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Dual spaces of weak Orlicz–Hardy spaces for vector-valued martingales

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ABSTRACT

The dual spaces of weak Orlicz–Hardy spaces for **B**-valued martingales, which are defined by quasi-concave functions, are identified. To obtain the results, the so-called weak-type generalized Campanato spaces are introduced and the classical technique of atomic decompositions is employed. The results obtained here extend the corresponding known results in the literature from real-valued setting to vector-valued setting and from strong-type spaces to weak-type spaces.

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1. Introduction

In recent years, due to their important role to know more precise properties of functions and martingales, some weak-type spaces have attracted more and more attention in harmonic analysis, martingale theory and functional space theory, e.g. [6,19] for weak Hardy spaces wH_p, [4,8,10,18] for Lorentz–Hardy spaces $H_{p,q}$ and [1,2,11,24] for Orlicz–Lorentz spaces $L_{F,G}$. In particular, the study of the so-called weak Orlicz spaces wL_Φ as well as weak Orlicz–Hardy spaces wH_Φ was initiated in the fundamental paper [12]. Later on and in particular in the very last years the theory has been developed rapidly. We refer to [7,9,13,16,22,23] and the references given there. For one thing, the weak Orlicz space is a more extensive class than weak Lebesgue space, classical Lebesgue space and classical Orlicz space; for another, it is also an extreme subclass of Orlicz–Lorentz spaces.

The purpose of this paper is to characterize the dual space of weak Orlicz–Hardy space $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$ for Banach space-valued martingales, which is defined by the conditional p-variation $s_p(\cdot)$ and associated with a quasi-concave function Φ. The results obtained here can be regarded as the extensions of that in Weisz [19], Miyamoto, Nakai and Sadasue [14] and Jiao, Wu and Peng [9] from real-valued setting to vector-valued setting, and that in Yu [21] from strong-type spaces to weak-type spaces.

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In the setting of real-valued martingales, the H_p-BMO duality theory is nowadays classical. It was proved by Herz in [3] and Weisz in [17,18] that the dual space of martingale Hardy space $H_p^{s_2}$ defined by the conditional square function for $0 < p \leq 1$ is the α-ordered squared Lipschitz class $\mathcal{L}_{2,\alpha}$ with $\alpha = \frac{1}{p} - 1$. Moreover, it was shown recently by Miyamoto, Nakai and Sadasue in [14] that the dual space of martingale Orlicz–Hardy space $H_\Phi^{s_2}$ is the generalized Campanato space $\mathcal{L}_{2,\phi}$ with $\phi(x) = 1/(x\Phi^{-1}(1/x))$, where Φ is equivalent to a concave function.

On weak-type martingale spaces, very recently, the Lorentz–Hardy–Karamata spaces $H_{p,q,b}^{s_2}$ (including Lorentz–Hardy spaces $H_{p,q}^{s_2}$ as special case when $b(t) \equiv 1$) and their dual spaces were investigated by K.-P. Ho in [4] and Jiao, Xie and Zhou in [10]. For weak martingale Hardy spaces $wH_p^{s_2} = H_{p,\infty}^{s_2}$, an extreme subclass of Lorentz–Hardy spaces $H_{p,q}^{s_2}$, let $\overline{H}_p^{s_2}$ be the $wH_p^{s_2}$ closure of $H_\infty^{s_2}$, the dual space of $\overline{H}_p^{s_2}$ for $0 < p \leq 1$ was identified by Weisz in [19] as weak Lipschitz space $w\mathcal{L}_{2,\alpha}$ with $\alpha = \frac{1}{p} - 1$.

As for the vector-valued case, it was proved by the author in [20] that the dual space of $\mathbf{B}$-valued martingale Hardy space $H_r^{s_p}(\mathbf{B})$ for $1 \leq r < \infty$ is the generalized Garsia space $\mathcal{K}_{r'}^{s_q}(\mathbf{B}^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Furthermore, recently, when $0 < r \leq 1$, the dual space of Hardy space $H_r^{s_p}(\mathbf{B})$ was identified by the author in [21] as the so-called generalized Lipschitz space $\mathcal{L}_{q,\beta}^{s_q}(\mathbf{B}^*)$ where $\beta = \frac{1}{r} - 1$.

It is natural to consider the characterization of the dual space of $\mathbf{B}$-valued martingale weak Orlicz–Hardy space $wH_\Phi^{s_p}(\mathbf{B})$ (including $wH_r^{s_p}(\mathbf{B})$ for $0 < r \leq 1$ as a special case), where Φ is a concave function.

In order to give the answer to this question, a new generalized weak Campanato space named $w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B})$ will be introduced and a more generalized theorem of atomic decompositions will be established in this paper.

2. Preliminaries and notation

Let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space with norm $\|\cdot\|$, $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability measure space, and $\{\mathcal{F}_n\}_{n\geq 0}$ a sequence of non-decreasing sub-σ-algebras of $\mathcal{F}$. The expectation operator is denoted by $E(\cdot)$, and the conditional expectation operator relative to $\mathcal{F}_n$ by $E(\cdot|\mathcal{F}_n)$.

Let $f = \{f_n\}_{n\geq 0}$ be a $\mathbf{B}$-valued martingale adapted to $\{\mathcal{F}_n\}_{n\geq 0}$ such that $f_0 = 0$. Denote its martingale difference by $df_n = f_n - f_{n-1}$ ($n \geq 0$, with convention $df_0 = 0$).

For $0 < p < \infty$, the p-variation and the conditional p-variation of a $\mathbf{B}$-valued martingale f are defined, respectively, by

$$S_{p,n}(f) := \left(\sum_{i=1}^{n} \|df_i\|^p\right)^{\frac{1}{p}}, \quad S_p(f) := \left(\sum_{i=1}^{\infty} \|df_i\|^p\right)^{\frac{1}{p}},$$

$$s_{p,n}(f) := \left(\sum_{i=1}^{n} E(\|df_i\|^p|\mathcal{F}_{i-1})\right)^{\frac{1}{p}}, \quad s_p(f) := \left(\sum_{i=1}^{\infty} E(\|df_i\|^p|\mathcal{F}_{i-1})\right)^{\frac{1}{p}}.$$

For the sake of convenience, $L_p(\mathbf{B})$ will denote the space of $\mathbf{B}$-valued functions or martingales which are defined on the probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\|f\|_{L_p(\mathbf{B})} := [E(\|f\|^p)]^{1/p} < \infty$ for functions or $\|f\|_{L_p(\mathbf{B})} := \sup_{n\geq 0}[E(\|f_n\|^p)]^{1/p} < \infty$ for martingales.

Let $\mathcal{G}$ be the set of all functions $\Phi : (0, +\infty) \to (0, +\infty)$ such that $\lim_{t\to 0^+} \Phi(t) = \Phi(0) = 0$ and $\lim_{t\to+\infty} \Phi(t) = +\infty$.

For any $\Phi \in \mathcal{G}$, we define the Orlicz space $L_\Phi = L_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ as the collection of all measurable functions g such that $\|g\|_{L_\Phi} < \infty$, where

$$\|g\|_{L_\Phi} := \inf\left\{c > 0 : \ E\left[\Phi\left(\frac{|g|}{c}\right)\right] \leq 1\right\}.$$

We define the weak Orlicz space $wL_\Phi = wL_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ as the set of all measurable functions g such that $\|g\|_{wL_\Phi} < \infty$, where

$$\|g\|_{wL_\Phi} := \inf\Big\{c > 0 : \sup_{t>0} \Phi\big(\frac{t}{c}\big)\mathbb{P}(|g| > t) \le 1\Big\}.$$

It is known (see [12]) that wL_Φ is a quasi-normed space with

$$\|f + g\|_{wL_\Phi} \le \kappa_\Phi(\|f\|_{wL_\Phi} + \|g\|_{wL_\Phi})$$

where $\kappa_\Phi \ge 1$ is a constant only depending on Φ.

Unlike the classical Orlicz spaces, it was proved by [13] that each space wL_Φ (even $\Phi(x) = x^p$ with $1 \le p < \infty$) has no absolutely continuous quasi-norm. Hence, let $w\mathcal{L}_\Phi$ be the set of all functions $g \in wL_\Phi$ having the absolute continuous quasi-norm defined by

$$w\mathcal{L}_\Phi := \big\{g \in wL_\Phi : \lim_{\mathbb{P}(A)\to 0} \|g\chi_A\|_{wL_\Phi} = 0\big\}.$$

It is easy to check that $w\mathcal{L}_\Phi$ is a closed linear subspace of wL_Φ, and a control convergence theorem in wL_Φ was shown by Liu and Wang in [13] as below:

Lemma 2.1. *(See [13].) Let $\Phi \in \Delta_2$, $f_n, f \in wL_\Phi$, $|f_n| \le g$ $(n \ge 1)$ and $g \in w\mathcal{L}_\Phi$. If $\{f_n\}$ converges to f almost everywhere, then $\lim_{n\to\infty} \|f_n - f\|_{wL_\Phi} = 0$.*

For $\mathbf{B}$-valued martingale, define the Orlicz–Hardy spaces and weak Orlicz–Hardy spaces, respectively, by

$$H_\Phi^{s_p}(\mathbf{B}) := \big\{f = \{f_n\}_{n\ge 0} : \|f\|_{H_\Phi^{s_p}(\mathbf{B})} = \|s_p(f)\|_{L_\Phi} < \infty\big\},$$
$$wH_\Phi^{s_p}(\mathbf{B}) := \big\{f = \{f_n\}_{n\ge 0} : \|f\|_{wH_\Phi^{s_p}(\mathbf{B})} = \|s_p(f)\|_{wL_\Phi} < \infty\big\}.$$

Define the closed linear subspace of $wH_\Phi^{s_p}(\mathbf{B})$, in which the conditional p-variation has the absolute continuous quasi-norm, by

$$w\mathcal{H}_\Phi^{s_p}(\mathbf{B}) := \big\{f = \{f_n\}_{n\ge 0} \in wH_\Phi^{s_p}(\mathbf{B}) : \|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})} = \|s_p(f)\|_{w\mathcal{L}_\Phi} < \infty\big\}.$$

