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Finite fractal dimensions of random attractors for stochastic FitzHugh–Nagumo system with multiplicative white noise ☆

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ABSTRACT

In this paper, we consider the asymptotic behavior of solutions for stochastic non-autonomous FitzHugh–Nagumo system with multiplicative white noise. First we prove the existence of random attractor of the random dynamical system generated by the solutions of considered system. Then we present some conditions for estimating an upper bound of the fractal dimension of a random invariant set of a random dynamical system on a separable Banach space and apply these conditions to prove the finiteness of fractal dimension of random attractor.

1. Introduction

Recently, the random attractor and its various properties, including existence, dimension and upper-semi-continuity, etc., for infinite dimensional random dynamical systems and stochastic evolution equations have been studied by many authors, see [4,5,7,9–18,20–28,37–45] and the references therein. In this article, we consider the random attractor for the following stochastic non-autonomous stochastic FitzHugh–Nagumo system with multiplicative white noise

$$\begin{cases} du_1 + (-\Delta u_1 + \alpha u_2 + f(u_1))dt = g_1(x,t)dt + bu_1 \circ dW(t) \\ du_2 + (\sigma u_2 - \beta u_1)dt = g_2(x,t)dt + bu_2 \circ dW(t) \end{cases} \quad \text{in} \quad U \times (\tau, +\infty), \quad \tau \in \mathbb{R}, \tag{1.1}$$

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where $u_1 = u_1(x,t)$, $u_2 = u_2(x,t)$ are real-valued functions on $U \times [0, +\infty)$, U is an open bounded set of $\mathbb{R}^3$ with a smooth boundary ∂U, α, σ, $\beta > 0$, $b \in \mathbb{R}$, $g_1(x,\cdot) \in C_b(\mathbb{R}, L^2(U))$, $g_2(x,\cdot) \in C_b(\mathbb{R}, H_0^1(U))$, $W(t)$ is a one-dimensional two-sided Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, the Borel σ-algebra $\mathcal{F}$ on Ω being generated by the compact open topology, and $\mathbb{P}$ being the corresponding Wiener measure on $\mathcal{F}$, "$\circ$" denotes the Stratonovich sense in the stochastic term. The function $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following condition:

(H) *there exist real constants* $c_0, c_1, c_2, c_3, c_4 \geq 0$ *such that*

$$|f(u)| \leq c_0 + |u|^3, \quad f(u)u \geq -c_1, \quad -c_2 \leq f'(u) \leq c_3 u^2 + c_4, \quad \forall u \in \mathbb{R}. \tag{1.2}$$

For example, the function

$$f(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3, \quad a_3 > 0, a_i \in \mathbb{R}, i = 0, 1, 2, 3 \tag{1.3}$$

satisfies **(H)**.

FitzHugh–Nagumo system is a model for describing the signal transmission across axons in neurobiology, see [6,19,31,33]. The global attractor and inertial manifolds for the deterministic FitzHugh–Nagumo system have been studied in the literature, see [28–30,34]. The existence of random attractor of the stochastic FitzHugh–Nagumo system driven by white noise has been studied in [1,2,15,20–22,41]. Notice that "additive white noise term" is independent of the state variable and "multiplicative white noise term" includes the state variable, which implies that the impact of these two types of noise on the solutions of evolution equations should be different.

Notice that equation (1.1) is a non-autonomous stochastic equation with the time-dependent external term g and white noise. For this case, Wang have established a good abstract theory about the existence and upper semicontinuity of random attractors by introducing two parametric spaces, and to prove the existence and upper semicontinuity of random attractors for non-autonomous stochastic reaction–diffusion equation and wave equations, see [37–40]. Motivated by the ideas of [13,15,40], we will consider the random attractor for stochastic system (1.1) from the following two respects:

First we prove the existence of random attractor of the random dynamical system associated with (1.1). The main problem here is the lack of "higher regularity" of second component u_2 of (1.1) which makes the difficulty to construct a compact attracting set of the random dynamical system. We decompose the second component u_2 into a sum of two parts to overcome this difficulty like dealing with the wave equation in [8,35].

Second, we are devoted to obtain an upper bound of fractal dimension of random attractor for a suitable small coefficient b of the random term, which implies that the random attractor of (1.1) can be embedded in a finite dimensional Euclidean space. It is known that there are some useful methods to estimate the upper bound of Hausdorff and fractal dimensions of random attractors for random dynamical systems in [12–14,25,42] but which are requiring the differentiability of random dynamical system or requiring that the Lipschitz constant of system and the "contraction" coefficient of the infinite dimensional part of system be independent of the sample points. However, a random attractor is generally not uniformly bounded along the sample path of sample points. Based on the idea of [13], we give some conditions for bounding the fractal dimension of a random invariant set for a random dynamical system, which doesn't require the differentiability of random dynamical system and just require the boundedness of expectation of some random variables. And we apply these conditions to obtain the finiteness of fractal dimension of random attractor.

Throughout this article, for simplicity, we identify "*a.e.* $\omega \in \Omega$" with "$\omega \in \Omega$".

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2. Mathematical setting

Consider the initial boundary value problem of (1.1):

$$\begin{cases} du_1 + (-\Delta u_1 + \alpha u_2 + f(u_1))dt = g_1(x,t)dt + bu_1 \circ dW(t), \\ du_2 + (\sigma u_2 - \beta u_1)dt = g_2(x,t)dt + bu_2 \circ dW(t) \quad \text{in} \quad U \times (\tau, +\infty), \\ u_1(t,x)|_{x\in\partial U} = u_2(t,x)|_{x\in\partial U} = 0, \quad t \geq \tau, \\ u_1(\tau,x) = u_{1,\tau}(x), \quad u_2(\tau,x) = u_{2,\tau}(x), \quad x \in U. \end{cases} \tag{2.1}$$

For any $t \in \mathbb{R}$, define θ_t on Ω: $\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t)$, then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system [3,9], $W(t)$ acting at $\omega \in \Omega$ is identified with $\omega(t)$, i.e., $W(t,\omega) = W(t)(\omega) = \omega(t)$ for $t \in \mathbb{R}$, $\omega \in \Omega$.

The operator $-\Delta$, where $D(-\Delta) = H^2(U) \cap H_0^1(U)$, is a self-adjoint positive linear operator with eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots, \quad \lambda_m \to +\infty \quad \text{as} \quad m \to +\infty.$$

Denote the inners and norms of $L^2(U)$ and $H_0^1(U)$ as $(\cdot,\cdot)$, $||\cdot||$ and $(\cdot,\cdot)_1$, $||\cdot||_1$, respectively. Let $E = L^2(U) \times L^2(U)$ be a Hilbert space with inner product

$$(v_1, v_2)_E = \beta(v_{11}, v_{12}) + \alpha(v_{21}, v_{22}), \quad \forall v_i = (v_{i1}, v_{i2}) \in E, \quad i = 1, 2. \tag{2.2}$$

To show that problem (2.1) generates a random dynamical system, we transfer (2.1) into a random system without noise term. Let $z(\theta_t\omega) = -\int_{-\infty}^0 e^s(\theta_t\omega)(s)ds$ $(t \in \mathbb{R})$ be an Ornstein–Uhlenbeck stationary process which solves the equation $dz + zdt = dW(t)$. From [3,7], for $\omega \in \Omega$, $t \mapsto z(\theta_t\omega)$ is continuous in t and

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = \lim_{t\to\pm\infty} \frac{1}{t} \int_0^t z(\theta_s\omega)ds = \lim_{t\to+\infty} e^{-\epsilon t}|z(\theta_{-t}\omega)| = 0, \quad \forall \epsilon > 0. \tag{2.3}$$

Introduce a variable transformation

$$(v_1(t,\omega,x), v_2(t,\omega,x)) = (e^{-bz(\theta_t\omega)}u_1(t,x), e^{-bz(\theta_t\omega)}u_2(t,x)). \tag{2.4}$$

Then (2.1) can be written as the following equivalent deterministic random system

$$\begin{cases} \frac{dv_1}{dt} - \Delta v_1 - bz(\theta_t\omega)v_1 + \alpha v_2 + e^{-bz(\theta_t\omega)}f(e^{bz(\theta_t\omega)}v_1) = e^{-bz(\theta_t\omega)}g_1(x,t), \quad t > \tau, \\ \frac{dv_2}{dt} - bz(\theta_t\omega)v_2 + \sigma v_2 - \beta v_1 = e^{-bz(\theta_t\omega)}g_2(x,t), \quad t > \tau, \\ v_1(t,\omega,x)|_{x\in\partial U} = v_2(t,\omega,x)|_{x\in\partial U} = 0, \quad t \geq \tau, \\ v_i(\tau,\omega,x) = e^{-bz(\theta_\tau\omega)}u_{i,\tau}(x) = v_{i,\tau}(\omega,x), \quad \tau \in \mathbb{R}, x \in U, i = 1,2. \end{cases} \tag{2.5}$$

It follows from Galerkin method or [32] and Lemma 3.2 below that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varphi_\tau(\omega) = (v_{1,\tau}(\omega,x), v_{2,\tau}(\omega,x)) \in E$, (2.5) has a unique global solution $\varphi(\cdot,\tau,\omega,\varphi_\tau(\omega)) \in C([\tau,+\infty); L^2(U)\times L^2(U)) \cap L^2_{loc}([\tau,+\infty); H_0^1(U) \times L^2(U))$, which defines a continuous random dynamical system:

$$\Phi : \mathbb{R}^+ \times \Omega \times E \to E,$$

$$\begin{aligned} \Phi(t,\tau,\omega,\varphi_\tau(\omega)) &= \Phi(t,\tau,\omega)\varphi_\tau(\omega) = \varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau(\theta_{-\tau}\omega)) \\ &= (v_1(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau(\theta_{-\tau}\omega)), v_2(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau(\theta_{-\tau}\omega))) \end{aligned}$$

over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ [39], where $\Phi(0,\tau,\omega)\varphi_\tau(\omega) = \varphi_\tau(\theta_{-\tau}\omega)$.

