J. Math. Anal. Appl. 441 (2016) 844–861

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of infinitely many spike solutions for a critical Hénon type biharmonic equation ☆

Yajing Zhang [a,*], Xinfu Chen [b], Jianghao Hao [a]

[a] *School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China*
[b] *Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*

A R T I C L E I N F O

Article history:
Received 12 December 2015
Available online 22 April 2016
Submitted by M. Musso

Keywords:
Bi-harmonic operator
Hénon problem
Critical Sobolev exponents
Spike solution

A B S T R A C T

Following the idea of Wei and Yan [32], with new ingredients to take care of the critical dimension $n = 5$, we construct infinitely many solutions of the biharmonic equation $\Delta^2 v = |x|^\alpha v^{\frac{n+4}{n-4}}$ in the unit ball of $\mathbb{R}^n$ ($n \geqslant 5$, $\alpha > 0$) with the Navier conditions $v = \Delta v = 0$ on the boundary.

1. Introduction

Consider the following Hénon type biharmonic problem: for $u = u(x)$,

$$\Delta^2 u = |x|^\alpha u^p, \quad u > 0 \quad \text{in} \quad B, \qquad u = \Delta u = 0 \quad \text{on} \quad \partial B, \tag{1.1}$$

where $\alpha > 0$ is a constant, $p = \frac{n+4}{n-4}$ is the critical Sobolev exponent for the imbedding $H^2 \hookrightarrow L^{p+1}$, and B is the unit ball in $\mathbb{R}^n$ centered at the origin. The purpose of this paper is to establish the following:

Theorem 1. *If $\alpha > 0$, $n \geqslant 5$, and $p = \frac{n+4}{n-4}$, problem (1.1) admits infinitely many solutions.*

The above theorem originates from a striking result of Wei and Yan [32] who demonstrated that there are infinitely many solutions of

$$-\Delta u = |x|^\alpha u^q, \ u > 0 \quad \text{in} \quad B, \qquad u = 0 \quad \text{on} \quad \partial B, \tag{1.2}$$

☆ This work is partially supported by NSF DMS-1008905, NNSFC (No. 61374089), NSF of Shanxi Province (No. 2014011005-2) and International Cooperation Projects of Shanxi Province (No. 2014081026).

* Corresponding author.
E-mail addresses: zhangyj@sxu.edu.cn (Y. Zhang), xinfu@pitt.edu (X. Chen), hjhao@sxu.edu.cn (J. Hao).

http://dx.doi.org/10.1016/j.jmaa.2016.04.028

when $q = \frac{n+2}{n-2}, \alpha > 0$ and $n \geqslant 4$. Based on Wei and Yan's idea, recently we proved in [19] an analogous result for the biharmonic equation (1.1), for the case $n \geqslant 6$. In both papers [19,32], the method does not cover the critical dimension $n = 3$ for (1.2) and $n = 5$ for (1.1). Recently, in [20] we extended the result of Wei and Yan to the critical dimension $n = 3$ for (1.2). The main purpose of this paper is therefore to extend the result of [19] to the critical dimension $n = 5$ for (1.1).

The original Hénon problem, modeling mass distribution in spherical symmetric clusters of stars, was formulated in 1973 by Hénon in [21] in terms of the second order elliptic problem (1.2). In 1982, Ni [24], for the first time proved rigorously the existence of radial solutions of (1.2) for each $\alpha > 0$ and $q \in (1, \frac{n+2+2\alpha}{n-2})$. Since then there are many works for (1.2) for either $q \approx \frac{n+2}{n-2}$ or $\alpha \gg 1$; see, for example, [2,4,8–12,18, 20,24–29,31,32], and the references therein. In particular, Wei and Yan [32], using a finite dimensional reduction argument (cf. [14,23,25,26,31]) equipped with a carefully chosen weighted L^∞ norm, proved that (1.2) admits infinitely many solutions when $n \geqslant 4$, $q = \frac{n+2}{n-2}$, and $\alpha > 0$. The problem for the smallest dimension $n = 3$ is left as an open problem in [32]. In [20], we solved the open problem.

Following the study of (1.2), in recent years biharmonic equations with nonlinear source terms have attracted quite a lot of attention; see, for example, [1,5,6,13,15–19,22,30,33–35] and the references therein. In [30], Wang proved that problem (1.1) possesses at least one non-radial solution when $n \geqslant 6$, $p = \frac{n+4}{n-4}$, and, additionally, α is large enough. With $p = \frac{n+4}{n-4}$, replacing $|x|^\alpha$ by a positive smooth function $K(x)$ and B by a general bounded domain in $\mathbb{R}^6$, Ayed and Hammami [1] proved, among many beautiful estimates, the existence of a solution under certain conditions on K. Figueiredo, Santos, and Miyagaki [13] proved that (1.1) admits a classical solution if and only if $1 < p < \frac{n+4+2\alpha}{n-4}$ (see also [30]), and established the existence of solutions of the form $v = v(|y|, |z|)$ with $y \in \mathbb{R}^\ell, z \in \mathbb{R}^{n-\ell}$ for (ℓ, p) in certain ranges. Quite recently, in [19] we developed the method of Wei and Yan [32] and proved that (1.1) admits infinitely many solutions when $p = \frac{n+4}{n-4}$ and $n \geqslant 6$. In this paper we shall further complete the method developed in [19,20,32] to prove Theorem 1.

The solutions constructed in [19,20,32] admit arbitrarily prescribed large number of peaks evenly distributed on a circle with radius close to 1. The possibility of the existence of multi-peak solutions is due to the local stability (modulo dilation and translation) of radially symmetric ground state in $\mathbb{R}^n$; see Bianchi and Egnell [7] for the Laplace operator and Bartsch, Weth, and Willem [3] for polyharmonic operators. We shall prove Theorem 1 by establishing the existence of a k peak solution for any large integer k.

Comparing with the original work of Wei and Yan [32], not only we provided a simpler, complete, and systematic method, but also we introduced at least three new ingredients that are essential to the problem solving:

1. A fundamentally different technique of local estimate for the difference R between approximation W and the ground state U_0; this new estimate, together with an explicit dependence of necessary estimates on a function $L(\sigma)$, allow us to extend Theorem 1 to the critical dimension $n = 5$.
2. Instead of using fixed norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$, we use variable norms $\|\cdot\|_\rho$ and $\|\cdot\|_{\rho+4}$ for $\rho \in [\tau, n-4]$, where $\|\cdot\|_* = \|\cdot\|_{m+\tau}$ and $\|\cdot\|_{**} = \|\cdot\|_{m+\tau+4}$ with $m = (n-4)/2$ and $\tau = (n-4)/(n-3)$. This modification is essential to the proof of Theorem 1 for the case when $n \geqslant 20$ (or $n \geqslant 10$ for (1.2)).
3. Differently from [32], our proof shows that the parameter σ is the solution of a transcendental equation $L(\sigma) = 0$, instead of an explicit expression.

The rest of the paper is organized as follows. In Section 2, we construct a two dimensional k-spike approximate solution manifold and prepare all necessary estimates needed for the existence of a k-spike solution of (1.1). In Section 3, we follow a routine technique (cf. [32]) using the estimates established in Section 2 to prove the existence of a k-spike solution, for each large enough integer k.

2. Multi-peak approximate solution

In this section we construct multi-peak approximation solutions of (1.1), following ideas that we developed in [20] for (1.2). The approximations are essentially the same as that in [19] which are constructed based on the original idea of Wei and Yan [32]; however, the estimates are different.

2.1. The approximate solution manifold

In the sequel we use the following notation:

$$p = \frac{n+4}{n-4}, \qquad m = \frac{n-4}{2}, \qquad \tau = \frac{n-4}{n-3}, \qquad a = \frac{\alpha}{1-p}, \qquad \Phi(x) = \frac{[n(n^2-4)(n-4)]^{m/4}}{(1+|x|^2)^m}.$$

Note that $\Delta^2\Phi = \Phi^p$ in $\mathbb{R}^n$. We denote by (λ^*, σ^*) the unique positive root, for (λ, σ), of

$$L(\sigma) = 0, \qquad \alpha\lambda^{n-4} = AL'(\sigma), \tag{2.1}$$

where

$$L(\sigma) = \sum_{i\neq 0}\Big(\frac{1}{|i\pi|^{2m}} - \frac{1}{[(i\pi)^2+\sigma^2]^m}\Big) - \frac{1}{\sigma^{2m}}, \qquad L'(\sigma) = \sum_{i=-\infty}^{\infty}\frac{2m\sigma}{[\sigma^2+(i\pi)^2]^{2m+2}}, \tag{2.2}$$

$$A = \frac{A_2}{A_1}, \quad A_1 = \frac{m}{n}\int_{\mathbb{R}^n}\Phi^{p+1}(y)dy, \quad A_2 = 2^{3-n}\int_{\mathbb{R}^n}\Phi(0)\Phi^p(y)dy. \tag{2.3}$$

Note that $L'(\sigma) > 0$ for $\sigma \in (0, \infty)$, $L(0+) = -\infty$ and $L(\infty) > 0$. Hence, (2.1) admits a unique root.

In the sequel, (λ, σ, k) are independent parameters and (ε, r) are dependent parameters satisfying

$$\lambda \in \Big[\frac{1}{2}\lambda^*,\ 2\lambda^*\Big], \quad \sigma \in \Big[\frac{1}{2}\sigma^*,\ 2\sigma^*\Big], \quad k \in \mathbb{N}\cap[1+2\sigma, \infty), \quad \varepsilon = \frac{1}{\lambda k^{1/\tau}}, \quad r = 1 - \frac{\sigma}{k}. \tag{2.4}$$

For each positive integer k, the two-dimensional approximate solution manifold is parameterized by (λ, σ) with element $W = W(\cdot, \lambda, \sigma, k)$ defined by

$$W = \sum_{i=0}^{k-1}\Big[U_i - U_{i*} - \zeta_i\Big], \tag{2.5}$$

where

$$U_i = \frac{r^a}{\varepsilon^m}\Phi\Big(\frac{x - r\mathbf{e}_i}{\varepsilon}\Big), \qquad U_{i*} = \frac{r^a}{\varepsilon^m}\Phi\Big(\frac{rx - \mathbf{e}_i}{\varepsilon}\Big), \qquad \mathbf{e}_i = \left(\cos\frac{2\pi i}{k}, \sin\frac{2\pi i}{k}, \mathbf{0}'\right), \tag{2.6}$$

and ζ_i is the unique solution of

$$\Delta^2\zeta_i = 0 \text{ in } \ B, \qquad \zeta_i = 0, \quad \Delta\zeta_i = \Delta U_i - \Delta U_{i*} \quad \text{on } \ \partial B.$$

In the sequel, we shall suppress the dependence on (λ, σ, k), writing $W(x, \lambda, \sigma, k)$ simply as $W(x)$. Note that

$$\Delta^2 U_i = r^\alpha U_i^p, \quad \Delta^2 U_{i*} = r^{\alpha+4}U_{i*}^p \quad \text{in } \ B, \qquad U_i = U_{i*}, \quad \Delta U_i = r^{-2}\Delta U_{i*} \quad \text{on } \ \partial B;$$

here the boundary values are derived from the identity $|rx - \mathbf{e}_i|^2 - |x - r\mathbf{e}_i|^2 = (1-r^2)(1-|x|^2)$.

