

Locally recurrent functions, density topologies and algebraic genericity



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ABSTRACT

We study the algebraic and topological genericity of certain subsets of locally recurrent functions, obtaining (among other results) algebraicity and spaceability within these classes.

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1. Introduction

This paper contributes to the ongoing search for linear structures of mathematical objects enjoying certain special or *unexpected* properties. This search began at the beginning of this century and, since then, many authors have become interested in this direction of research. Many different fields in Mathematics were influenced by this, from Linear Chaos to Real and Complex Analysis, passing through Set Theory and Linear and Multilinear Algebra, or even Operator Theory, Topology, Measure Theory, Abstract Algebra and Probability Theory. We refer the interested reader to the recent survey paper [10] and to the monograph [1] for a full detailed study of this modern area of research.

Let us now present some, by now, well known definitions on algebraic genericity (see, e.g. [1,9,10,23,37]).

Definition 1.1. Let κ be a cardinal number.

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- i) Let L be a vector space and a set $A \subset L$. We say that A is κ -lineable if $A \cup \{0\}$ contains a κ -dimensional vector space.
- ii) Let L be a Banach space and a set $A \subset L$. We say that A is κ -spaceable if $A \cup \{0\}$ contains a κ -dimensional closed vector space. (We say that A is spaceable if A is κ -spaceable for some infinite κ .)
- iii) Let L be a linear commutative algebra and a set $A \subset L$. We say that A is κ -algebrable if $A \cup \{0\}$ contains a κ -generated algebra B (i.e. there exists a minimal system of generators of B with cardinality κ).
- iv) Let L be a linear commutative algebra and a set $A \subset L$. We say that A is κ -strongly algebrable if $A \cup \{0\}$ contains a κ -generated algebra B that is isomorphic with a free algebra (i.e. if $X = \{x_\alpha, \alpha < \kappa\}$ denotes the set of generators of B then the set \tilde{X} of all elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of the elements from \tilde{X} are in $A \cup \{0\}$).

This article is arranged in two main sections. Section 2 focuses on continuous locally recurrent functions and its algebraic and topological genericity within subsets of continuous functions, obtaining spaceability and strong algebrability within this class of functions. Section 3 deals with lineability, algebrability and spaceability results within classes of locally recurrent functions that are continuous with respect to certain density topologies.

2. Strong algebrability and spaceability of continuous locally recurrent functions

The following definition was first introduced in [12] (see also [8]).

Definition 2.1. Let $I \subset \mathbb{R}$ be a non-trivial closed interval and let $x \in I$. A function $f : I \rightarrow \mathbb{R}$ is said to be *right (left) recurrent* at x if, given any $\varepsilon > 0$, there exists $y \in I$ such that $0 < y - x < \varepsilon$ ($0 < x - y < \varepsilon$) and $f(y) = f(x)$.

The function f is called *locally recurrent* on I , if it is (left or right) recurrent at each $x \in I$. We will denote the set of all such functions by $\text{LR}(I)$.

The property of being locally recurrent is linked to *everywhere surjectivity*:

Definition 2.2 ([2]). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *everywhere surjective* ($f \in \text{ES}(\mathbb{R})$ from now on) if, given any interval $I \subset \mathbb{R}$, $f(I) = \mathbb{R}$.

It is clear that any everywhere surjective function is locally recurrent (on any non-trivial subinterval of \mathbb{R}). Furthermore, the class of everywhere surjective functions enjoys the following interesting properties: Any $f \in \mathbb{R}^{\mathbb{R}}$ can be expressed as both, a sum of two everywhere surjective functions and a limit of a sequence of everywhere surjective functions (see [25,38]). Furthermore, in [2] the authors show that the class $\text{ES}(\mathbb{R})$ (and therefore the class $\text{LR}(\mathbb{R})$) is $2^{\mathfrak{c}}$ -lineable, which is the best possible result in terms of dimension (see [1,10,14,26] for more lineability-related results on everywhere surjective functions).

In the rest of this section we shall focus on a more concrete subset of LR functions. Namely, we study the set of non-constant, continuous, locally recurrent functions with derivative zero almost everywhere. The first construction of such a function appeared in [34]. While the original construction relies heavily on the properties of decimal expansion, we present another construction that only requires the use of elementary properties of real functions and, in our opinion, allows for easier verification of local recurrence. This construction first appeared in [30, Theorem 23.16, p. 179]. Since the original source is not readily accessible, we present it here in detail.