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3. Existence of random attractor

First we prove the existence of random attractor for Φ. Let $\mathcal{D} = \mathcal{D}(E)$ be the collection of the tempered families $B = \{B(\tau, \omega) \subseteq E : \tau \in \mathbb{R}, \omega \in \Omega\}$ of nonempty subsets of E with respect to $(\theta_t)_{t\in\mathbb{R}}$, where $\lim_{t\to\infty} e^{-\epsilon|t|} \sup_{\varphi\in B(\tau+t,\theta_t\omega)} ||\varphi||_E = 0$ for any $\epsilon > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. A family $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called *a $\mathcal{D}$-pullback random attractor* for Φ if (i) $A(\tau, \omega)$ is measurable in ω with respect to $\mathcal{F}$ and compact in E for all $\tau \in \mathbb{R}$, $\omega \in \Omega$; (ii) A is invariant, i.e., for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\Phi(t, \tau, \omega, A(\tau, \omega)) = A(t + \tau, \theta_t\omega)$, $\forall t \geq 0$; (iii) A attracts every $B \in \mathcal{D}$: for $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\lim_{t\to+\infty} \mathrm{d}_H(\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)), A(\tau, \omega)) = 0$.

As a direct consequence of [9–11,39], we have the following existence criteria of random attractor for Φ.

Lemma 3.1. *If Φ has a compact measurable (w.r.t. $\mathcal{F}$) $\mathcal{D}$-pullback attracting set K in $\mathcal{D}$ such that for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for every $B \in \mathcal{D}$, $\lim_{t\to+\infty} \mathrm{d}_H(\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)), K(\tau, \omega)) = 0$, then Φ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}$ in $\mathcal{D}$ given by*

$$\mathcal{A}(\tau, \omega) = \bigcap_{r\geq 0} \overline{\bigcup_{t\geq r} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega))}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.$$

For the estimates of bound of solutions for (2.5), we have the following lemmas.

Lemma 3.2. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B \in \mathcal{D}(E)$, there exist $T(\tau, \omega, B) \geq 0$ and a tempered positive valued random variable*

$$M_0^2(\omega) = 1 + \left(\frac{2\beta}{\lambda_1}\|g_1\|^2 + \frac{\alpha}{\sigma}\|g_2\|^2 + 2c_1|U|\beta\right) \int_{-\infty}^{0} e^{-2bz(\theta_s\omega)+\int_s^0 2bz(\theta_r\omega)dr+\frac{\rho}{2}s} ds \tag{3.1}$$

such that the solution $\varphi(r, \tau, \omega, \varphi_\tau(\omega)) \in E$ of (2.5) with $\varphi_\tau(\omega) \in B(\tau, \omega) \in B$ satisfies:

$$||\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))|| \leq M_0(\omega), \ \ \forall t \geq T(\tau, \omega, B). \tag{3.2}$$

Proof. For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, let $\varphi(r) = (v_1(r), v_2(r)) = \varphi(r, \tau - t, \omega, \varphi_{\tau-t}(\omega)) \in E$ $(r \geq \tau - t)$ be a solution of (2.5) with initial value $\varphi_{\tau-t}(\omega) \in E$. Taking the inner product of the first equation and second equation of (2.5) with βv_1 and αv_2, respectively, and by

$$e^{-bz(\theta_r\omega)} \int_U f(e^{bz(\theta_r\omega)}v_1)v_1 dx \geq -c_1|U|e^{-2bz(\theta_r\omega)}, \quad \text{(by (1.2))}$$

$$2e^{-bz(\theta_r\omega)}(g_1, v_1) \leq \frac{2}{\lambda_1}e^{-2bz(\theta_r\omega)}\|g_1\|^2 + \frac{\lambda_1}{2}\|v_1\|^2,$$

$$2e^{-bz(\theta_r\omega)}(g_2, v_2) \leq \frac{1}{\sigma}e^{-2bz(\theta_r\omega)}\|g_2\|^2 + \sigma\|v_2\|^2,$$

we have

$$\frac{d}{dt}\|\varphi\|_E^2 + (\rho - 2bz(\theta_r\omega))\,\|\varphi\|_E^2 + \beta\|\nabla v_1\|^2 + \frac{\sigma\alpha}{2}||v_2||^2 \leq c_5 e^{-2bz(\theta_r\omega)}, \tag{3.3}$$

where

$$c_5 = \left(\frac{2\beta}{\lambda_1}||g_1||^2 + \frac{\alpha}{\sigma}||g_2||^2 + 2c_1|U|\beta\right), \quad \rho = \frac{1}{2}\min\{\lambda_1, \sigma\}, \quad ||g_i||^2 = \sup_{t\in\mathbb{R}} ||g_i(x,t)||^2, i = 1, 2.$$

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By applying Gronwall inequality to (3.3) on $[\tau-t, r]$ ($r \geq \tau-t$) and replacing ω by $\theta_{-\tau}\omega$, we have

$$\begin{aligned}\|\varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \leq{}& e^{\int_{\tau-t}^{r} 2bz(\theta_{s-\tau}\omega)ds-\rho(r+t-\tau)}\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2\\ &+c_5\int_{\tau-t}^{r} e^{-2bz(\theta_{s-\tau}\omega)+\int_s^r 2bz(\theta_{s-\tau}\omega)ds-\rho(r-s)}ds, \end{aligned} \tag{3.4}$$

thus,

$$\begin{aligned}\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2 \leq{}& e^{\int_{-t}^{0} 2bz(\theta_{s}\omega)ds-\rho t}\|\varphi_{\tau-t}(\theta_{-\tau}\omega))\|_E^2\\ &+c_5\int_{-\infty}^{0} e^{-2bz(\theta_{s}\omega)+\int_s^0 2bz(\theta_{r}\omega)dr+\rho s}ds. \end{aligned} \tag{3.5}$$

By (2.3) and $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in B(\tau-t,\theta_{-t}\omega)$,

$$\int_{-\infty}^{0} e^{-2bz(\theta_{s}\omega)+\int_s^0 2bz(\theta_{r}\omega)dr+\rho s}ds < +\infty, \quad \lim_{t\to+\infty} e^{\int_{-t}^{0} 2bz(\theta_{s}\omega)ds-\rho t}\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|^2 = 0.$$

Choose $M_0(\omega)$ being as in (3.1), then $M_0(\omega)$ is independent of $\tau \in \mathbb{R}$ and is tempered and (3.5) implies (3.2). The proof is completed. □

Lemma 3.3. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B \in \mathcal{D}(E)$, there exist $T_1(\tau,\omega,B)$, $T_2(\omega,B) \geq 0$ and tempered random variables*

$$M_1^2(\omega) = 1 + c_5 e^{-\int_{-1}^{0} 2bz(\theta_s\omega)ds+\rho}\int_{-\infty}^{0} e^{-2bz(\theta_{s}\omega)+\int_s^0 2bz(\theta_{s}\omega)ds+\rho s}ds, \tag{3.6}$$

$$M_2^2(\omega) = \left(\frac{1}{\beta}(1+2c_2+2|b|\max_{0\leq s\leq 1}|z(\theta_s\omega)|)+\frac{2\alpha}{\sigma}\right)M_1^2(\omega)+\|g\|^2\int_{-1}^{0} e^{-2bz(\theta_s\omega)}ds, \tag{3.7}$$

$$M_3^2(\omega) = 1+2\left(\beta c_5+\frac{||\nabla g_2||^2}{\sigma}\right)\int_{-\infty}^{0} e^{-2bz(\theta_{r}\omega)+\int_r^0 2bz(\theta_{s}\omega)ds+\rho r}dr, \quad ||\nabla g_2||^2 = \sup_{t\in\mathbb{R}}||\nabla g_2(x,t)||^2, \tag{3.8}$$

such that the solution $\varphi(r,\tau-t,\omega,\varphi_{\tau-t}) = (v_1,v_2) \in E$ of (3.1) with $\varphi_{\tau-t}(\omega) \in B(\tau-t,\omega) \in B$ has the following properties:

(i) the first component v_1 satisfies:

$$||\nabla v_1(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}(\theta_{-\tau}\omega))|| \leq M_2(\omega), \ \ \forall t \geq T_1(\tau,\omega,B); \tag{3.9}$$

(ii) the second component v_2 can be decomposed into a sum $v_2 = v_{21}+v_{22}$, where

$$\|v_{21}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2 \leq e^{\int_{-t}^{0} 2bz(\theta_s\omega)ds-\sigma t}||v_{2,\tau-t}(\omega)||^2, \quad \forall t \geq 0, \tag{3.10}$$

and

$$\|\nabla v_{22}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\| \leq M_3(\omega), \quad \forall t \geq T_2(\tau,\omega,B). \tag{3.11}$$

