing that $(L + \Lambda^\tau) \circ \Lambda^{-1}$ has an exact symbol with respect to the Kohn–Nirenberg quantization which is equal to $(a(x, \xi) + \langle \xi \rangle^\tau) \langle \xi \rangle^{-1} = \lambda_g(x, \xi)$, we deduce that $\lambda_g(x, D)$ is invertible in the algebra of pseudodifferential operators $OpS(m, g)$. More precisely, $\lambda_g(x, D) : H(\lambda_g, g) \rightarrow L^2$ is an isomorphism. Thus, g is regular. \square

As a consequence we obtain the following theorem for the metric [\(1.4\)](#) associated to the subelliptic operator L with principal symbol $a(x, \xi)$.

Theorem 3.7. *Let g be the Hörmander metric defined by [\(1.4\)](#). Let $s \in \mathbb{R}$, $T > 0$, $0 < \gamma \in C^\infty(\mathbb{R})$, $f_0 \in H(\lambda_g^s, g)$, $f_1 \in H(\lambda_g^{s-1}, g)$ and $w \in L^2([0, T], H(\lambda_g^{s-1}, g))$. Then, there exists a unique solution $u \in C([0, T], H(\lambda_g^s, g))$ of the Cauchy problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = -\gamma(t) a(x, D)^2 \Lambda^{-2} u + w, & (\text{in the sense of } \mathcal{D}'([0, T] \times \mathbb{R}^n)) \\ u(0) = f_0, \\ \frac{\partial u}{\partial t}(0) = f_1. \end{cases} \quad (3.11)$$

Moreover, the solution u satisfies the inequality [\(3.8\)](#). If $w \in C^\infty([0, T], H_g^{+\infty})$ and $f \in H_g^{+\infty}$ then $u \in C^\infty([0, T], H_g^{+\infty})$.

Proof. We note that for the metric (1.4), the operator $\Delta_{(g)}$ is of the form

$$\Delta_{(g)} = (L + \Lambda^\tau)^2 \Lambda^{-2} + R(x, D),$$

where $R(x, D) \in OpS(\lambda_g, g)$.

We consider the case $\tau = 1$ which is the worst situation, and observe that the symbol $\lambda_g(x, \xi)^2$ can be written as

$$\begin{aligned} \lambda_g(x, \xi)^2 &= (a(x, \xi) + \langle \xi \rangle)^2 \langle \xi \rangle^{-2} \\ &= a(x, \xi)^2 \langle \xi \rangle^{-2} + 2a(x, \xi) \langle \xi \rangle^{-1} + 1. \end{aligned}$$

Since $a(x, \xi) \langle \xi \rangle^{-1} \in S(\lambda_g, g)$, the operator $a(x, D)^2 \Lambda^{-2}$ is of the form $\Delta_{(g)}$:

$$a(x, D)^2 \Lambda^{-2} = \lambda_g^2(x, D) + R_1(x, D) = \Delta_{(g)},$$

where $R_1(x, D) \in OpS(\lambda_g, g)$ and $R_1(x, \xi) = 2a(x, \xi) \langle \xi \rangle^{-1} + 1$.

By Lemma 3.6 the metric g is regular and it is clear that $g_X(y, \eta) = g_X(y, -\eta)$ for all $X = (x, \xi)$, $Y = (y, \eta)$. Now by applying Theorem 3.5 to the operator $\Delta_{(g)} = a(x, D)^2 \Lambda^{-2}$ we conclude the proof. \square

Remark 3.8. In the case, $g = g^{\rho, \delta}$ where $\delta < \rho$, one has $\lambda_g(x, \xi) = \langle \xi \rangle^{(\rho - \delta)}$ and an operator $\Delta_{(g)}$ is of the form

$$\Delta_{(g)} = \Lambda^{2(\rho - \delta)} + R(x, D),$$

where $R(x, D) \in OpS(\langle \xi \rangle^{(\rho - \delta)}, g^{\rho, \delta})$.