Theorem 2.3 ([30], Theorem 23.16, p. 179). *There exists a function $f : [0, 1] \rightarrow \mathbb{R}$ such that f is non-constant, continuous, locally recurrent and $f'(x) = 0$ for almost every $x \in [0, 1]$.*

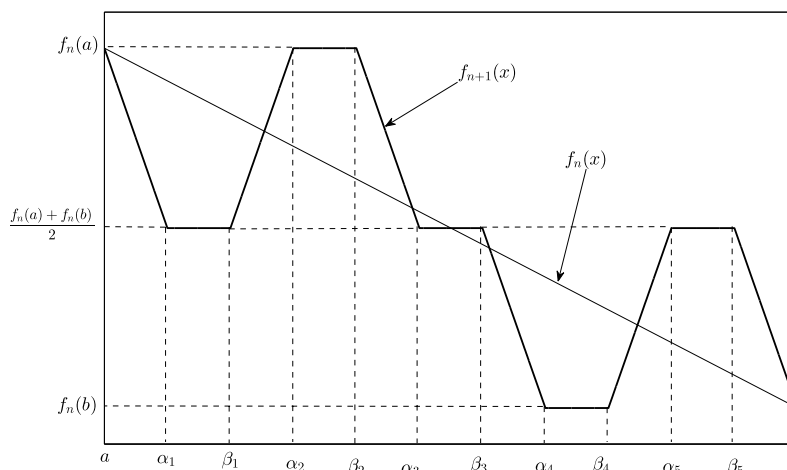


Fig. 1. This picture shows f_{n+1} in comparison to f_n in an adjacent interval $[a, b]$ to $\cup_{n \in \mathbb{N}} G_n$ where the graph of f_n is a line of negative slope.

Proof. Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of open subsets of $[0, 1]$ with the following properties:

- i) $G_1 = \bigcup_{i=1}^3 (a_i, b_i)$, $a_1 = 0$, $b_3 = 1$, $b_i < a_{i+1}$, $i = 1, 2$.
- ii) For any $a, b \in \overline{G_n}$, $a < b$, such that $(a, b) \cap G_n = \emptyset$ we have

$$(a, b) \cap G_{n+1} = \bigcup_{i=1}^5 (\alpha_i, \beta_i), \quad a < \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_5 < b$$

and

$$\frac{a+b}{2} \in (\alpha_3, \beta_3).$$

- iii) $G_n \subset G_{n+1}$, $n \in \mathbb{N}$.
- iv) $\lambda(\bigcup_{n \in \mathbb{N}} G_n) = 1$.

Let $f_1 : [0, 1] \rightarrow \mathbb{R}$ be such that

$$\begin{aligned} f_1(x) &= 0, & x &\in [a_i, b_i], \quad i = 1, 3, \\ f_1(x) &= 1, & x &\in [a_2, b_2], \\ f_1 &\text{ is linear on both } [b_1, a_2] \text{ and } [b_2, a_3]. \end{aligned}$$

Next, define functions f_n , $n \in \mathbb{N}$ such that

- 1) $f_{n+1}(x) = f_n(x)$ for every $x \in \overline{G_n}$.
- 2) For any $a, b \in \overline{G_n}$, $a < b$, $(a, b) \cap G_n = \emptyset$ we have

$$\begin{aligned} f_{n+1}(x) &= f_n(a), & x &\in [\alpha_2, \beta_2], \\ f_{n+1}(x) &= f_n(b), & x &\in [\alpha_4, \beta_4], \\ f_{n+1}(x) &= \frac{f_n(a) + f_n(b)}{2}, & x &\in [\alpha_1, \beta_1] \cup [\alpha_3, \beta_3] \cup [\alpha_5, \beta_5], \\ f_{n+1} &\text{ is linear elsewhere.} \end{aligned}$$

The reader can find in Fig. 1 how f_{n+1} changes with respect to f_n in $[a, b]$.

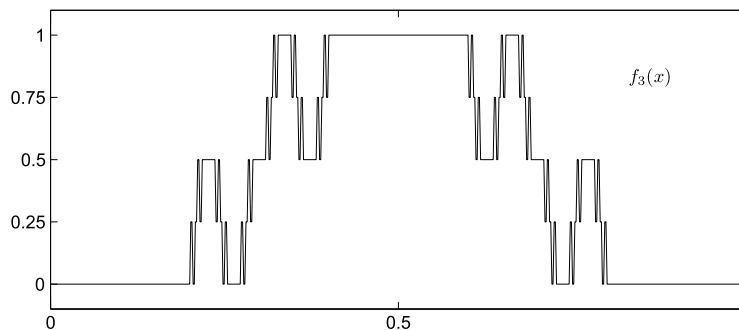


Fig. 2. This picture shows the graph of f_3 , which gives already a very precise idea about the graph of f .

We want to show that the functions f_n converge uniformly to a function with all the desired properties. First of all, it follows from the construction that

$$\|f_{n+1} - f_n\| \leq 2^{-n}, \quad n \in \mathbb{N},$$

which shows that the sequence $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy and therefore converges to some continuous $f : [0, 1] \rightarrow \mathbb{R}$.