2.2. Idea of the construction of a k-spike solution

We work on the following spaces of functions with symmetry:

$$\mathcal{H} = \Big\{\phi \in C\big(\overline{B}\big) \;:\; \phi(se^{\mathbf{i}(\pm\theta+\frac{2\pi}{k})},\, x') = \phi(se^{\mathbf{i}\theta},\; |x'|\,, 0, \cdots, 0)\Big\},$$
$$\mathcal{H}_0 = \big\{\phi \in \mathcal{H} \cap C^2(\bar{B}) \;:\; \phi = \Delta\phi = 0 \;\text{ on }\; \partial B, \quad \langle \phi, \Delta^2 W_\lambda\rangle = 0, \quad \langle \phi, \Delta^2 W_\sigma\rangle = 0\big\}.$$

Here we have used $(Re^{\mathbf{i}\theta}, x') \in \mathbb{C}\times\mathbb{R}^{n-2} \cong \mathbb{R}^n$ and $\langle f, g\rangle = \int_B fg dx$. We consider, for (c_1, c_2, v),

$$(c_1, c_2, v - W) \in \mathbb{R}^2 \times \mathcal{H}_0, \qquad \Delta^2 v = |x|^\alpha |v|^p + c_1\Delta^2 W_\lambda + c_2 \Delta^2 W_\sigma \;\text{ in }\; B. \tag{2.7}$$

Note that $u := v$ is a solution of (1.1) if $c_1 = c_2 = 0$. We shall choose appropriate λ and σ such that $c_1 = c_2 = 0$. Let $\phi = v - W$. The above equation for (c_1, c_2, v) can be written as the equation for

$$(c_1, c_2, \phi) \in \mathbb{R}^2 \times \mathcal{H}_0, \qquad \mathcal{L}\phi - c_1\Delta^2 W_\lambda - c_2\Delta^2 W_\sigma = F + N(\phi), \tag{2.8}$$

where

$$\mathcal{L}\phi = \Delta^2\phi - p|x|^\alpha W^{p-1}\phi, \quad F = -\Delta^2 W + |x|^\alpha W^p, \quad N(\phi) = |x|^\alpha\big\{|W+\phi|^p - W^p - pW^{p-1}\phi\big\}. \tag{2.9}$$

To solve the nonlinear problem (2.8), we first investigate the linear problem: given $f \in \mathcal{H}$, find $(c_1, c_2, \phi) \in \mathbb{R}^2 \times \mathcal{H}_0$ such that $\mathcal{L}\phi - c_1\Delta^2 W_\lambda - c_2\Delta^2 W_\sigma = f$ in B. After showing the invertibility of $\mathcal{L}$ and the smallness of F, we can use Newton's method to obtain a locally unique solution of (2.9). Differently from [32], for the smallness of F, we shall fully utilize the cancelation in $U_i - U_{i*}$ when $k|x - r\mathbf{e}_i| > 1$.

For convenience, we introduce the following:

$$d_i(x) = \sqrt{1 + \frac{|x - r\mathbf{e}_i|^2}{\varepsilon^2}}, \quad d_{i*}(x) = \sqrt{1 + \frac{|rx - \mathbf{e}_i|^2}{\varepsilon^2}}, \quad \mathbf{x}_i = r\mathbf{e}_i,$$
$$\omega_\rho(x) = \sum_{i=0}^{k-1} \frac{1}{d_i^\rho(x)}, \quad \omega_\rho^*(x) = \sum_{i=0}^{k-1}\frac{1}{d_{i*}^\rho(x)}, \quad \|f\|_\rho = \sup_{x\in B}\frac{|f(x)|}{\omega_\rho(x)}.$$

In the sequel, we shall use the following basic facts (first summarized in [20]):

1. For each $\varrho \in (0, n)$ and $\rho > n - \varrho$,

$$\int\limits_{\mathbb{R}^n} \frac{|X-Y|^{-\varrho}}{(1+|Y|)^\rho} dY = O(1)\begin{cases}(1+|X|)^{n-\varrho-\rho} & \text{if } \;\rho < n,\\ (1+|X|)^{-\varrho}\ln[2+|X|] & \text{if } \;\rho = n,\\ (1+|X|)^{-\varrho} & \text{if } \;\rho > n;\end{cases} \tag{2.10}$$

$$\int\limits_B \frac{\omega_\rho(y)}{|x-y|^\varrho}\frac{dy}{\varepsilon^{n-\varrho}} = O(1)\begin{cases}\omega_{\rho+\varrho-n}(x) & \text{if } \;\rho < n,\\ \omega_\varrho(x)\ln k & \text{if } \;\rho = n,\\ \omega_\varrho(x) & \text{if } \;\rho > n;\end{cases} \tag{2.11}$$

$$\int\limits_B \frac{1}{d_{i*}^\rho(x) d_0^\varrho(x)}\frac{dx}{\varepsilon^n} = O(1)\begin{cases} d_{i*}^{n-\rho-\varrho}(\mathbf{x}_0) & \text{if } \;\rho < n,\\ d_{i*}^{-\varrho}(\mathbf{x}_0)\;\ln k & \text{if } \;\rho = n,\\ d_{i*}^{-\varrho}(\mathbf{x}_0)\;(\varepsilon k)^{\rho-n} & \text{if } \;\rho > n.\end{cases} \tag{2.12}$$

Here and in the sequel, $O(f) = O(1)f$ where $O(1)$ is a function or a constant bounded by a constant depending only on α, n, ρ, and ϱ.

2. Denote by $(-\Delta)^{-2}$ the inverse of Δ^2 subject to the Navier boundary condition on B. Since the Green's function is bounded by $O(1)|x-y|^{4-n}$, for any $\eta > 0$, by (2.11) with $\varrho = n-4$,

$$\Big\|(-\Delta)^{-2}f\Big\|_\rho = O(\varepsilon^4)\begin{cases} \|f\|_{\rho+4} & \text{if } \ 0<\rho<n-4, \\ \min\{\|f\|_n \ln k, \ \|f\|_{n+\eta}\} & \text{if } \ \rho = n-4. \end{cases} \tag{2.13}$$

3. Since $1 \leqslant d_i \leqslant 2/\varepsilon$ and $\lambda^\tau \varepsilon^\tau k = 1$, for any $\rho \geqslant \tau$, $\varrho \geqslant \tau$, and $\eta \geqslant 0$,

$$\omega^*_\rho = O(1)\begin{cases} (\varepsilon k)^\rho & \text{if } \ \rho > 1, \\ \varepsilon k \ln k & \text{if } \ \rho = 1, \\ \varepsilon^\rho k & \text{if } \ \rho < 1; \end{cases} \qquad \omega_\rho = O(1)\begin{cases} \omega_\varrho^{\rho/\varrho} & \text{if } \ \varrho \leqslant \rho, \\ \omega_\varrho^{\frac{\rho-\tau}{\varrho-\tau}} & \text{if } \ \rho < \varrho, \\ 1 & \text{if } \ \rho = \tau\,; \end{cases} \tag{2.14}$$

$$\frac{1}{2\lambda^\tau}\ \|\phi\|_\tau \leqslant \|\phi\|_{L^\infty(B)} = O(1)\|\phi\|_\rho, \qquad \|\phi\|_\rho \leqslant \|\phi\|_{\rho+\eta} \leqslant \Big(\frac{2}{\varepsilon}\Big)^\eta \|\phi\|_\rho. \tag{2.15}$$

4. Denote $\Omega_i := \{x \in B : |x-\mathbf{x}_i| \leqslant |x-\mathbf{x}_j| \ \forall j\}$. Then, for $\rho \in \mathbb{R}$,

$$\frac{1}{\varepsilon k} \leqslant d_i \leqslant d_{i*} \leqslant [1+2\sigma]d_i, \qquad \frac{1}{d_i^\rho} - \frac{1}{d_{i*}^\rho} = \frac{O(1)}{\varepsilon k d_{i*}^{\rho+1}} \qquad \forall\, x \in B \setminus \Omega_i. \tag{2.16}$$

2.3. The decomposition $W = U_0 + R$

For $x \in \Omega_0 = \{x \in B : x\cdot \mathbf{e}_0 = \min_i x \cdot \mathbf{e}_i\}$, we define

$$R(x) := \sum_{i=1}^{k-1}\Big(U_i - U_{i*} - \zeta_i\Big) - U_{0*} - \zeta_0 \quad \Longrightarrow \quad W = U_0 + R, \tag{2.17}$$

$$\mathbf{E}(x) := \sum_{i=1}^{k-1} \frac{1}{(\varepsilon k d_i)^{n-4}} - \sum_{i=0}^{k-1}\frac{1}{(\varepsilon k d_{i*})^{n-4}} \quad \Longrightarrow \quad R(x) = \frac{\Phi(0) r^a}{\varepsilon^m \lambda^{n-4} k}\mathbf{E}(x) - \sum_{i=0}^{k-1}\zeta_i. \tag{2.18}$$

Lemma 2.1. *For $x \in \bar\Omega_0$, there is the decomposition $W = U_0 + R$ where R satisfies*

$$\varepsilon^m\Big\{|R| + |R_\lambda| + |R_\sigma|\Big\} = \frac{O(1)}{d_{0*}^{n-4}} + \sum_{i=1}^{k-1}\frac{O(1)}{\varepsilon k d_{i*}^{n-3}}, \tag{2.19}$$

$$\varepsilon^m\Big\{k|R| + |\nabla R| + \frac{|\nabla\otimes\nabla R|}{k}\Big\} = O(1). \tag{2.20}$$

