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On Valdivia strong version of Nikodym boundedness property ☆

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Dedicated to the memory of Professor Manuel Valdivia (1928–2014)

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ABSTRACT

Following Schachermayer, a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of Ω is said to have the N*-property* if a $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is uniformly bounded on $\mathcal{A}$, where $ba(\mathcal{A})$ is the Banach space of the real (or complex) finitely additive measures of bounded variation defined on $\mathcal{A}$. Moreover $\mathcal{B}$ is said to have the *strong* N*-property* if for each increasing countable covering $(\mathcal{B}_m)_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has the N-property. The classical Nikodym–Grothendieck's theorem says that each σ-algebra $\mathcal{S}$ of subsets of Ω has the N*-property*. The Valdivia's theorem stating that each σ-algebra $\mathcal{S}$ has the strong N-property motivated the main measure-theoretic result of this paper: We show that if $(\mathcal{B}_{m_1})_{m_1}$ is an increasing countable covering of a σ-algebra $\mathcal{S}$ and if $(\mathcal{B}_{m_1,m_2,\ldots,m_p,m_{p+1}})_{m_{p+1}}$ is an increasing countable covering of $\mathcal{B}_{m_1,m_2,\ldots,m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p$, then there exists a sequence $(n_i)_i$ such that each $\mathcal{B}_{n_1,n_2,\ldots,n_r}$, $r \in \mathbb{N}$, has the strong N-property. In particular, for each increasing countable covering $(\mathcal{B}_m)_m$ of a σ-algebra $\mathcal{S}$ there exists $\mathcal{B}_n$ which has the strong N-property, improving mentioned Valdivia's theorem. Some applications to localization of bounded additive vector measures are provided.

1. Introduction

Let $\mathcal{B}$ be a subset of an algebra $\mathcal{A}$ of subsets of a set Ω (in brief, set-algebra $\mathcal{A}$). The normed space $L(\mathcal{B})$ is the *span*$\{\chi_C : C \in \mathcal{B}\}$ of the characteristic functions of each set $C \in \mathcal{B}$ with the supremum norm $\|\cdot\|$ and $ba(\mathcal{A})$ is the Banach space of finitely additive measures on $\mathcal{A}$ with bounded variation endowed with the variation norm, i.e., $|\cdot| := |\cdot|(\Omega)$. If $\{C_i : 1 \leqslant i \leqslant n\}$ is a measurable partition of $C \in \mathcal{A}$ and $\mu \in ba(\mathcal{A})$ then $|\mu|(C) = \Sigma_i |\mu|(C_i)$ and, as usual, we represent also by μ the linear form in $L(\mathcal{A})$ determined by

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$\mu(\chi_C) := \mu(C)$, for each $C \in \mathcal{A}$. By this identification we get that the dual of $L(\mathcal{A})$ with the dual norm is isometric to $ba(\mathcal{A})$ (see e.g., [2, Theorem 1.13]).

Polar sets are considered in the dual pair $< L(\mathcal{A}), ba(\mathcal{A}) >$, M° means the polar of a set M and if $\mathcal{B} \subset \mathcal{A}$ the topology in $ba(\mathcal{A})$ of pointwise convergence in $\mathcal{B}$ is denoted by $\tau_s(\mathcal{B})$. $(E', \tau_s(E))$ is the vector space of all continuous linear forms defined on a locally convex space E endowed with the topology $\tau_s(E)$ of the pointwise convergence in E. In particular, the topology $\tau_s(L(\mathcal{A}))$ in $ba(\mathcal{A})$ is $\tau_s(\mathcal{A})$.

The convex (absolutely convex) hull of a subset M of a topological vector space is denoted by $co(M)$ ($absco(M)$) and $absco(M) = co(\cup\{rM : |r| = 1\})$. An equivalent norm to the supremum norm in $L(\mathcal{A})$ is the Minkowski functional of $absco(\{\chi_C : C \in \mathcal{A}\})$ ([14, Propositions 1 and 2]) and its dual norm is the $\mathcal{A}$-supremum norm, i.e., $\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$, $\mu \in ba(\mathcal{A})$. The closure of a set is marked by an overline, hence if $P \subset L(\mathcal{A})$ then $\overline{span(P)}$ is the closure in $L(\mathcal{A})$ of the linear hull of P. $\mathbb{N}$ is the set $\{1, 2, \dots\}$ of positive integers.

Recall the classical Nikodym–Dieudonné–Grothendieck theorem (see [1, page 80, named as Nikodym–Grothendieck boundedness theorem]): *If $\mathcal{S}$ is a σ-algebra of subsets of a set Ω and M is a $\mathcal{S}$-pointwise bounded subset of $ba(\mathcal{S})$ then M is a bounded subset of $ba(\mathcal{S})$ (i.e., $\sup\{|\mu(C)| : \mu \in M, C \in \mathcal{S}\} < \infty$,* or, equivalently, $\sup\{|\mu|(\Omega) : \mu \in M\} < \infty$). This theorem was firstly obtained by Nikodym in [11] for a subset M of countably additive complex measures defined on $\mathcal{S}$ and later on by Dieudonné for a subset M of $ba(2^\Omega)$, where 2^Ω is the σ-algebra of all subsets of Ω, see [3].

It is said that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set Ω has the *Nikodym property*, N-property in brief, if the Nikodym–Dieudonné–Grothendieck theorem holds for $\mathcal{B}$, i.e., *if each $\mathcal{B}$-pointwise bounded subset M of $ba(\mathcal{A})$ is bounded in $ba(\mathcal{A})$* (see [12, Definition 2.4] or [15, Definition 1]). Let us note that in this definition we may suppose that M is $\tau_s(\mathcal{A})$-closed and absolutely convex. If $\mathcal{B}$ has N-property then the polar set $\{\chi_C : C \in \mathcal{B}\}^\circ$ is bounded in $ba(\mathcal{A})$, hence $\{\chi_C : C \in \mathcal{B}\}^{\circ\circ} = \overline{absco\{\chi_C : C \in \mathcal{B}\}}$ is a neighborhood of zero in $L(\mathcal{A})$, whence $L(\mathcal{B})$ is dense in $L(\mathcal{A})$.