The function f is non-constant since we have $f(x) = f_1(x)$, $x \in G_1$ by construction.

Fix $x \in \bigcup_{n \in \mathbb{N}} G_n$. There exist $m \in \mathbb{N}$ and an open interval $I \subset G_m$ such that $x \in I \subset \bar{I} \subset G_m$. It now follows from 1) and iii) that f is constant on \bar{I} and $f'(x) = 0$. Thus $f' = 0$ almost everywhere according to property iv).

To show that f is locally recurrent on $[0, 1]$ notice first that f is constant on any maximal interval of $\bigcup_{n \in \mathbb{N}} G_n$ and therefore it is locally recurrent at any point $x \in \bigcup_{n \in \mathbb{N}} \overline{G_n}$. Thus we are left to show that f is recurrent in points of $[0, 1] \setminus \bigcup_{n \in \mathbb{N}} \overline{G_n}$. To this end, let $x \in [0, 1] \setminus \bigcup_{n \in \mathbb{N}} \overline{G_n}$ and $\varepsilon > 0$. Find $n \in \mathbb{N}$ such that $x \in (a, b) \subset [a, b] \subset (x - \varepsilon, x + \varepsilon)$, where $a, b \in \overline{G_n} \setminus G_n$. By the construction of the sequences $\{G_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ there exist disjoint intervals $J_1 = [c_1, d_1]$ and $J_2 = [c_2, d_2]$ such that:

- a) $c_i, d_i \in \overline{G_{n+1}}$, $i = 1, 2$,
- b) $(c_i, d_i) \subset [0, 1] \setminus G_{n+1}$, $i = 1, 2$,
- c) $J_i \subset [a, b]$, $j = 1, 2$,
- d) $x \in J_1$,
- e) $f_{n+1}(J_1) = f_{n+1}(J_2)$.

We thus have

$$f(x) \in f(J_1) = f_{n+1}(J_1) = f_{n+1}(J_2) = f(J_2).$$

This finishes the proof. In order to have an idea about what the graph of f looks like, the reader can find a sketch of f_3 in Fig. 2. \square

Remark 2.4. The function f constructed above actually has the following additional property: For any $0 < \varepsilon < 1$ there exists an open set $G_\varepsilon \subset [0, 1]$ such that f is constant on any maximal subinterval of G_ε and $\lambda(G_\varepsilon) > 1 - \varepsilon$.

Let us now introduce some additional notation at this point:

Definition 2.5. Let $I \subset \mathbb{R}$ be a non-trivial closed interval. We denote by $\text{LRC}(I)$ the set of all continuous, non-constant, locally recurrent functions on I . Furthermore, we denote by $\text{LRCS}(I)$ the set of all continuous, non-constant, locally recurrent functions on I with derivative zero almost everywhere on I .

The following lemma shows that one cannot hope to use the Baire category theorem to obtain a non-constructive proof of existence of a locally recurrent non-constant continuous function.

Lemma 2.6. *The set $LRC([0, 1])$ is of first category in $C([0, 1])$.*

Proof. Denote by $P([0, 1])$ the set of all functions $f \in C([0, 1])$ that attain its global maximum at more than one point. It follows directly from the definition that

$$LRC([0, 1]) \subset P([0, 1]).$$

Thus we only need to show that $P([0, 1])$ is of first category in $C([0, 1])$ to prove our assertion. This is precisely [24, Theorem 1] (see also [11, Lemma 3.1, p. 215] for a more general result). \square

The above lemma shows that, from the topological point of view, the set of continuous locally recurrent functions is “a small set” (it is an element of the σ -ideal of meager sets). It is therefore quite interesting that from the point of lineability it is the largest possible. We show this in the two main theorems of this section, the first one of which is the following:

Theorem 2.7. *The set $LRCS([0, 1])$ is strongly \mathfrak{c} -algebrable.*

Before we move on to the proof, we need a definition and an additional result.

Definition 2.8. We say that a function $g : \mathbb{R} \rightarrow \mathbb{R}$ is *exponential-like* (of rank m) whenever g is given by $g(x) = \sum_{i=1}^m a_i e^{\beta_i x}$ for some distinct non-zero real numbers β_1, \dots, β_m and some non-zero real numbers a_1, \dots, a_m .