Proof. (i) Applying Gronwall inequality to (3.3) over $[\tau-1,\tau]$ and by (3.4), we have

$$\int_{\tau-1}^{\tau} e^{\int_s^\tau 2bz(\theta_{s-\tau}\omega)ds-\rho(\tau-s)}\left(\beta\|\nabla v_1(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2+\frac{\alpha\sigma}{2}||v_2(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})||^2\right)ds$$
$$\le e^{\int_{-t}^0 2bz(\theta_s\omega)ds-\rho t}\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|^2+c_5\int_{-\infty}^0 e^{-2bz(\theta_s\omega)+\int_s^0 2bz(\theta_s\omega)ds+\rho s}ds.$$

Thus, there exists $T_1(\tau,\omega,B)\ge 1$ such that for all $t\geqslant T_1(\tau,\omega,B)$, we get

$$\int_{\tau-1}^{\tau}\left(\beta\|\nabla v_1(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)\|^2+\frac{\alpha\sigma}{2}||v_2(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)||^2\right)ds$$
$$\le 1+c_5e^{-\int_{-1}^0 2bz(\theta_s\omega)ds+\rho}\int_{-\infty}^0 e^{-2bz(\theta_s\omega)+\int_s^0 2bz(\theta_s\omega)ds+\rho s}ds\doteq M_1^2(\omega). \tag{3.12}$$

Taking (formally) the inner product of first equation of (2.5) with $-\Delta v_1$ in $L^2(U)$ (which can be made rigorous by considering the Galerkin approximation) and by

$$e^{-bz(\theta_{r-\tau}\omega)}\int_U f(e^{bz(\theta_{r-\tau}\omega)}v_1)\Delta v_1dx\le c_2\|\nabla v_1\|^2,\quad (\text{by } (1.2))$$

$$2\alpha(v_2,\Delta v_1)+2e^{-bz(\theta_{r-\tau}\omega)}(g_1,-\Delta v_1)\le\alpha^2||v_2||^2+e^{-2bz(\theta_{r-\tau}\omega)}\|g_1\|^2+2\|\Delta v_1\|^2,$$

we have

$$\frac{d}{dt}\|\nabla v_1\|^2\le(2bz(\theta_{r-\tau}\omega)+2c_2)\|\nabla v_1\|^2+\alpha^2||v_2||^2+e^{-2bz(\theta_{r-\tau}\omega)}\|g_1\|^2,\quad r\ge\tau-t. \tag{3.13}$$

Take $t\geqslant T_1(\tau,\omega,B)\ge 1$ and $s\in[\tau-1,\tau]$. Integrate (3.13) over $[s,\tau]$ to get

$$\|\nabla v_1(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2$$
$$\leqslant\|\nabla v_1(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2+\|g_1\|^2\int_{\tau-1}^{\tau}e^{-2bz(\theta_{s-\tau}\omega)}ds$$
$$+\int_{\tau-1}^{\tau}(2|b||z(\theta_{s-\tau}\omega)|+2c_2)\|\nabla v_1(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2ds$$
$$+\alpha^2\int_{\tau-1}^{\tau}\|v_2(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2ds. \tag{3.14}$$

Again, integrating (3.14) with respect to s over $[\tau-1,\tau]$, we find that for $t\geqslant T_1(\tau,\omega,B)$,

$$\|\nabla v_1(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2$$
$$\leqslant\int_{\tau-1}^{\tau}(2b|z(\theta_{s-\tau}\omega)|+2c_2+1)\|\nabla v_1(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2ds$$

$$+\alpha^2\int_{\tau-1}^{\tau}||v_2(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))||^2ds+\|g_1\|^2\int_{\tau-1}^{\tau}e^{-2bz(\theta_{s-\tau}\omega)}ds$$

$$\le\left(\frac{1}{\beta}(1+2c_2+2|b|\max_{0\le s\le 1}|z(\theta_s\omega)|)+\frac{2\alpha}{\sigma}\right)M_1^2(\omega)+\|g_1\|^2\int_{-1}^{0}e^{-2bz(\theta_s\omega)}ds$$

$$\doteq M_2^2(\omega), \tag{3.15}$$

where $M_2^2(\omega)$ is tempered.

(ii) Decompose $v_2=v_{21}+v_{22}$, where

$$\frac{dv_{21}(t)}{dt}+[\sigma-bz(\theta_t\omega)]v_{21}=0,\quad v_{21}(\tau)=v_2(\tau,\omega,x)=e^{-bz(\theta_\tau\omega)}u_{2,\tau}(x),\quad t>\tau \tag{3.16}$$

and

$$\frac{dv_{22}(t)}{dt}+[\sigma-bz(\theta_t\omega)]v_{22}(t)=\beta v_1(t)+e^{-bz(\theta_t\omega)}g_2(x,t),\quad v_{22}(\tau)=0. \tag{3.17}$$

Taking the inner product of (3.16) with $-v_{21}$ in $L^2(U)$, we obtain (3.10). Taking the inner product of (3.17) with $-\Delta v_{22}$ in $L^2(U)$, we find that

$$\frac{d}{dt}\|\nabla v_{22}\|^2+[\rho-2bz(\theta_t\omega)]\|\nabla v_{22}\|^2\le\frac{2\beta^2}{\sigma}||\nabla v_1(t)||^2+\frac{2}{\sigma}e^{-2bz(\theta_t\omega)}||\nabla g_2||^2. \tag{3.18}$$

Applying Gronwall Lemma to (3.18) and (3.3) on $[\tau-t,\tau]$, respectively, we find that

$$\|\nabla v_{22}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2$$
$$\le\int_{\tau-t}^{\tau}e^{\int_r^\tau 2bz(\theta_{s-\tau}\omega)ds-\rho(\tau-r)}\left(\frac{2\beta^2}{\sigma}||\nabla v_1(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}\omega)||^2+\frac{2}{\sigma}e^{-2bz(\theta_{r-\tau}\omega)}||\nabla g_2||^2\right)dr$$

and

$$\int_{\tau-t}^{\tau}e^{\int_r^\tau 2bz(\theta_{s-\tau}\omega)ds-\rho(\tau-r)}\beta\|\nabla v_1(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2dr$$
$$\leqslant e^{\int_{-t}^0 2bz(\theta_s\omega)ds-\rho t}\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2+c_5\int_{\tau-t}^{\tau}e^{-2bz(\theta_{r-\tau}\omega)+\int_r^\tau 2bz(\theta_{s-\tau}\omega)ds-\rho(\tau-r)}dr,$$

thus,

$$\|\nabla v_{22}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))\|^2$$
$$\le\frac{2\beta}{\sigma}e^{\int_{-t}^0 2bz(\theta_s\omega)ds-\rho t}\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|_E^2+2\left(\beta c_5+\frac{||\nabla g_2||^2}{\sigma}\right)\int_{-\infty}^{0}e^{-2bz(\theta_r\omega)+\int_r^0 2bz(\theta_s\omega)ds+\rho r}dr.$$

So, there exist $T_2(\omega,B)\ge 0$ and a tempered random variable $M_3^2(\omega)$ defined by (3.8) such that (3.11) holds. This completes the proof. □

By the Lemmas 3.1–3.3, we have the existence of random attractor for Φ.

Theorem 3.4. *For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the RDS Φ possesses a $\mathcal{D}(E)$-pullback random attractor $\mathcal{A}(\tau, \omega) \subseteq B_0(\omega) \in \mathcal{D}(E)$ with*

$$\sup_{(v_1,v_2)\in\mathcal{A}(\tau,\omega)} ||v_1||_{H_0^1(U)} \leq M_2(\omega), \qquad \sup_{(v_1,v_2)\in\mathcal{A}(\tau,\omega)} ||v_2||_{H_0^1(U)} \leq M_3(\omega). \tag{3.19}$$

Proof. For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, let $\Lambda_1(\omega)$, $\Lambda_2(\omega)$ be two balls of $H_0^1(U)$ centered at 0 with radius $M_2(\omega)$ and $M_3(\omega)$, respectively, then by the compactness of embedding $H_0^1(U) \times H_0^1(U) \hookrightarrow L^2(U) \times L^2(U)$, $B_1(\omega) = \Lambda_1(\omega) \times \Lambda_2(\omega)$ is a compact measurable set of E. From Lemma 3.3, for any $\tau \in \mathbb{R}, \omega \in \Omega$ and $B(\tau, \omega) \in \mathcal{D}(E)$, it holds that for $t \geq \max\{T_1(\omega, B), T_2(\omega, B)\}$,

$$\mathrm{d}_H(\varphi(\tau, \tau - t, \theta_{-\tau}\omega, B(\tau - t, \theta_{-t}\omega)), B_1(\omega)) \leq e^{\int_{-t}^0 2bz(\theta_s\omega)ds - \sigma t} ||v_{2,\tau-t}(\theta_{-\tau}\omega)||^2 \overset{t\to+\infty}{\to} 0,$$

thus, $B_1(\omega)$ is a compact measurable $\mathcal{D}(E)$-pullback attracting set of Φ in E. By Lemma 3.1, Φ possesses a $\mathcal{D}(E)$-pullback random attractor $\mathcal{A}(\tau, \omega) \subseteq B_1(\omega)$ satisfying (3.19). The proof is completed. $\square$

4. Fractal dimension of random attractor

In this section, we prove the boundedness of fractal dimension of $\mathcal{A}(\tau, \omega)$ for Φ based on the following result.