Proof. Since $\Delta^2\zeta_i = 0 < \Delta^2 U_{i*}$ in B and $\Delta\zeta_i = [r^{-2}-1]\Delta U_{i*}$ on ∂B, by comparison, $0 > \Delta\zeta_i \geqslant [r^{-2}-1]\Delta U_i^*$ in B. As $\zeta_i = 0 < U_{i*}$ on ∂B, by comparison, $0 < \zeta_i < [r^{-2}-1]U_{i*}$ in B. Consequently,

$$0 \leqslant \varepsilon^m \sum_{i=1}^k \zeta_i = \frac{O(\varepsilon^m)}{k}\sum_{i=1}^k U_{i*} = O(\varepsilon^m)U_{0*} = \frac{O(1)}{d_{0*}^{n-4}} \qquad \text{in } \ \bar\Omega_0.$$

Next, using Green's function for $(-\Delta)^{-1}$ and (2.10) and (2.14) we find that

$$\begin{aligned}\varepsilon^m \sum_{i=1}^k |\nabla\zeta_i| &= O(\varepsilon^m[r^{-2}-1])\sum_{i=1}^k \int_B \frac{|\Delta U_{i*}(y)|}{|x-y|^{n-1}}dy = \frac{O(1)}{k\varepsilon}\int_B \frac{\omega^*_{n-2}(y)}{|x-y|^{n-1}}\frac{dy}{\varepsilon} \\ &= \frac{O(1)\omega^*_{n-3}}{\varepsilon k} = O(1)(\varepsilon k)^{n-4} = \frac{O(1)}{k}. \end{aligned} \tag{2.21}$$

In particular, since $\zeta_i = 0$ on ∂B, we find that

$$0 < \varepsilon^m \sum_{i=1}^k \zeta_i(\mathbf{x}_0) = \varepsilon^m \Big\| \sum_{i=1}^k \nabla \zeta_i \Big\|_{L^\infty(B)} \Big[1 - |\mathbf{x}_0| \Big] = \frac{O(1)}{k^2}. \tag{2.22}$$

Also, using L^p estimate we have, for any $p > 1$, with $O(1)$ depending only on n, α and p,

$$\begin{aligned} \frac{\varepsilon^m}{k} \Big\| \sum_{i=1}^k \nabla \otimes \nabla \zeta_i \Big\|_{L^p(B)} &= \frac{O(1)\varepsilon^m}{k} \Big\| \sum_{i=1}^k \Delta \zeta_i \Big\|_{L^\infty(B)} = \frac{O(1)\varepsilon^m}{k^2} \Big\| \sum_{i=1}^k |\Delta U_i^*| \Big\|_{L^\infty(B)} \\ &= \frac{O(1)\|\omega_{n-2}^*\|_{L^\infty(B)}}{k^2\varepsilon^2} = O(1)(\varepsilon k)^{n-4} = \frac{O(1)}{k}. \end{aligned}$$

Finally, using Schauder estimates we can derive $\| \sum_{i=1}^k \zeta_i \|_{C^{5/2}(\bar{B})} = O(1)k^s$ for some $s > 0$. We then can use interpolation to bound the L^∞ norm by the L^p norm and the $C^{1/2}$ norm with $p \gg 1$ to obtain

$$\frac{\varepsilon^m}{k} \Big\| \sum_{i=1}^k \nabla \otimes \nabla \zeta_i \Big\|_{L^\infty(B)} = O(1).$$

The estimates for $R, \nabla R, \nabla \otimes \nabla R$ then follow from (2.16) and (2.14).

Next, when $x \in \partial B$, $\zeta_{i\lambda} = 0, \zeta_{i\sigma} = 0$, and

$$\frac{\Delta \zeta_\lambda}{\Delta \zeta} = \frac{\partial}{\partial \lambda} \ln[-\Delta(U_i - U_{i*})] = O(1), \qquad \frac{\Delta \zeta_\sigma}{\Delta \zeta} = \frac{\partial}{\partial \sigma} \ln[-\Delta(U_i - U_{i*})] = O(1).$$

Hence, by comparison,

$$\zeta_{i\lambda} = O(1)\zeta_i = O(k^{-1})U_{i*}, \qquad \zeta_{i\sigma} = O(1)\zeta_i = O(k^{-1})U_{i*} \qquad \text{in} \quad B.$$

Note that, since $\varepsilon^{-1} = \lambda k^{1/\tau}$ and $r = 1 - \sigma k^{-1}$,

$$U_{i\lambda} = \frac{mU_i}{\lambda} \Big\{ \frac{2}{d_i^2} - 1 \Big\} = O(U_i), \quad U_{i\sigma} = \frac{2mU_i}{\varepsilon k} \Big\{ \frac{r\mathbf{e}_i - x}{\varepsilon} \cdot \frac{\mathbf{e}_i}{d_i^2} - \frac{a\varepsilon}{2mr} \Big\} = \frac{O(U_i)}{\varepsilon k\, d_i}, \tag{2.23}$$

$$U_{i*\lambda} = \frac{mU_{i*}}{\lambda} \Big\{ \frac{2}{d_{i*}^2} - 1 \Big\} = O(U_{i*}), \quad U_{i*\sigma} = \frac{2mU_{i*}}{\varepsilon k} \Big\{ \frac{rx - \mathbf{e}_i}{\varepsilon} \cdot \frac{x}{d_{i*}^2} - \frac{a\varepsilon}{2mr} \Big\} = \frac{O(U_{i*})}{\varepsilon k d_{i*}}. \tag{2.24}$$

The estimates for R_λ and R_σ then follow from (2.16). This completes the proof. □

2.4. *The residue $F = -\Delta^2 W + |x|^\alpha W^p$*

Lemma 2.2. *Let $F = -\Delta^2 W + |x|^\alpha W^p$ and $L(\cdot)$ be as in (2.2). Then for each $\rho \in [0, n)$ and $x \in \bar{\Omega}_0$,*

$$\varepsilon^{mp} F(x) = \frac{O(1)}{d_0^{\rho+4}} \Big\{ \frac{|L(\sigma)|}{k} + (\varepsilon k)^{\min\{n-3, n-\rho\}} \Big\}. \tag{2.25}$$

Consequently, for $\rho \in [0, n)$, $\|\varepsilon^{mp} F\|_{\rho+4} = O(1)(\varepsilon k)^{\mu(\rho)}$ where $\mu(\rho) = \min\{n-4, n-\rho\}$.

Proof. Assume that $x \in \bar{\Omega}_0$ and $\rho \in [0, n)$. Using $W = U_0 + R$ and $\Delta^2 U_0 = r^\alpha U_0^p$, we obtain

$$F = \big(|x|^\alpha - r^\alpha \big) U_0^p + p|x|^\alpha U_0^{p-1} R - \Delta^2 R + F_1, \quad F_1 = |x|^\alpha \big(W^p - U_0^p - pU_0^{p-1} R \big).$$

1. Since $(r^a - r^{4+a})\sum_{i=0}^{k-1} U_{i*}^p < 2\sigma U_{0*}^p$, we obtain from (2.16) and (2.14) that

$$\Delta^2 R = r^\alpha \sum_{i=1}^{k-1} U_i^p - r^{4+\alpha}\sum_{i=0}^{k-1} U_{i*}^p = \frac{O(1)}{\varepsilon^{mp}}\left\{\frac{1}{d_{0*}^{n+4}} + \sum_{i=1}^{k-1}\frac{(\varepsilon k)^{-1}}{d_{i*}^{n+5}}\right\}$$
$$= \frac{O(1)}{\varepsilon^{mp} d_{0*}^{\rho+4}}\left\{\frac{1}{d_{0*}^{n-\rho}} + \sum_{i=0}^{k-1}\frac{(\varepsilon k)^{-1}}{d_{i*}^{1+n-\rho}}\right\} = \frac{O(1)}{\varepsilon^{mp}}\frac{(\varepsilon k)^{n-\rho}}{d_{0*}^{\rho+4}}.$$

2. Since $d_{i*}^{4-n} \leqslant d_{0*}^{-\theta} d_{i*}^{4+\theta-n}$ for $\theta \in [0, n-4)$, by (2.19) and (2.14), we derive that

$$|\varepsilon^m R| = \frac{O(1)(\varepsilon k)^{n-4-\theta}}{d_{0*}^{\theta}} \qquad \forall\, \theta \in [0, n-4). \tag{2.26}$$

(i) Suppose $n \geqslant 12$. Then $1 < p \leqslant 2$, so setting $\theta = \frac{\rho+4}{p} = \frac{\rho+4}{n+4}(n-4)$ in (2.26) we obtain

$$F_1 = O(1)|R|^p = \frac{O(1)(\varepsilon k)^{n-\rho}}{\varepsilon^{mp} d_{0*}^{\rho+4}}\,.$$

(ii) Suppose (a) $8 \leqslant n \leqslant 11$, or (b) $n = 5$ and $\rho \in [3, n)$, or (c) $n = 6$ and $\rho \in [2, n)$, or (d) $n = 7$ and $\rho \in [1, n)$. Then, $p > 2$ and $n + \rho \geqslant 8$. Using (2.26) with $\theta = (n + \rho - 8)/2$, we obtain

$$F_1 = O(1)|R|^p + O(1)U_0^{p-2}R^2 = \frac{O(1)(\varepsilon k)^{n-\rho}}{\varepsilon^{mp} d_{0*}^{\rho+4}} + \frac{O(1)}{d_0^{12-n}}\frac{(\varepsilon k)^{2(n-4)-2\theta}}{\varepsilon^{mp}\, d_{0*}^{2\theta}} = \frac{O(1)(\varepsilon k)^{n-\rho}}{\varepsilon^{mp}\, d_0^{\rho+4}}.$$

(iii) Suppose $n = 5$ and $\rho \in [0, 3)$, or $n = 6$ and $\rho \in [0, 2)$, or $n = 7$ and $\rho \in [0, 1)$. Then $12 - n > \rho + 4$. By (2.19), we have

$$\begin{aligned} F_1 &= O(1)|R|^p + O(1)U_0^{p-2}R^2 \\ &= \frac{O(1)}{\varepsilon^{mp}}\frac{(\varepsilon k)^{n-\rho}}{d_{0*}^{\rho+4}} + \frac{O(1)}{\varepsilon^{mp}}\frac{1}{d_0^{12-n}}\left[\frac{1}{d_{0*}^{n-4}} + \sum_{i=1}^{k-1}\frac{1}{\varepsilon k d_{i*}^{n-3}}\right]^2 \\ &= \frac{O(1)[(\varepsilon k)^{n-\rho} + (\varepsilon k)^{2(n-4)}]}{\varepsilon^{mp}\, d_0^{\rho+4}} = \frac{O(1)(\varepsilon k)^{\min\{2n-8, n-\rho\}}}{\varepsilon^{mp} d_0^{\rho+4}}. \end{aligned}$$