The following simple lemma can be found in [3, Lemma 8]:

Lemma 2.9. *For every positive integer m and any exponential-like function g of rank m and each $c \in \mathbb{R}$ the preimage $g^{-1}(\{c\})$ has at most m elements. In particular, there exists a finite decomposition of \mathbb{R} into intervals such that g is strictly monotone on each of them.*

The following theorem (see [3, Proposition 7] or [6]) is an essential ingredient in our proof of Theorem 2.7:

Theorem 2.10. *Let \mathfrak{F} be a family of real functions on $[0, 1]$. Suppose that there exists $f \in \mathfrak{F}$ such that $g \circ f \in \mathfrak{F} \setminus \{0\}$ for every exponential-like function $g : \mathbb{R} \rightarrow \mathbb{R}$. Then \mathfrak{F} is strongly \mathfrak{c} -algebrable.*

Proof (of Theorem 2.7). As we mentioned above, we aim to use Theorem 2.10. To this end, let f be as in Theorem 2.3 and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary exponential-like function. The function $g \circ f$ is obviously continuous and locally recurrent on $[0, 1]$.

Furthermore, $(g \circ f)'(x)$ exists for any $x \in (0, 1)$ such that $f'(x)$ exists and thus $(g \circ f)' = 0$ almost everywhere on $[0, 1]$.

Suppose, for the sake of contradiction, that $g \circ f$ is constant. According to Lemma 2.9 the set $\{f(x) : x \in [0, 1]\}$ is then finite and since f is continuous, f is constant as well, which is a contradiction.

Thus we have that $g \circ f \in LRCS[0, 1]$ and the claim follows from Theorem 2.10. \square

The strong \mathfrak{c} -algebrability of LRCS complements several known results. For instance, in [3, Theorem 9] the authors show that the set of continuous functions of bounded variation on $[0, 1]$ that are non-constant on any open subinterval and have derivative zero almost everywhere is strongly \mathfrak{c} -algebrable. The algebra

constructed in Theorem 2.7 consists, on the other hand, of functions with unbounded variation (see [33, Proposition 9]) and derivative zero almost everywhere. In addition, these functions are “constant almost everywhere” in the sense of Remark 2.4.

In [27] the authors prove \mathfrak{c} -algebrability of the set of continuous functions on \mathbb{R} such that the sets of their proper local minima and maxima, respectively, are dense subsets of \mathbb{R} . In direct contrast, the functions in LRC have no isolated local extrema.

The following is the second main theorem of this section. Notice that, while strong \mathfrak{c} -algebrability implies \mathfrak{c} -lineability, it does not (in general) imply spaceability (see [4] and [5] for an example of a strong \mathfrak{c} -algebrable, non-spaceable set).

Theorem 2.11. *The set $\text{LRCS}([0, 1])$ is spaceable.*

Proof. First, recall that an infinite-dimensional Banach space must have an uncountable basis (see, e.g., [7]). Spaceability and \mathfrak{c} -spaceability of subsets of $C([0, 1])$ are, thus, equivalent terms.

It was proved in [28] that while the set of differentiable functions on $[0, 1]$ is not spaceable as a subset of $C([0, 1])$, the set

$$C_{\text{diff}}((0, 1)) = \{f \in C([0, 1]), f \text{ is differentiable on } (0, 1)\}$$

is spaceable as subset of $C([0, 1])$. It follows that $C_{\text{diff}}((0, 1))$ is also spaceable as a subset of

$$C_0([0, 1]) = \{g \in C([0, 1]) : g(0) = 0\}.$$

Let $V \subset C_{\text{diff}}((0, 1)) \cap C_0([0, 1])$ denote such a closed vector subspace and let $f : [0, 1] \rightarrow [0, 1]$ be as in Theorem 2.3. Set

$$W = V \circ f = \{g \circ f : [0, 1] \rightarrow \mathbb{R}, g \in V\}.$$

We claim that W has all the desired properties. Indeed, as $f([0, 1]) = [0, 1]$ and V is closed, W is closed as well. It follows directly from the definition, that every function $w \in W$ is a continuous locally recurrent function with derivative zero almost everywhere on $[0, 1]$. Furthermore, since $f(0) = 0$ and $V \subset C_0([0, 1])$, we have that $w \in W$ is a constant function if and only if $w \equiv 0$. Thus $W \setminus \{0\} \subset \text{LRCS}([0, 1])$ and the proof is finished. \square

3. \mathcal{I} -density continuous, density continuous LRC functions

In this section we aim to study classes of LRC functions that are continuous with respect to the density and \mathcal{I} -density topologies. It is organized in the following fashion:

Firstly, we shall recall the necessary background on density and \mathcal{I} -density topologies. In second place, we provide results concerning classes of density-continuous functions. These results serve as a motivation for the final part of this section. There we show that the class of functions in $\text{LRC}([0, 1])$ that are simultaneously density and \mathcal{I} -density continuous is strongly \mathfrak{c} -algebrable (but not spaceable!). Most of the results presented in the rest of this section can be found in [22] (which represents a detailed seminar source on the topics of density topologies and the classes of functions that are continuous with respect to these topologies, see also [19, 31]).