Theorem 4.1. *Let $\{\Psi(t, \tau, \omega)\}_{t\geq 0,\tau\in\mathbb{R},\omega\in\Omega}$ be a random dynamical system on a separable Banach space X over $\mathbb{R}$ and ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\omega\}_{t\in\mathbb{R}})$. Assume that there exists a family of bounded closed random subsets $\{\chi(\tau, \omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ of X satisfying the following conditions: for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

(H1) *there exists a tempered random variable R_ω (independent of τ) such that the diameter $||\chi(\tau, \omega)||_X$ of $\chi(\tau, \omega)$ is bounded by R_ω, i.e., $\sup_{\tau\in\mathbb{R}} \sup_{u\in\chi(\tau,\omega)} ||u||_X \leq R_\omega < \infty$, and $R_{\theta_t\omega}$ is continuous in t for all $t \in \mathbb{R}$;*

(H2) *invariance: $\chi(t + \tau, \theta_t\omega) = \Psi(t, \tau, \omega)\chi(\tau, \omega)$ for all $t \geq 0$;*

(H3) *there exist positive numbers λ, δ, t_0, random variables $C_0(\omega) \geq 0$, $C_1(\omega) \geq 0$ and m-dimensional projector P_m: $X \to P_mX$ ($\dim(P_mX) = m$) such that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any u, $v \in \chi(\tau, \omega)$, it holds that*

$$||P_m\Psi(t_0, \tau, \omega)u - P_m\Psi(t_0, \tau, \omega)v||_X \leq e^{\int_0^{t_0} C_0(\theta_s\omega)ds} ||u - v||_X \tag{4.1}$$

and

$$||(I - P_m)\Psi(t_0, \tau, \omega)u - (I - P_m)\Psi(t_0, \tau, \omega)v||_X \leq (e^{-\lambda t_0 + \int_0^{t_0} C_1(\theta_s\omega)ds} + \delta e^{\int_0^{t_0} C_0(\theta_s\omega)ds})||u - v||_X, \tag{4.2}$$

where λ, δ, t_0, m are independent of τ and ω.

(H4) *λ, t_0, δ, $C_0(\omega)$, $C_1(\omega)$ satisfy conditions:*

$$\begin{cases} 0 \leq \mathbf{E}[C_1(\omega)] < \lambda, \quad t_0 \geq \frac{\ln\frac{5}{3}}{\lambda - \mathbf{E}[C_1(\omega)]} > 0, \\ 0 \leq \mathbf{E}[C_0^2(\omega)] < \infty, \\ 0 < \delta \leq \min\left\{\frac{1}{20}, \frac{1}{8} e^{-\frac{2}{\ln\frac{3}{2}} t_0^2 (\mathbf{E}[C_0^2(\omega)] + \lambda\mathbf{E}[C_0(\omega)])}\right\}, \end{cases} \tag{4.3}$$

where "$\mathbf{E}$" denotes the expectation.

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Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, the fractal dimension of $\chi(\tau,\omega)$ has an upper bound:

$$\dim_f \chi(\tau,\omega) = \lim \sup_{\varepsilon \to 0+} \frac{\ln N_\varepsilon(\chi(\tau,\omega))}{-\ln \varepsilon} \le \frac{2m \ln\left(\frac{\sqrt{m}}{\delta}+1\right)}{\ln \frac{4}{3}} < \infty, \tag{4.4}$$

where $N_\varepsilon(\chi(\tau,\omega))$ is the minimal number of balls with radius $\varepsilon > 0$ covering $\chi(\tau,\omega)$ in X.

(For the proof of this theorem, see appendix.)

It is easy to see that $\mathcal{A}(\tau,\omega)$ satisfies conditions (H1) and (H2) of Theorem 4.1. Now we show the Lipschitz property of Φ on $\mathcal{A}(\tau,\omega)$. For any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varphi_{j\tau}(\omega) = (v_{1\tau}^{(j)}(\omega), v_{2\tau}^{(j)}(\omega)) \in \mathcal{A}(\tau,\omega)$, $j = 1,2$, let

$$\begin{aligned} \varphi_j(r) &= \varphi_j(r,\tau,\omega,\varphi_\tau^{(j)}(\omega)) = (v_1^{(j)}(r), v_2^{(j)}(r)), \quad j = 1,2, \\ y(r) &= \varphi_1(r) - \varphi_2(r) = (y_1(r), y_2(r)), \quad r \ge \tau, \end{aligned}$$

then

$$\begin{cases} \frac{dy_1}{dt} - \Delta y_1 - bz(\theta_t\omega)y_1 + \alpha y_2 + e^{-bz(\theta_t\omega)}[f(e^{bz(\theta_t\omega)}v_1^{(1)}(t)) - f(e^{bz(\theta_t\omega)})v_1^{(2)}(t))] = 0, \quad t > \tau, \\ \frac{dy_2}{dt} - bz(\theta_t\omega)y_2 + \sigma y_2 - \beta y_1 = 0, \quad t > \tau, \\ y_i(t,\omega,x)|_{x\in\partial U} = 0; \quad t \ge \tau, \quad i = 1,2, \\ y_i(\tau,\omega,x) = v_{i\tau}^{(1)}(\omega,x) - v_{i\tau}^{(2)}(\omega,x), \quad \tau \in \mathbb{R}, x \in U, i = 1,2. \end{cases} \tag{4.5}$$

By the invariance of $\mathcal{A}(\tau,\omega)$, the cocycle property of Φ and Theorem 3.4, for $r \ge \tau$, it holds that $v_1(r)$, $v_2(r) \in \mathcal{A}(r,\theta_r\omega) \subseteq B_1(\theta_r\omega)$ and by (3.19),

$$||v_1^{(j)}(r)||_1 \le M_2(\theta_r\omega), \quad \forall r \ge \tau, \quad j = 1,2. \tag{4.6}$$

Lemma 4.2. *For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \ge 0$ and any $\varphi_{j\tau}(\omega) \in \mathcal{A}(\tau,\omega)$, $j = 1,2$, it holds that*

$$||\varphi_1(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_{1\tau}(\theta_{-\tau}\omega)) - \varphi_2(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_{2\tau}(\theta_{-\tau}\omega))||_E \le e^{\int_0^t |bz(\theta_s\omega)|ds + c_2 t}||\varphi_{1\tau} - \varphi_{2\tau}||_E. \tag{4.7}$$

Proof. Taking the inner product of the first equation and second equation of (4.5) with βy_1 and αy_2, respectively, and summing them, and by

$$e^{-bz(\theta_r\omega)} \int_U [f(e^{bz(\theta_r\omega)}v_1^{(1)}) - f(e^{bz(\theta_r\omega)}v_1^{(2)})]y_1 dx \ge -c_2||y_1(r)||^2,$$

we have

$$\frac{d}{dt}\|y(r)\|_E^2 \le 2[bz(\theta_r\omega) + c_2]||y(r)||_E^2, \quad \forall r \ge \tau. \tag{4.8}$$

Then by Gronwall inequality to (4.8) on $[\tau, r]$ $(r \ge \tau)$ and $\omega \to \theta_{-\tau}\omega$, $r \to t+\tau$, (4.7) holds. The proof is completed. □

Let $\{e_j\}_{j\in\mathbb{N}} \subset D(-\Delta)$ be the eigenvectors of operator $-\Delta$ corresponding to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ with $-\Delta e_j = \lambda_j e_j$ for $j \in \mathbb{N}$, then $\{e_j\}_{j\in\mathbb{N}}$ form an orthonormal base of $L^2(U)$ and $H_0^1(U)$. Write

$$L_n^2(U) = \text{span}\{e_1, e_2, \cdots, e_n\}, \quad L_n^2(U)^\perp = \text{span}\{e_{n+1}, e_{n+2}, \cdots\}, \quad n \in \mathbb{N},$$

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and let

$$P_n : L^2(U) \to L^2_n(U), \quad Q_n = I - P_n : L^2(U) \to L^2_n(U)^\perp$$

be the orthonormal projector defined by

$$P_n v = v_n \quad \text{for} \quad v \in L^2(U) \quad \text{with} \quad v = v_n + v_n^\perp \quad \text{where} \quad v_n \in L^2_n(U), v_n^\perp \in L^2_n(U)^\perp,$$

then

$$\lambda_{n+1}||Q_n v||^2 \le ||v||_1^2, \quad \forall v \in H_0^1(U). \tag{4.9}$$