It is well known that *the algebra of finite and co-finite subsets of* $\mathbb{N}$ *fails N-property* [2, Example 5 in page 18] and that Schachermayer proved that *the algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has N-property* (see [12, Corollary 3.5] and a generalization in [4, Corollary]). A recent improvement of this result for the algebra $\mathcal{J}(K)$ of Jordan measurable subsets of a compact k-dimensional interval $K := \Pi\{[a_i, b_i] : 1 \leqslant i \leqslant k\}$ in $\mathbb{R}^k$ has been provided in [15, Theorem 2], where Valdivia proved that *if $\mathcal{J}(K)$ is the increasing countable union $\cup_m \mathcal{B}_m$ there exists a positive integer n such that $\mathcal{B}_n$ has N-property* (see [8, Theorem 1] for a strong result in $\mathcal{J}(K)$). This fact motivated to say that a subset $\mathcal{B}$ of a set-algebra $\mathcal{A}$ has the *strong Nikodym property*, sN-property in brief, if for each increasing covering $\cup_m \mathcal{B}_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has N-property. As far as we know this result suggested the following very interesting Valdivia's open question (2013):

Problem 1 *([15, Problem 1])*. Let $\mathcal{A}$ be an algebra of subsets of Ω. Is it true that N-property of $\mathcal{A}$ implies sN-property?

Note that the Nikodym–Dieudonné–Grothendieck stating that every σ-algebra $\mathcal{S}$ of subsets of a set Ω has property N is a particular case of the following Valdivia's theorem.

Theorem 1 *([14, Theorem 2]). Each σ-algebra $\mathcal{S}$ of subsets of Ω has sN-property.*

Following [7, Chapter 7, 35.1] a family $\{B_{m_1,m_2,\dots,m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ of subsets of A is an *increasing web in A* if $(B_{m_1})_{m_1}$ is an increasing covering of A and $(B_{m_1,m_2,\dots,m_p,m_{p+1}})_{m_{p+1}}$ is an increasing covering of $B_{m_1,m_2,\dots,m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p$. We will say that *a set-algebra $\mathcal{A}$ of subsets of Ω has the web strong N-property* (*web-sN-property*, in brief) *if for each increasing web $\{\mathcal{B}_{m_1,m_2,\dots,m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ in $\mathcal{A}$ there exists a sequence $(n_i)_i$ in $\mathbb{N}$ such that each $\mathcal{B}_{n_1,n_2,\dots,n_i}$ has sN-property, for each $i \in \mathbb{N}$.*

The main measure-theoretic result of this paper is the following theorem, motivated by Theorem 1 and covering all mentioned results for σ-algebras.

Theorem 2. *Each σ-algebra $\mathcal{S}$ of subsets of Ω has web-sN-property.*

In particular, if $\mathcal{B}_{m_1,m_2,\ldots,m_p} = \mathcal{B}_{m_1}$ for each $p \in N$, we have the following improvement of Theorem 1: If $(\mathcal{B}_m)_m$ *is an increasing covering of a σ-algebra $\mathcal{S}$ of subsets of Ω there exists an index n so that $\mathcal{B}_n$ has sN-property.*

Next section provides properties concerning N-property of subsets of a set-algebra $\mathcal{A}$ and unbounded subsets of $ba(\mathcal{A})$. These results will be used in Section 3 to provide necessary facts to complete the proof of our main result (Theorem 2).

Last section deals with applications of Theorem 2 to localizations of bounded finite additive vector measures.

A characterization of sN-property of a set-algebra $\mathcal{A}$ by a locally convex property of $L(\mathcal{A})$ was obtained in [15, Theorem 3]. Analogously a characterization of *web*-sN-property of a set-algebra $\mathcal{A}$ by a locally convex property of $L(\mathcal{A})$ may be found easily following [5] and [10].

2. Nikodym property and deep unbounded sets

To keep the paper self-contained we provided a short proof of the next (well known) proposition.

Proposition 3. *Let $\mathcal{A}$ be an algebra of subsets of Ω and let M be an absolutely convex $\tau_s(\mathcal{A})$-closed subset of $ba(\mathcal{A})$. The following properties are equivalent:*

1. *For each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$ the set $M \cap \mathcal{Q}^\circ$ is an unbounded subset of $ba(\mathcal{A})$.*
2. *For each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$ such that $span\{M^\circ\} \cap span\{\mathcal{Q}\} = \{0\}$ the set $M \cap \mathcal{Q}^\circ$ is unbounded in $ba(\mathcal{A})$.*
3. *M° is not a neighborhood of zero in $span\{M^\circ\}$ or the codimension of $span\{M^\circ\}$ in $L(\mathcal{A})$ is infinite.*

If M is unbounded and $\overline{span\{M^\circ\}} = L(\mathcal{A})$ then M verifies the previous properties.

Proof. To prove these equivalences recall that if M is a $\tau_s(\mathcal{A})$-closed and absolutely convex subset of ba$(\mathcal{A})$ then $M^{\circ\circ} = M$ [7, Chapter 4 20.8.5].

(1) $\iff$ (2). Let $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$. First we prove that if there exists $m_1 \in M^\circ$ such that $\chi_{Q_1} = h_1 m_1 + \Sigma_{2\leqslant i\leqslant r} h_i \chi_{Q_i}$ and if $h := 2 + \Sigma_{1\leqslant i\leqslant r} |h_i|$ then

$$\operatorname{absco}(M^\circ \cup \mathcal{Q}) \subset h \operatorname{absco}(M^\circ \cup \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\}). \tag{1}$$

In fact, if $x \in \operatorname{absco}(M^\circ \cup \mathcal{Q})$ then $x = \lambda_0 m_0 + \Sigma_{1\leqslant i\leqslant r}\lambda_i \chi_{Q_i}$, with $m_0 \in M^\circ$ and $\Sigma_{0\leqslant i\leqslant r} |\lambda_i| \leqslant 1$, whence $x = \lambda_0 m_0 + \lambda_1 h_1 m_1 + \Sigma_{2\leqslant i\leqslant r}(\lambda_1 h_i + \lambda_i)\chi_{Q_i}$. From $m_2 := (1 + |\lambda_0| + |\lambda_1 h_1|)^{-1}(\lambda_0 m_0 + \lambda_1 h_1 m_1) \in M^\circ$ we get the representation $x = (1 + |\lambda_0| + |\lambda_1 h_1|)m_2 + \Sigma_{2\leqslant i\leqslant r}(\lambda_1 h_i + \lambda_i)\chi_{Q_i}$ which verifies the inequality $1 + |\lambda_0| + |\lambda_1 h_1| + \Sigma_{2\leqslant i\leqslant r} |\lambda_1 h_i + \lambda_i| \leqslant h$, whence $x \in h \operatorname{absco}(M^\circ \cup \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\})$. Taking polar sets in (1) we obtain that

$$M \cap \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\}^\circ \subset h(M \cap \mathcal{Q}^\circ),$$

hence if $M \cap \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\}^\circ$ is unbounded one gets that $M \cap \mathcal{Q}^\circ$ is also unbounded. The rest of this equivalence is obvious.