For later application, we write our estimate as, taking $\rho = 2$ and using $\varepsilon^m R(x) = \varepsilon^m R(\mathbf{x}_0) + O(\varepsilon)d_0$ and $\varepsilon^m R = O(1)k^{-1}$ for the special case $n = 5$ to obtain

$$F = \big(|x|^\alpha - r^\alpha\big)U_0^p + p|x|^\alpha U_0^{p-1}R + 36U_0^7 R^2(\mathbf{x}_0)\mathbf{1}_{\{n=5\}} + \frac{O(1)(\varepsilon k)^{n-2}}{\varepsilon^{mp} d_0^6}. \tag{2.27}$$

3. Since $r^\alpha - |x|^\alpha = O(1)|r\mathbf{e}_0 - x| = O(1)\varepsilon d_0$ and $\varepsilon = O(1)(\varepsilon k)^{n-3}$, we have

$$(r^\alpha - |x|^\alpha)U_0^p = \frac{O(1)\varepsilon}{\varepsilon^{mp} d_0^{n+3}} = \begin{cases} \frac{O(1)(\varepsilon k)^{n-3}}{\varepsilon^{mp} d_0^{\rho+4}} & \text{if } \rho \in [0, n-1), \\ \frac{O(1)\varepsilon^{n-\rho}[\varepsilon d_0]^{1+\rho-n}}{\varepsilon^{mp} d_0^{4+\rho}} = \frac{O(1)\varepsilon^{n-\rho}}{\varepsilon^{mp} d_0^{\rho+4}} & \text{if } \rho \in [n-1, n). \end{cases}$$

4. Finally we estimate $U_0^{p-1}R$. We use $2m(p-1) = 8$.
(i) If $\rho \in [4, n)$ or $\rho \in [0, 4)$ and $\varepsilon k d_0 \geqslant 1$, then taking $\theta = \max\{0, \rho - 4\}$ we obtain

$$\varepsilon^{mp} U_0^{p-1} R = \frac{O(1)}{d_0^8}\frac{(\varepsilon k)^{n-4-\theta}}{d_{0*}^{\theta}} = \frac{O(1)(\varepsilon k)^{n-\rho}}{d_0^{\rho+4}}.$$

(ii) If $\rho \in [0,4)$ and $\varepsilon k d_0 \leqslant 1$, then $1 \leqslant d_0 \leqslant (\varepsilon k)^{-1}$ so by (2.18), (2.22), and (2.20), we obtain

$$\varepsilon^{mp} U_0^{p-1} R = \frac{O(1)}{d_0^8}\Big\{\frac{|\mathbf{E}(\mathbf{x}_0)|}{k} + \frac{1}{k^2} + \varepsilon d_0\Big\} = \frac{O(1)}{d_0^{\rho+4}}\Big\{\frac{|\mathbf{E}(\mathbf{x}_0)|}{k} + (\varepsilon k)^{\min\{n-3,n-\rho\}}\Big\}.$$

Later on we shall show that $\mathbf{E}(\mathbf{x}_0) = 2^{4-n}L(\sigma) + O(\varepsilon k)$ (cf. (2.30)). Collecting all these estimates and using $k^{-1} = O(1)(\varepsilon k)^{n-4}$ we obtain the assertion of the lemma. □

2.5. Energy and its variation

The existence of spike solutions is based on the following facts:

Lemma 2.3. *Let W be as in (2.5) where (λ, σ, k) satisfies (2.4). Define*

$$J(\lambda, \sigma, k) = \int_B \Big(\frac{1}{2}\big|\Delta W\big|^2 - \frac{|x|^\alpha}{p+1} W^{p+1}\Big)\, dx.$$

Let $L(\cdot)$, A, A_1 and A_2 be defined by (2.2)–(2.3). Then

$$\frac{\partial J(\lambda,\sigma,k)}{\partial \lambda} = (n-4)\lambda^{3-n} A_2 L(\sigma) + O(\varepsilon k),$$
$$\frac{\partial J(\lambda,\sigma,k)}{\partial \sigma} = \alpha A_1 - \lambda^{4-n} A_2 L'(\sigma) + O(\varepsilon k).$$

Note that J admits a critical point in an $O(\varepsilon k)$ neighborhood of (λ^*, σ^*), the unique root of (2.1).

Proof. Set $\mathbf{x}_0 = r\mathbf{e}_0$ and $\sigma_0 = \min\{1, \sigma\}$. Then, $B_0 := \{x : |x - \mathbf{x}_0| < \sigma_0/k\} \subset \Omega_0$. For $t = \lambda$ or σ,

$$\begin{aligned}\frac{\partial J}{\partial t} &= \int_B \Big\{\Delta W \cdot \Delta W_t - |x|^\alpha W^p W_t\Big\}\, dx = -\int_B F W_t dx = -k\int_{\Omega_0} F(U_{0t} + R_t) dx \\ &= -k\int_{\Omega_0} F R_t dx - k\int_{\Omega_0\setminus B_0} F U_{0t} dx - k\int_{B_0} F U_{0t} dx.\end{aligned} \tag{2.28}$$

1. Using $m(p+1) = n$, (2.25) with $\rho = 4$, $k(\varepsilon k)^{n-4} = O(1)$, (2.19), (2.12), and (2.14), we obtain

$$\begin{aligned}k\int_{\Omega_0} F R_t dx &= \int_{\Omega_0} \frac{O(1)}{d_0^8}\Big(\frac{1}{d_{0*}^{n-4}} + \frac{1}{\varepsilon k}\sum_{i=1}^{k-1}\frac{1}{d_{i*}^{n-3}}\Big)\frac{dx}{\varepsilon^n} \\ &= O(1)\Big\{\frac{1 + \mathbf{1}_{\{n=8\}}\ln k}{d_{0*}^{\min\{n-4,4\}}(\mathbf{x}_0)} + \frac{1}{\varepsilon k}\sum_{i=1}^{k-1}\frac{1 + \mathbf{1}_{\{n=8\}}\ln k}{d_{i*}^{\min\{n-3,5\}}(\mathbf{x}_0)}\Big\} = O(\varepsilon k).\end{aligned}$$

Next, since $\varepsilon k d_0 \geqslant \sigma_0$ in $\Omega_0 \setminus B_0$, by (2.23), we see that $\varepsilon^m U_{0t} = O(1) d_0^{4-n}$. Hence, using (2.25) with $\rho = 4$ we obtain

$$k\int_{\Omega_0\setminus B_0} F U_{0t} dx = \int_{\Omega_0\setminus B_0} \frac{O(1)}{d_0^{n+4}}\frac{dx}{\varepsilon^n} = \int_{|Y| > \frac{\sigma_0}{\varepsilon k}} \frac{O(1)\, dY}{|Y|^{n+4}} = O(1)(\varepsilon k)^4.$$

To study the last integral in (2.28), we use (2.27). Using $\varepsilon^m U_{0\lambda} = O(1)d_0^{4-n}$, $\varepsilon^m U_{0\sigma} = O(1)(\varepsilon k)^{-1}d_0^{3-n}$, and $\int_{B_0} U_{0\sigma}(x)U_0^7(x)R^2(\mathbf{x}_0)dx = O((\varepsilon k)^3)$ when $n = 5$ we obtain

$$\frac{\partial J}{\partial t} = k\int\limits_{B_0}(r^\alpha - |x|^\alpha)U_0^p U_{0t}dx - kp\int\limits_{B_0}|x|^\alpha R U_0^{p-1}U_{0t}dx + O(\varepsilon k). \tag{2.29}$$

2. It remains to evaluate the two integrals in (2.29). From (2.23), one can derive the following:

$$U_0^p U_{0\lambda} = \frac{m}{\lambda n}\mathrm{div}\Big([x-\mathbf{x}_0]\ U_0^{p+1}\Big), \qquad U_0^p U_{0\sigma} = \frac{m}{nk}\mathbf{e}_0\cdot\nabla U_0^{p+1} - \frac{aU_0^{p+1}}{kr},$$
$$U_0^{p-1}U_{0\lambda} = \frac{1}{p\lambda}\mathrm{div}\Big([x-\mathbf{x}_0]U_0^p\Big) - \frac{m}{p\lambda}U_0^p, \qquad U_0^{p-1}U_{0\sigma} = \frac{1}{pk}\mathbf{e}_0\cdot\nabla U_0^p - \frac{aU_0^p}{kr}.$$

First, we calculate

$$\begin{aligned} k\int\limits_{B_0}(r^\alpha - |x|^\alpha)U_0^pU_{0\lambda}dx &= \frac{mk}{\lambda n}\int\limits_{B_0}(r^\alpha - |x|^\alpha)\mathrm{div}([x-\mathbf{x}_0]U_0^{p+1})dx \\ &= \frac{O(1)}{k}\int\limits_{\partial B_0}U_0^{p+1}dS_x + \frac{\alpha mk}{\lambda n}\int\limits_{B_0}\frac{x\cdot(x-\mathbf{x}_0)}{|x|^{2-\alpha}}U_0^{p+1}dx \\ &= O((\varepsilon k)^n) + O(k)\int\limits_{B_0}\Big(\mathbf{x}_0\cdot(x-\mathbf{x}_0) + \frac{O(1)}{k^2}\Big)U_0^{p+1}dx = \frac{O(1)}{k}. \end{aligned}$$