Lemma 4.3. *Let n be large enough such that $\lambda_{n+1} > \sigma + 1$, then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \ge 0$, there exist a random variable $C_3(\omega) \ge 0$ and a n-dimensional finite dimensional projector P_n: $L^2(U) \to L^2_n(U)$ such that for any $\varphi_{j\tau}(\omega) \in \mathcal{A}(\tau, \omega)$, $j = 1, 2$, it holds that for $t \ge \frac{2}{\sigma} \ln\left(1 + \frac{\alpha\beta}{\sigma}\right)$,*

$$\begin{aligned} &||(I - P_n)\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{1\tau}(\theta_{-\tau}\omega) - (I - P_n)\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{2\tau}(\theta_{-\tau}\omega)|| \\ &\le \left(e^{-\frac{\sigma}{4}t + \int_0^t |b||z(\theta_s\omega)|ds} + \delta_n e^{\int_0^t C_3(\theta_s\omega)ds}\right) \|\varphi_{1\tau} - \varphi_{2\tau}\|, \end{aligned} \tag{4.10}$$

and

$$||P_n\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{1\tau}(\theta_{-\tau}\omega) - P_n\Phi(t, \tau, \theta_{-\tau}\omega)\varphi_{2\tau}(\theta_{-\tau}\omega)|| \le e^{\int_0^t C_3(\theta_s\omega)ds}||\varphi_{1\tau} - \varphi_{2\tau}||, \tag{4.11}$$

where

$$\delta_n = \sqrt{\left(1 + \frac{\alpha\beta}{\sigma^2}\right)\frac{\beta}{\sqrt{2}}}\lambda_{n+1}^{-\frac{1}{4}}. \tag{4.12}$$

Proof. Taking the inner product of the first equation of (4.5) with $y_{1,n} = Q_n y_1$ in $L^2(U)$, and by

$$\begin{aligned} &||e^{-bz(\theta_t\omega)}[f(e^{bz(\theta_t\omega)}v_1^{(1)}(t)) - f(e^{bz(\theta_t\omega)}v_1^{(2)}(t))]||^2_{L^{\frac{6}{5}}} \\ &\le c_6 e^{-2bz(\theta_t\omega)}\left(e^{4bz(\theta_t\omega)}M_2^4(\theta_r\omega) + 1\right) \cdot ||\varphi_1(t) - \varphi_2(t)||_E^2, \end{aligned}$$

and

$$2\alpha(y_2, y_{1,n}) \le 2\sqrt{\frac{\alpha}{\beta}}\sqrt{\alpha}||y_2|| \cdot \sqrt{\beta}||y_1|| \le \sqrt{\frac{\alpha}{\beta}}||\varphi_1(t) - \varphi_2(t)||_E^2,$$

we have that

$$\begin{aligned} &2\int_U e^{-bz(\theta_t\omega)}[f(e^{bz(\theta_t\omega)}v_1^{(1)}(t)) - f(e^{bz(\theta_t\omega)}v_1^{(2)}(t))]y_{1,n}dx \\ &\le 2||e^{-bz(\theta_t\omega)}[f(e^{bW(t)}v_1^{(1)}(t)) - f(e^{bW(t)}v_1^{(2)}(t))]||_{L^{\frac{6}{5}}} \cdot ||y_{1,n}||_{L^6} \\ &\le c_6 e^{-2bz(\theta_t\omega)}\left(e^{4bz(\theta_t\omega)}M_2^4(\theta_r\omega) + 1\right) \cdot ||\varphi_1(t) - \varphi_2(t)||_E^2 + ||y_{1,n}||_1^2, \end{aligned}$$

and

$$
\begin{aligned}
&\frac{d}{dt}\|y_{1,n}\|^2 + [\lambda_{n+1} - 2bz(\theta_t\omega)]\|y_{1,n}\|^2 \\
&\le \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\theta_t\omega)} M_2^4(\theta_r\omega) + c_6 e^{-2bz(\theta_t\omega)}\right) \cdot ||\varphi_1(t) - \varphi_2(t)||_E^2,
\end{aligned}
\tag{4.13}
$$

where c_6 is a positive constant. By Gronwall inequality to (4.13) on $[\tau, \tau+t]$ $(t \ge 0)$ and (4.7), we have that

$$
\begin{aligned}
&||y_{1,n}(\tau+t, \tau, \theta_{-\tau}\omega, y_\tau(\theta_{-\tau}\omega))||^2 \\
&\le ||y_{1,n}(\tau)||^2 e^{\int_\tau^{\tau+t}(2bz(\theta_{s-\tau}\omega)-\lambda_{n+1})ds} \\
&\quad + \int_\tau^{\tau+t} \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\theta_{r-\tau}\omega)} M_2^4(\theta_{r-\tau}\omega) + c_6 e^{-2bz(\theta_{r-\tau}\omega)}\right) \cdot ||\varphi_1(r) - \varphi_2(r)||_E^2 \\
&\quad \cdot e^{\int_r^{\tau+t}(2bz(\theta_{s-\tau}\omega)-\lambda_{n+1})ds} dr \\
&\le ||y_{1,n}(\tau)||^2 e^{\int_0^t (2bz(\theta_s\omega)-\lambda_{n+1})ds} \\
&\quad + ||\varphi_{1\tau} - \varphi_{2\tau}||_E^2 \int_\tau^{\tau+t} \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\theta_{r-\tau}\omega)} M_2^4(\theta_{r-\tau}\omega) + c_6 e^{-2bz(\theta_{r-\tau}\omega)}\right) e^{2\int_\tau^r (bz(\theta_{s-\tau}\omega)+c_2)ds} \\
&\quad \cdot e^{\int_r^{\tau+t}(2bz(\theta_{s-\tau}\omega)-\lambda_{n+1})ds} dr \\
&\le ||y_{1,n}(\tau)||^2 e^{\int_0^t (2bz(\theta_s\omega)-\lambda_{n+1})ds} \\
&\quad + ||\varphi_{1\tau} - \varphi_{2\tau}||_E^2 e^{\int_0^t (2bz(\theta_s\omega)+2c_2)ds} \int_0^t \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\theta_r\omega)} M_2^4(\theta_r\omega) + c_6 e^{-2bz(\theta_r\omega)}\right) \\
&\quad \cdot e^{-\lambda_{n+1}(t-r)} dr.
\end{aligned}
\tag{4.14}
$$

Since $\sqrt{x} \le e^x$ for all $x \ge 0$, it follows that

$$
\begin{aligned}
&\int_0^t \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\theta_r\omega)} M_2^4(\theta_r\omega) + c_6 e^{-2bz(\theta_r\omega)}\right) \cdot e^{-\lambda_{n+1}(t-r)} dr \\
&\le \left(\int_0^t \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\theta_r\omega)} M_2^4(\theta_r\omega) + c_6 e^{-2bz(\theta_r\omega)}\right)^2 dr\right)^{\frac{1}{2}} \left(\int_0^t e^{-2\lambda_{n+1}(t-r)} dr\right)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{2\lambda_{n+1}}} e^{\int_0^t \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\theta_r\omega)} M_2^4(\theta_r\omega) + c_6 e^{-2bz(\theta_r\omega)}\right)^2 dr}
\end{aligned}
$$

and by (4.14),

$$
\begin{aligned}
&||y_{1,n}(\tau+t, \tau, \theta_{-\tau}\omega, \varphi_{1\tau}(\theta_{-\tau}\omega))||^2 \\
&\le ||y_{1,n}(\tau)||^2 e^{\int_0^t 2bz(\theta_s\omega)ds - \lambda_{n+1}t} + \frac{1}{\sqrt{2\lambda_{n+1}}} e^{\int_0^t [2bz(\theta_s\omega) + C_4(\theta_s\omega)]ds} ||\varphi_{1\tau} - \varphi_{2\tau}||_E^2,
\end{aligned}
\tag{4.15}
$$

where

$$
C_4(\omega) = 2c_2 + \left(\sqrt{\frac{\alpha}{\beta}} + c_6 e^{2bz(\omega)} M_2^4(\omega) + c_6 e^{-2bz(\omega)}\right)^2. \tag{4.16}
$$

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Taking the inner product of the second equation of (4.5) with $y_{2,n} = Q_n y_2$ in $L^2(U)$, we have that

$$\frac{d}{dt}\|y_{2,n}\|^2 + [\sigma - 2bz(\theta_t\omega)]\|y_{2,n}\|^2 \le \frac{\beta^2}{\sigma}\|y_{1,n}\|^2. \tag{4.17}$$

For $\lambda_{n+1} > \sigma + 1$, by (4.15), we have

$$\begin{aligned}
&\|y_{2,n}(\tau+t,\tau,\theta_{-\tau}\omega, y_\tau(\theta_{-\tau}\omega))\|^2 \\
&\le ||y_{2,n}(\tau)||^2 e^{\int_\tau^{\tau+t}(2bz(\theta_{s-\tau}\omega)-\sigma)ds} \\
&\quad + \frac{\beta^2}{\sigma}\int_\tau^{\tau+t} \|y_{1,n}(s,\tau,\theta_{-\tau}\omega, y_\tau(\theta_{-\tau}\omega))\|^2 \cdot e^{\int_s^{\tau+t}(2bz(\theta_{r-\tau}\omega)-\sigma)dr} ds \\
&\le ||y_{2,n}(\tau)||^2 e^{\int_\tau^{\tau+t}(2bz(\theta_{s-\tau}\omega)-\sigma)ds} \\
&\quad + \frac{\beta^2}{\sigma}\int_0^t \left(||y_{1,n}(\tau)||^2 e^{\int_0^s 2bz(\theta_r\omega)dr - \lambda_{n+1}s} + \frac{1}{\sqrt{2\lambda_{n+1}}} e^{\int_0^s [2bz(\theta_r\omega)+C_4(\theta_r\omega)]dr} ||\varphi_{1\tau} - \varphi_{2\tau}||_E^2 \right) \\
&\quad \cdot e^{\int_s^t (2bz(\theta_r\omega)-\sigma)dr} ds \\
&\le \left(||y_{2,n}(\tau)||^2 + \frac{\beta^2}{\sigma}||y_{1,n}(\tau)||^2 \right) e^{\int_0^t 2bz(\theta_r\omega)dr - \sigma t} \\
&\quad + \frac{\beta^2}{\sigma^2}\frac{1}{\sqrt{2\lambda_{n+1}}} e^{\int_0^t [2bz(\theta_r\omega)+C_4(\theta_r\omega)]dr - \sigma t} ||\varphi_{1\tau} - \varphi_{2\tau}||_E^2,
\end{aligned} \tag{4.18}$$