(2) $\iff$ (3). If M° is a neighborhood of zero in span$\{M^\circ\}$ and if $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$ is a cobase of span$\{M^\circ\}$ in $L(\mathcal{A})$ then absco$(M^\circ \cup \mathcal{Q})$ is a neighborhood of zero in $L(\mathcal{A})$, hence

$$(\mathrm{absco}(M^\circ \cup \mathcal{Q}))^\circ = M \cap \mathcal{Q}^\circ$$

is a bounded subset of $\mathrm{ba}(\mathcal{A})$.

If M° is not a neighborhood of zero in $\mathrm{span}\{M^\circ\}$ or if the codimension of $\mathrm{span}\{M^\circ\}$ in $L(\mathcal{A})$ is infinite, then for each finite set $\mathcal{Q} := \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$ such that $\mathrm{span}\{M^\circ\} \cap \mathrm{span}\{\mathcal{Q}\} = \{0\}$ the set $\mathrm{absco}(M^\circ \cup \mathcal{Q})$ is not a neighborhood of zero in $L(\mathcal{A})$, whence the set $(\mathrm{absco}(M^\circ \cup \mathcal{Q}))^\circ = M \cap \mathcal{Q}^\circ$ is unbounded in $\mathrm{ba}(\mathcal{A})$.

If M is an unbounded subset of $\mathrm{ba}(\mathcal{A})$ then M° is not a neighborhood of zero in $L(\mathcal{A})$. If, additionally, $\overline{\mathrm{span}\{M^\circ\}} = L(\mathcal{A})$ we have, by denseness, that M° is not a neighborhood of zero in $\mathrm{span}\{M^\circ\}$ and we obtain that M verifies (3). $\square$

The fact that if a subset M of $ba(\mathcal{A})$ verifies (1) in Proposition 3 then its subsets $M \cap \mathcal{Q}^\circ$ are unbounded, for each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$, motivates the following definition.

Definition 1. Let B be an element of the algebra $\mathcal{A}$ of subsets of Ω. A subset M of $\mathrm{ba}(\mathcal{A})$ is deep B-unbounded if each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$ verifies that

$$\sup\{|\mu(C)| : \mu \in M \cap \mathcal{Q}^\circ,\, C \in \mathcal{A},\, C \subset B\} = \infty, \tag{2}$$

or, equivalently, $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} = \infty$.

In particular, a subset M of $ba(\mathcal{A})$ is deep Ω-unbounded if $M \cap \mathcal{Q}^\circ$ is an unbounded subset of $ba(\mathcal{A})$, for each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$. Therefore an absolutely convex $\tau_s(\mathcal{A})$-closed subset M of $ba(\mathcal{A})$ is deep Ω-unbounded if and only if M verifies condition (2) or (3) in Proposition 3. If, additionally, $\overline{span\{M^\circ\}} = L(\mathcal{A})$ then M is deep Ω-unbounded if and only if it is unbounded.

Next proposition furnishes sequences of deep Ω-unbounded subsets of $ba(\mathcal{A})$. The particular case $\cup_m \mathcal{B}_m = \mathcal{A}$ is Theorem 1 in [15].

Proposition 4. *Let $\mathcal{A}$ be an algebra of subsets of Ω and let $(\mathcal{B}_m)_m$ be an increasing sequence of subsets of $\mathcal{A}$ such that each $\mathcal{B}_m$ does not have N-property and $\overline{span\{\chi_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$. There exists $n_0 \in \mathbb{N}$ such that for each $m \geqslant n_0$ there exists a deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset M_m of $ba(\mathcal{A})$ which is pointwise bounded in $\mathcal{B}_m$, i.e., $\sup\{|\mu(C)| : \mu \in M_m\} < \infty$ for each $C \in \mathcal{B}_m$. In particular this proposition holds if $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has N-property.*

Proof. If for each $m \in \mathbb{N}$ the subspace $H_m := \overline{\mathrm{span}\{\chi_C : C \in \mathcal{B}_m\}}$ has infinite codimension in $L(\mathcal{A})$ then, by (3) in Proposition 3, the polar set of $P_m := \overline{\mathrm{absco}\{\chi_C : C \in \mathcal{B}_m\}}$ is the deep Ω-unbounded set $M_m := P_m^\circ$. The definition of polar set implies that $\sup\{|\mu(C)| : \mu \in M_m\} \leqslant 1$, for each $C \in \mathcal{B}_m$. Whence we get the proposition with $n_0 = 1$.

If there exists p such that the codimension of $F := \overline{\mathrm{span}\{\chi_C : C \in \mathcal{B}_p\}}$ in $L(\mathcal{A}) = \mathrm{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\}$ is the finite positive number q then $\{\chi_C : C \in \cup_m \mathcal{B}_m\} \not\subset F$, whence there exists $m_1 \in \mathbb{N}$ and $D \in \mathcal{B}_{p+m_1}$ such that $\chi_D \notin F$ and then the codimension of $\overline{\mathrm{span}\{\chi_C : C \in \mathcal{B}_{p+m_1}\}}$ in $L(\mathcal{A})$ is less or equal than $q-1$. Therefore there exists n_0 such that $\overline{\mathrm{span}\{\chi_C : C \in \mathcal{B}_m\}} = L(\mathcal{A})$, for each $m \geqslant n_0$. As for each $m \geqslant n_0$ the set $\mathcal{B}_m$ does not have N-property there exists an absolutely convex $\tau_s(\mathcal{A})$-closed unbounded subset M_m of $\mathrm{ba}(\mathcal{A})$ such that $\sup\{|\mu(C)| : \mu \in M_m\} < k_C < \infty$, for each $C \in \mathcal{B}_m$, and then it follows that $\{k_C^{-1}\chi_C : C \in \mathcal{B}_m\} \subset M_m^\circ$. This inclusion implies that $\mathrm{span}\{\chi_C : C \in \mathcal{B}_m\} \subset \mathrm{span}\{M_m^\circ\}$, whence $\overline{\mathrm{span}\{M_m^\circ\}} = L(\mathcal{A})$, because $\overline{\mathrm{span}\{\chi_C : C \in \mathcal{B}_m\}} = L(\mathcal{A})$. Then, by Proposition 3, the unbounded set M_m is deep Ω-unbounded for each $m \geqslant n_0$.