For any $1 \le p < \infty$ and any function $\phi : (0, +\infty) \to (0, +\infty)$, we denote

$$t_\phi^p(f, x) := \frac{1}{\phi(x)} x^{-\frac{1}{p}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|f - f^\tau\|_{L_p(\mathbf{B})},$$
$$u_\phi^p(f, x) := \frac{1}{\phi(x)} x^{-\frac{1}{p}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_p(f - f^\tau)\|_{L_p(\mathbb{R})},$$

where τ is a stopping time and f^τ is the stopped martingale. Furthermore, the generalized weak Campanato spaces for $\mathbf{B}$-valued martingales are defined by

$$w\mathcal{L}_{p,\phi}(\mathbf{B}) := \Big\{f = \{f_n\}_{n\ge 0} \in L_p : \|f\|_{w\mathcal{L}_{p,\phi}(\mathbf{B})} := \int_0^\infty \frac{t_\phi^p(f, x)}{x}\mathrm{d}x < \infty\Big\},$$

$$w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B}) := \Big\{f = \{f_n\}_{n\ge 0} \in L_p : \|f\|_{w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B})} := \int_0^\infty \frac{u_\phi^p(f, x)}{x}\mathrm{d}x < \infty\Big\}.$$

Remark 2.1. For real martingales, the space $\mathcal{L}_{2,\phi}(\mathbb{R})$ was introduced by Miyamoto, Nakai and Sadasue in [14] and the weak-type space $w\mathcal{L}_{2,\alpha}(\mathbb{R})$ was introduced by Weisz in [19] as the dual spaces of Orlicz–Hardy space $H^{s_2}_{\Phi}(\mathbb{R})$ and weak Hardy space $w\mathcal{H}^{s_2}_{r}(\mathbb{R})$, where $\phi(x) = 1/(x\Phi^{-1}(1/x))$ and $\alpha = \frac{1}{r} - 1$, respectively. However, it was proved by the author in [21] that the dual space of $\mathbf{B}$-valued martingale Hardy space $H^{s_p}_r(\mathbf{B})$ $(0 < r \leq 1)$ is not $\mathcal{L}_{q,\beta}(\mathbf{B}^*)$ but $\mathcal{L}^{s_q}_{q,\beta}(\mathbf{B}^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta = \frac{1}{r} - 1$. For this reason, we introduce the weak-type Campanato space $w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B})$ as above. It will be seen in the final section that the relation between $w\mathcal{L}_{p,\phi}(\mathbf{B})$ and $w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B})$ is connected closely with the geometrical properties of the underlying Banach space.

Let α_0 and α_1 be real numbers such that $0 < \alpha_0 < \alpha_1$. The class $Q[\alpha_0, \alpha_1]$ consists of all functions $\Phi \in \mathcal{G}$ such that $\Phi(t)t^{-\alpha_0}$ are non-decreasing and $\Phi(t)t^{-\alpha_1}$ are non-increasing. Moreover, we say that $\Phi(t)$ belongs to the class $Q(\alpha_0, \alpha_1]$, whenever $\Phi(t) \in Q[\alpha_0 + \varepsilon, \alpha_1]$ for some $\varepsilon > 0$ (see e.g. Persson [15]).

Remark 2.2. Let $\Phi(t) \in Q[\alpha_0, \alpha_1]$, it is easy to verify that $\Phi(st) \leq \max(s^{\alpha_0}, s^{\alpha_1})\Phi(t)$ for $s, t \in (0, +\infty)$.

A slightly generalized class, which is called $\mathcal{G}[\alpha_0, \alpha_1]$ with $0 < \alpha_0 < \alpha_1$, was introduced by Miyamoto, Nakai and Sadasue [14]. Let $\mathcal{G}[\alpha_0, \alpha_1]$ be the set of all functions $\Phi \in \mathcal{G}$, if there exists a constant $c_\Phi \in [1, +\infty)$ such that

$$\Phi(st) \leq c_\Phi \max(s^{\alpha_0}, s^{\alpha_1})\Phi(t) \quad \text{for } s, t \in (0, +\infty).$$

For the relation between $Q[\alpha_0, \alpha_1]$ and $\mathcal{G}[\alpha_0, \alpha_1]$, the following result was proved by Persson in [15].

Lemma 2.2. *(See [15].) Let $\Phi \in \mathcal{G}$, the following conditions are equivalent:*
(i) $\Phi(st) \leq c_\Phi \max(s^{\alpha_0}, s^{\alpha_1})\Phi(t)$ *for* $s, t \in (0, +\infty)$, *i.e.* $\Phi \in \mathcal{G}[\alpha_0, \alpha_1]$.
(ii) $\Phi(t)$ *is equivalent to some function* $\widetilde{\Phi} \in Q[\alpha_0, \alpha_1]$.

Remark 2.3. For any $0 < \varepsilon < 1$, we note the fact that if $\Phi(t) \in \mathcal{G}[\varepsilon, 1]$, then $\Phi(t)$ is equivalent to some concave function $\widetilde{\Phi}(t) \in Q[\varepsilon, 1]$, $\Phi(t)$ is bijective and $\Phi(t) \in \Delta_2$ (see [14,15]).

The following elementary but useful lemma was also given by Persson in [15].

Lemma 2.3. *(See [15].) Let $\Phi(t) \in Q[\alpha_0, \alpha_1]$. Then the inverse $\Phi^{-1}(t)$ exists and $\Phi^{-1}(t) \in Q[\alpha_1^{-1}, \alpha_0^{-1}]$, whenever $\alpha_0 > 0$.*

The well known lemmas below were proved by Hoffmann-Jørgensen and Pisier in [5].

Lemma 2.4. *(See [5].) Let $\mathbf{B}$ be a Banach space, $1 < p \leqslant 2$. Then the following are equivalent:*
(i) $\mathbf{B}$ *is isomorphic to a p-uniformly smooth Banach space;*
(ii) *There exists a constant $C_p > 0$ only depending on p, such that*

$$E(\|f_m - f_n\|^p | \mathscr{F}_n) \leqslant C_p E\Big(\sum_{i=n+1}^{m} \|df_i\|^p \Big| \mathscr{F}_n \Big), \quad \forall\ 0 \leqslant n \leqslant m$$

for every $\mathbf{B}$-valued martingale $f = (f_n, n \in \mathbb{N})$.
(iii) *For every $\mathbf{B}$-valued martingale $f = (f_n, n \in \mathbb{N})$, if $E\big(\sum_{n=1}^{\infty} \|df_n\|^p\big) < \infty$, then $\{f_n\}_{n \geqslant 1}$ converges in probability.*

Lemma 2.5. *(See [5].) Let* $\mathbf{B}$ *be a Banach space,* $2 \leqslant q < \infty$. *Then the following are equivalent:*

(i) $\mathbf{B}$ *is isomorphic to a* q*-uniformly convex Banach space;*

(ii) *There exists a constant* $C_q > 0$ *only depending on* q, *such that*

$$E\Big(\sum_{i=n+1}^{m} \|df_i\|^q \Big| \mathscr{F}_n\Big) \leqslant C_q E(\|f_m - f_n\|^q | \mathscr{F}_n), \quad \forall\ 0 \leqslant n \leqslant m$$

for every $\mathbf{B}$*-valued martingale* $f = (f_n, n \in \mathbb{N})$.

(iii) *For every uniformly bounded* $\mathbf{B}$*-valued martingale* $f = (f_n, n \in \mathbb{N})$, $S^{(q)}(f) < \infty$, *a.e.*

3. Weak atomic decompositions

Definition 3.1. Let $p > 1$, a $\mathbf{B}$-valued martingale $a = \{a_n\}_{n\geq 0}$ is said to be a weak atomic martingale relative to the conditional p-variation $s_p(\cdot)$, if there exists a stopping time τ such that (i) $a_n = 0$, if $\tau \geq n$, (ii) $\|s_p(a)\|_\infty < +\infty$.

Definition 3.2. Let $p > 1$, a $\mathbf{B}$-valued measurable function a is said to be a weak atomic function relative to the conditional p-variation $s_p(\cdot)$, if there exists a stopping time τ such that (i) $a_n = E(a|\mathcal{F}_n) = 0$, if $\tau \geq n$, (ii) $\|s_p(a)\|_\infty < +\infty$.

Remark 3.1. It is obvious that a weak atomic function is a wake atomic martingale, but the converse is not true. So Definition 3.1 is a slight generalization of Definition 3.2. In the scalar-valued setting, atoms are usually defined as functions. However, it will be shown in Theorem 3.2 that the existence of the decompositions as atomic functions essentially depends on the uniform smoothness of the underlying Banach space in vector-valued case. In order to completely characterize the dual space of the weak Orlicz–Hardy space for $\mathbf{B}$-valued martingales, replacing the functions, it is more suitable to define the atoms as martingales.

Theorem 3.1. *Let* $\Phi(t) \in \mathcal{G}[\varepsilon, 1]$ *with* $0 < \varepsilon < 1$, *and let* $\mathbf{B}$ *be a Banach space,* $1 < p < \infty$. *Then for any martingale* $f = \{f_n\}_{n\geq 0} \in w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$, *there exist a sequence of weak atomic martingales* $\{a^{(k)} = \{a_n^{(k)}\} : n \geq 0\}_{k\in\mathbb{Z}}$ *and corresponding stopping times* $\{\tau_k\}_{k\in\mathbb{Z}}$, *such that*

(i) $f_n = \sum_{k\in\mathbb{Z}} a_n^{(k)}$, *a.e. for* $n \geq 0$.