3.1. Density and \mathcal{I} -density topologies

Let us first recall some definitions:

Definition 3.1. Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. We call x a *density point* of A if

$$\lim_{h \searrow 0} \frac{\lambda(A \cap (x-h, x+h))}{2h} = 1.$$

We denote the set of density points of A by $\Phi_{\mathcal{D}}(A)$.

Definition 3.2. We call a set $A \subset \mathbb{R}$ *density open*, if

$$A \subseteq \Phi_{\mathcal{D}}(A).$$

We further denote:

$$\mathcal{T}_{\mathcal{D}} = \{A \in \mathbb{R} : A \subseteq \Phi_{\mathcal{D}}(A)\}.$$

$\mathcal{T}_{\mathcal{D}}$ is a topology on \mathbb{R} called the *density topology*. Note that if $A \in \mathcal{T}_{\mathcal{D}}$, then A is Lebesgue measurable (see [22, Theorem 1.2.2] for both claims). The density topology has several interesting properties: it is neither separable, nor does it have the Lindelöf property; a set is compact with respect to $\mathcal{T}_{\mathcal{D}}$ if and only if it is finite and a set is closed and discrete with respect to $\mathcal{T}_{\mathcal{D}}$ if and only if it is a Lebesgue-null set (see [22, Theorem 1.2.3] for a more detailed list of properties of $\mathcal{T}_{\mathcal{D}}$).

Before we move on to the definition of the \mathcal{I} -density topology, let us present some additional notation:

For $A \subseteq \mathbb{R}$ we denote by χ_A the characteristic (or indicator) function of A . Furthermore, given $x \in \mathbb{R}$ we denote

$$A - x = \{a - x; a \in A\}.$$

By \mathcal{I} we denote the σ -ideal of meager subsets of \mathbb{R} . We say that a property is satisfied *\mathcal{I} -almost everywhere* (\mathcal{I} -a.e.) if and only if the set of points in which this property does not hold belongs to \mathcal{I} .

By \mathcal{B} we denote the set of all Borel subsets of \mathbb{R} . We say that a sequence $\{f_n\}_{n \in \mathbb{N}}$ of \mathcal{B} -measurable real functions converges with respect to \mathcal{I} to some \mathcal{B} -measurable real function f if for every subsequence $\{f_{n_k}\}_{k \in \mathbb{N}} \subset \{f_n\}_{n \in \mathbb{N}}$ there exists a subsequence $\{f_{n_{k_l}}\}_{l \in \mathbb{N}} \subset \{f_{n_k}\}_{k \in \mathbb{N}}$ which converges to f \mathcal{I} -a.e. We use the following notation:

$$\lim_{n \rightarrow \infty} f_n = f, \mathcal{I}\text{-a.e.}$$

Definition 3.3. Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. We call x a *\mathcal{I} -density point* of A , if

$$\lim_{n \rightarrow \infty} \chi_{n(A-x) \cap [-1,1]} = \chi_{[-1,1]}, \mathcal{I}\text{-a.e.}$$

We denote the set of all \mathcal{I} -density points of A by $\Phi_{\mathcal{I}}(A)$.

Definition 3.4. We call a set $B \in \mathcal{B}$ *\mathcal{I} -density open*, if

$$B \subseteq \Phi_{\mathcal{I}}(B).$$

We further denote

$$\mathcal{T}_{\mathcal{I}} = \{B \in \mathcal{B} : B \subseteq \Phi_{\mathcal{I}}(B)\}.$$

$\mathcal{T}_{\mathcal{I}}$ is a topology on \mathbb{R} (see [22, Theorem 2.3.2]) called the \mathcal{I} -density topology. It was first introduced by Wilczyński in [36,39] as a category-analogue of the density topology $\mathcal{T}_{\mathcal{D}}$. It is neither separable, nor is it regular; a set is closed and discrete with respect to $\mathcal{T}_{\mathcal{I}}$ if, and only if, it is an element of \mathcal{I} (i.e., a meager set); and a set is compact with respect to $\mathcal{T}_{\mathcal{I}}$ if, and only if, it is finite (for a more detailed list of properties of $\mathcal{T}_{\mathcal{I}}$ see [22, Theorem 2.6.2]).

3.2. Spaces of density and \mathcal{I} -density continuous functions

In this section, we focus on properties of spaces of real (continuous) functions that are continuous with respect to one or both of the density-type topologies presented in the previous section.

As before, we begin by introducing the notation relevant to this section:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real function and let \mathcal{T} be a topology on \mathbb{R} . We say that f is \mathcal{T} -continuous if it is continuous with respect to \mathcal{T} on the domain and on the range. We use the following notation:

$$C_{\mathcal{T}}([0, 1]) = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is } \mathcal{T}\text{-continuous}\}.$$

We denote the class of real analytic functions on $[0, 1]$ by \mathcal{A} .