If $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has N-property then $\overline{\mathrm{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$ and this proposition holds. $\square$

Next Proposition 5 follows from [15, Proposition 1]. We give a simplified proof according to our current notation.

Proposition 5. *Let B be an element of an algebra $\mathcal{A}$ and $\{C_1, C_2, \ldots, C_q\}$ a finite partition of B by elements of $\mathcal{A}$. If M is a deep B-unbounded subset of $ba(\mathcal{A})$ there exists C_i, $1 \leqslant i \leqslant q$, such that M is deep C_i-unbounded.*

Proof. If for each i, $1 \leqslant i \leqslant q$, there exists a finite set $\mathcal{Q}^i$ of characteristic functions of elements of $\mathcal{A}$ such that $\sup\{|\mu|(C_i) : \mu \in M \cap (\mathcal{Q}^i)^\circ\} < H_i$, $i \in \{1, 2, \ldots, q\}$, then we get the contradiction that the set $\mathcal{Q} = \cup_{1 \leqslant i \leqslant q} \mathcal{Q}^i$ verifies that $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} < \Sigma_{1 \leqslant i \leqslant q} H_i$. □

If $t = (t_1, t_2, \ldots, t_p)$, $s = (s_1, s_2, \ldots, s_q)$, T and U are two elements and two subsets of $\cup_s \mathbb{N}^s$ we define $t(i) := (t_1, t_2, \ldots, t_i)$ if $1 \leqslant i \leqslant p$, $t(i) := \emptyset$ if $i > p$, $T(m) := \{t(m) : t \in T\}$, for each $m \in \mathbb{N}$, $t \times s := (t_1, t_2, \ldots, t_p, t_{p+1}, t_{p+2}, \ldots, t_{p+q})$, with $t_{p+j} := s_j$, for $1 \leqslant j \leqslant q$, and $T \times U := \{t \times u : t \in T, \ u \in U\}$. We simplify (t_1), (n) and $T \times \{(n)\}$ by t_1, n and $T \times n$. The *length of* $t = (t_1, t_2, \ldots, t_p)$ *is* p and the cardinal of a set C is denoted by $|C|$.

If $v \in \cup_s \mathbb{N}^s$ and $t \times v \in U$ then $t \times v$ *is an extension of* t *in* U. A sequence $(t^n)_n$ of elements $t^n = (t_1^n, t_2^n, \ldots, t_n^n, \ldots) \in \cup_s \mathbb{N}^s$ is an *infinite chain* if for each $n \in \mathbb{N}$ the element t^{n+1} is an extension of the section $t^n(n)$, i.e., $\emptyset \neq t^n(n) = t^{n+1}(n)$.

A subset U of $\cup_n \mathbb{N}^n$ is *increasing* at $t = (t_1, t_2, \ldots, t_p) \in \cup_s \mathbb{N}^s$ if U contains p elements $t^1 = (t_1^1, t_2^1, \ldots)$ and $t^i = (t_1, t_2, \ldots, t_{i-1}, t_i^i, t_{i+1}^i, \ldots)$, $1 < i \leqslant p$, such that $t_i < t_i^i$, for each $1 \leqslant i \leqslant p$. *A non-void subset U of $\cup_s \mathbb{N}^s$ is increasing (increasing respect to a subset V of $\cup_s \mathbb{N}^s$) if U is increasing at each $t \in U$ (at each $t \in V$)*, hence U is increasing if $|U(1)| = \infty$ and $|\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty$, for each $t = (t_1, t_2, \ldots, t_p) \in U$ and $1 \leqslant i < p$.

If $\{B_u : u \in \cup_s \mathbb{N}^s\}$ is an increasing web in A and U is an increasing subset of $\cup_{s \in \mathbb{N}} \mathbb{N}^s$ then $\mathcal{B} := \{B_{u(i)} : u \in U, \ 1 \leqslant i \leqslant \text{length } u\}$ verifies that $(B_{u(1)})_{u \in U}$ is an increasing covering of A and for each $u = (u_1, u_2, \ldots, u_p) \in U$ and each $i < p$ the sequence $(B_{u(i) \times n})_{u(i) \times n \in U(i+1)}$ is an increasing covering of $B_{u(i)}$. If, additionally, each element $u \in U$ has an extension in U then renumbering the indexes in the elements of $\mathcal{B}$ we get an increasing web.

The Definition 2 deals with increasing subsets of $\cup_{s \in \mathbb{N}} \mathbb{N}^s$ and it is motivated by the technical Example 1 which will be used onwards to complete the proof of Theorem 2. A particular class of increasing trees, named NV-trees – surely reminding Nikodym and Valdivia –, is considered in [9, Definition 1].

Definition 2. An increasing tree T is an increasing subset of $\cup_{s \in \mathbb{N}} \mathbb{N}^s$ without infinite chains.

An increasing tree T is *trivial* if $T = T(1)$; then T is an infinite subset of $\mathbb{N}$. The sets $\mathbb{N}^i$, $i \in \mathbb{N} \backslash \{1\}$, and the set $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are nontrivial increasing trees.

An increasing subset S of an increasing tree T is an increasing tree. From this observation it follows the Claim 6.

Claim 6. *If $(S_n)_n$ is a sequence of non-void subsets of an increasing tree T such that for each $n \in \mathbb{N}$ the set S_{n+1} is increasing respect to S_n, then $S := \cup_n S_n$ is an increasing tree.*

Proof. It is enough to notice that S is an increasing subset of T. □

Example 1. Let $\mathcal{B} := \{\mathcal{B}_{m_1, m_2, \ldots, m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in an algebra $\mathcal{A}$ of subsets of Ω with the property that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $q \in \mathbb{N}$ such that $\mathcal{B}_{m_1, m_2, \ldots, m_q}$ does not have sN-property. Then there exists an increasing web $\mathcal{C} := \{\mathcal{C}_{m_1, m_2, \ldots, m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ in $\mathcal{A}$

and an increasing tree T such that for each $(t_1, t_2, \ldots, t_p) \in T$ there exists a deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_{t_1,t_2,\ldots,t_p}$ of $ba(\mathcal{A})$ which is pointwise bounded in $\mathcal{C}_{t_1,t_2,\ldots,t_p}$, i.e.,

$$\sup\{|\mu(C)| : \mu \in M_{t_1,t_2,\ldots,t_p}\} < \infty, \tag{3}$$

for each $C \in \mathcal{C}_{t_1,t_2,\ldots,t_p}$.