Here we use the fact $U_0^{p+1} = O(1)(\varepsilon k)^{2n}\varepsilon^{-n}$ on ∂B_0 and that U_0 is a function of $|x-\mathbf{x}_0|$. Similarly,

$$\begin{aligned} k\int\limits_{B_0}(r^\alpha - |x|^\alpha)U_0^pU_{0\sigma}dx &= \frac{m}{n}\int\limits_{B_0}(r^\alpha - |x|^\alpha)\mathbf{e}_0\cdot\nabla U_0^{p+1}dx + O(1)\int\limits_{B_0}\varepsilon d_0 U_0^{p+1}dx \\ &= \frac{\alpha m}{n}\int\limits_{B_0}\frac{(x - r\mathbf{e}_0 + r\mathbf{e}_0)\cdot\mathbf{e}_0}{|x|^{2-\alpha}}U_0^{p+1}dx + O(1)\Big[(\varepsilon k)^n + \varepsilon\Big] \\ &= \frac{\alpha m}{n}\int\limits_{\mathbb{R}^n}\Phi^{p+1}(y)dy + \frac{O(1)}{k} \ = \ \alpha A_1 + \frac{O(1)}{k}; \end{aligned}$$

$$\begin{aligned} -kp\int\limits_{B_0}RU_0^{p-1}U_{0\lambda}dx &= \frac{k}{\lambda}\int\limits_{B_0}R\Big\{mU_0^p - \mathrm{div}\big([x-\mathbf{x}_0]U_0^p\big)\Big\}dx \\ &= \frac{mk}{\lambda}\int\limits_{B_0}\Big(R(\mathbf{x}_0) + O(1)\|\nabla R\|_{L^\infty}\varepsilon d_0\Big)U_0^pdx + O(1)(\varepsilon k)^4 \\ &= \frac{mkR(\mathbf{x}_0)}{\lambda}\int\limits_{\mathbb{R}^n}\Phi^p(y)dy + O(\varepsilon k) = \frac{2^{n-3}mA_2}{\lambda^{n-3}}\mathbf{E}(\mathbf{x}_0) + O(\varepsilon k); \end{aligned}$$

here we have used the following: on ∂B_0, $|x-\mathbf{x}_0| = \sigma_0/k$, $R = O(1)/k$ and $U_0^p = O(1)(\varepsilon k)^{n+4}$; in B_0, $\varepsilon^m|\nabla R| = O(1)$ and $|x-\mathbf{x}_0| \leqslant \varepsilon d_0$; and $k\varepsilon^m R(\mathbf{x}_0) = \lambda^{4-n}r^a\Phi(0)\mathbf{E}(\mathbf{x}_0) + O(k^{-1})$ (cf. (2.18) and (2.22)). Finally,

$$
\begin{aligned}
-kp\int\limits_{B_0} RU_0^{p-1}U_{0\sigma}dx &= \int\limits_{B_0} R\Big\{\frac{apU_0^p}{r} - \mathbf{e}_0\cdot\nabla U_0^p\Big\}dx \\
&= \int\limits_{B_0}\Big(\nabla R\cdot\mathbf{e}_0 + \frac{ap}{r}R\Big)U_0^p dx + O(1)(\varepsilon k)^4 \\
&= \mathbf{e}_0\cdot\nabla R(\mathbf{x}_0)\int\limits_{\mathbb{R}^n}\Phi^p(y)dy + O(\varepsilon k) \\
&= \frac{2^{n-3}A_2}{\lambda^{n-4}}\frac{\mathbf{e}_0\cdot\nabla\mathbf{E}(\mathbf{x}_0)}{k} + O(\varepsilon k);
\end{aligned}
$$

here we use $\varepsilon^m\mathbf{e}_0\cdot\nabla R = \varepsilon^m\mathbf{e}_0\cdot\nabla R(\mathbf{x}_0) + O(1)\varepsilon k d_0$, and $\varepsilon^m\nabla R(\mathbf{x}_0) = \lambda^{4-n}r^a k^{-1}\Phi(0)\nabla\mathbf{E}(\mathbf{x}_0) + O(k^{-1})$ (cf. (2.18) and (2.21)). In conclusion, we have

$$
\frac{\partial J}{\partial\lambda} = \frac{2^{n-3}mA_2}{\lambda^{n-3}}\mathbf{E}(\mathbf{x}_0) + O(\varepsilon k), \qquad \frac{\partial J}{\partial\sigma} = \alpha A_1 + \frac{2^{n-3}A_2}{\lambda^{n-4}}\frac{\mathbf{e}_0\cdot\mathbf{E}(\mathbf{x}_0)}{k} + O(\varepsilon k).
$$

3. Finally, using the asymptotic expansion presented in [20] we find that, for $L(\cdot)$ defined in (2.2),

$$
\mathbf{E}(\mathbf{x}_0) = 2^{4-n}L(\sigma) + O(\varepsilon k), \quad \mathbf{e}_0\cdot\nabla\mathbf{E}(\mathbf{x}_0) = k\big\{-2^{3-n}L'(\sigma) + O(\varepsilon k)\big\}. \tag{2.30}
$$

Collecting all estimates, we then obtain the assertion of the lemma. □

2.6. Smallness of $\mathcal{L}W_\lambda$ and $\mathcal{L}W_\sigma$ as measure

Note that W_λ and W_σ are tangent vectors of the approximate solution manifold. They are approximate eigenfunctions of $\mathcal{L}$ associated with small eigenvalues. Here we study the sizes of $\mathcal{L}W_\lambda$ and $\mathcal{L}W_\sigma$.

Lemma 2.4. *Let $\mathcal{L}$ be defined in (2.9) where W is defined in (2.5). Then $\mathcal{L}W_\lambda$ and $\mathcal{L}W_\sigma$ are small in the sense that there exists a universal constant C_0 (depending only on α and n) such that*

$$
\sup_{\phi\in\mathcal{H}_0}\frac{\big|\langle\phi,\mathcal{L}W_\lambda\rangle\big|}{k\,\|\varepsilon^m\phi\|_{L^\infty(B)}} \leqslant C_0\varepsilon k, \qquad \sup_{\phi\in\mathcal{H}_0}\frac{\big|\langle\phi,\mathcal{L}W_\sigma\rangle\big|}{k\,\|\varepsilon^m\phi\|_{L^\infty(B)}} \leqslant C_0\Big[\varepsilon k + |L(\sigma)|\mathbf{1}_{\{n=5\}}\Big]. \tag{2.31}
$$

Proof. We can assume, without loss of generality, that $\|\varepsilon^m\phi\|_{L^\infty(B)} = 1$. For $t = \lambda$ or σ, by symmetry,

$$
\frac{1}{k}\langle\phi,\mathcal{L}W_t\rangle = \frac{1}{k}\int\limits_B\phi\big(\Delta^2W - |x|^\alpha W^p\big)_t dx = -\int\limits_{\Omega_0}\phi F_t dx = O(1)\int\limits_{\Omega_0}|F_t|\frac{dx}{\varepsilon^m}.
$$

Note that

$$
F_t = \sum_{i=0}^{k-1}\Big(r^{\alpha+4}U_{i*}^p - r^\alpha U_i^p\Big)_t + p|x|^\alpha W^{p-1}(U_{0t} + R_t) = g_1 + g_2 + g_3 + g_4 + g_5,
$$

where

$$
\begin{aligned}
&g_1 = \sum_{i=0}^{k-1}\Big[\frac{\alpha+4}{\alpha}r^4U_{i*}^p - U_i^p\Big](r^\alpha)_t, \quad g_2 = \sum_{i=0}^{k-1}pr^{\alpha+4}U_{i*}^{p-1}U_{i*t} - \sum_{i=1}^{k-1}pr^\alpha U_i^{p-1}U_{it}, \\
&g_3 = p|x|^\alpha W^{p-1}R_t, \quad g_4 = p\big(|x|^\alpha - r^\alpha\big)W^{p-1}U_{0t}, \quad g_5 = pr^\alpha\big(W^{p-1} - U_0^{p-1}\big)U_{0t}.
\end{aligned}
$$

We estimate each term as follows.

1. Since $r_\lambda = 0$ and $r_\sigma = -1/k$, we obtain, by symmetry

$$\int\limits_{\Omega_0} |g_1| \frac{dx}{\varepsilon^m} = \frac{O(1)}{k} \sum_{i=0}^{k-1} \int\limits_{\Omega_0} U_i^p \frac{dx}{\varepsilon^m} = \frac{O(1)}{k} \sum_{i=0}^{k-1} \int\limits_{\Omega_i} U_0^p \frac{dx}{\varepsilon^m} = \frac{O(1)}{k} \int\limits_{\mathbb{R}^n} \Phi^p(z)dz = \frac{O(1)}{k}.$$

2. Since $|U_{i*t}| = O(1)U_{i*}$ for $x \in B$ and $|U_{it}| = O(1)U_{i*}$ for $x \notin \Omega_i$, by symmetry,

$$\int\limits_{\Omega_0} |g_2| \frac{dx}{\varepsilon^m} = \sum_{i=0}^{k-1} \int\limits_{\Omega_0} \frac{O(1)}{d_{i*}^{n+4}} \frac{dx}{\varepsilon^n} = \int\limits_{B} \frac{O(1)}{d_{0*}^{n+4}} \frac{dx}{\varepsilon^n} = O(1)(\varepsilon k)^4.$$

3. Using $W = U_0 + R$ and (2.26), we have, for any $\theta \in [0, n-4)$, $\delta \in [0, 8)$, and $x \in \Omega_0$,

$$\varepsilon^m W = \frac{O(1)}{d_0^{n-4}} + \frac{(\varepsilon k)^{n-4-\theta}}{d_{0*}^{\theta}}, \qquad \Big(\varepsilon^m W\Big)^{p-1} = \frac{O(1)}{d_0^8} + \frac{O(1)(\varepsilon k)^{8-\delta}}{d_{0*}^{\delta}}. \tag{2.32}$$

Hence, fixing $\delta = 9/2$ and using (2.19) and (2.12) we have

$$\begin{aligned}
\int\limits_{\Omega_0} |g_3| \frac{dx}{\varepsilon^m} &= O(1) \int\limits_{\Omega_0} \Big(\frac{(\varepsilon k)^{8-\delta}}{d_{0*}^{\delta}} + \frac{1}{d_0^8} \Big) \Big(\frac{1}{d_{0*}^{n-4}} + \sum_{i=1}^{k-1} \frac{(\varepsilon k)^{-1}}{d_{i*}^{n-3}} \Big) \frac{dx}{\varepsilon^n} \\
&= O(1)(\varepsilon k)^{8-\delta} \Big(\frac{1}{d_{0*}^{\delta-4}(\mathbf{x}_0)} + \sum_{i=1}^{k-1} \frac{(\varepsilon k)^{-1}}{d_{i*}^{\delta-3}(\mathbf{x}_0)} \Big) \\
&\quad + O(1) \Big(\frac{1}{d_{0*}^{\min\{n-4,4\}}(\mathbf{x}_0)} + \sum_{i=1}^{k-1} \frac{(\varepsilon k)^{-1}}{d_{i*}^{\min\{n-3,5\}}(\mathbf{x}_0)} \Big) \Big(1 + \mathbf{1}_{\{n=8\}} \ln k \Big) \\
&= O(1)(\varepsilon k)^4 + O(1)(\varepsilon k)^{\min\{n-4,4\}}(1 + \mathbf{1}_{\{n=8\}} \ln k) = O(\varepsilon k).
\end{aligned}$$