4. Parabolic type equations

We shall now consider a parabolic type case by proving a version of Theorem 3.2 when K is of order 2 and scalar. We first establish a corresponding energy estimate. In the following theorems we will be considering operators $P(x, D_x) \in OpS(\lambda_g^\nu, g)$ satisfying the following condition with respect to a Hörmander metric g and $\nu \in \mathbb{R}$:

(E, g, ν) There exists a constant C_ν such that

$$ReP(x, \xi) \geq C_\nu (\lambda_g - 1)^\nu. \quad (4.1)$$

We note that $\lambda_g - 1 \geq 0$ due to the uncertainty principle. The condition (E, g, ν) is an extension of the notion of strongly ellipticity in the classes $S_{1,0}^\nu$. For the metric $g = g^{1,0}$ and $\nu = 2$, we have $\lambda_g^2 = \langle \xi \rangle^2$, and the ellipticity of order 2 for $K \in S_{1,0}^2$ is equivalent to the condition $(E, g^{1,0}, 2)$. Some other examples are given at the end of this section.

Theorem 4.1. Let g be a Hörmander metric, $s \in \mathbb{R}$, $T > 0$, $0 \leq -K(t, x, \xi) \in S(\lambda_g^2, g)$ depending smoothly on t and satisfying the condition $(E, g, 2)$. Let $Q = \frac{\partial}{\partial t} - K(t, x, D_x)$, $v \in C^1([0, T], H(\lambda_g^s, g)) \cap C([0, T], H(\lambda_g^{s+1}, g))$. Then, there exists a constant $C > 0$ such that v satisfies the energy estimate:

$$\begin{aligned} &\|v(t)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|v(\tau)\|_{H(\lambda_g^{s+1}, g)}^2 d\tau \\ &\leq C \left(\|v(0)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|Qv(\tau)\|_{H(\lambda_g^{s-1}, g)}^2 d\tau \right) \end{aligned} \quad (4.2)$$

for all $t \in [0, T]$. The same conclusion holds for the operator Q^* .

Proof. First we will obtain some L^2 estimates. We write $Qv = \omega$, as in the proof of [Theorem 3.1](#) one can prove that

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{L^2}^2 &= 2\operatorname{Re}\langle \partial_t v, v \rangle \\ &= 2\operatorname{Re}\langle K(t)v, v \rangle \end{aligned} \quad (4.3)$$

$$+ 2\operatorname{Re}\langle \omega, v \rangle. \quad (4.4)$$

The term (4.3) can be estimated by applying the Fefferman–Phong inequality ([\[16\]](#), Theorem 18.6.8) to the nonnegative symbol $-K(t) + C_2 - C_2\lambda_g^2$. We have

$$\begin{aligned} \operatorname{Re}\langle K(t)v, v \rangle &= \operatorname{Re}\langle (K(t) - C_2 + C_2\Lambda_g^2 + C_2 - C_2\Lambda_g^2)v, v \rangle \\ &\leq C\|v\|_{L^2}^2 + C_3\|v\|_{L^2}^2 - C_2\|v\|_{H(\lambda_g, g)}^2 \\ &\leq C\|v\|_{L^2}^2 - C_2\|v\|_{H(\lambda_g, g)}^2. \end{aligned} \quad (4.5)$$

For (4.4), we have

$$\begin{aligned} |\langle v, \omega \rangle| &\leq \|v\|_{H(\lambda_g, g)} \|\omega\|_{H(\lambda_g^{-1}, g)} \\ &\leq \frac{1}{2}(\|v\|_{H(\lambda_g, g)}^2 + \|\omega\|_{H(\lambda_g^{-1}, g)}^2). \end{aligned} \quad (4.6)$$

Then, by (4.5), (4.6) we obtain

$$\frac{d}{dt} \|v(t)\|_{L^2}^2 + \|v(t)\|_{H(\lambda_g, g)}^2 \leq C\|v(t)\|_{L^2}^2 + C_0\|\omega(t)\|_{H(\lambda_g^{-1}, g)}^2.$$

Integrating between 0 and t the above inequality we get

$$\begin{aligned} &\|v(t)\|_{L^2}^2 + \int_0^t \|v(\tau)\|_{H(\lambda_g, g)}^2 d\tau \\ &\leq C \left(\|v(0)\|_{L^2}^2 + \int_0^t (\|v(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{H(\lambda_g^{-1}, g)}^2) d\tau \right). \end{aligned} \quad (4.7)$$