Proof. If each $\mathcal{B}_{m_1}$, $m_1 \in \mathbb{N}$, does not have N-property then the example is given by $\mathcal{C} := \mathcal{B}$ and $T := \mathbb{N} \backslash \{1, 2, \ldots, n_0 - 1\}$, where n_0 is the natural number obtained in Proposition 4 applied to the increasing covering $(\mathcal{B}_{m_1})_{m_1}$ of $\mathcal{A}$. Hence we may suppose that there exists $m_1 \in \mathbb{N}$ such that $\mathcal{B}_{t_1}$ has N-property for each $t_1 \geqslant m_1$ and then:

(i_1) Either $\mathcal{B}_{t_1}$ does not have sN-property for each $t_1 \in \mathbb{N}$ and the inductive process finish defining $T_0 := \{t_1 \in \mathbb{N} : t_1 \geqslant m_1\}$.

(ii_1) Or there exists $m'_1 \in \mathbb{N}$ such that $\mathcal{B}_{t_1}$ has sN-property for each $t_1 \geqslant m'_1$. Then we write $Q_1 := \emptyset$ and $Q'_1 := \{t_1 \in \mathbb{N} : t_1 \geqslant m'_1\}$.

Let us assume that for each j, with $2 \leqslant j \leqslant i$, we have obtained by induction two disjoint subsets Q_j and Q'_j of $\mathbb{N}^j$ such that each $t = (t_1, t_2, \ldots, t_j) \in Q_j \cup Q'_j$ verifies:

1. $t(j-1) = (t_1, t_2, \ldots, t_{j-1}) \in Q'_{j-1}$.
2. If $t \in Q_j$ the set $\mathcal{B}_t$ has N-property but it does not have sN-property and $S_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is a cofinite subset of $\mathbb{N}$ such that $t(j-1) \times S_{t(j-1)} \subset Q_j$.
3. If $t \in Q'_j$ the set $\mathcal{B}_t$ has sN-property and $S'_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is a cofinite subset of $\mathbb{N}$ such that $t(j-1) \times S'_{t(j-1)} \subset Q'_j$.

If $t := (t_1, t_2, \ldots, t_i) \in Q'_i$ then $\mathcal{B}_{t_1,t_2,\ldots,t_i}$ has sN-property and $(\mathcal{B}_{t_1,t_2,\ldots,t_i,n})_n$ is an increasing covering of $\mathcal{B}_{t_1,t_2,\ldots,t_i}$, hence there exists m_{i+1} such that $\mathcal{B}_{t_1,t_2,\ldots,t_i,n}$ has N-property for each $n \geqslant m_{i+1}$. Then we may have two possible cases:

(i_{i+1}) Either $\mathcal{B}_{t_1,t_2,\ldots,t_i,n}$ does not have sN-property for each $n \in \mathbb{N}$ and we define $S_{t_1,t_2,\ldots,t_i} := \{n \in \mathbb{N} : m_{i+1} \leqslant n\}$ and $S'_{t_1,t_2,\ldots,t_i} := \emptyset$,

(ii_{i+1}) or there exists $m'_{i+1} \in \mathbb{N}$ such that $\mathcal{B}_{t_1,t_2,\ldots,t_i,n}$ has sN-property for each $n \geqslant m'_{i+1}$. In this case let $S_{t_1,t_2,\ldots,t_i} := \emptyset$ and $S'_{t_1,t_2,\ldots,t_i} := \{n \in \mathbb{N} : m'_{i+1} \leqslant n\}$.

We finish this induction procedure by setting $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$. By construction Q_{i+1} and Q'_{i+1} verify the properties 1., 2. and 3. with $j = i + 1$.

The fact that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $j \in \mathbb{N}$ such that $\mathcal{B}_{m_1,m_2,\ldots,m_j}$ does not have sN-property imply that $T_0 := \cup\{Q_i : i \in \mathbb{N}\}$ does not contain infinite chains, because if $(m_1, m_2, \ldots, m_p) \in Q_p$ then $\mathcal{B}_{m_1,m_2,\ldots,m_{p-1}}$ has sN-property, whence for each $(t_1, t_2, \ldots, t_k) \in Q'_k$ there exists $q \in \mathbb{N}$ and $(t_{k+1}, \ldots, t_{k+q}) \in \mathbb{N}^q$ such that $(t_1, t_2, \ldots, t_k, t_{k+1}, \ldots, t_{k+q}) \in Q_{k+q}$ and then $T_0(k) = Q_k \cup Q'_k$, for each $k \in \mathbb{N}$. These equalities imply that T_0 is increasing, because $|T_0(1)| = |Q'_1| = \infty$ and if $t = (t_1, t_2, \ldots, t_p) \in T_0$ the sets $S'_{t(i-1)}$, $1 < i < p$, and $S_{t(p-1)}$ are cofinite subsets of $\mathbb{N}$.

This increasing tree T_0 as well as the trivial increasing tree obtained in (i_1), also named T_0, verify that for each $t = (t_1, t_2, \ldots, t_p) \in T_0$ the family $\mathcal{B}_{t_1,t_2,\ldots,t_p}$ has N-property and it does not have sN-property, whence $\mathcal{B}_{t_1,t_2,\ldots,t_p}$ has an increasing covering $(\mathcal{B}'_{t_1,t_2,\ldots,t_p,n})_n$ such that each $\mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$ does not have N-property. By Proposition 4 there exists $n_0 \in \mathbb{N}$ such that for each $n \geqslant n_0$ there exists a deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_{t_1,t_2,\ldots,t_p,n}$ of $\mathrm{ba}(\mathcal{A})$ which is $\mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$ pointwise bounded, i.e., $\sup\{|\mu(C)|$:

$\mu \in M_{t_1,t_2,\ldots,t_p,n}\} < \infty$, for each $C \in \mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$. We assume $n_0 = 1$, removing $\mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$ when $n < n_0$ and changing n by $n - n_0 + 1$.

Then we get the example with the increasing tree $T := T_0 \times \mathbb{N}$ and with the increasing web $\mathcal{C} := \{\mathcal{C}_t : t \in \cup_s \mathbb{N}^s\}$ in the algebra $\mathcal{A}$ such that for each $t = (t_1, t_2, \ldots, t_p) \in \cup_s \mathbb{N}^s$ either $\mathcal{C}_t := \mathcal{B}'_{t(i)}$ if $i \leqslant p$ and $t(i) \in T$ or $\mathcal{C}_t := \mathcal{B}_t$ if $\{t(i) : 1 \leqslant i \leqslant p\} \cap T = \emptyset$. $\square$

Let U be a subset of $\cup_s \mathbb{N}^s$. *An element $t \in \cup_s \mathbb{N}^s$ admits increasing extension in U if the set of $\{v \in \cup_s \mathbb{N}^s : t \times v \in U\}$ contains an increasing subset.* We need the following obvious properties (a), (b_1) and (b_2) to prove Proposition 7, stating that if a subset U of an increasing tree T does not contain an increasing tree then $T \backslash U$ contains an increasing tree.