(ii) $s_p(a^{(k)}) \leq A \cdot 2^k$, *for some* $A > 0$ *and*

$$\inf\Big\{c > 0 : \ \sup_{k\in\mathbb{Z}} \Phi\Big(\frac{2^k}{c}\Big)\mathbb{P}(\tau_k < \infty) \leq 1\Big\} < \infty. \tag{1}$$

(iii)

$$\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})} \approx \inf\inf\Big\{c > 0 : \ \sup_{k\in\mathbb{Z}} \Phi\Big(\frac{2^k}{c}\Big)\mathbb{P}(\tau_k < \infty) \leq 1\Big\}, \tag{2}$$

where the first infimum is taken over all the preceding decompositions of f.

Moreover, for each $n \geq 0$, *the sum* $\sum_{k\in\mathbb{Z}} a_n^{(k)}$ *converges to* f_n *in* $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$.

Proof. By Lemma 2.2, since for every $\Phi \in \mathcal{G}[\varepsilon, 1]$, there exists some function $\widetilde{\Phi} \in Q[\varepsilon, 1]$ such that Φ is equivalent to $\widetilde{\Phi}$ and the functionals $\|\cdot\|_{wL_\Phi}$ and $\|\cdot\|_{wL_{\widetilde{\Phi}}}$ are equivalent, it suffices to prove the theorem for every $\Phi \in Q[\varepsilon, 1]$.

Let $\mathbf{B}$-valued martingale $f = \{f_n\}_{n\geq 0} \in w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$. For every integer $k \in \mathbb{Z}$ and $n \geq 0$, we define

$$\tau_k := \inf\{n \geq 0 : \ s_{p,n+1}(f) > 2^k\}, \quad a_n^{(k)} := f_n^{\tau_{k+1}} - f_n^{\tau_k}.$$

Then $a^{(k)}=\{a_n^{(k)}\}_{n\geq 0}$ is a $\mathbf{B}$-valued martingale for any fixed $k\in\mathbb{Z}$, and we have

$$\sum_{k\in\mathbb{Z}} a_n^{(k)}=\sum_{k\in\mathbb{Z}}(f_n^{\tau_{k+1}}-f_n^{\tau_k})=f_n,\ \text{a.e. for } n\geq 0.$$

From $s_p(f^{\tau_k})=s_{p,\tau_k}(f)\leq 2^k$, we get that

$$s_p(a^{\tau_k})\leq s_p(f^{\tau_{k+1}})+s_p(f^{\tau_k})\leq 2^{k+1}+2^k=3\cdot 2^k.$$

When $\tau_k\geq n$, $a_n^{(k)}:=f_n^{\tau_{k+1}}-f_n^{\tau_k}=f_n-f_n=0$. Then, $a^{(k)}=\{a_n^{(k)}\}_{n\geq 0}$ is a weak atomic martingale and (i) holds.

Since the set of $\{\tau_k<\infty\}=\{s_p(f)>2^k\}$ for each $k\in\mathbb{Z}$, we have

$$\Phi\Big(\frac{2^k}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}\Big)\mathbb{P}(\tau_k<\infty)=\Phi\Big(\frac{2^k}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}\Big)\mathbb{P}(s_p(f)>2^k)\leq 1.$$

Then

$$\inf\Big\{c>0:\ \sup_{k\in\mathbb{Z}}\Phi\big(\frac{2^k}{c}\big)\mathbb{P}(\tau_k<\infty)\leq 1\Big\}\leq\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}<\infty, \tag{3}$$

that is (1) and (ii) is proved.

On the other hand, suppose that (i) and (ii) are satisfied, and let

$$D:=\inf\Big\{c>0:\ \sup_{k\in\mathbb{Z}}\Phi\big(\frac{2^k}{c}\big)\mathbb{P}(\tau_k<\infty)\leq 1\Big\}.$$

Without loss of generality, set $A=2^{m_A}$, where $m_A\geq 0$ is a nonnegative integer. For any $\lambda>0$, choose $j\in\mathbb{Z}$ such that $2^j\leq\lambda<2^{j+1}$. Then

$$f_n=\sum_{k=-\infty}^{j-1}a_n^{(k)}+\sum_{k=j}^{\infty}a_n^{(k)}:=g_n+h_n,\quad n\geq 0.$$

By the sub-linearity of $s_p(f)$, we have $s_p(f)\leq s_p(g)+s_p(h)$, and thus

$$\mathbb{P}(s_p(f)>2A\lambda)\leq\mathbb{P}(s_p(g)>A\lambda)+\mathbb{P}(s_p(h)>A\lambda). \tag{4}$$

From (ii), we obtain that

$$s_p(g)\leq\sum_{k=-\infty}^{j-1}s_p(a^{(k)})\leq\sum_{k=-\infty}^{j-1}A\cdot 2^k\leq A\cdot 2^j\leq A\lambda,$$

hence $\mathbb{P}(s_p(g)>A\lambda)=0$.

Since $a_n^{(k)}=0$ on the set of $\{\tau_k\geq n\}$, thus $s_p(a^{(k)})=0$ on the set of $\{\tau_k=\infty\}$. From (4) and the definition of the number D, we have

$$\begin{aligned}\Phi\Big(\frac{2A\lambda}{2D}\Big)\mathbb{P}(s_p(f)>2A\lambda)&\leq\Phi\Big(\frac{A\lambda}{D}\Big)\mathbb{P}(s_p(h)>A\lambda)\\&\leq\Phi\Big(\frac{A\lambda}{D}\Big)\sum_{k=j}^{\infty}\mathbb{P}(\tau_k<\infty)\leq\Phi\Big(\frac{A\lambda}{D}\Big)\sum_{k=j}^{\infty}\Phi\Big(\frac{2^k}{D}\Big)^{-1}.\end{aligned}$$

By the assumption $A=2^{m_A}$, we obtain $2^j\leq 2^{m_A+j}\leq A\lambda=2^{m_A}\lambda<2^{m_A+j+1}$. Then we write

$$\Phi\Big(\frac{2A\lambda}{2D}\Big)\mathbb{P}(s_p(f)>2A\lambda)\le \Phi\Big(\frac{A\lambda}{D}\Big)\sum_{k=j}^{m_A+j}\Phi\Big(\frac{2^k}{D}\Big)^{-1}+\Phi\Big(\frac{A\lambda}{D}\Big)\sum_{k=m_A+j+1}^{\infty}\Phi\Big(\frac{2^k}{D}\Big)^{-1}$$
$$:=I_1+I_2.$$

Because the function $\Phi\in Q[\varepsilon,1]$ with $0<\varepsilon<1$, then $\Phi(t)t^{-1}$ is non-increasing and $\Phi(t)t^{-\varepsilon}$ is non-decreasing. Thus

$$I_1=\sum_{k=j}^{m_A+j}\frac{\Phi\Big(\frac{A\lambda}{D}\Big)}{\Phi\Big(\frac{2^k}{D}\Big)}\le\sum_{k=j}^{m_A+j}\frac{A\lambda}{2^k}\le\sum_{k=j}^{m_A+j}\frac{2^{m_A+j+1}}{2^k}\le\frac{2^{m_A+1}}{1-2^{-1}}=2^{m_A+2}:=C_1,$$

and

$$I_2=\sum_{k=m_A+j+1}^{\infty}\frac{\Phi\Big(\frac{A\lambda}{D}\Big)}{\Phi\Big(\frac{2^k}{D}\Big)}\le\sum_{k=m_A+j+1}^{\infty}\Big(\frac{A\lambda}{2^k}\Big)^{\varepsilon}\le 2^{(m_A+j+1)\varepsilon}\sum_{k=m_A+j+1}^{\infty}\Big(\frac{1}{2^{\varepsilon}}\Big)^k=\frac{1}{1-2^{-\varepsilon}}:=C_2.$$

Let $C_0:=C_1+C_2$, it is easy to see that $C_0>1$. Then $I_1+I_2\le C_0$ and

$$\Phi\Big(\frac{2A\lambda}{C_0^{1/\varepsilon}\cdot 2D}\Big)\mathbb{P}(s_p(f)>2A\lambda)\le\frac{1}{C_0}\Phi\Big(\frac{2A\lambda}{2D}\Big)\mathbb{P}(s_p(f)>2A\lambda)\le\frac{1}{C_0}\cdot C_0=1.$$

This means that

$$\|f\|_{w\mathcal{H}_{\Phi}^{s_p}(\mathbf{B})}=\|s_p(f)\|_{wL_{\Phi}}\le 2C_0^{1/\varepsilon}\inf\Big\{c>0:\ \sup_{k\in\mathbb{Z}}\Phi\big(\frac{2^k}{c}\big)\mathbb{P}(\tau_k<\infty)\le 1\Big\}.\tag{5}$$

Combining (3) and (5), we get (2) and (iii) is proved.

To verify that the sum of $\sum_{k\in\mathbb{Z}}a_n^{(k)}$ converges to f_n in $w\mathcal{H}_{\Phi}^{s_p}(\mathbf{B})$ for every $n\ge 0$. For any $k\in\mathbb{Z}$ and $n\ge 0$, by the equality

$$s_{p,n}(f-f^{\tau_k})^p=s_{p,n}(f)^p-s_{p,n}(f^{\tau_k})^p,$$

we have

$$s_{p,n}(f-f^{\tau_k}),\ s_{p,n}(f^{\tau_k})\le s_{p,n}(f),$$

and

$$s_{p,n}(f-f^{\tau_k}),s_{p,n}(f^{\tau_{-k}})\to 0,\ \text{a.e. as } k\to\infty.$$

The martingale $f\in w\mathcal{H}_{\Phi}^{s_p}(\mathbf{B})$ implies that $s_p(f)\in w\mathcal{L}_{\Phi}$ has absolutely continuous quasi-norm. Thus, by Lemma 2.1, for each $n\ge 0$, we obtain

$$\|f_n-\sum_{k=-M}^{N}a_n^{(k)}\|_{w\mathcal{H}_{\Phi}^{s_p}(\mathbf{B})}=\|s_{p,n}(f-f^{\tau_{N+1}}+f^{\tau_{-M}})\|_{wL_{\Phi}}$$
$$\le\kappa_{\Phi}(\|s_{p,n}(f-f^{\tau_{N+1}})\|_{wL_{\Phi}}+\|s_{p,n}(f^{\tau_{-M}})\|_{wL_{\Phi}})\to 0,\ \text{as } M,N\to\infty,$$

which shows that $\sum_{k\in\mathbb{Z}}a_n^{(k)}$ converges to f_n in $w\mathcal{H}_{\Phi}^{s_p}(\mathbf{B})$. The proof of the theorem is complete. □

Moreover, replacing the weak atomic martingales, if the martingales in $w\mathcal{H}_{\Phi}^{s_p}(\mathbf{B})$ are asked to be decomposed as weak atomic functions, we have the result as below.