thus,

$$\begin{aligned}
&\beta\|y_{1,n}(\tau+t,\tau,\theta_{-\tau}\omega, y_\tau(\theta_{-\tau}\omega))\|^2 + \alpha\|y_{2,n}(\tau+t,\tau,\theta_{-\tau}\omega, y_\tau(\theta_{-\tau}\omega))\|^2 \\
&\le \left(\left(1 + \frac{\alpha\beta}{\sigma}\right) e^{-\frac{\sigma}{2}t}\beta\|y_{1,n}(\tau)\|^2 + \alpha\|y_{2,n}(\tau)\|^2 \right) e^{\int_0^t (2|bz(\theta_r\omega)| - \frac{\sigma}{2})dr} \\
&\quad + \left(1 + \frac{\alpha\beta}{\sigma^2}\right) \frac{\beta}{\sqrt{2\lambda_{n+1}}} e^{\int_0^t [2b|z(\theta_s\omega)| + C_4(\theta_s\omega)]dr} ||\varphi_{1\tau} - \varphi_{2\tau}||_E^2.
\end{aligned} \tag{4.19}$$

For $t \ge \frac{2}{\sigma}\ln\left(1 + \frac{\alpha\beta}{\sigma}\right)$, that is, $\left(1 + \frac{\alpha\beta}{\sigma}\right)e^{-\frac{\sigma}{2}t} \le 1$, we have

$$\begin{aligned}
&\|y_n(\tau+t,\tau,\theta_{-\tau}\omega, y_\tau(\theta_{-\tau}\omega))\|_E^2 \\
&\le \left(e^{\int_0^t (2|b||z(\theta_r\omega)| - \frac{\sigma}{2})dr} + \left(1 + \frac{\alpha\beta}{\sigma^2}\right)\frac{\beta}{\sqrt{2\lambda_{n+1}}} e^{2\int_0^t C_3(\theta_r\omega)dr} \right) ||\varphi_{1\tau} - \varphi_{2\tau}||_E^2,
\end{aligned} \tag{4.20}$$

i.e., (4.10) holds, where $C_3(\omega) = |b||z(\omega)| + \frac{1}{2}C_4(\omega)$. From (4.7) and $C_3(\omega) > |b||z(\omega)| + c_2 \ge 0$, it follows that (4.11) holds. The proof is completed. □

To show the finiteness of the expectations of $C_3^2(\omega)$ and $C_3(\omega)$, we have recourse to the following result in [16].

Lemma 4.4. *(See [16].) The Ornstein–Uhlenbeck process $z(\theta_t\omega)$ satisfies*

$$\mathbf{E}\left[e^{\gamma\int_\tau^{\tau+t}|z(\theta_s\omega)|ds}\right] \le e^{\gamma t}, \quad \textit{for } 0 \le \gamma^2 \le 1,\ \tau \in \mathbb{R}, t \ge 0; \tag{4.21}$$

$$\mathbf{E}[|z(\theta_t\omega)|^r] = \frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi}}, \quad \forall r > 0, t \in \mathbb{R}, \tag{4.22}$$

where Γ *is the Gamma function.*

Lemma 4.5. *If the coefficient* b *of the random term of* (2.1) *is small such that*

$$|b| < \min\left\{\frac{1}{256}, \frac{\rho}{4}\sqrt{\pi}\right\}, \tag{4.23}$$

then

$$0 \leq \mathbf{E}[C_3(\omega)], \quad \mathbf{E}[C_3^2(\omega)] < \infty. \tag{4.24}$$

Proof. By the property of Gamma function Γ,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(n) = (n-1)!, \quad \Gamma(r+1) = r\Gamma(r), \quad \forall r \in \mathbb{R}. \tag{4.25}$$

By the recursion,

$$\Gamma(\frac{1+2k}{2}) = \frac{(2k-1)(2k-3)\cdots 3\cdot 1}{2^k}\sqrt{\pi}, \quad \Gamma(\frac{1+(2k+1)}{2}) = k!, \quad \forall k \in \mathbb{N}. \tag{4.26}$$

For any $|q| \leq 1$, by (4.22) and (4.26),

$$\begin{aligned}
\mathbf{E}[e^{qz(\omega)}] &\leq \mathbf{E}\left[1 + \frac{1}{2!}|z(\omega)|^2 + \frac{1}{4!}|z(\omega)|^4 + \cdots + \frac{1}{(2k)!}|z(\omega)|^{2k} + \cdots\right] \\
&\quad + \mathbf{E}\left[|z(\omega)| + \frac{1}{3!}|z(\omega)|^3 + \frac{1}{5!}|z(\omega)|^5 + \cdots + \frac{1}{(2k+1)!}|z(\omega)|^{2k+1} + \cdots\right] \\
&\leq 1 + \frac{1}{2^2} + \frac{1}{2}\frac{1}{2^4} + \cdots + \frac{1}{k!}\frac{1}{2^{2k}} + \cdots \\
&\quad + \frac{1}{\sqrt{\pi}}\left(1 + \frac{1}{3!} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{(k+1)\cdot(k+2)\cdot(2k+1)} + \cdots\right) \\
&< \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^k} + \cdots\right) + \frac{1}{\sqrt{\pi}}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \cdots\right) \\
&= \frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}}.
\end{aligned} \tag{4.27}$$

By (4.16),

$$C_3(\omega) = \bar{c}\left(1 + |z(\omega)| + e^{8bz(\omega)} + e^{-4bz(\omega)} + M_2^{16}(\omega)\right), \tag{4.28}$$

and

$$C_3^2(\omega) = c_7 + b^2|z(\omega)|^2 + c_8e^{16bz(\omega)} + c_9e^{-8bz(\omega)} + c_{10}M_2^{32}(\omega), \tag{4.29}$$

where $\bar{c}$, c_7, c_8, c_9, c_{10} are positive constants. Now let us prove the boundedness of the expectations of each term in the expression of $C_3(\omega)$ in (4.29). From (3.6) and (3.7), we have that

$$M_2^{32}(\omega) \leq c_{11} + c_{12}M_1^{64}(\omega) + c_{13}\max_{0\leq s\leq 1}|z(\theta_s\omega)|^{32} + c_{14}\int_{-1}^{0} e^{-32bz(\theta_s\omega)}ds, \tag{4.30}$$

and

$$M_1^{64}(\omega) \leq c_{15} + c_{16} e^{-128\int_{-1}^{0} bz(\theta_s\omega)ds} + c_{17}\int_{-\infty}^{0} e^{-256bz(\theta_s\omega)+32\rho s} ds$$
$$+ c_{18}\int_{-\infty}^{0} e^{\int_s^0 128bz(\theta_s\omega)ds+32\rho s} ds, \tag{4.31}$$

where c_i, $i = 11$–18, are constants. By (4.21), (4.22), (4.23) and (4.27), we have

$$\mathbf{E}\left(e^{-128b\int_{-1}^{0} z(\theta_s\omega)ds}\right) \leq \mathbf{E}\left(e^{128|b|\int_{-1}^{0} |z(\theta_s\omega)|ds}\right) \leq e^{128|b|} < \infty,$$

$$\mathbf{E}\left(\int_{-\infty}^{0} e^{-256bz(\theta_s\omega)+32\rho s} ds\right) \leq \frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}\cdot 32\rho} < \infty,$$

$$\mathbf{E}\left(\int_{-\infty}^{0} e^{\int_s^0 128bz(\theta_s\omega)ds+32\rho s}\right) \leq \int_{-\infty}^{0} e^{(32\rho+128|b|)s} ds \leq \frac{1}{32\rho+128|b|} < \infty,$$

$$\mathbf{E}\left(\max_{0\leq s\leq 1} |z(\theta_s\omega)|^{32}\right) = \mathbf{E}\left(|z(\theta_s\omega)|^{32}\right) = \frac{15\cdot 13\cdot \ldots \cdot 3\cdot 1}{2^{16}} < \infty,$$

$$\mathbf{E}\left(\int_{-1}^{0} e^{-32bz(\theta_s\omega)} ds\right) = \int_{-1}^{0} \mathbf{E}[e^{-32bz(\theta_s\omega)}]ds \leq \frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}}.$$

So, by (4.30),

$$\mathbf{E}[M_2^{32}(\omega)] \leq c_{11} + c_{12}\left(c_{15} + c_{16}e^{128|b|} + c_{17}\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}\cdot 32\rho} + c_{18}\frac{1}{32\rho+128|b|}\right)$$
$$+ c_{13}\frac{15\cdot 13\cdot \ldots \cdot 3\cdot 1}{2^{16}} + c_{14}\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}}$$
$$< \infty$$

and by (4.29),

$$\mathbf{E}[C_3^2(\omega)] \leq c_7 + \frac{b^2}{2} + (c_8+c_9)\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}} + c_{10}\mathbf{E}[M_2^{32}(\omega)] < \infty. \tag{4.32}$$

The proof is completed. □

As a consequence of Theorem 4.1 and Lemma 4.3, Lemma 4.5, we have the following main result of this section.