Therefore, replacing v by $\Lambda_g^s v$ we have

$$\begin{aligned} &\|v(t)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|v(\tau)\|_{H(\lambda_g^{s+1}, g)}^2 d\tau \\ &\leq C \left(\|v(0)\|_{H(\lambda_g^s, g)}^2 + \int_0^t (\|v(\tau)\|_{H(\lambda_g^s, g)}^2 + \|Q\Lambda_g^s v(\tau)\|_{H(\lambda_g^{-1}, g)}^2) d\tau \right). \end{aligned} \quad (4.8)$$

We now observe that $\Lambda_g^{-1}Q\Lambda_g^s v = \Lambda_g^{s-1}Qv + O(\|v\|_{H(\lambda_g^s, g)})$. Thus, applying this identity to second term in the integral of the right-hand side of (4.8) we get

$$\|v(t)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|v(\tau)\|_{H(\lambda_g^{s+1}, g)}^2 d\tau$$

$$\leq C \left(\|v(0)\|_{H(\lambda_g^s, g)}^2 + \int_0^t (\|v(\tau)\|_{H(\lambda_g^s, g)}^2 + \|\omega(\tau)\|_{H(\lambda_g^{s-1}, g)}^2) d\tau \right). \quad (4.9)$$

Now by Lemma 4.4 of [9] we obtain

$$\begin{aligned} & \|v(t)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|v(\tau)\|_{H(\lambda_g^{s+1}, g)}^2 d\tau \\ & \leq C \left(\|v(0)\|_{H(\lambda_g^s, g)}^2 + \int_0^t \|\omega(\tau)\|_{H(\lambda_g^{s-1}, g)}^2 d\tau \right). \end{aligned} \quad (4.10)$$

This proves the estimate (4.2). The fact that the estimate also holds for Q^* follows analogously taking into account that K^* also satisfies the condition $(E, g, 2)$. \square

Reasoning analogously as in the proof of Theorem 3.2 by using instead the above energy inequality we obtain the following theorem.

Theorem 4.2. *Let g be a Hörmander metric, $s \in \mathbb{R}$, $T > 0$, $0 \leq -K(t, x, \xi) \in S(\lambda_g^2, g)$ depending smoothly on t and satisfying the condition $(E, g, 2)$. Let $f \in H(\lambda_g^s, g)$ and $\omega \in L^2([0, T], H(\lambda_g^{s-1}, g))$. Then, there exists a unique solution $v \in C([0, T], H(\lambda_g^s, g))$ of the Cauchy problem*

$$\begin{cases} \frac{\partial v}{\partial t} = K(t)v + \omega, & (\text{in the sense of } \mathcal{D}'([0, T] \times \mathbb{R}^n)) \\ v(0) = f. \end{cases} \quad (4.11)$$

Moreover, $v \in L^2([0, T], H(\lambda_g^{s+1}, g))$ the solution v satisfies the energy estimate (4.2). If $\omega \in C^\infty([0, T], H_g^{+\infty})$ and $f \in H_g^{+\infty}$ then $v \in C^\infty([0, T], H_g^{+\infty})$.

Example. (i) By taking $g = g^{1,0}$ on $\mathbb{R}^n \times \mathbb{R}^n$, and $K(t, x, \xi) = -|\xi^2|$. Then $K = K(D) = \Delta$, where Δ is the Laplacian on \mathbb{R}^n and one recovers a classical Cauchy problem for the heat equation.

(ii) If $g = \frac{dx^2}{1+|x|^2+|\xi|^2} + \frac{d\xi^2}{1+|x|^2+|\xi|^2}$ and $\mu > 0$, then $(-\Delta + |x|^2 + I)^\mu \in OpS(\lambda_g^\mu, g)$. Hence, one can consider K of the form $K = \gamma(t)(-\Delta + |x|^2 + I)^2$ with $0 \leq \gamma \in C^\infty(\mathbb{R})$.

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