Theorem 3.2. *Let $\Phi(t) \in \mathcal{G}[\varepsilon, 1]$ with $0 < \varepsilon < 1$, and let $\mathbf{B}$ be a Banach space, $1 < p \leq 2$. Then the following statements are equivalent:*

(a) $\mathbf{B}$ *is isomorphic to a p-uniformly smooth Banach space.*

(b) *For every $\mathbf{B}$-valued martingale $f = \{f_n\}_{n\geq 0}$ in $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$, there exists a sequence of weak atomic functions $\{a^{(k)}\}_{k\in\mathbb{Z}}$ such that* (i), (ii) *and* (iii) *in Theorem 3.1 hold. Moreover, for any $0 < r \leq p$ we have*

$$\|a^{(k)}\|_r \leq A \cdot C_p \cdot 2^k \mathbb{P}(\tau_k < \infty)^{\frac{1}{r}} \quad \text{for all } k \in \mathbb{Z}. \tag{6}$$

Proof. (a) $\Rightarrow$ (b). Let $\mathbf{B}$ be isomorphic to a p-uniformly smooth Banach space. By Theorem 3.1, the only two things need to be proved are that every $a^{(k)}$ is a weak atomic function and the inequality (6) holds.

Applying Lemma 2.4, as the consequence of (ii) in Theorem 3.1, for any fixed $k \in \mathbb{Z}$, we have that

$$\begin{aligned}\|a^{(k)}\|_p &= \sup_{n\geq 0} \|a_n^{(k)}\|_p \leq C_p \|s_p(a^{(k)}) \cdot \chi_{\{\tau_k<\infty\}}\|_p \\ &\leq C_p \|s_p(a^{(k)})\|_\infty \mathbb{P}(\tau_k < \infty)^{\frac{1}{p}} \leq A \cdot C_p \cdot 2^k \mathbb{P}(\tau_k < \infty)^{\frac{1}{p}} < \infty.\end{aligned}$$

So, $a^{(k)} = \{a_n^{(k)}\}_{n\geq 0}$ is a L_p-bounded martingale. Since the uniformly smoothness implies Radon–Nikodym property, the almost sure convergence of the martingale $\{a_n^{(k)}\}_{n\geq 0}$ follows at once from the classical martingale theorem of Doob, and we still denote its limit function by $a^{(k)}$. Noticing that $1 < p \leq 2$, it is clear that $a_n^{(k)} = E(a^{(k)}|\mathcal{F}_n)$ and $\chi_{\{\tau_k\geq n\}} E(a^{(k)}|\mathcal{F}_n) = 0$ for all $n \geq 0$. Hence, $a^{(k)}$ is a weak atomic function.

Furthermore, by Hölder's inequality and the previous estimate of $\|a^{(k)}\|_p$, for all $k \in \mathbb{Z}$, we have

$$\|a^{(k)}\|_r = \|a^{(k)}\chi_{\{\tau_k<\infty\}}\|_r \leq \|a^{(k)}\|_p \cdot \mathbb{P}(\tau_k < \infty)^{\frac{1}{r}-\frac{1}{p}} \leq A \cdot C_p \cdot 2^k \mathbb{P}(\tau_k < \infty)^{\frac{1}{r}},$$

which proves (6).

(b) $\Rightarrow$ (a). Suppose (b) is true. Let $f = \{f_n\}_{n\geq 0}$ be a $\mathbf{B}$-valued martingale and satisfy that $\sum_{n=1}^\infty \|df_n\|^p < \infty$. Then we have

$$\begin{aligned}\|s_p(f)\|_1 \leq \|s_p(f)\|_p &= \Big[E\Big(\sum_{n=1}^\infty E\big(\|df_n\|^p \big| \mathscr{F}_{n-1}\big)\Big)\Big]^{\frac{1}{p}} \\ &= \Big[E\Big(\sum_{n=1}^\infty \|df_n\|^p\Big)\Big]^{\frac{1}{p}} \leq \Big\|\sum_{n=1}^\infty \|df_n\|^p\Big\|_\infty^{\frac{1}{p}} < \infty.\end{aligned}$$

So, the martingale $f \in \mathcal{H}_1^{s_p}(\mathbf{B}) \subset w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$. Consequently, f has the weak atomic decomposition of (i) in Theorem 3.1, hence

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k a_n^{(k)} = \sum_{k\in\mathbb{Z}} \mu_k E(a^{(k)}|\mathcal{F}_n), \text{ a.e.}, \quad \forall\ n \geqslant 1.$$

In consequences of (6), on the one hand, noticing $\{\tau_k < \infty\} = \{s_p(f) > 2^k\}$ and by Abel's transformation, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \|a^{(k)}\|_1 &\leq A \cdot C_p \sum_{k\in\mathbb{Z}} 2^k \mathbb{P}(\tau_k < \infty) = A \cdot C_p \sum_{k\in\mathbb{Z}} 2^k \mathbb{P}(s_p(f) > 2^k) \\ &= A \cdot C_p \sum_{k\in\mathbb{Z}} (2^{k+1} - 2^k)\mathbb{P}(s_p(f) > 2^k) \\ &= 2A \cdot C_p \sum_{k\in\mathbb{Z}} 2^{k-1}\mathbb{P}(2^{k-1} < s_p(f) \leq 2^k) \\ &\leq 2A \cdot C_p E[s_p(f)] = 2A \cdot C_p \|f\|_{\mathcal{H}_1^{s_p}(\mathbf{B})} < \infty.\end{aligned}$$

On the other hand, it is easy to see that, for any $k \in \mathbb{Z}$, $a_n^{(k)} \to a^{(k)}$ in L_1 as $n \to \infty$.

Therefore, for any $\varepsilon > 0$, there exist an integer $k_0 > 0$ such that $\sum_{|k|>k_0} \|a^{(k)}\|_1 < \frac{\varepsilon}{4}$, and a natural number $N > 0$ such that $\sup_{|k|\le k_0} \|a_m^{(k)} - a_n^{(k)}\|_1 < \frac{\varepsilon}{4(k_0+1)}$ when $m, n > N$. Thus

$$\begin{aligned}\|f_m - f_n\|_1 &= \|\sum_{k\in\mathbb{Z}}(a_m^{(k)} - a_n^{(k)})\|_1 \le \sum_{k\in\mathbb{Z}} \|a_m^{(k)} - a_n^{(k)}\|_1 \\ &= \sum_{|k|>k_0} \|a_m^{(k)} - a_n^{(k)}\|_1 + \sum_{|k|\le k_0} \|a_m^{(k)} - a_n^{(k)}\|_1 \\ &\le 2\sum_{|k|>k_0} \|a^{(k)}\|_1 + \sum_{|k|\le k_0} \Big(\sup_{|k|\le k_0} \|a_m^{(k)} - a_n^{(k)}\|_1\Big) < \varepsilon \quad (m, n > N),\end{aligned}$$

which implies that $\{f_n\}_{n\geqslant 0}$ is a Cauchy sequence in Banach space $L_1(\mathbf{B})$. Hence $\{f_n\}_{n\geqslant 0}$ is convergent in probability. By Lemma 2.4, $\mathbf{B}$ is isomorphic to a p-uniformly smooth space. □

Remark 3.2. Take note of the differences between Theorems 3.1 and 3.2. From Theorem 3.1, it is known that if we only ask the atoms to be martingales then no geometry of Banach spaces is requested. However, if we want additionally the atoms to be functions, Theorem 3.2 shows that the p-uniform smoothness of the Banach spaces must be employed.

4. Duality theorem for weak Orlicz–Hardy spaces

The main result of this section is the identification of the dual space of weak Orlicz–Hardy space for $\mathbf{B}$-valued martingales. To prove that the dual spaces of $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$ are the generalized weak Campanato spaces $w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*)$, we use the atomic decompositions obtained in the previous section. We start with two simple but useful lemmas in preparation.

Lemma 4.1. *Let $\Phi(t) \in \mathcal{G}[\varepsilon, 1]$ with $0 < \varepsilon < 1$, and let $\mathbf{B}$ be a Banach space, $1 < p < \infty$. Then $H_p^{s_p}(\mathbf{B})$ is dense in $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$.*

Proof. Firstly, it is known that $H_p^{s_p}(\mathbf{B}) \subset H_\Phi^{s_p}(\mathbf{B}) \subset w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$.

For every martingale $f = \{f_n\}_{n\ge 0} \in w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$, by Theorem 3.1, there exist a sequence $\{a^{(k)}\}_{k\in\mathbb{Z}}$ of weak atomic martingales and the corresponding stopping times $\{\tau_k\}_{k\in\mathbb{Z}}$, such that, for any $n \ge 0$

$$f_n = \sum_{k\in\mathbb{Z}} a_n^{(k)} = \sum_{k\in\mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}) \ \text{ in } w\mathcal{H}_\Phi^{s_p}(\mathbf{B}).$$

For any natural number N, define martingale $g^{(N)} = \{g_n^{(N)}\}_{n\ge 0}$ by

$$g_n^{(N)} := \sum_{k=-N}^{N} a_n^{(k)}, \quad \text{for } n \ge 0.$$

On the one hand, since $s_{p,n}(f^{\tau_k}) = s_p(f_n^{\tau_k}) \le 2^k$, we obtain

$$\begin{aligned}s_{p,n}(g^{(N)}) &= s_p\Big(\sum_{k=-N}^{N} a_n^{(k)}\Big) = s_p\Big(\sum_{k=-N}^{N} (f_n^{\tau_{k+1}} - f_n^{\tau_k})\Big) = s_p(f_n^{\tau_{N+1}} - f_n^{\tau_{-N}}) \\ &\le s_p(f_n^{\tau_{N+1}}) + s_p(f_n^{\tau_{-N}}) \le 2^{N+1} + 2^{-N} < \infty \ \text{ for all } n \ge 0.\end{aligned}$$

Hence $\|g^{(N)}\|_{H_p^{s_p}(\mathbf{B})} = \|s_p(g^{(N)})\|_{L_p} \le 2^{N+1} + 2^{-N} < \infty$, therefore, $g^{(N)} \in H_p^{s_p}(\mathbf{B})$ for each fixed $N \in \mathbb{N}^+$.