Theorem 4.6. *Suppose* (H) *and* (4.23) *hold. Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, the fractal dimension of $\mathcal{A}(\tau,\omega)$ has finite upper bound:*

$$\dim_f \mathcal{A}(\tau,\omega) \leq \frac{4n_0 \ln\left(\frac{\sqrt{2n_0}\lambda_{n_0+1}^{\frac{1}{4}}}{\sqrt{\left(1+\frac{\alpha\beta}{\sigma^2}\right)\frac{\beta}{\sqrt{2}}}} + 1\right)}{\ln\frac{4}{3}} < \infty, \tag{4.33}$$

where

$$n_0 = \min\left\{n : \lambda_{n+1}^{-\frac{1}{4}} \le \left(\left(1+\frac{\alpha\beta}{\sigma^2}\right)\frac{\beta}{\sqrt{2}}\right)^{-\frac{1}{2}} \min\left\{\frac{1}{20}, \frac{1}{8}e^{-\frac{2}{\ln\frac{3}{2}}t_0^2(\mathbf{E}[C_3^2(\omega)]+\frac{\rho}{4}\mathbf{E}[C_3(\omega)])}\right\}\right\} < \infty \tag{4.34}$$

and

$$t_0 = \max\left\{\frac{\ln\frac{5}{3}}{\frac{\rho}{4}-\frac{|b|}{\sqrt{\pi}}}, \quad \frac{2}{\sigma}\ln\left(1+\frac{\alpha\beta}{\sigma}\right)\right\} > 0. \tag{4.35}$$

Proof. From (4.23) and (4.32),

$$0 \le \mathbf{E}[|b||z(\omega)|] = \frac{|b|}{\sqrt{\pi}} < \frac{\rho}{4}, \tag{4.36}$$

and

$$0 < \mathbf{E}[C_3^2(\omega)] + \frac{\rho}{4}\mathbf{E}[C_3(\omega)] < \infty. \tag{4.37}$$

Take t_0 as in (4.35). Comparing (4.2) and (4.10), we see that

$$0 < \delta = \sqrt{\left(1+\frac{\alpha\beta}{\sigma^2}\right)\frac{\beta}{\sqrt{2}}}\lambda_{n+1}^{-\frac{1}{4}} \overset{n\to\infty}{\to} 0. \tag{4.38}$$

Then by Theorem 4.1, (4.33) holds. The proof is completed. $\square$

Appendix A

Proof of Theorem 4.1. Let $\tau \in R$ and $\omega \in \Omega$, by (H1), taking $u_0 \in \chi(\tau,\omega)$, then

$$\chi(\tau,\omega) \subseteq B(u_0, R_\omega). \tag{A.1}$$

For any $u \in \chi(\tau,\omega) = \chi(\tau,\omega) \cap B(u_0, R_\omega)$, by (4.1)–(4.2), we have

$$||P_m\Phi(t_0,\tau,\omega)u - P_m\Phi(t_0,\tau,\omega)u_0||_X \le e^{\int_0^{t_0} C_0(\theta_s\omega)ds}R_\omega, \tag{A.2}$$

and

$$||(I-P_m)\Phi(t_0,\tau,\omega)u - (I-P_m)\Phi(t_0,\tau,\omega)u_0||_X \le (e^{-\lambda t_0+\int_0^{t_0} C_1(\theta_s\omega)ds} + \delta e^{\int_0^{t_0} C_0(\theta_s\omega)ds})R_\omega. \tag{A.3}$$

By Lemma 1.2 in [13], there exist $v_{01}^1, ..., v_{01}^{n_1} \in P_mX$ such that

$$B_{P_mX}(P_m\Phi(t_0,\tau,\omega)u_0, e^{\int_0^{t_0} C_0(\theta_s\omega)ds}R_\omega) \subseteq \bigcup_{j=1}^{n_1} B_{P_mX}(v_{01}^j, \delta e^{\int_0^{t_0} C_0(\theta_s\omega)ds}R_\omega), \tag{A.4}$$

where $B_{P_mX}(v,r)$ denotes the ball in P_mX of radius r and center v, and

$$n_1 \le \left(\frac{\sqrt{m}}{\delta}+1\right)^m. \tag{A.5}$$

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Take

$$u_{01}^j = v_{01}^j + (I - P_m)\Phi(t_0,\tau,\omega)u_0 \in X, \quad j = 1, \dots, n_1, \tag{A.6}$$

then there exists a $j \in \{1, \dots, n_1\}$ such that

$$\begin{aligned} & ||\Phi(t_0,\tau,\omega)u - u_{01}^j||_X \\ \le\ & ||P_m\Phi(t_0,\tau,\omega)u - v_{01}^j||_X + ||(I-P_m)\Phi(t_0,\tau,\omega)u - (I-P_m)\Phi(t_0,\tau,\omega)u_0||_X \\ \le\ & \left(e^{-\lambda t_0 + \int_0^{t_0} C_1(\theta_s\omega)ds} + 2\delta e^{\int_0^{t_0} C_0(\theta_s\omega)ds}\right) R_\omega. \end{aligned} \tag{A.7}$$

Thus, by (A.7) and (H2), it follows that

$$\begin{aligned} \chi(t_0+\tau,\theta_{t_0}\omega) &= \Phi(t_0,\tau,\omega)(\chi(\tau,\omega)\cap B(u_0,R_\omega)) \\ &\subseteq \bigcup_{j=1}^{n_1} B(u_{01}^j, \left(e^{-\lambda t_0 + \int_0^{t_0} C_1(\theta_s\omega)ds} + 2\delta e^{\int_0^{t_0} C_0(\theta_s\omega)ds}\right) R_\omega). \end{aligned} \tag{A.8}$$

Making the recursion to the inclusion (A.8), we have

$$\chi(kt_0+\tau,\theta_{kt_0}\omega) = \Phi(kt_0,\tau,\omega)\chi(\tau,\omega) \subseteq \bigcup_{j=1}^{n_1\cdot\ldots\cdot n_k} B(u_{0k}^j, (\prod_{l=1}^{k}\sigma_l)R_\omega), \quad k \ge 1, \tag{A.9}$$

where

$$n_l \le \left(\frac{\sqrt{m}}{\delta}+1\right)^m, \quad \sigma_l = e^{-\lambda t_0 + \int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds} + 2\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds}, \quad l = 1, \dots k. \tag{A.10}$$

Thus, by (A.10), the minimal number $N_{r_k}(\chi(kt_0+\tau,\theta_{kt_0}\omega))$ of balls with radius $r_k = (\prod_{l=1}^k \sigma_l)R_\omega$ covering $\chi(kt_0+\tau,\theta_{kt_0}\omega)$ in X satisfies

$$N_{r_k}(\chi(kt_0+\tau,\theta_{kt_0}\omega)) \le n_1\cdot\ldots\cdot n_k \le \left(\frac{\sqrt{m}}{\delta}+1\right)^{km}. \tag{A.11}$$

Set

$$J = \left\{\omega\in\Omega : \int_0^{t_0} [C_0(\theta_s\omega) - C_1(\theta_s\omega)]ds + \lambda t_0 > \ln\frac{1}{8\delta}\right\}. \tag{A.12}$$

(a) If $\theta_{(l-1)t_0}\omega \in J$, then we have

$$\begin{aligned} & \int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds - \int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds + \lambda t_0 \\ &= \int_0^{t_0} [C_0(\theta_s(\theta_{(l-1)t_0}\omega) - C_1(\theta_s(\theta_{(l-1)t_0}\omega))ds + \lambda t_0 \\ &> \ln\frac{1}{8\delta}, \end{aligned} \tag{A.13}$$

that is,

$$8\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds} > e^{-\lambda t_0+\int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds}, \tag{A.14}$$

thus, by (4.3), (A.10) and (A.14),

$$\sigma_l = e^{-\lambda t_0+\int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds} + 2\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds} < 10\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds} \le \frac{1}{2} e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds},\ l \ge 1$$

and

$$\prod_{l=1}^{k} \sigma_l \le \frac{1}{2^k} e^{\sum_{l=1}^{k} \zeta_J(\theta_{(l-1)t_0}\omega)\cdot\int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds}, \tag{A.15}$$

where

$$\zeta_J(\omega) = \begin{cases} 1,\ \omega \in J, \\ 0,\ \omega \notin J. \end{cases} \tag{A.16}$$

Since $(\theta_t)_{t\in\mathbb{R}}$ is measure-preserving and ergodic on $(\Omega, \mathcal{F}, \mathbb{P})$, by Birkhoff ergodic theorem [36], we have that

$$\mathbf{E}[C_i(\theta_s\omega) = \mathbf{E}[C_i(\omega)], \quad \mathbf{E}[C_0^2(\theta_s\omega) = \mathbf{E}[C_0^2(\omega)], \quad \forall s \in \mathbb{R}, i = 0, 1 \tag{A.17}$$

and for all $\omega \in \Omega$ (in fact, for $a.e.\ \omega \in \Omega$),

$$\frac{1}{k}\sum_{l=1}^{k} \zeta_J(\theta_{(l-1)t_0}\omega) \cdot \int\limits_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds \to \mathbf{E}\left(\zeta_J(\omega)\cdot\int\limits_0^{t_0} C_0(\theta_s\omega)ds\right) \quad \text{as} \quad k \to \infty. \tag{A.18}$$