The following theorem first appeared in [21, Theorem 1] (see also [22, Theorem 3.6.7]).

Theorem 3.5. *Let \tilde{C} denote the set of functions in $C([0, 1])$ that are $\mathcal{T}_{\mathcal{D}}$ -continuous in at least one point of $[0, 1]$. Then \tilde{C} is a first category subset of $C([0, 1])$.*

The next theorem consists of several claims that can all be found (in some cases in a more general form) in [22, Theorem 3.7.5]. We present them here in a form that is relevant to our aim.

Theorem 3.6. *The following statements hold:*

- i) *The classes $C_{\mathcal{T}_{\mathcal{I}}}([0, 1])$ and $C_{\mathcal{T}_{\mathcal{D}}}([0, 1])$ are not closed under uniform convergence.*
- ii) *The spaces $C_{\mathcal{T}_{\mathcal{I}}}([0, 1])$ and $C_{\mathcal{T}_{\mathcal{D}}}([0, 1])$ equipped with the standard topology of uniform convergence are both of first category in themselves.*
- iii) *The classes $C_{\mathcal{T}_{\mathcal{I}}}([0, 1])$ and $C_{\mathcal{T}_{\mathcal{D}}}([0, 1])$ are not closed under addition.*

Remark 3.7. Claim ii) in Theorem 3.6 appeared, originally, in [17, Corollary 2 & Theorem 8]. Claim iii) was first proved in [16, Example 2] for the class $C_{\mathcal{T}_{\mathcal{D}}}([0, 1])$ and in [13, Example 1] for $C_{\mathcal{T}_{\mathcal{I}}}([0, 1])$.

The following simple corollary of Theorem 3.6 ii) will be of further use to us:

Corollary 3.8. *The spaces $C_{\mathcal{T}_{\mathcal{I}}}([0, 1])$ and $C_{\mathcal{T}_{\mathcal{D}}}([0, 1])$ equipped with the standard topology of uniform convergence are not spaceable.*

Proof. Let X denote either one of the above mentioned spaces. Suppose, for the sake of contradiction, that there exists an infinite-dimensional Banach space $V \subseteq X$. Then V , being a closed subset of X , is of first category in itself. This, however, is a contradiction since V is a Baire space. \square

The following theorem presents an outline of the various inclusions between classes of $\mathcal{T}_{\mathcal{D}}$ -continuous functions that are at the same time continuous, $\mathcal{T}_{\mathcal{I}}$ -continuous, or both. It originally appeared in a more general form in [18, Theorem 4.2] (see also [22, Theorem 3.7.4]).

Theorem 3.9. *The following proper inclusions hold (here $A \rightarrow B$ means $A \subsetneq B$):*

$$\begin{array}{ccccc}
C_{\mathcal{T}_D}([0, 1]) \cap C_{\mathcal{T}_I}([0, 1]) & \xrightarrow{\quad\quad\quad} & C_{\mathcal{T}_D}([0, 1]) \\
\uparrow & & \uparrow \\
\mathcal{A} \longrightarrow C([0, 1]) \cap C_{\mathcal{T}_D}([0, 1]) \cap C_{\mathcal{T}_I}([0, 1]) & \longrightarrow & C_{\mathcal{T}_D}([0, 1]) \cap C([0, 1]).
\end{array}$$

From now on, we will use the following simpler notation:

$$\text{CDI}([0, 1]) = C([0, 1]) \cap C_{\mathcal{T}_D}([0, 1]) \cap C_{\mathcal{T}_I}([0, 1]).$$

One of the implications of the previous theorem is that $\text{CDI}([0, 1])$ is \mathfrak{c} -strongly algebrable. This follows from the inclusion $\mathcal{A} \subsetneq \text{CDI}([0, 1])$ and the fact that every exponential-like function is real analytic.

However, the following theorem shows that $\text{CDI}([0, 1])$ is not closed under addition. To our best knowledge, this results does not appear in the existing literature even though it is an easy consequence of several already known results.

Theorem 3.10. *The class $\text{CDI}([0, 1])$ is not closed under addition.*

Proof. In [22, Theorem 3.2.5 and Corollary 3.2.6] the authors prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x > 0 \\ 0 & x = 0 \\ -e^{\frac{-1}{x^2}} & x < 0 \end{cases}$$

is not \mathcal{I} -density continuous. More specifically, it follows from the proof of [22, Theorem 3.2.5] that f is not right \mathcal{I} -density continuous at 0.