On the other hand, for every $n \geq 0$, by Theorem 3.1, we have that $g_n^{(N)} \to f_n$ in $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$ as $N \to \infty$. In other words, the martingale sequence $\{g^{(N)}\}_{N\geq 1}$ converges to f in $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$, this is desired. □

Lemma 4.2. *(See [21].) Let* $\mathbf{B}$ *be a Banach space,* $1 < p < \infty$. *Then* $(H_p^{s_p}(\mathbf{B}))^* \approx H_q^{s_q}(\mathbf{B}^*)$, *where* $\frac{1}{p}+\frac{1}{q} = 1$.

Theorem 4.1. *Let* $\Phi(t) \in \mathcal{G}[\varepsilon, 1]$ *with* $0 < \varepsilon < 1$, *and let* $\mathbf{B}$ *be a reflexive Banach space,* $1 < p < \infty$. *Then* $(w\mathcal{H}_\Phi^{s_p}(\mathbf{B}))^* \approx w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*)$, *where* $\frac{1}{p}+\frac{1}{q} = 1$ *and* $\phi(t) = \frac{1}{t\Phi^{-1}(1/t)}$.

Proof. (i) To prove that $w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*) \subset (w\mathcal{H}_\Phi^{s_p}(\mathbf{B}))^*$. Given a $\mathbf{B}^*$-valued martingale $g = \{g_n\}_{n\geq 0} \in w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*)$, then $s_q(g) \in L_q(\mathbb{R})$, and for every $\mathbf{B}$-valued martingale $f = \{f_n\}_{n\geq 0} \in H_p^{s_p}(\mathbf{B})$, then $s_p(f) \in L_p(\mathbb{R})$. Since $(L_p(\mathbf{B}))^* \approx L_q(\mathbf{B}^*)$, and for any $i, j \in \mathbb{N}$, $df_i \in L_p(\mathbf{B})$, $dg_j \in L_q(\mathbf{B}^*)$, $df_i \cdot dg_j := \langle df_i, dg_j\rangle$ makes sense and satisfies that

$$|E(df_i \cdot dg_j)| = |E(\langle df_i, dg_j\rangle)| \leq \|df_i\|_{L_p(\mathbf{B})} \cdot \|dg_j\|_{L_q(\mathbf{B}^*)} < +\infty.$$

Moreover, by Hölder's inequality, for $n \geq 0$, we have that

$$\begin{aligned}
|E(f_n g_n)| &= \Big|E\Big[\Big(\sum_{i=1}^n df_i\Big)\Big(\sum_{j=1}^n dg_j\Big)\Big]\Big| = \Big|E\Big[\Big(\sum_{i=1}^n df_i \cdot dg_i\Big)\Big]\Big| \\
&= \Big|E\Big[\Big(\sum_{i=1}^n E(df_i \cdot dg_i|\mathcal{F}_{i-1})\Big)\Big]\Big| \\
&\leq E\Big[\Big(\sum_{i=1}^n \big(E(\|df_i\|^p|\mathcal{F}_{i-1})\big)^{1/p} \cdot \big(E(\|dg_i\|^q|\mathcal{F}_{i-1})\big)^{1/q}\Big)\Big] \\
&\leq E\Big[\Big(\sum_{i=1}^n E(\|df_i\|^p|\mathcal{F}_{i-1})\Big)^{1/p} \cdot \Big(\sum_{i=1}^n E(\|dg_i\|^q|\mathcal{F}_{i-1})\Big)^{1/q}\Big] \\
&= E(s_{p,n}(f)s_{q,n}(g)) \leq \|s_p(f)\|_p \|s_q(g)\|_q < +\infty. \qquad (7)
\end{aligned}$$

Thus, we can define a bounded lineal functional $l_g(\cdot)$ on $H_p^{s_p}(\mathbf{B})$ by

$$l_g(f) := E(fg) := \lim_{n\to\infty} E(f_n g_n), \quad \text{for all} \quad f \in H_p^{s_p}(\mathbf{B}).$$

On the other hand, for any $\mathbf{B}$-valued martingale $f = \{f_n\}_{n\geq 0} \in H_p^{s_p}(\mathbf{B}) \subset w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$, owing to Theorem 3.1, there exists a sequence of weak atomic martingales $\{a^{(k)}\}_{k\in\mathbb{Z}}$ and corresponding stopping times $\{\tau_k\}_{k\in\mathbb{Z}}$, such that

$$f_n = \sum_{k\in\mathbb{Z}} a_n^{(k)}, \quad \text{a.e. for } n \geq 0,$$

and

$$s_p(a^{(k)}) \leq A \cdot 2^k, \quad \Phi\Big(\frac{2^k}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}\Big)\mathbb{P}(\tau_k < \infty) \leq 1, \quad \text{for } k \in \mathbb{Z}. \qquad (8)$$

Hence, $E(f_n g_n) = \sum_{k\in\mathbb{Z}} E(a_n^{(k)} \cdot g_n)$ for $n \geq 0$. Moreover, by Hölder's inequality, similar to the proof of (7), for $k \in \mathbb{Z}$, we have

$$|E(a_n^{(k)} \cdot g_n)| = \big|E\big(a_n^{(k)}(g_n - g_n^{\tau_k})\big)\big| \leq \|s_{p,n}(a^{(k)})\|_p \Big(\int_{\{\tau_k<\infty\}} s_{q,n}(g - g^{\tau_k})^q \mathrm{d}\mathbb{P}\Big)^{\frac{1}{q}}. \qquad (9)$$

From (8), we get

$$\|s_{p,n}(a^{(k)})\|_p = \|s_{p,n}(a^{(k)})\chi_{\{\tau_k<\infty\}}\|_p \le A\cdot 2^k \mathbb{P}(\tau_k<\infty)^{\frac{1}{p}} \le A\cdot 2^k \Phi\Big(\frac{2^k}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}\Big)^{-\frac{1}{p}}. \tag{10}$$

For every $k\in\mathbb{Z}$, denote $A_k = \Phi\Big(\frac{2^k}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}\Big)^{-1}$, then $A_k>0$ and the sequence $\{A_k\}_{k\in\mathbb{Z}}$ is decreasing. From (9) and (10), thus

$$\begin{aligned}
|E(a_n^{(k)}\cdot g_n)| &\le A\cdot 2^k\cdot A_k^{\frac{1}{p}}\|s_{q,n}(g-g^{\tau_k})\cdot\chi_{\{\tau_k<\infty\}}\|_q\\
&\le A\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}\frac{2^k}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}A_k^{\frac{1}{p}}\|s_q(g-g^{\tau_k})\cdot\chi_{\{\tau_k<\infty\}}\|_q\\
&= A\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}\Phi^{-1}\big(\frac{1}{A_k}\big)A_k^{\frac{1}{p}}\|s_q(g-g^{\tau_k})\cdot\chi_{\{\tau_k<\infty\}}\|_q\\
&= A\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}\cdot A_k\Phi^{-1}\big(\frac{1}{A_k}\big)A_k^{\frac{1}{p}-1}\|s_q(g-g^{\tau_k})\cdot\chi_{\{\tau_k<\infty\}}\|_q\\
&= A\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}\frac{1}{\phi(A_k)}A_k^{-\frac{1}{q}}\|s_q(g-g^{\tau_k})\cdot\chi_{\{\tau_k<\infty\}}\|_q\\
&\le A\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}\frac{1}{\phi(A_k)}A_k^{-\frac{1}{q}}\sup_{\mathbb{P}(\tau_k<\infty)\le A_k}\|s_q(g-g^{\tau_k})\|_q\\
&= A\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}u_\phi^q(g,A_k).
\end{aligned}$$

Since $\Phi(t)t^{-\varepsilon}$ is non-decreasing, we have

$$\frac{A_{k+1}}{A_k} = \frac{\Phi\Big(\frac{2^k}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}\Big)}{\Phi\Big(\frac{2^{k+1}}{\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}}\Big)} \le \Big(\frac{2^k}{2^{k+1}}\Big)^\varepsilon = \Big(\frac{1}{2}\Big)^\varepsilon \quad \text{for all } k\in\mathbb{Z}.$$

Thus

$$\sum_{k=-\infty}^{\infty}u_\phi^q(g,A_k) = \sum_{k=-\infty}^{\infty}\frac{u_\phi^q(g,A_k)}{A_k-A_{k+1}}(A_k-A_{k+1}) \le \frac{1}{1-2^{-\varepsilon}}\sum_{k=-\infty}^{\infty}\frac{u_\phi^q(g,A_k)}{A_k}(A_k-A_{k+1}).$$

It is easy to see that the function $\frac{u_\phi^q(g,x)}{x} = \Phi^{-1}\big(\frac{1}{x}\big)x^{-\frac{1}{q}}\sup_{\mathbb{P}(\tau_k<\infty)\le x}\|s_q(g-g^{\tau_k})\|_q$ is non-increasing, noticing that $\lim_{k\to-\infty}A_k=\infty$ and $\lim_{k\to\infty}A_k=0$, then

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}u_\phi^q(g,A_k) &\le \frac{1}{1-2^{-\varepsilon}}\sum_{k=-\infty}^{\infty}\int_{A_{k+1}}^{A_k}\frac{u_\phi^q(g,x)}{x}\mathrm{d}x\\
&= \frac{1}{1-2^{-\varepsilon}}\int_0^{\infty}\frac{u_\phi^q(g,x)}{x}\mathrm{d}x = \frac{1}{1-2^{-\varepsilon}}\|g\|_{w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*)}.
\end{aligned}$$

Consequently, we have

$$|l_g(f)| \le \frac{A}{1-2^{-\varepsilon}}\|f\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})}\|g\|_{w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*)}.$$

Since $H_p^{s_p}(\mathbf{B})$ is dense in $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$ by Lemma 4.1, $l_g(\cdot)$ can be uniquely extended to a bounded linear functional on $w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$. That is $w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*) \subset (w\mathcal{H}_\Phi^{s_p}(\mathbf{B}))^*$.