By (H4),

$$\mathbf{E}[C_0(\omega)] \le \frac{1}{2}(1 + \mathbf{E}[C_0^2(\omega)]) < \infty.$$

Thus, for every $\omega \in \Omega$, there exists a large integer $k_0(\omega) \in \mathbb{N}$ such that

$$\sum_{l=1}^{k} \zeta_J(\theta_{(l-1)t_0}\omega) \cdot \int\limits_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds \le 2k\mathbf{E}\left(\zeta_J(\omega)\cdot\int\limits_0^{t_0} C_0(\theta_s\omega)ds\right), \quad \forall k \ge k_0(\omega). \tag{A.19}$$

By (A.12), (A.16) and Hölder inequality, we have

$$\begin{aligned}
&\mathbf{E}\left(\zeta_J(\omega)\int\limits_0^{t_0} C_0(\theta_s\omega)ds\right) \\
&= \mathbf{E}\left(\frac{\zeta_J(\omega)\int_0^{t_0} C_0(\theta_s\omega)ds\left(\int_0^{t_0}[C_0(\theta_s\omega) - C_1(\theta_s\omega)]ds + \lambda t_0\right)}{\int_0^{t_0}[C_0(\theta_s\omega) - C_1(\theta_s\omega)]ds + \lambda t_0}\right) \\
&\le \mathbf{E}\left(\frac{\zeta_J(\omega)\left[\left(\int_0^{t_0} C_0(\theta_s\omega)ds\right)^2 - \int_0^{t_0} C_0(\theta_s\omega)ds\int_0^{t_0} C_1(\theta_s\omega)ds + \lambda t_0\int_0^{t_0} C_0(\theta_s\omega)ds\right]}{\ln\frac{1}{8\delta}}\right)
\end{aligned}$$

$$\le \frac{1}{\ln\frac{1}{8\delta}}\mathbf{E}\left(\zeta_J(\omega)\left[\left(\int_0^{t_0} C_0(\theta_s\omega)ds\right)^2 + \lambda t_0\int_0^{t_0} C_0(\theta_s\omega)ds\right]\right)$$
$$\le \frac{1}{\ln\frac{1}{8\delta}}\mathbf{E}\left(t_0^2\int_0^{t_0} C_0^2(\theta_s\omega)ds + \lambda t_0\int_0^{t_0} C_0(\theta_s\omega)ds\right)$$
$$\le \frac{1}{\ln\frac{1}{8\delta}}t_0^2\left(\mathbf{E}[C_0^2(\omega)] + \lambda\mathbf{E}[C_0(\omega)]\right). \tag{A.20}$$

Thus, by (4.3), (A.15) and (A.20), we have

$$\prod_{l=1}^k \sigma_l \le \frac{1}{2^k}e^{k\frac{2}{\ln\frac{1}{8\delta}}t_0^2(\mathbf{E}[C_0^2(\omega)]+\lambda\mathbf{E}[C_0(\omega)])} \le \frac{1}{2^k}e^{k\ln\frac{3}{2}} = \left(\frac{3}{4}\right)^k. \tag{A.21}$$

(b) If $\theta_{(l-1)t_0}\omega \notin J$, by (A.12), we have

$$\int_{(l-1)t_0}^{lt_0} [C_0(\theta_s\omega) - C_1(\theta_s\omega)]ds + \lambda t_0 \le \ln\frac{1}{8\delta}, \quad \text{i.e.,} \quad 2\delta e^{\int_{(l-1)t_0}^{lt_0} C_0(\theta_s\omega)ds} \le \frac{1}{4}e^{-\lambda t_0+\int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds}, \tag{A.22}$$

then by (A.10) and (A.22),

$$\sigma_l \le \frac{5}{4}e^{-\lambda t_0+\int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds}, \quad \prod_{l=1}^k \sigma_l \le \left(\frac{5}{4}\right)^k e^{-\lambda k t_0+\sum_{l=1}^k\int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds}. \tag{A.23}$$

By the ergodic theorem and (A.17), for any $\omega \in \Omega$,

$$\frac{1}{k}\sum_{l=1}^k \int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds \to \mathbf{E}\left(\int_0^{t_0} C_1(\theta_s\omega)ds\right) = t_0\mathbf{E}[C_1(\omega)] \quad \text{as} \quad k\to\infty. \tag{A.24}$$

It follows that for every $\omega \in \Omega$, there exists $k_1(\omega)$ such that for $k \ge k_1(\omega)$,

$$\sum_{l=1}^k \int_{(l-1)t_0}^{lt_0} C_1(\theta_s\omega)ds \le \frac{1}{2}kt_0\left(\mathbf{E}[C_1(\omega)] + \lambda\right). \tag{A.25}$$

Thus, by (4.3), (A.23) and (A.25), for all $k \ge k_1(\omega)$,

$$\prod_{l=1}^k \sigma_l \le \left(\frac{5}{4}\right)^k \left(e^{-\frac{1}{2}t_0(\lambda-\mathbf{E}[C_1(\omega)])}\right)^k \le \left(\frac{3}{4}\right)^k. \tag{A.26}$$

Therefore, combining (A.21) of (a) and (A.26) of (b), for every $\omega \in \Omega$ and $k \ge k_2(\omega) = \max\{k_0(\omega), k_1(\omega)\}$, we have

$$\prod_{l=1}^k \sigma_l \le \left(\frac{3}{4}\right)^k \stackrel{k\to\infty}{\to} 0. \tag{A.27}$$

Note that the right side of (A.9) is independent of τ, replacing ω by $\theta_{-kt_0}\omega$ in (A.9), we have

$$\chi(\tau,\omega)=\Phi(kt_0,\tau-kt_0,\theta_{-kt_0}\omega)\chi(\tau-kt_0,\theta_{-kt_0}\omega)\subseteq\bigcup_{j=1}^{n_1\dots n_k}B(u_{0k}^j,(\prod_{l=1}^k\sigma_l)R_{\theta_{-kt_0}\omega}),\quad k\geq 1. \tag{A.28}$$

By (H1) and [3], there exists a tempered random variable $\bar{b}_\omega$ (>0) such that

$$R_{\theta_{-kt_0}\omega}\leq \bar{b}_\omega e^{\frac{k}{2}\ln\frac{4}{3}},\quad \forall k\geq k_2(\omega). \tag{A.29}$$

By (A.27) and (A.29), we have

$$0<r_k(\omega)=(\prod_{l=1}^k\sigma_l)R_{\theta_{-kt_0}\omega}\leq\left(\frac{3}{4}\right)^k\bar{b}_\omega e^{\frac{k}{2}\ln\frac{4}{3}}=\bar{b}_\omega e^{-\frac{k}{2}\ln\frac{4}{3}}\overset{k\to\infty}{\to}0. \tag{A.30}$$

Now let us estimate the upper bound of fractal dimension $\dim_f\chi(\tau,\omega)=\limsup_{\varepsilon\to 0+}\frac{\ln N_\varepsilon(\chi(\tau,\omega))}{-\ln\varepsilon}$ of $\chi(\tau,\omega)$.

Let $0<\varepsilon<1$ and let $n_\varepsilon=n_\varepsilon(\omega)\in\mathbb{N}$ be an integer such that $r_{n_\varepsilon-1}(\omega)\leq\varepsilon<r_{n_\varepsilon}(\omega)$. Then $n_\varepsilon\to+\infty$ as $\varepsilon\to 0_+$. Taking ε small such that $n_\varepsilon-1\geq k_2(\omega)$, then by (A.30),

$$\frac{1}{-\ln\varepsilon}\leq\frac{1}{-\ln r_{n_\varepsilon}(\omega)}\leq\frac{1}{\frac{n_\varepsilon}{2}\ln\frac{4}{3}-\ln\bar{b}_\omega}, \tag{A.31}$$

and by (A.11),

$$N_\varepsilon(\chi(\tau,\omega))\leq N_{r_{n_\varepsilon-1}(\omega)}(\chi(\tau,\omega))\leq n_1\cdot\ldots\cdot n_{n_\varepsilon-1}\leq\left(\frac{\sqrt{m}}{\delta}+1\right)^{(n_\varepsilon-1)m}. \tag{A.32}$$

Then it follows from (A.31)–(A.32) that

$$\begin{aligned}\dim_f\chi(\tau,\omega)&=\limsup_{\varepsilon\to 0+}\frac{\ln N_\varepsilon(\chi(\tau,\omega))}{-\ln\varepsilon}\leq\limsup_{n_\varepsilon\to+\infty}\frac{\ln N_{r_{n_\varepsilon-1}(\omega)}(\chi(\tau,\omega))}{-\ln r_{n_\varepsilon}(\omega)}\\&\leq\limsup_{n_\varepsilon\to+\infty}\frac{(n_\varepsilon-1)m\ln\left(\frac{\sqrt{m}}{\delta}+1\right)}{\frac{n_\varepsilon}{2}\ln\frac{4}{3}-\ln\bar{b}_\omega}=\frac{2m\ln\left(\frac{\sqrt{m}}{\delta}+1\right)}{\ln\frac{4}{3}}<\infty.\end{aligned} \tag{A.33}$$

The proof is completed. □

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