On the other hand, the function $g : [0, 1] \rightarrow \mathbb{R}$ defined as

$$g(x) = f(x) + x, \quad x \in [0, 1]$$

is \mathcal{I} -density continuous (see [22, Corollary 3.2.12]). The derivative of g is bounded on $[0, 1]$ and the second derivative of g is zero at finitely many points. Thus, according to [16, Corollary 1 and Corollary 2] we have that g is \mathcal{T}_D -continuous.

Denote $h(x) = -x$, $x \in [0, 1]$. Then h is real analytic and therefore $h \in \text{CDI}([0, 1])$ by Theorem 3.9. However, $g + h = f|_{[0, 1]}$ is not \mathcal{I} -density continuous. This finishes the proof. \square

In the following we will focus on a new class of functions, namely:

$$\text{LRCDI}([0, 1]) = \text{LRC}([0, 1]) \cap C_{\mathcal{T}_D}([0, 1]) \cap C_{\mathcal{T}_I}([0, 1]).$$

Note that

$$\text{LRCDI}([0, 1]) \cap \mathcal{A} \subset \text{LRC}([0, 1]) \cap C^1([0, 1]) = \text{span}\{1\}. \quad (3.1)$$

Strong \mathfrak{c} -algebrability of $\text{LRCDI}([0, 1])$ is thus non-trivial according to Theorem 3.10. In the following section we will show that $\text{LRCDI}([0, 1])$ is, however, strongly \mathfrak{c} -algebrable.

Let us note here that strong \mathfrak{c} -algebrability of functions continuous with respect to density topologies was previously studied in [29], where the authors show (among other results) that there exist \mathfrak{c} -many classes of functions that are continuous with respect to certain density-type topologies such that every non-empty difference of two such classes is strongly \mathfrak{c} -algebrable.

3.3. Strong \mathfrak{c} -algebrability of $\text{LRCDI}([0, 1])$

The first order of business is to show that $\text{LRCDI}([0, 1])$ is non-empty. The following theorem shows that this is indeed the case. While the function in question has already been studied on several different occasions, this is (up to our best knowledge) the first time its local recurrence is established.

Theorem 3.11. *There exists a non-constant function $f \in \text{LRCDI}([0, 1])$.*

Proof. In [32] the author shows that $f : [0, 1] \rightarrow [0, 1]$ defined as the x -coordinate of the Peano area-filling curve is a continuous, $\mathcal{T}_{\mathcal{D}}$ -continuous function. In [15, Theorem 1] the authors prove that f is also $\mathcal{T}_{\mathcal{I}}$ -continuous.

Thus we only need to show that f is locally recurrent on $[0, 1]$. Let us first recall its construction:

Let $g : [0, 9] \rightarrow [0, 3]$ be a continuous function defined as follows:

$$\begin{aligned} g(0) &= 0, & g(1) &= 1, \\ g(2) &= 0, & g(3) &= 1, \\ g(4) &= 2, & g(5) &= 1, \\ g(6) &= 2, & g(7) &= 3, \\ g(8) &= 2, & g(9) &= 3, \\ g &\text{ is linear otherwise.} \end{aligned}$$

Let

$$f_1(x) = \frac{1}{3}g(9x), \quad x \in [0, 1].$$

For every $n \in \mathbb{N}$, $k \in \{0, 1, \dots, 9^n - 1\}$, $t \in [0, 9^{-n}]$ let

$$f_{n+1}\left(\frac{k}{9^n} + t\right) = f_n\left(\frac{k}{9^n}\right) + \frac{1}{3}g(9^{n+1}t)\left(f_n\left(\frac{k+1}{9^n}\right) - f_n\left(\frac{k}{9^n}\right)\right). \quad (3.2)$$

The sequence $\{f_n\}_{n \in \mathbb{N}}$ then converges uniformly to f (see [32]). Let us further denote

$$I_k^n = \left[\frac{k}{9^n}, \frac{k+1}{9^n}\right], \quad n \in \mathbb{N}, \quad k \in \{0, 1, \dots, 9^n - 1\}.$$

Note that for any such interval I_k^n we have by construction of $\{f_n\}_{n \in \mathbb{N}}$

$$f_n(I_k^n) = f_{n+m}(I_k^n) = f(I_k^n), \quad m \in \mathbb{N}. \quad (3.3)$$

We aim to show that f is locally recurrent on $[0, 1]$. To this end, pick $x \in [0, 1]$ and $\varepsilon > 0$. Find $n \in \mathbb{N}$ such that $9^{-n} < 2\varepsilon$ and a suitable integer k such that $x \in I_k^n$. Consider the graph of the function $f_{n+1} \upharpoonright I_k^n$. It follows from (3.2) that it consists of a rescaled (and possibly translated) graph of the function