(a) If U is a subset of $\cup_s \mathbb{N}^s$ and U does not contain an increasing tree then there exists $m_1 \in \mathbb{N}$ such that each $n \in \mathbb{N} \backslash \{1, 2, \ldots, m_1\}$ does not admit increasing extension in U.

(b) Let $t \in \cup_s \mathbb{N}^s$ and let U be a subset of the increasing tree T. Suppose that t does not admit increasing extension in U and that $T_t := \{v \in \cup_s \mathbb{N}^s : t \times v \in T\} \neq \emptyset$. Then

(b_1) if the increasing tree T_t is trivial there exists $m_{i+1} \in \mathbb{N}$ such that the set

$$(t \times \{\mathbb{N} \backslash \{1, 2, \ldots, m_{i+1}\}) \cap T$$

is an infinite subset of $T \backslash U$,

(b_2) if T_t is non-trivial there exists $m'_{i+1} \in \mathbb{N}$ such that each element of

$$\left(t \times \{\mathbb{N} \backslash \{1, 2, \ldots, m'_{i+1}\}\right) \cap T(i+1)$$

does not admit increasing extension in U.

Proposition 7. *Let U be a subset of an increasing tree T. If U does not contain an increasing tree then $T \backslash U$ contains an increasing tree.*

Proof. It is enough to prove that $T \backslash U$ contains an increasing subset W. Now we follow the scheme of the proof in Example 1. In fact, if T is a trivial increasing tree the proposition is obvious. Hence we may suppose that T is a non-trivial increasing tree. Then we define $Q_1 := \emptyset$ and by (a) there exists $m'_1 \in \mathbb{N}$ such that each element of the set $Q'_1 := \{n \in T(1) : m'_1 \leqslant n\}$ does not admit increasing extension in U. Notice that $Q'_1 \subset T(1) \backslash T$.

Let us suppose that we have obtained for each j, with $2 \leqslant j \leqslant i$, two disjoint subsets Q_j and Q'_j such that $Q_j \subset T(j) \cap (T \backslash U)$, $Q'_j \subset T(j) \backslash T$ and each $t \in Q_j \cup Q'_j$ verifies the following properties:

1. $t(j-1) \in Q'_{j-1}$.
2. If $t \in Q_j$ then the cardinal of $S_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is infinite and $t(j-1) \times S_{t(j-1)} \subset Q_j$.
3. If $t \in Q'_j$ then t does not admit increasing extension in U, the cardinal of $S'_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is infinite and $t(j-1) \times S'_{t(j-1)} \subset Q'_j$.

If $t \in Q'_i$ then $t \in T(i) \backslash T$ and it does not admit increasing extension in U. If $T_t = \{v \in \cup_s \mathbb{N}^s : t \times v \in T\}$ then, by (b_1) and (b_2), it follows that the following two cases may happen:

i. If T_t is trivial then there exists $m_{i+1} \in \mathbb{N}$ such that the infinite set $S_t := \{n \in \mathbb{N} : m_{i+1} \leqslant n, t \times n \in T(i+1)\}$ verifies that $t \times S_t \subset T \backslash U$ and we define $S'_t := \emptyset$.

ii. If T_t is non-trivial then there exists $m'_{i+1} \in \mathbb{N}$ such that the infinite set $S'_t := \{n \in \mathbb{N} : m'_{i+1} < n, t \times n \in T(i+1)\}$ verifies that $t \times S'_t \subset T(i+1)\backslash T$ and each element of $t \times S'_t$ does not admit increasing extension in U. Now we define $S_t := \emptyset$.

We finish this induction procedure by setting $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$.

By construction $Q_{i+1} \subset T(i+1) \cap (T\backslash U)$, $Q'_{i+1} \subset T(i+1)\backslash T$, and each $t \in Q_{i+1} \cup Q'_{i+1}$ verifies the properties 1., 2. and 3. changing j by $i+1$.

As T does not contain infinite chains we deduce from 1. that for each $(t_1, t_2, \dots, t_i) \in Q'_i$ there exists $q \in \mathbb{N}$ and $(t_{i+1}, \dots, t_{i+q}) \in \mathbb{N}^q$ such that $(t_1, t_2, \dots, t_i, t_{i+1}, \dots, t_{i+q}) \in Q_{i+q}$. Whence, for each $i \in \mathbb{N}$, $(\cup_{j>i} Q_j)(i) = Q'_i$ and then $W := \cup\{Q_j : j \in \mathbb{N}\}$ is a subset of $T\backslash U$.

W has the increasing property because from $W(k) = Q_k \cup Q'_k$, for each $k \in \mathbb{N}$, it follows that $|W(1)| = |Q'_1| = \infty$ and if $t = (t_1, t_2, \dots, t_p) \in W$ then $(t_1, t_2, \dots, t_i) \in Q'_i$, if $1 < i < p$, and $(t_1, t_2, \dots, t_p) \in Q_p$, hence the infinite subsets $S'_{t(i-1)}$ and $S_{t(p-1)}$ of $\mathbb{N}$ verify that $t(i-1) \times S'_{t(i-1)} \subset Q'_i \subset W(i)$ and $t(p-1) \times S_{t(p-1)} \subset Q_p \subset W$. □

Next Proposition 8 follows from [15, Propositions 2 and 3] and we give a simplified proof according to our current notation for the sake of completeness.