4. Using $r^\alpha - |x|^\alpha = O(1)\varepsilon d_0$, (2.32) with $\delta = 5$, and $U_{0t} = O(1)(\varepsilon k)^{-1} U_0$ we obtain

$$\int\limits_{\Omega_0} |g_4| \frac{dx}{\varepsilon^m} = \int\limits_{\Omega_0} O(1)\varepsilon d_0 \Big(\frac{1}{d_0^8} + \frac{(\varepsilon k)^3}{d_{0*}^5} \Big) \frac{1}{(\varepsilon k) d_0^{n-4}} \frac{dx}{\varepsilon^n} = \frac{O(1)}{k} + O(1)(\varepsilon k)^2 \varepsilon \ln k = \frac{O(1)}{k}.$$

5. Finally we estimate g_5. By considering cases $U_0 > 2|R|$ and $U_0 \leqslant 2|R|$ we can show that

$$W^{p-1} - U_0^{p-1} = \big[U_0 + R\big]^{p-1} - U_0^{p-1} = O(1)(\max\{U_0, |R|\})^{p-2}|R|.$$

Taking for $\theta = 9(n-4)/16$ in (2.26) and using $\varepsilon^m U_{0t} = O(1)(\varepsilon k)^{-1} d_0^{4-n}$ and (2.12) we obtain

$$\int\limits_{\Omega_0} |R|^{p-1} |U_{0t}| \frac{dx}{\varepsilon^m} = \int\limits_{\Omega_0} \frac{O(\varepsilon k)^{7/2}}{d_{0*}^{9/2}} \frac{1}{\varepsilon k d_0^{n-4}} \frac{dx}{\varepsilon^n} = \frac{O(1)(\varepsilon k)^{5/2}}{d_{0*}^{1/2}(\mathbf{x}_0)} = O((\varepsilon k)^3) = O(\varepsilon k). \tag{2.33}$$

(i) Suppose $n \geqslant 12$. Then $p \leqslant 2$, so $(\max\{U_0, |R|\})^{p-2}|R| \leqslant |R|^{p-1}$. This implies, by (2.33), that

$$\int\limits_{\Omega_0} |g_5| \frac{dx}{\varepsilon^m} = O(\varepsilon k).$$

(ii) Suppose $5 \leqslant n \leqslant 11$ and $t = \lambda$. Using (2.33), $U_{0\lambda} = O(1)U_0$ and (2.19) we obtain

$$\int_{\Omega_0} |g_5|\frac{dx}{\varepsilon^m} = O(1)\int_{\Omega_0} \left[|R|^{p-2} + U_0^{p-2}\right]|U_{0\lambda}R|\frac{dx}{\varepsilon^m} = O(\varepsilon k) + \int_{\Omega_0} \frac{O(1)}{d_0^8}\Big(\frac{1}{d_{0*}^{n-4}} + \sum_{i=1}^{k-1}\frac{(\varepsilon k)^{-1}}{d_{i*}^{n-3}}\Big)\frac{dx}{\varepsilon^n} = O(\varepsilon k).$$

(iiia) Suppose $6 \leqslant n \leqslant 11$ and $t = \sigma$. Using (2.33), $\varepsilon^m U_{0\sigma} = O(1)(\varepsilon k)^{-1}d_0^{3-n}$ and (2.19) we obtain

$$\int_{\Omega_0} |g_5|\frac{dx}{\varepsilon^m} = O(1)\int_{\Omega_0} \left[|R|^{p-2} + U_0^{p-2}\right]|U_{0\sigma}R|\frac{dx}{\varepsilon^m} = O(\varepsilon k) + \int_{\Omega_0} \frac{O(1)}{\varepsilon k\, d_0^9}\Big(\frac{1}{d_{0*}^{n-4}} + \sum_{i=1}^{k-1}\frac{(\varepsilon k)^{-1}}{d_{i*}^{n-3}}\Big)\frac{dx}{\varepsilon^n} = O(\varepsilon k).$$

(iiib) Suppose $n = 5$ and $t = \sigma$. We use the expansion

$$W^{p-1} - U_0^{p-1} = [U_0 + R]^8 - U_0^8 = 8U_0^7 R + O(1)|R|^2 U_0^6 + O(1)|R|^8.$$

Using $\varepsilon^m U_{0\sigma} = O(1)(\varepsilon k)^{-1}d_0^{3-n}$, (2.26) with $\theta = 0$, and (2.33), we have

$$\int_{\Omega_0} \Big(U_0^6 R^2|U_{0\sigma}| + |R|^8|U_{0\sigma}|\Big)\frac{dx}{\varepsilon^m} = \frac{O(1)}{\varepsilon k}\int_{\Omega_0} \frac{(\varepsilon k)^2}{d_0^8}\frac{dx}{\varepsilon^n} + O(\varepsilon k) = O(\varepsilon k).$$

Finally we expand (with $p = 9$, $n = 5$, and (2.22)),

$$\varepsilon^{mp}U_0^{p-2}|R||U_{0\sigma}| = \frac{O(\varepsilon^m)}{d_0^7}\Big[|R(\mathbf{x}_0)| + O(1)\|\nabla R\|_{L^\infty}\varepsilon d_0\Big]\frac{1}{\varepsilon k d_0^2} = \frac{O(1)}{\varepsilon k d_0^9}\Big(\frac{|\mathbf{E}(\mathbf{x}_0)|}{k} + \frac{1}{k^2}\Big) + \frac{O(1)}{k d_0^8}.$$

Since $\varepsilon k^2 = 1/\lambda$ for $n = 5$, we have

$$\int_{\Omega} |g_5|\frac{dx}{\varepsilon^m} = \int_{\Omega_0} \frac{O(1)}{d_0^9}\Big(|\mathbf{E}(\mathbf{x}_0)| + \frac{1}{k}\Big)\frac{dx}{\varepsilon^n} + \int_{\Omega_0} \frac{O(1)}{k\, d_0^8}\frac{dx}{\varepsilon^n} + O(\varepsilon k) = O(1)\Big[|\mathbf{E}(\mathbf{x}_0)| + \varepsilon k\Big].$$

Finally, using (2.30), we then obtain the assertion of the lemma. □

2.7. *Concentration of the masses of W, W_λ, W_σ*

The norm $\|\cdot\|_\rho$ is specially designed for L^∞ functions with mass concentrated near centers of spikes. The following result is critical for the study of the inverse of $\mathcal{L}$; it essentially reduces the study of $\mathcal{L}^{-1}$ to $(-\Delta)^{-2}$. We know from (2.13) that if $(-\Delta)^2\phi = f$, then the $\|\phi\|_\rho$ norm $(0 < \rho < n-4)$ can be bounded by the $\|f\|_{\rho+4}$ norm. If f depends on ϕ via a linear combinations of $\phi W^{p-1}, \Delta^2 W_\lambda$, and $\Delta^2 W_\sigma$, we have a boot strap argument to bound the norm $\|\phi\|_\varrho$ by $\|\phi\|_\rho$ for some $\varrho > \rho$ if the $\|f\|_{\varrho+4}$ norm can be bounded by the $\|\phi\|_\rho$ norm. The following result makes such bootstrap possible.

Lemma 2.5. *Let W be as in (2.5). Then*

(i) *for any $\rho \geqslant \tau$ and $\phi \in \mathcal{H}$, $\|\varepsilon^{mp}\phi W^{p-1}\|_{\rho+7} = O(1)\|\varepsilon^m\phi\|_\rho$;*
(ii) $\|\varepsilon^{mp}\Delta^2 W_\lambda\|_{n+4} + \varepsilon k\|\varepsilon^{mp}\Delta^2 W_\sigma\|_{n+5} = O(1)$.

Proof. (i) Let $x \in \Omega_0$. Set $\delta = 7 + \tau$. Then $\rho + 7 \geqslant \delta \geqslant \tau$. Hence, by (2.32) and (2.14),

$$\frac{\varepsilon^{mp}\phi W^{p-1}}{\|\varepsilon^m \phi\|_\rho} = \frac{O(1)\omega_\rho}{d_0^\delta} = O(1)\omega_\delta\omega_\rho = O(1)\omega_{\rho+7}^{\frac{\delta-\tau}{\rho+7-\tau}}\omega_{\rho+7}^{\frac{\rho-\tau}{\rho+7-\tau}} = O(1)\omega_{\rho+7}.$$

The first assertion of the lemma thus follows.

(ii) Note that $\Delta^2 W_\lambda = (\Delta^2 W)_\lambda = \sum_{i=0}^{k-1}[r^\alpha U_i^p - r^{\alpha+4}U_{i*}^p]_\lambda = O(1)\varepsilon^{-mp}\sum_{i=0}^{k-1} d_i^{-(n+4)}$, we hence find that $\|\varepsilon^{mp}\Delta^2 W_\lambda\|_{n+4} = O(1)$. Similarly, we find that $\varepsilon k\|\varepsilon^{mp}\Delta W_\sigma\|_{n+5} = O(1)$. This completes the proof. $\square$

Finally, we show that W_λ and W_σ are almost orthogonal.