$$\tilde{g}(t) = g(9^{n+1}t), \quad t \in [0, 9^{-n}].$$

A closer examination of the graph of \tilde{g} yields the existence of $l_1, l_2 \in \{0, 1, \dots, 9^{n+1} - 1\}$, $l_1 \neq l_2$ such that $I_{l_1}^{n+1}, I_{l_2}^{n+1} \subset I_k^n$,

$$x \in I_{l_1}^{n+1} \setminus I_{l_2}^{n+1}$$

and

$$f_{n+1}(I_{l_1}^{n+1}) = f_{n+1}(I_{l_2}^{n+1}).$$

Using (3.3) we get

$$f(x) \in f(I_{l_1}^{n+1}) = f_{n+1}(I_{l_1}^{n+1}) = f_{n+1}(I_{l_2}^{n+1}) = f(I_{l_2}^{n+1}).$$

The function f is continuous and therefore Darboux. Thus we obtain the existence of $y \in I_{l_2}^{n+1} \subset (x - \varepsilon, x) \cup (x, x + \varepsilon)$ such that $f(x) = f(y)$ and the proof is finished. \square

We are now ready to prove the main result of this chapter.

Theorem 3.12. *The class $\text{LRCDI}([0, 1])$ is strongly \mathfrak{c} -algebrable.*

Proof. We aim to use Theorem 2.10. To this end, let f be as in Theorem 3.11 and let g be an exponential-like function. The function $g \circ f$ is obviously in $\text{LRC}([0, 1])$. Since $g \in \mathcal{A} \subsetneq \text{CDI}(\mathbb{R})$ by Theorem 3.9 we immediately get that $g \circ f \in \text{CDI}([0, 1])$ as well. Thus $g \circ f \in \text{LRCDI}([0, 1])$ and the claim follows by Theorem 2.10. \square

Having seen that $\text{LRCDI}([0, 1])$ is \mathfrak{c} -algebrable, one could also ask whether this class is \mathfrak{c} -spaceable. However, it follows directly from Corollary 3.8 that this is not the case. Thus $\text{LRCDI}([0, 1])$, $C_{\mathcal{T}_D}([0, 1])$ and $C_{\mathcal{T}_I}([0, 1])$ present an example of function classes that are \mathfrak{c} -strongly algebrable, but not spaceable.

3.4. Differentiability properties

In this section, we discuss differentiability properties of functions from the algebra constructed in Theorem 3.12. Let us first recall the necessary definitions:

Definition 3.13. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *approximately differentiable* at a point x , if there exists a real number $D^{\mathcal{D}}f(x)$ and a Lebesgue measurable set E which has density 1 at x such that

$$\lim_{y \in E, y \rightarrow x} \frac{f(y) - f(x)}{y - x} = D^{\mathcal{D}}f(x).$$

Definition 3.14. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *\mathcal{I} -approximately differentiable* at a point x , if there exists a real number $D^{\mathcal{I}}f(x)$ such that for every $\varepsilon > 0$, x is an \mathcal{I} -density point of some Baire subset of

$$\left\{ t \in \mathbb{R} : \frac{f(t) - f(x)}{t - x} \in (D^{\mathcal{I}}f(x) - \varepsilon, D^{\mathcal{I}}f(x) + \varepsilon) \right\}.$$

The function constructed in Theorem 3.11 is nowhere approximately differentiable [20] and nowhere \mathcal{I} -approximately differentiable [15].

Lemma 3.15. *Let $h = g \circ f$, where f is the function from Theorem 3.11 and g is an exponential-like function. Then the function h is approximately non-differentiable \mathcal{I} -a.e., and it is also \mathcal{I} -approximately non-differentiable \mathcal{I} -almost everywhere.*

Proof. Let

$$E_h = \{x \in [0, 1] : g'(f(x)) = 0\}.$$

The set E_h is a preimage of a finite set under f and thus it is nowhere dense (see proof of [15, Theorem 1]) and closed. We aim to show that h is not \mathcal{I} -approximately differentiable at any point of $[0, 1] \setminus E_h$. To this end, let $x \in [0, 1] \setminus E_h$ and suppose, for the sake of contradiction, that $D^{\mathcal{I}}h(x)$ exists. Find $\delta > 0$ such that $f((x - \delta, x + \delta)) \subset \{g' \neq 0\}$. We can write

$$f(t) = g^{-1}(h(t)) \circ h(t), \quad t \in (x - \delta, x + \delta).$$

Thus, according to [22, Lemma 4.3.11], $D^{\mathcal{I}}f(x)$ exists as well, which is a contradiction. Non-approximate differentiability of h at points of $[0, 1] \setminus E_h$ can be proved analogously with the use of [35, Theorem 5.2]. \square

Note that the previous lemma implies that the algebra constructed in Theorem 3.12 consists of nowhere monotone functions.

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