Proposition 8. *Let $\{B, Q_1, \dots, Q_r\}$ be a subset of the algebra $\mathcal{A}$ of subsets of Ω and let M be a deep B-unbounded absolutely convex subset of $ba(\mathcal{A})$. Then given a positive real number α and a natural number $q > 1$ there exists a finite partition $\{C_1, C_2, \dots, C_q\}$ of B by elements of $\mathcal{A}$ and a subset $\{\mu_1, \mu_2, \dots, \mu_q\}$ of M such that $|\mu_i(C_i)| > \alpha$ and $\Sigma_{1 \leqslant j \leqslant r} |\mu_i(Q_j)| \leqslant 1$, for $i = 1, 2, \dots, q$.*

Proof. Let $\mathcal{Q} = \{\chi_B, \chi_{Q_1}, \chi_{Q_2}, \dots, \chi_{Q_r}\}$. The deep B-unboundedness of M and the inclusion $M \subset rM$ imply that

$$\sup\{|\mu(D)| : \mu \in rM \cap \mathcal{Q}^\circ,\ D \subset B,\ D \in \mathcal{A}\} = \infty.$$

Hence there exists $P_1 \subset B$, with $P_1 \in \mathcal{A}$, and $\mu \in rM \cap \mathcal{Q}^\circ$ such that $|\mu(P_1)| > r(1+\alpha)$. Clearly $\mu_1 = r^{-1}\mu \in M$, $|\mu_1(P_1)| > 1 + \alpha$ and $|\mu_1(f)| = r^{-1}|\mu(f)| \leqslant r^{-1}$ for each $f \in \mathcal{Q}$, hence $|\mu_1(B)| \leqslant r^{-1} \leqslant 1$ and $\Sigma_{1 \leqslant j \leqslant r} |\mu_1(Q_j)| \leqslant r^{-1} r = 1$. The set $P_2 := B\backslash P_1$ verifies that

$$|\mu_1(P_2)| \geqslant |\mu_1(P_1)| - |\mu_1(B)| > 1 + \alpha - 1 = \alpha.$$

From Proposition 5 there exists $i \in \{1, 2\}$ such that M is deep P_i-unbounded. To finish the first step of the proof let $C_1 := P_1$ if M is deep P_2-unbounded and let $C_1 := P_2$ if M is deep P_1-unbounded. Then M is deep $B\backslash C_1$-unbounded.

Apply the same argument in $B\backslash C_1$ to obtain a measurable set $C_2 \subset B\backslash C_1$ and a measure $\mu_2 \in M$ such that $|\mu_2(C_2)| > \alpha$, $|\mu_2(B\backslash(C_1 \cup C_2))| > \alpha$ and $\Sigma\{|\mu_2(Q_j)| : 1 \leqslant j \leqslant r\} \leqslant 1$, being M deep $B\backslash(C_1 \cup C_2)$-unbounded. Hence the proof is provided by applying $q-1$ times this argument. In the last step we define $\mu_q := \mu_{q-1}$ and $C_q = B\backslash(C_1 \cup \dots \cup C_{q-1})$. □

Proposition 9. *Let B be an element of an algebra $\mathcal{A}$ and $\{M_t : t \in T\}$ a family of deep B-unbounded subsets of $ba(\mathcal{A})$ indexed by an increasing tree T. If $t^j := (t^j_1, t^j_2, \dots, t^j_{p_j}) \in T$, for each $1 \leqslant j \leqslant k$, and $q = 2 + \Sigma\{p_j : 1 \leqslant j \leqslant k\}$ then for each finite partition $\{C_1, C_2, \dots, C_q\}$ of B by elements of $\mathcal{A}$ there exists $h \in \{1, 2, \cdots, q\}$ and an increasing tree T_1 such that $\{t^1, t^2, \dots, t^k\} \subset T_1 \subset T$ and $\{M_t : t \in T_1\}$ is a family of deep $B\backslash C_h$-unbounded subsets.*

Proof. Let $\{C_1, C_2, \dots, C_q\}$ be a finite partition of B by elements of $\mathcal{A}$ with $q = 2 + \Sigma\{p_j : 1 \leqslant j \leqslant k\}$. From Proposition 5 it follows that if $\{M_u : u \in U\}$ is a family of deep B-unbounded subsets of $\mathrm{ba}(\mathcal{A})$ indexed by an increasing tree U and $V_i := \{u \in U : M_u$ is deep C_i-unbounded$\}$, $1 \leqslant i \leqslant q$, then $U = \cup_{1\leqslant i\leqslant q} V_i$ and, by Proposition 7, there exists l, with $1 \leqslant l \leqslant q$, such that V_l contains an increasing tree U_l. Therefore

(a) If $\{M_u : u \in U\}$ is a family of deep B-unbounded subsets indexed by an increasing tree U there exists $l \in \{1, 2, \ldots, q\}$ and an increasing tree U_l contained in U such that $\{M_u : u \in U_l\}$ is a family of deep C_l-unbounded subsets.

In particular, for the increasing tree T and for each element $t^j \in T$, with $1 \leqslant j \leqslant k$, there exist by (a) and Proposition 5:

(1) $i_0 \in \{1, 2, \ldots, q\}$ and an increasing tree T_{i_0} contained in T such that $\{M_t : t \in T_{i_0}\}$ is a family of deep C_{i_0}-unbounded subsets,
(2) $i^j \in \{1, 2, \ldots, q\}$ such that M_{t^j} is deep C_{i^j}-unbounded.

Let $S := \{j : 1 \leqslant j \leqslant k,\ t^j \notin T_{i_0}\}$. For each $j \in S$ and each section $t^j(m-1)$ of $t^j = (t^j_1, t^j_2, \ldots, t^j_{p_j})$, with $2 \leqslant m \leqslant p_j$, the set $W^j_m := \{v \in \cup_s \mathbb{N}^s : t^j(m-1) \times v \in T\}$ is an increasing tree such that $\{M_{(t^j_1, t^j_2, \ldots, t^j_{m-1})\times w} : w \in W^j_m\}$ is a family of deep B-unbounded subsets. By (a) there exists:

(3) $i^j_m \in \{1, 2, \ldots, q\}$ and an increasing tree V^j_m contained in W^j_m such that

$$\{M_{(t^j_1, t^j_2, \ldots, t^j_{m-1})\times v} : v \in V^j_m\}$$

is a family of deep $C_{i^j_m}$-unbounded subsets. Clearly $(t^j_1, t^j_2, \ldots, t^j_{m-1}) \times V^j_m \subset T$.

As the number of sets C_{i_0}, C_{i^j}, $C_{i^j_m}$, with $j \in S$ and $2 \leqslant m \leqslant p_j$, is less or equal than $q-1$, there exists $h \in \{1, 2, \cdots, q\}$ such that

$$D := C_{i_0} \cup (\cup\{C_{i^j} \cup C_{i^j_m} : j \in S,\ 2 \leqslant m \leqslant p_j\}) \subset B \backslash C_h.$$

Let T_1 be the union of the sets T_{i_0}, $\{t^j : j \in S\}$ and $\{(t^j_1, t^j_2, \ldots, t^j_{m-1})\} \times V^j_m$, with $j \in S$ and $2 \leqslant m \leqslant p_j$. Clearly for each $t \in T_1$ the set M_t is deep D-unbounded, whence M_t is also deep $B \backslash C_h$-unbounded. By construction $\{t^1, t^2, \ldots, t^k\} \subset T_1$ and T_1 has the increasing property and it is a subset of the increasing tree T. Whence T_1 is an increasing tree. □

We finish this section with a combination of Propositions 8 and 9. The obtained Proposition 10 is a fundamental tool for the next section.