Lemma 2.6. *There are positive constants $a_1(n)$ and $a_2(n)$ that depend only on n such that*

$$M := \frac{1}{k}\begin{bmatrix} \lambda^2 \langle \Delta W_\lambda, \Delta W_\lambda\rangle & \lambda\varepsilon k\, \langle \Delta W_\sigma, \Delta W_\lambda\rangle \\ \lambda\varepsilon k\, \langle \Delta W_\lambda, \Delta W_\sigma\rangle & (\varepsilon k)^2\, \langle \Delta W_\sigma, \Delta W_\sigma\rangle \end{bmatrix} = \begin{bmatrix} a_1(n) & 0 \\ 0 & a_2(n) \end{bmatrix} + \frac{O(1)}{k}. \tag{2.34}$$

Proof. For $\ell, t = \lambda$ or σ, by integration by parts and symmetry,

$$\frac{1}{k}\langle \Delta W_\ell, \Delta W_t\rangle = \frac{1}{k}\int\limits_B (\Delta^2 W_\ell) W_t dx = \int\limits_B \left(r^\alpha U_0^p - r^{4+\alpha}U_{0*}^p\right)_\ell W_t\, dx. \tag{2.35}$$

Note that d_0 and d_{0*} are proportional in $B \setminus B_0$. Also $|U_{i\lambda}| + \varepsilon k|U_{i\sigma}| + |U_{i*\lambda}| + |U_{i*\sigma}| = O(1)U_i$ in B. Hence, for $\ell = \lambda$ or σ, by (2.12),

$$\begin{aligned}
&\int\limits_{B\setminus B_0} \left|(r^\alpha U_0^p - r^{4+\alpha}U_{0*}^p)_\ell\right| \left\{|W_\lambda| + \varepsilon k|W_\sigma|\right\} dx \\
&= O(1)\sum_{i=0}^{k-1}\int\limits_{B\setminus B_0} \frac{1}{d_{0*}^{n+4} d_i^{n-4}}\frac{dx}{\varepsilon^n} = \sum_{i=0}^{k-1}\frac{O(1)(\varepsilon k)^4}{d_{0*}^{n-4}(\mathbf{x}_i)} = \sum_{i=0}^{k-1}\frac{O(1)(\varepsilon k)^4}{d_{i*}^{n-4}(\mathbf{x}_0)} = O(1)(\varepsilon k)^n \ln k.
\end{aligned}$$

Next, using (2.19) we obtain, for $t = \lambda$ or σ,

$$\int\limits_{B_0}\left\{\left|(r^\alpha U_0^p - r^{4+\alpha}U_{0*}^p)_\lambda\right| + \varepsilon k\left|(r^\alpha U_0^p - r^{4+\alpha}U_{0*}^p)_\sigma\right|\right\}|R_t| dx = \frac{O(1)}{k}\int\limits_{B_0}\frac{1}{d_0^{n+4}}\frac{dx}{\varepsilon^n} = \frac{O(1)}{k}.$$

The assertion of the lemma thus follows from the identities, ignoring the r^a factor, $\int_{\mathbb{R}^n} U_0^{p-1}U_{0\lambda}U_{0\sigma}\, dx = 0$,

$$\begin{aligned}
\lambda^2 \int\limits_{\mathbb{R}^n} U_0^{p-1}U_{0\lambda}^2\, dx &= a_1(n) := \int\limits_{\mathbb{R}^n} \Phi^{p-1}(y)|m\Phi(y) + y\cdot\nabla\Phi(y)|^2 dy, \\
(\varepsilon k)^2 \int\limits_{\mathbb{R}^n} U_0^{p-1}U_{0\sigma}^2\, dx &= a_2(n) := \frac{1}{n}\int\limits_{\mathbb{R}^n} \Phi^{p-1}(y)|\nabla\Phi(y)|^2 dy. \quad \square
\end{aligned}$$

3. Construction of spike solutions

In this section, we prove Theorem 1 by showing the existence of a k-spike solution for each large enough integer k. We first study the linearized Euler–Lagrange equation associated with an energy functional. Then we solve the Euler–Lagrange equation. Finally we show that there is a critical point at which the solution of the Euler–Lagrange equation is indeed a solution of (1.1).

3.1. The linearized Euler–Lagrange equation

Let $\mathcal{L}\phi := \Delta^2\phi - p|x|^\alpha W^{p-1}\phi$. Here we solve the linear problem: Given $f \in \mathcal{H}$, find (c_1, c_2, ϕ) such that

$$(c_1, c_2, \phi) \in \mathbb{R} \times \mathbb{R} \times \mathcal{H}_0, \qquad \mathcal{L}\phi - c_1\Delta^2 W_\lambda - c_2\Delta^2 W_\sigma = f \ \text{ in } \ B. \tag{3.1}$$

Lemma 3.1. *Assume that $n \geqslant 5$. There exists a positive constant k_0 that depends only on n and α such that for each integer $k \geqslant k_0$ and real parameters (λ, σ) satisfying (2.4), problem (3.1) with $f \in \mathcal{H}$ admits a unique solution. Moreover, for each $\rho \in [\tau, n-4)$, there exists a constant $C(n, \alpha, \rho)$ that depends only on n, α, and ρ such that the solution satisfies*

$$\|\varepsilon^m \phi\|_\rho + |c_1| + (\varepsilon k)^{-1}|c_2| \leqslant C(n, \alpha, \rho)\, \|\varepsilon^{mp} f\|_{\rho+4}. \tag{3.2}$$

The proof can be found in [19].

3.2. The Euler–Lagrange equation

Here we solve (2.8) by using Lemma 3.1 and a contraction mapping theorem. We need the following:

Lemma 3.2. *Let $N(\phi) = |x|^\alpha\big[|W+\phi|^p - W^p - pW^{p-1}\phi\big]$. If $\rho \geqslant \tau$, then for every $\phi, \psi \in \mathcal{H}$ ($\phi \not\equiv \psi$),*

$$\frac{\|\varepsilon^{mp}(N(\phi) - N(\psi))\|_{\rho+4}}{\|\varepsilon^m(\phi-\psi)\|_\rho} = O(1)\max\big\{\|\varepsilon^m\phi\|_{\rho^*}, \|\varepsilon^m\psi\|_{\rho^*}, \|\varepsilon^m\phi\|_{\rho^*}^{p-1}, \|\varepsilon^m\psi\|_{\rho^*}^{p-1}\big\}, \tag{3.3}$$

$$\|\varepsilon^{mp}N(\phi)\|_{\rho+3} = O(1)\max\{\|\varepsilon^m\phi\|_{\hat\rho}, \|\varepsilon^m\phi\|_{\hat\rho}^{p-1}\}\|\varepsilon^m\phi\|_\rho, \tag{3.4}$$

where

$$\rho^* = m + \tau\ \max\Big\{1,\ \frac{m}{\rho+4-\tau}\Big\}, \qquad \hat\rho = \frac{3m}{4} + \tau\ \max\Big\{1,\ \frac{3m/4}{\rho+3-\tau}\Big\}. \tag{3.5}$$

The proof can be found in [19,20].

Theorem 2. *There exists a large integer k_1 such that for every integer $k \geqslant k_1$ and (λ, σ) satisfying (2.4), (2.8) admits a unique solution (c_1, c_2, v) satisfying*

$$|c_1| + |(\varepsilon k)^{-1}c_2| + \|\varepsilon^m\phi\|_\tau = O(1)\big\{|L(\sigma)|k^{-1} + \varepsilon\big\}, \tag{3.6}$$

$$\|\varepsilon^m\phi\|_\rho = O(1)\big\{|L(\sigma)|k^{-1} + (\varepsilon k)^{\min\{n-3,n-\rho\}}\big[1 + \mathbf{1}_{\{\rho=n-4\}}\ln k\big]\big\} \quad \forall\, \rho \in [\tau, n-4]. \tag{3.7}$$

Proof. 1. Let $\nu = n/(n+4)$, η be a small positive number, say $\eta = 1/(n+4)^2$, $\varrho = n-4-\eta$,

$$\mu(\rho) = \min\{n-4, n-\rho\}, \qquad \mathbf{X} = \big\{\phi \in \mathcal{H}\ :\ \|\varepsilon^m\phi\|_\varrho \leqslant (\varepsilon k)^{\nu\mu(\varrho)}\big\}.$$

Fix an arbitrary $\phi \in \mathbf{X}$. Since $\varrho^* \leqslant 2m = n-4$, by (2.15), we see that

$$\|\varepsilon^m\phi\|_{\varrho^*} \leqslant \|\varepsilon^m\phi\|_{n-4} \leqslant 2^\eta\varepsilon^{-\eta}\|\varepsilon^m\phi\|_\varrho \leqslant (2\lambda^{n-2})^\eta(\varepsilon k)^{\nu\mu(\varrho)-(n-3)\eta}. \tag{3.8}$$

Now set $f = F + N(\phi)$. By (2.25) and (3.3) with $\psi \equiv 0$ we find that

$$\|\varepsilon^{mp}f\|_{\varrho+4} = O(1)\big\{(\varepsilon k)^{\mu(\varrho)} + (\varepsilon k)^{[\nu\mu(\varrho)-(n-3)\eta]\min\{p,2\}}\big\} = O(1)(\varepsilon k)^{\mu(\varrho)}.$$

Define (c_1, c_2, ψ) in $\mathbb{R}^2 \times \mathcal{H}_0$ as the unique solution of $\mathcal{L}\psi - c_1\Delta^2 W_\lambda - c_2\Delta^2 W_\sigma = f$. By Lemma 3.1,

$$\|\varepsilon^m \psi\|_\varrho \leqslant C\|\varepsilon^{mp} f\|_{\varrho+4} \leqslant C_1[\varepsilon k]^{\mu(\varrho)} \leqslant (\varepsilon k)^{\nu\mu(\varrho)},$$

if k is large enough. We now define $\mathbf{T}\phi := \psi$. Then $\mathbf{T}$ maps $\mathbf{X}$ to itself if $k \gg 1$. In addition, for any $\phi_1, \phi_2 \in \mathbf{X}$, we have, by (3.3) and (3.8),

$$\|\varepsilon^m \mathbf{T}(\phi_1 - \phi_2)\|_\varrho = O(1)\varepsilon^{mp}\|N(\phi_1) - N(\phi_2)\|_{\varrho+4} = o(1)\|\varepsilon^m(\phi_1 - \phi_2)\|_\varrho.$$

Hence, $\mathbf{T}$ is a contraction if $k \gg 1$. Consequently, when $k \gg 1$, by the contraction mapping theorem, there exists a unique fixed point of $\mathbf{T}$ in $\mathbf{X}$, which gives a unique solution, denoted by (c_1, c_2, ϕ), of (2.8) in $\mathbf{X}$.

2. Next, we apply (3.2) to $\mathcal{L}\phi = F + N(\phi)$ with $\rho = \tau$ to obtain

$$\|\varepsilon^m \phi\|_\tau + |c_1| + |(\varepsilon k)^{-1} c_2| = O(1)\|\varepsilon^{mp} F\|_{\tau+4} + \|\varepsilon^{mp} N(\phi)\|_{\tau+4}.$$

By (3.3) and (3.8), $\|\varepsilon^{mp} N(\phi)\|_{\tau+4} = o(1)\|\varepsilon^m \phi\|_\tau$. Thus by (2.25), we obtain (3.6).