(ii) Conversely, to prove $(w\mathcal{H}_\Phi^{s_p}(\mathbf{B}))^* \subset w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*)$. Given a bounded linear functional $l(\cdot) \in (w\mathcal{H}_\Phi^{s_p}(\mathbf{B}))^*$, since $H_p^{s_p}(\mathbf{B}) \subset w\mathcal{H}_\Phi^{s_p}(\mathbf{B})$, $l(\cdot)$ is also a bounded linear functional on $H_p^{s_p}(\mathbf{B})$. As the consequence of $(H_p^{s_p}(\mathbf{B}))^* \approx H_q^{s_q}(\mathbf{B}^*)$, there exists a $\mathbf{B}^*$-valued martingale $g = \{g_n\}_{n\geq 0}$ in $H_q^{s_q}(\mathbf{B}^*)$, such that

$$l(f) = l_g(f) := E(fg) = \lim_{n\to\infty} E(f_n g_n) \quad \text{for} \quad f \in H_p^{s_p}(\mathbf{B}).$$

For any stopping time τ, set $b = g - g^\tau$, then $b \in H_q^{s_q}(\mathbf{B}^*)$, too.

On the one hand, by the reflexivity of $\mathbf{B}$ and applying Lemma 4.2, we have $(H_q^{s_q}(\mathbf{B}^*))^* \approx H_p^{s_p}(\mathbf{B}^{**}) \approx H_p^{s_p}(\mathbf{B})$. Hence, for any $\delta > 0$, there exists some $\mathbf{B}$-valued martingale $a_\delta \in H_p^{s_p}(\mathbf{B})$ with $\|a_\delta\|_{H_p^{s_p}(\mathbf{B})} < 1+\delta$, such that

$$\|b\|_{H_q^{s_q}(\mathbf{B}^*)} = \|b - b^\tau\|_{H_q^{s_q}(\mathbf{B}^*)} = E\big[(b - b^\tau)a_\delta\big] = E\big[b(a_\delta - a_\delta^\tau)\big].$$

In particular, there exists a $\mathbf{B}$-valued martingale $a \in H_p^{s_p}(\mathbf{B})$ with $\|a\|_{H_p^{s_p}(\mathbf{B})} \leq 2$, such that

$$\|b\|_{H_q^{s_q}(\mathbf{B}^*)} = E\big[b(a - a^\tau)\big].$$

Let τ_k be stopping times satisfying

$$\mathbb{P}(\tau_k < \infty) \leq 2^{-k} \quad \text{for} \quad k \in \mathbb{Z},$$

and define $\mathbf{B}$-valued functions by

$$\xi^{(k)} := (2^{-k})^{\frac{1}{p}} \Phi^{-1}(2^k)(a - a^{\tau_k}) \quad \text{for} \quad k \in \mathbb{Z}.$$

For any fixed $k \in \mathbb{Z}$, the function $\xi^{(k)}$ is not necessary a weak atom, however, $\xi_n^{(k)} = E(\xi^{(k)}|\mathcal{F}_n) = 0$ on the set $\{\tau_k \geq n\}$.

For any $\lambda > 0$, choose $j \in \mathbb{Z}$ such that $2^j \leq \lambda < 2^{j+1}$. Let the martingales f^N, g^N and h^N $(N \in \mathbb{N})$ be defined by

$$f_n^N := \sum_{k=-N}^{N} \xi_n^{(k)}, \quad g_n^N := \sum_{k=-N}^{j-1} \xi_n^{(k)}, \quad \text{and} \quad h_n^N := \sum_{k=j}^{N} \xi_n^{(k)} \quad (n \geq 0).$$

We will show that $\|f^N\|_{w\mathcal{H}_\Phi^{s_p}(\mathbf{B})} \leq C_{p,\Phi}$ $(N \in \mathbb{N})$, where the constant $C_{p,\Phi} > 0$ is independent of N. We have

$$\begin{aligned}\|s_p(g^N)\|_p &\leq \sum_{k=-N}^{j-1} \|s_p(\xi^{(k)})\|_p \leq \sum_{k=-N}^{j-1} (2^{-k})^{\frac{1}{p}} \Phi^{-1}(2^k) \|s_p(a - a^{\tau_k})\|_p \\ &\leq 4 \sum_{k=-N}^{j-1} (2^{-k})^{\frac{1}{p}} \Phi^{-1}(2^k).\end{aligned}$$

Applying Lemma 2.3, from $\Phi \in Q[\varepsilon, 1]$, we have $\Phi^{-1} \in Q[1, \frac{1}{\varepsilon}]$, then $\frac{\Phi^{-1}(t)}{t}$ is non-decreasing. Since $2^k < 2^j \leq \lambda$ when $k \leq j-1$, we have $\frac{\Phi^{-1}(2^k)}{\Phi^{-1}(\lambda)} \leq \frac{2^k}{\lambda}$. Thus

$$\begin{aligned}\mathbb{P}(s_p(g^N) > \Phi^{-1}(\lambda)) &\le \frac{1}{\Phi^{-1}(\lambda)}\|s_p(g^N)\|_1 \le \frac{1}{\Phi^{-1}(\lambda)}\|s_p(g^N)\|_p \\ &\le 4\sum_{k=-N}^{j-1}(2^{-k})^{\frac{1}{p}}\frac{\Phi^{-1}(2^k)}{\Phi^{-1}(\lambda)} \le 4\sum_{k=-N}^{j-1}(2^{-k})^{\frac{1}{p}}\frac{2^k}{\lambda} \\ &= 4\lambda^{-1}\sum_{k=-N}^{j-1}(2^{1-\frac{1}{p}})^k \le 4\frac{2^{(1-\frac{1}{p})(j-1)}}{1-2^{\frac{1}{p}-1}}\lambda^{-1} := C_{p,\Phi}\lambda^{-1}.\end{aligned}$$

Denote $K_1 := (2\max\{C_{p,\Phi}, 1\})^{\frac{1}{\varepsilon}}$, then

$$\begin{aligned}&\Phi\Big(\frac{\Phi^{-1}(\lambda)}{K_1}\Big)\mathbb{P}(s_p(g^N) > \Phi^{-1}(\lambda)) \le \big(\frac{1}{K_1}\big)^{\varepsilon}\cdot\lambda\cdot\mathbb{P}(s_p(g^N) > \Phi^{-1}(\lambda)) \\ \le\ &\frac{1}{2\max\{C_{p,\Phi}, 1\}}\cdot\lambda\cdot\mathbb{P}(s_p(g^N) > \Phi^{-1}(\lambda)) \le \frac{1}{2}.\end{aligned} \tag{11}$$

Noticing that $\xi_n^{(k)} = 0$ on the set $\{\tau_k \ge n\}$ and $\mathbb{P}(\tau_k < \infty) \le 2^{-k}$, we get

$$\mathbb{P}(s_p(h^N) < \Phi^{-1}(\lambda)) \le \sum_{k=j}^{N}\mathbb{P}(\tau_k < \infty) \le \sum_{k=j}^{N}2^{-k} \le 2^{1-j} = 4\cdot 2^{-(j+1)} < 4\cdot\lambda^{-1}.$$

Denote $K_2 := 8^{\frac{1}{\varepsilon}}$, then

$$\Phi\Big(\frac{\Phi^{-1}(\lambda)}{K_2}\Big)\mathbb{P}(s_p(h^N) > \Phi^{-1}(\lambda)) \le \big(\frac{1}{K_2}\big)^{\varepsilon}\cdot\lambda\mathbb{P}(s_p(h^N) > \Phi^{-1}(\lambda)) \le \frac{1}{2}. \tag{12}$$

Applying the sub-linearity of $s_p(\cdot)$, we have $s_p(f^N) \le s_p(g^N) + s_p(h^N)$. Thus

$$\mathbb{P}(s_p(f^N) > 2\Phi^{-1}(\lambda)) \le \mathbb{P}(s_p(g^N) > \Phi^{-1}(\lambda)) + \mathbb{P}(s_p(h^N) > \Phi^{-1}(\lambda)). \tag{13}$$

Consequently, taking $K_0 := 2\max\{K_1, K_2\}$, from (11)–(13), we have

$$\begin{aligned}&\Phi\Big(\frac{2\Phi^{-1}(\lambda)}{K_0}\Big)\mathbb{P}(s_p(f^N) > 2\Phi^{-1}(\lambda)) \\ \le\ &\Phi\Big(\frac{2\Phi^{-1}(\lambda)}{K_0}\Big)\big[\mathbb{P}(s_p(g^N) > \Phi^{-1}(\lambda)) + \mathbb{P}(s_p(h^N) > \Phi^{-1}(\lambda))\big] \\ \le\ &\Phi\Big(\frac{\Phi^{-1}(\lambda)}{K_1}\Big)\mathbb{P}(s_p(g^N) > \Phi^{-1}(\lambda)) + \Phi\Big(\frac{\Phi^{-1}(\lambda)}{K_2}\Big)\mathbb{P}(s_p(h^N) > \Phi^{-1}(\lambda)) \\ \le\ &\frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$

which implies there exists a constant $C = C_{p,\Phi}$ depending only on p and Φ, such that $\|f^N\|_{w\mathcal{H}_{\Phi}^{s_p}(\mathbf{B})} \le C$.