3. Finally, we notice from (2.25) that $F = F_1 + F_2$ where $\|\varepsilon^{mp} F_1\|_{n+1} = O(1)|L(\sigma)|k^{-1}$ and $\|\varepsilon^{mp} F_2\|_{\rho+4} = O(1)(\varepsilon k)^{\min\{n-3, n-\rho\}}$ for $\rho \in [0, n)$. Hence, from

$$\phi = (-\Delta)^{-2}\big[F_1 + F_2 + p|x|^\alpha W^{p-1}\phi + c_1\Delta^2 W_\lambda + c_2\Delta^2 W_\sigma + N(\phi)\big],$$

we obtain from (2.13) that for any $\rho \in [\tau, n-4]$ and small positive η,

$$\begin{aligned}
\|\varepsilon^m \phi\|_\rho = O(1)\varepsilon^{mp}\big\{&\|F_1\|_{n+1} + \|F_2\|_{\rho+4}(1 + \mathbf{1}_{\{\rho=n-4\}}\ln k) + \\
&\|\phi W^{p-1}\|_{\rho+5} + |c_1|\|\Delta W_\lambda\|_{n+1} + |c_2|\|\Delta W_\sigma\|_{n+1} + O(1)\|N(\phi)\|_{\rho+4+\eta}\big\} \\
= O(1)\big\{&|L(\sigma)|k^{-1} + (\varepsilon k)^{\min\{n-3, n-\rho\}}\big(1 + \mathbf{1}_{\{\rho=n-4\}}\ln k\big)\big\} \\
&+ O(1)\big\{\|\phi\|_{\max\{\tau,\rho-2\}} + |c_1| + |(\varepsilon k)^{-1}c_2|\big\} + C_*(\|\varepsilon^m\phi\|_{n-4})^{\min\{p-1,1\}}\big(\|\varepsilon^m\phi\|_\rho \varepsilon^{-\eta}\big),
\end{aligned}$$

by Lemma 2.5 and (3.3). Note that C_* depending only on η. Hence, by (3.8), there exist $\eta \in (0, 1/2)$ and $k_1 \gg 1$ such that when $k \geqslant k_1$, $C_*(\|\varepsilon^m\phi\|_{n-4})^{\min\{p-1,1\}}\varepsilon^{-\eta} \leqslant 1/2$. This gives (3.7) for $\rho \in [\tau, \min\{2+\tau, n-4\}]$. Also by induction, one can establish the assertion (3.7) step by step for $\rho \in [2i+\tau, \min\{2i+2+\tau, n-4\}]$ for $i = 0, 1, \cdots, [n/2-2]$. This completes the proof of the theorem. □

3.3. Proof of Theorem 1

For given (λ, σ, k) satisfying (2.4) and $k \geqslant k_1$, let (c_1, c_2, v) be the solution of (2.7) given by Theorem 2. We now find (λ, σ) such that $(c_1, c_2) = (0, 0)$. The pair (c_1, c_2) satisfies the following:

$$kM\begin{bmatrix} c_1/\lambda \\ c_2/(\varepsilon k) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \varepsilon k \end{bmatrix}\begin{bmatrix} \langle\phi, \mathcal{L}W_\lambda\rangle - \langle F, W_\lambda\rangle - \langle N(\phi), W_\lambda\rangle \\ \langle\phi, \mathcal{L}W_\sigma\rangle - \langle F, W_\sigma\rangle - \langle N(\phi), W_\sigma\rangle \end{bmatrix}, \tag{3.9}$$

where M is given in Lemma 2.6. The system is obtained by integrating (2.8) multiplied by $[\lambda W_\lambda, \varepsilon k W_\sigma]$ and using integration by parts. We want to show that the right-hand side vanishes at some point $(\lambda, \sigma) \sim (\lambda^*, \sigma^*)$.

(i) First of all, direct calculation shows that, for $t = \lambda$ or σ,

$$-\langle F, W_t\rangle = \frac{\partial J(\lambda, \sigma, k)}{\partial t}.$$

(ii) Next, from (3.6) we obtain

$$\|\varepsilon^m \phi\|_{L^\infty(B)} = O(1)\|\varepsilon^m\phi\|_\tau = O(1)\big\{|L(\sigma)|k^{-1} + \varepsilon\big\}.$$

Hence, by Lemma 2.4,

$$\langle \phi, \mathcal{L}W_\lambda\rangle = O(\varepsilon k^2)\|\varepsilon^m\phi\|_{L^\infty} = O(\varepsilon k),$$
$$\langle \phi, \mathcal{L}W_\sigma\rangle = O(|L(\sigma)| + \varepsilon k)\big(|L(\sigma)|\mathbf{1}_{\{n=5\}} + \varepsilon k\big) = O(1)L^2(\sigma)\mathbf{1}_{\{n=5\}} + O(\varepsilon k).$$

(iii) Finally, since $(U_0 - U_{0*} - \zeta_0)_\lambda = O(1)U_0 = O(1)\varepsilon^{-m}d_0^{4-n} = O(1)\varepsilon^m|x - \mathbf{x}_0|^{4-n}$ and $(U_0 - U_{0*} - \zeta_0)_\sigma = O(1)U_0/(\varepsilon k d_0) = \varepsilon^{-m}(\varepsilon k)^{-1}d_0^{3-n} = O(k^{-1})\varepsilon^m|x - \mathbf{x}_0|^{3-n}$, for any chosen $\rho \in [\tau, n-4)$, we can use symmetry and (2.11) to derive

$$\begin{aligned}
\langle N(\phi), W_\lambda\rangle &= k\ \langle N(\phi), (U_0 - U_{0*} - \zeta_0)_\lambda\rangle \\
&= O(k)\int_B \frac{\varepsilon^m|N(\phi)|}{|x-\mathbf{x}_0|^{n-4}}dx = O(k)\int_B \frac{\varepsilon^{mp}|N(\phi)|}{|x-\mathbf{x}_0|^{n-4}}\frac{dx}{\varepsilon^4} \\
&= O(k)\|\varepsilon^{mp}N(\phi)\|_{\rho+4}\int_{B_1}\frac{\omega_{\rho+4}(x)}{|x-\mathbf{x}_0|^{n-4}}\frac{dx}{\varepsilon^4} \\
&= O(k)\|\varepsilon^{mp}N(\phi)\|_{\rho+4}\omega_\rho(\mathbf{x}_0) \\
&= O(k)\|\varepsilon^m\phi\|_\rho \max\{\|\varepsilon^m\phi\|_{\rho^*}, \|\varepsilon^m\phi\|_{\rho^*}^{p-1}\} \\
&= O(1)(\varepsilon k)^{\sigma_1}[1 + \ln k\mathbf{1}_{\{\rho^*=n-4\}}],
\end{aligned}$$

$$\begin{aligned}
\langle N(\phi), W_\sigma\rangle &= \frac{O(1)}{\varepsilon}\int_B \frac{|\varepsilon^{mp}N(\phi)|}{|x-\mathbf{x}_0|^{n-3}}\frac{dx}{\varepsilon^3} = \frac{O(1)}{\varepsilon}\|\varepsilon^{mp}N(\phi)\|_{\rho+3} \\
&= \frac{O(1)}{\varepsilon}\|\varepsilon^m\phi\|_\rho \max\{\|\varepsilon^m\phi\|_{\hat\rho}, \|\varepsilon^m\phi\|_{\hat\rho}^{p-1}\} = O(1)(\varepsilon k)^{\sigma_2},
\end{aligned}$$

where, by (3.7),

$$\sigma_1 = \min\{n-4, n-\rho\} + \min\{n-4, n-\rho^*\}\min\{p-1, 1\} + 4 - n\,,$$
$$\sigma_2 = \min\{n-4, n-\rho\} + \min\{n-4, n-\hat\rho\}\min\{p-1, 1\} + 3 - n\,.$$

For simplicity, we take $\rho = \tau$. When $n \geqslant 6$, one can verify that $\tau < \hat\rho < \rho^* < n-4$, $\sigma_1 \geqslant 2$, and $\sigma_2 \geqslant 1$.

Now we consider the case $n = 5$. Then $\varepsilon = 1/(\lambda k^2)$. In this case, $m = 1/2, \tau = 1/2, n-4 = 1$, and $m + \tau = 1$. The assertion of Theorem 2 can be stated as

$$\|\varepsilon^m\phi\|_\rho = \frac{O(1)}{k}\Big\{|L(\sigma)| + \frac{1}{k}\Big\}\quad \forall\,\rho \in [\tau, 1), \qquad \|\varepsilon^m\phi\|_1 = \frac{O(1)}{k}\Big\{|L(\sigma)| + \frac{\ln k}{k}\Big\}.$$

We take $\rho = \tau$. Then $\rho^* = 1$ and $\hat\rho = 7/8$. Using the explicit bounds of $\|\varepsilon^m\phi\|_\rho$, $\|\varepsilon^m\phi\|_{\hat\rho}$, and $\|\varepsilon^m\phi\|_{\rho^*}$ we obtain

$$\langle N[\phi], W_\lambda\rangle = O(\varepsilon k), \qquad \langle N[\phi], W_\sigma\rangle = O(1)L^2(\sigma)\mathbf{1}_{\{n=5\}} + O(\varepsilon k).$$

In conclusion, (3.9) can be written as,

$$M\begin{bmatrix} c_1/\lambda \\ c_2/(\varepsilon k)\end{bmatrix} = \begin{bmatrix} k^{-1} & 0 \\ 0 & \varepsilon\end{bmatrix}\begin{bmatrix} (n-4)\lambda^{4-n}A_2L(\sigma) + O(1)\varepsilon k \\ \alpha A_1 - \lambda^{4-n}A_2L'(\sigma) + O(1)|L^2(\sigma)|\mathbf{1}_{\{n=5\}} + O(\varepsilon k)\end{bmatrix}.$$

By continuity and a simple topological degree argument, for each large enough integer k, there exists at least one pair $(\lambda, \sigma) = (\lambda^*, \sigma^*) + O(\varepsilon k)$ such that $c_1 = c_2 = 0$, from which we obtain a solution of (1.1).

Since $v = U_0 + R + \phi$ in Ω_0, upon noting that $\|\varepsilon^m R\|_{L^\infty(\Omega_0)} = O(1)k^{-1}$ and $\|\varepsilon^m \phi\|_{L^\infty(B)} = O(1)k^{-1}$, we obtain the following, of which Theorem 1 is a direct consequence.

Theorem 3. *Suppose $n \geqslant 5, p = \frac{n+4}{n-4}$, and $\alpha > 0$. There exists a positive integer K such that for each integer $k \geqslant K$, (1.1) admits a solution of the form*

$$u(x) = \frac{1}{\varepsilon^m}\Big\{\max_i \Phi\Big(\frac{x - r\mathbf{e}_i}{\varepsilon}\Big) + \frac{O(1)}{k}\Big\},$$

where $\varepsilon = k^{-\frac{n-3}{n-4}}/\lambda$, $r = (1 - \sigma/k)$, $\mathbf{e}_i$ is defined in (2.6), and $(\lambda, \sigma) = (\lambda^*, \sigma^*) + O(k^{-\frac{1}{n-4}})$.

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