On the one hand, noticing that $\frac{u_\phi^q(g,x)}{x}$ is non-increasing on $(0, +\infty)$, then

$$\begin{aligned}\|g\|_{w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B}^*)} &= \int_0^{\infty}\frac{u_\phi^q(g,x)}{x}\mathrm{d}x = \sum_{k=-\infty}^{\infty}\int_{2^{-k}}^{2^{-(k-1)}}\frac{u_\phi^q(g,x)}{x}\mathrm{d}x \\ &\le \sum_{k=-\infty}^{\infty}\frac{u_\phi^q(g,2^{-k})}{2^{-k}}(2^{-(k-1)} - 2^{-k}) = \sum_{k=-\infty}^{\infty}u_\phi^q(g,2^{-k}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned} l(f^N) &= E\Big(\sum_{n=1}^{\infty} df_n^N \cdot dg_n\Big) = E\Big(\sum_{n=1}^{\infty}\sum_{k=-N}^{N} d\xi_n^{(k)} \cdot dg_n\Big) \\ &= \sum_{k=-N}^{N} E\Big(\sum_{n=1}^{\infty} d\xi_n^{(k)} \cdot dg_n\Big) = \sum_{k=-N}^{N} E(\xi^{(k)} \cdot g) = \sum_{k=-N}^{N} E\big[\xi^{(k)}(g - g^{\tau_k})\big] \\ &= \sum_{k=-N}^{N} E\big[\xi^{(k)} b\big] = \sum_{k=-N}^{N} E\Big[\frac{\Phi^{-1}(2^k)}{(2^k)^{\frac{1}{p}}}(a - a^{\tau_k})b\Big] \\ &= \sum_{k=-N}^{N} \frac{\Phi^{-1}(2^k)}{(2^k)^{\frac{1}{p}}} E\big[(a - a^{\tau_k})b\big] = \sum_{k=-N}^{N} \frac{\Phi^{-1}(2^k)}{(2^k)^{\frac{1}{p}}} \|s_q(g - g^{\tau_k})\|_q \\ &= \sum_{k=-N}^{N} \frac{1}{\phi(2^{-k})}(2^{-k})^{\frac{1}{p}-1} \|s_q(g - g^{\tau_k})\|_q \\ &= \sum_{k=-N}^{N} \frac{1}{\phi(2^{-k})}(2^{-k})^{-\frac{1}{q}} \|s_q(g - g^{\tau_k})\|_q. \end{aligned}$$

Taking over all $N \in \mathbb{N}$ and the supremum over all of such stopping times satisfying $\mathbb{P}(\tau_k < \infty) \le 2^{-k}$ $(k \in \mathbb{Z})$, we obtain

$$C_{p,\phi}\|l\| \ge \sup_{N\in\mathbb{N}} |l(f^N)| \ge \sum_{k=-\infty}^{\infty} u_\phi^q(g, 2^{-k}) \ge \|g\|_{w\mathcal{L}^{s_q}_{q,\phi}(\mathbf{B}^*)}.$$

This means that $(w\mathcal{H}^{s_p}_{\Phi}(\mathbf{B}))^* \subset w\mathcal{L}^{s_q}_{q,\phi}(\mathbf{B}^*)$. □

5. Relation between $w\mathcal{L}_{p,\phi}(\mathbf{B})$ and $w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B})$

In this section, we discuss the embedding between $w\mathcal{L}_{p,\phi}(\mathbf{B})$ and $w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B})$. As a consequence, we give a new characterization of the geometrical properties of a Banach space.

Theorem 5.1. *Let* $\mathbf{B}$ *be a Banach space. Then, for any* $1 < p \le 2$ *and function* $\phi : (0, +\infty) \to (0, +\infty)$*, the following statements are equivalent:*

(i) $\mathbf{B}$ *is isomorphic to a* p*-uniformly smooth Banach space.*

(ii) $w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B}) \subset w\mathcal{L}_{p,\phi}(\mathbf{B})$ *and there exists a constant* $C = C_p > 0$ *such that*

$$\|f\|_{w\mathcal{L}_{p,\phi}(\mathbf{B})} \le C\|f\|_{w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B})} \tag{14}$$

for every $\mathbf{B}$*-valued martingale* $f = \{f_n\}_{n\ge 0}$ *in* $w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B})$.

Proof. (i) ⇒ (ii). Let $\mathbf{B}$ is isomorphic to a p-uniformly smooth Banach space and martingale $f = \{f_n\}_{n\ge 0} \in w\mathcal{L}^{s_p}_{p,\phi}(\mathbf{B})$. Then by Lemma 2.4, for any stopping time τ, we have

$$\|f - f^\tau\|_{L_p(\mathbf{B})} \le C_p\|s_p(f - f^\tau)\|_{L_p(\mathbb{R})}.$$

Consequently, for any $x \in (0, +\infty)$ we have

$$t_\phi^p(f,x) = \frac{1}{\phi(x)}x^{-\frac{1}{p}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|f - f^\tau\|_p$$
$$\le C_p \frac{1}{\phi(x)}x^{-\frac{1}{p}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_p(f - f^\tau)\|_p = C_p \cdot u_\phi^p(f,x).$$

This implies that $w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B}) \subset w\mathcal{L}_{p,\phi}(\mathbf{B})$ and $\|f\|_{w\mathcal{L}_{p,\phi}(\mathbf{B})} \le C\|f\|_{w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B})}$.

(ii) $\Rightarrow$ (i). Suppose the inequality (14) is true, let $f = \{f_n\}_{n\ge 0}$ be a $\mathbf{B}$-valued martingale and satisfy $\sum_{n=1}^\infty \frac{\|df_n\|^p}{n^p} \in L_\infty$.

Set $\widetilde{f_n} = \sum_{k=1}^n \frac{df_k}{k}$, then $\widetilde{f} = \{\widetilde{f_n}\}_{n\ge 0}$ is also a $\mathbf{B}$-valued martingale, and then

$$S_p(\widetilde{f})^p = \sum_{n=1}^\infty \|d\widetilde{f_n}\|^p = \sum_{n=1}^\infty \left\|\frac{df_n}{n}\right\|^p = \sum_{n=1}^\infty \frac{\|df_n\|^p}{n^p} \in L_\infty.$$

For any stopping time τ, $S_p(\widetilde{f} - \widetilde{f}^\tau) \le S_p(\widetilde{f}) + S_p(\widetilde{f}^\tau) \le 2S_p(\widetilde{f})$, then $S_p(\widetilde{f} - \widetilde{f}^\tau) \in L_\infty$. Noticing that $S_p(\widetilde{f} - \widetilde{f}^\tau) = S_p(\widetilde{f} - \widetilde{f}^\tau)\chi_{\{\tau<\infty\}}$, we have

$$\|s_p(\widetilde{f} - \widetilde{f}^\tau)\|_p = \|S_p(\widetilde{f} - \widetilde{f}^\tau)\|_p = \left[E\big(S_p(\widetilde{f} - \widetilde{f}^\tau)^p\chi_{\{\tau<\infty\}}\big)\right]^{\frac{1}{p}}$$
$$\le \|S_p(\widetilde{f} - \widetilde{f}^\tau)\|_\infty \mathbb{P}(\tau<\infty)^{\frac{1}{p}} \le 2\|S_p(\widetilde{f})\|_\infty \mathbb{P}(\tau<\infty)^{\frac{1}{p}}. \tag{15}$$

Particularly, taking $\phi(x) = x^{-\frac{1}{2p}}$ on $(0,+\infty)$, by (15) and noticing $\|s_p(\widetilde{f} - \widetilde{f}^\tau)\|_p \le 2\|S_p(\widetilde{f})\|_\infty$, we have

$$\|\widetilde{f}\|_{w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B})} = \int_0^\infty \frac{u_\phi^p(\widetilde{f},x)}{x}\mathrm{d}x$$
$$= \int_0^1 x^{\frac{1}{2p}-1}x^{-\frac{1}{p}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_p(\widetilde{f} - \widetilde{f}^\tau)\|_p \mathrm{d}x + \int_1^\infty x^{\frac{1}{2p}-1}x^{-\frac{1}{p}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_p(\widetilde{f} - \widetilde{f}^\tau)\|_p \mathrm{d}x$$
$$\le \int_0^1 x^{\frac{1}{2p}-1} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_p(\widetilde{f} - \widetilde{f}^\tau)\|_p \mathbb{P}(\tau<\infty)^{-\frac{1}{p}} \mathrm{d}x + \int_1^\infty x^{-\frac{1}{2p}-1} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_p(\widetilde{f} - \widetilde{f}^\tau)\|_p \mathrm{d}x$$
$$\le 2\|S_p(\widetilde{f})\|_\infty \Big(\int_0^1 x^{\frac{1}{2p}-1}\mathrm{d}x + \int_1^\infty x^{-\frac{1}{2p}-1}\mathrm{d}x \Big)$$
$$= 2\|S_p(\widetilde{f})\|_\infty \big(2px^{\frac{1}{2p}}\big|_0^1 - 2px^{-\frac{1}{2p}}\big|_1^\infty\big) = 8p\|S_p(\widetilde{f})\|_\infty < \infty.$$

So $\widetilde{f} \in w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B})$ with $\phi(x) = x^{-\frac{1}{2p}}$. By (14), we have

$$\|\widetilde{f}\|_{w\mathcal{L}_{p,\phi}(\mathbf{B})} \le C_p\|\widetilde{f}\|_{w\mathcal{L}_{p,\phi}^{s_p}(\mathbf{B})} \le 8pC_p\|S_p(\widetilde{f})\|_\infty < \infty. \tag{16}$$

For any stopping time τ and $0 < x < 1$, we have

$$\|\widetilde{f} - \widetilde{f}^\tau\|_p \le \sup_{\mathbb{P}(\tau<\infty)\le x} \|\widetilde{f} - \widetilde{f}^\tau\|_p + \sup_{x<\mathbb{P}(\tau<\infty)\le 1} \|\widetilde{f} - \widetilde{f}^\tau\|_p.$$

Since $\int_0^1 x^{\frac{1}{2p}-1}\mathrm{d}x = \int_1^\infty x^{-\frac{1}{2p}-1}\mathrm{d}x = 2p$ and $x^{-\frac{1}{p}} > 1$ when $0 < x < 1$, from (16), we have

$$\|\widetilde{f}-\widetilde{f}^{\tau}\|_p=\frac{1}{2p}\int_0^1 x^{\frac{1}{2p}-1}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x$$

$$\leq\frac{1}{2p}\Big(\int_0^1 x^{\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x+\int_0^1 x^{\frac{1}{2p}-1}\sup_{x<\mathbb{P}(\tau<\infty)\leq 1}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x\Big)$$

$$\leq\frac{1}{2p}\Big(\int_0^1 x^{\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x+\int_0^1 x^{\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq 1}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x\Big)$$

$$=\frac{1}{2p}\Big(\int_0^1 x^{\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x+2p\cdot\sup_{\mathbb{P}(\tau<\infty)\leq 1}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\Big)$$

$$=\frac{1}{2p}\Big(\int_0^1 x^{\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x+\int_1^{\infty} x^{-\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq 1}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x\Big)$$

$$=\frac{1}{2p}\Big(\int_0^1 x^{\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x+\int_1^{\infty} x^{-\frac{1}{2p}-1}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x\Big)$$

$$\leq\frac{1}{2p}\Big(\int_0^1 x^{\frac{1}{2p}-1}x^{-\frac{1}{p}}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x+\int_1^{\infty} x^{\frac{1}{2p}-1}x^{-\frac{1}{p}}\sup_{\mathbb{P}(\tau<\infty)\leq x}\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\mathrm{d}x\Big)$$

$$=\frac{1}{2p}\int_0^{\infty}\frac{t_{\phi}^p(\widetilde{f},x)}{x}\mathrm{d}x=\frac{1}{2p}\|\widetilde{f}\|_{w\mathcal{L}_{p,\phi}(\mathbf{B})}\leq 4C_p\|S_p(\widetilde{f})\|_{\infty}<\infty.$$

Especially, taking $\tau=0$, a.e., we have

$$\|\widetilde{f}\|_p=\|\widetilde{f}-\widetilde{f}^{\tau}\|_p\leq 4C_p\|S_p(\widetilde{f})\|_{\infty}<\infty.$$

Therefore, $\widetilde{f_n}=\sum_{k=1}^n\frac{df_k}{k}$ converges almost everywhere as $n\to\infty$. From Kronecker's Lemma, $\frac{f_n}{n}\to 0$, a.e. By Lemma 2.4, this implies that $\mathbf{B}$ is isomorphic to a p-uniformly smooth Banach space. □

Theorem 5.2. *Let $\mathbf{B}$ be a Banach space. Then, for any $2\leq q<\infty$ and function $\phi:(0,+\infty)\to(0,+\infty)$, the following statements are equivalent:*

(i) $\mathbf{B}$ *is isomorphic to a q-uniformly convex Banach space.*

(ii) $w\mathcal{L}_{q,\phi}(\mathbf{B})\subset w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B})$ *and there exists a constant $C=C_q>0$ such that*

$$\|f\|_{w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B})}\leq C\|f\|_{w\mathcal{L}_{q,\phi}(\mathbf{B})} \tag{17}$$

for every $\mathbf{B}$-valued martingale $f=\{f_n\}_{n\geq 0}$ in $w\mathcal{L}_{q,\phi}(\mathbf{B})$.

Proof. (i) ⇒ (ii). Let $\mathbf{B}$ is isomorphic to a q-uniformly convex Banach space and martingale $f=\{f_n\}_{n\geq 0}\in w\mathcal{L}_{q,\phi}(\mathbf{B})$. Then by Lemma 2.5, for any stopping time τ, we have

$$\|s_q(f-f^{\tau})\|_q\leq C_q\|f-f^{\tau}\|_q,$$

furthermore, for any $x\in(0,+\infty)$ we have

$$u_\phi^p(f,x) = \frac{1}{\phi(x)} x^{-\frac{1}{q}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_q(f-f^\tau)\|_q$$
$$\le C_q \frac{1}{\phi(x)} x^{-\frac{1}{q}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|f-f^\tau\|_q = C_q \cdot t_\phi^q(f,x).$$

This implies that $w\mathcal{L}_{q,\phi}(\mathbf{B}) \subset w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B})$ and $\|f\|_{w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B})} \le C\|f\|_{w\mathcal{L}_{q,\phi}(\mathbf{B})}$.

(ii) $\Rightarrow$ (i). Suppose (17) is true. Let $\mathbf{B}$-valued martingale $f = \{f_n\}_{n\ge 0}$ be uniformly bounded, that is, $\|f\|_\infty = \sup_{n\ge 0}\|f_n\|\infty \le M < \infty$. Then for any stopping time τ, we have

$$\|f-f^\tau\|_q = \big[E\big(\|f-f^\tau\|^q \chi_{\{\tau<\infty\}}\big)\big]^{\frac{1}{q}} \le \|f-f^\tau\|_\infty \mathbb{P}(\tau<\infty)^{\frac{1}{q}}$$
$$\le 2\|f\|_\infty \mathbb{P}(\tau<\infty)^{\frac{1}{q}} \le 2M < \infty. \tag{18}$$

Particularly, taking $\phi(x) = x^{-\frac{1}{2q}}$ on $(0,+\infty)$, by (18) and $\|f-f^\tau\|_q \le \|f-f^\tau\|_\infty \le 2\|f\|_\infty$, we have

$$\|f\|_{w\mathcal{L}_{q,\phi}(\mathbf{B})} = \int_0^\infty \frac{t_\phi^q(f,x)}{x}\mathrm{d}x$$
$$= \int_0^1 x^{\frac{1}{2q}-1} x^{-\frac{1}{q}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|f-f^\tau\|_q \mathrm{d}x + \int_1^\infty x^{\frac{1}{2q}-1} x^{-\frac{1}{q}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|f-f^\tau\|_q \mathrm{d}x$$
$$\le \int_0^1 x^{\frac{1}{2q}-1} \sup_{\mathbb{P}(\tau<\infty)\le x} \|f-f^\tau\|_\infty \mathrm{d}x + \int_1^\infty x^{-\frac{1}{2q}-1} \sup_{\mathbb{P}(\tau<\infty)\le x} \|f-f^\tau\|_\infty \mathrm{d}x$$
$$\le 2\|f\|_\infty \Big(\int_0^1 x^{\frac{1}{2q}-1}\mathrm{d}x + \int_1^\infty x^{-\frac{1}{2q}-1}\mathrm{d}x \Big)$$
$$= 2\|f\|_\infty \big(2q x^{\frac{1}{2q}}\big|_0^1 - 2q x^{-\frac{1}{2q}}\big|_1^\infty\big) = 8q\|f\|_\infty \le 8qM < \infty.$$

So, $f \in w\mathcal{L}_{q,\phi}(\mathbf{B})$ with $\phi(x) = x^{-\frac{1}{2q}}$. Moreover, by (17), we get

$$\|f\|_{w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B})} \le C\|f\|_{w\mathcal{L}_{q,\phi}(\mathbf{B})} \le 8qC_qM < \infty. \tag{19}$$

Since $\int_0^1 x^{\frac{1}{2q}-1}\mathrm{d}x = \int_1^\infty x^{-\frac{1}{2q}-1}\mathrm{d}x = 2q$ and $x^{-\frac{1}{q}} > 1$ when $0 < x < 1$, from (19), for any stopping time τ, we have

$$\|S_q(f-f^\tau)\|_q = \|s_q(f-f^\tau)\|_q = \frac{1}{2q}\int_0^1 x^{\frac{1}{2q}-1}\|s_q(f-f^\tau)\|_q \mathrm{d}x$$
$$\le \frac{1}{2q}\Big(\int_0^1 x^{\frac{1}{2q}-1} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_q(f-f^\tau)\|_q \mathrm{d}x + \int_1^\infty x^{-\frac{1}{2q}-1} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_q(f-f^\tau)\|_q \mathrm{d}x \Big)$$
$$\le \frac{1}{2q}\Big(\int_0^1 x^{\frac{1}{2q}-1} x^{-\frac{1}{q}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_q(f-f^\tau)\|_q \mathrm{d}x + \int_1^\infty x^{\frac{1}{2q}-1} x^{-\frac{1}{q}} \sup_{\mathbb{P}(\tau<\infty)\le x} \|s_q(f-f^\tau)\|_q \mathrm{d}x \Big)$$
$$= \frac{1}{2q}\int_0^\infty \frac{u_\phi^q(f,x)}{x}\mathrm{d}x = \frac{1}{2q}\|f\|_{w\mathcal{L}_{q,\phi}^{s_q}(\mathbf{B})} \le \frac{1}{2q}\cdot 8qC_qM = 4C_qM < \infty.$$

Especially, take $\tau = 0$, a.e., we have

$$\|S_q(f)\|_q = \|S_q(f - f^\tau)\|_q \leq 4C_q M < \infty.$$

Therefore $S_q(f) < \infty$, a.e. By Lemma 2.5, it is shown that $\mathbf{B}$ is isomorphic to a q-uniformly convex Banach space. □

Corollary 5.1. *Let* $\mathbf{B}$ *be a Banach space. Then, for any function* $\phi : (0, +\infty) \to (0, +\infty)$*, the following statements are equivalent:*

(i) $\mathbf{B}$ *is isomorphic to a Hilbert space.*

(ii) $w\mathcal{L}_{2,\phi}(\mathbf{B}) \approx w\mathcal{L}^{s_2}_{2,\phi}(\mathbf{B})$ *and there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1\|f\|_{w\mathcal{L}_{2,\phi}(\mathbf{B})} \leq \|f\|_{w\mathcal{L}^{s_2}_{2,\phi}(\mathbf{B})} \leq C_2\|f\|_{w\mathcal{L}_{2,\phi}(\mathbf{B})}$$

for every $\mathbf{B}$*-valued martingale* $f = \{f_n\}_{n\geq 0}$.

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