Proposition 10. *Let $\{B, Q_1, \ldots, Q_r\}$ be a subset of an algebra $\mathcal{A}$ of subsets of Ω, and let $\{M_t : t \in T\}$ be a family of deep B-unbounded absolutely convex subsets of $ba(\mathcal{A})$, indexed by an increasing tree T. Then for each positive real number α and each finite subset $\{t^j : 1 \leqslant j \leqslant k\}$ of T there exist $\{B_j \in \mathcal{A} : 1 \leqslant j \leqslant k\}$, formed by k pairwise disjoint subsets B_j of B, $1 \leqslant j \leqslant k$, a set $\{\mu_j \in M_{t^j},\ 1 \leqslant j \leqslant k\}$ and an increasing tree T^* such that:*

1. $|\mu_j(B_j)| > \alpha$ *and* $\Sigma\{|\mu_j(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$, *for* $j = 1, 2, \ldots, k$,
2. $\{t^j : 1 \leqslant j \leqslant k\} \subset T^* \subset T$ *and* $\{M_t : t \in T^*\}$ *is a family of deep* $(B \backslash \cup_{1\leqslant j\leqslant k} B_j)$*-unbounded sets.*

Proof. Let $t^j := (t^j_1, t^j_2, \ldots, t^j_{p_j})$, for $1 \leqslant j \leqslant k$. By Proposition 8 applied to B, α, $q := 2 + \Sigma_{1\leqslant j\leqslant k} p_j$ and M_{t^1} there exist a partition $\{C^1_1, C^1_2, \ldots, C^1_q\}$ of B by elements of $\mathcal{A}$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_q\} \subset M_{t^1}$ such that:

$$\left|\lambda_k(C_k^1)\right| > \alpha \quad \text{and} \quad \Sigma_{1\leqslant i\leqslant r}\left|\lambda_k(Q_i)\right| \leqslant 1, \quad \text{for } k=1,2,\ldots,q, \tag{4}$$

hence Proposition 9 applied to the sets $\{C_1^1, C_2^1, \cdots, C_q^1\}$, $\{M_t : t \in T\}$ and $\{t^j : 1 \leqslant j \leqslant k\}$ gives $h \in \{1,2,\cdots,q\}$ and a family $\{M_t : t \in T_1\}$ of deep $B\backslash C_h^1$-unbounded subsets indexed by an increasing tree T_1 such that $\{t^1, t^2, \ldots, t^k\} \subset T_1 \subset T$. If $B_1 := C_h^1$ and $\mu_1 := \lambda_h$ then (4) holds with $\lambda_k = \mu_1$ and $C_k^1 = B_1$. Clearly $\{M_t : t \in T_1\}$ is a family of deep $B\backslash B_1$-unbounded subsets.

If we apply again Proposition 8 to $B\backslash B_1$, α, q and M_{t^2} we obtain a partition $\{C_1^2, C_2^2, \cdots, C_q^2\}$ of $B\backslash B_1$ by elements of $\mathcal{A}$ and $\{\zeta_1, \zeta_2, \cdots, \zeta_q\} \subset M_{t^2}$ such that

$$\left|\zeta_k(C_k^2)\right| > \alpha \quad \text{and} \quad \Sigma_{1\leqslant i\leqslant r}\left|\zeta_k(Q_i)\right| \leqslant 1, \quad \text{for } k=1,2,\ldots,q,$$

and then by Proposition 9 (applied to $\{C_1^2, C_2^2, \cdots, C_q^2\}$, $\{M_t : t \in T_1\}$ and $\{t^j : 1 \leqslant j \leqslant k\}$) there exists $l \in \{1,2,\cdots,q\}$ and a family $\{M_t : t \in T_2\}$ of deep $(B\backslash B_1)\backslash C_l^2$-unbounded subsets indexed by an increasing tree T_2 such that $\{t^1, t^2, \ldots, t^k\} \subset T_2 \subset T$. Now if $B_2 := C_l^2$ and $\mu_2 := \zeta_l$ then $|\mu_2(B_2)| > \alpha$, $\Sigma\{|\mu_2(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$ and $\{M_t : t \in T_2\}$ is a family of deep $B\backslash(B_1 \cup B_2)$-unbounded subsets. With $k-2$ new repetitions of this procedure we get the proof with $T^* := T_k$. □

3. Proof of Theorem 2

With a induction procedure based in Proposition 10 we obtain Proposition 12 that together with the next elementary covering property for families indexed by increasing trees enable to prove Theorem 2.

Proposition 11. *If $\mathcal{Y} = \{Y_{m_1,m_2,\ldots,m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ is an increasing web in Y and T is an increasing tree then $Y = \cup\{Y_y : y \in T\}$.*

Proof. Let us suppose that $y \in Y\backslash(\cup\{Y_t : t \in T\})$. As $\mathcal{Y}$ is an increasing web and T is an increasing tree then $Y = \cup\{Y_{t(1)} : t \in T\}$, whence there exists $u^1 = (u_1^1, u_2^1, \ldots) \in T$ such that

$$y \in Y_{u_1^1}\backslash(\cup\{Y_t : t \in T\}).$$

Assume that there exists $\{u^2, u^3, \ldots, u^n\} \subset T$ such that $\emptyset \neq u^{j-1}(j-1) = u^j(j-1)$ and $y \in Y_{u^j(j)}\backslash \cup\{Y_t : t \in T\}$, for $2 \leqslant j \leqslant n$. Then $y \in Y_{u^n(n)}\backslash \cup\{Y_t : t \in T\}$, with $u^n(n) = (u_1^n, u_2^n, \ldots, u_n^n)$. As $\mathcal{Y}$ is an increasing web and T is an increasing tree then $Y_{u^n(n)} = \cup\{Y_{u^n(n)\times s} : u^n(n) \times s \in T(n+1)\}$, hence there exists $u^{n+1} \in T$ such that $u^n(n) = u^{n+1}(n)$ and

$$y \in Y_{u^{n+1}(n+1)}\backslash(\cup\{Y_t : t \in T\}).$$

This induction procedure gives the contradiction that T contains the infinite chain $(u^n)_n$. Therefore $Y = \cup\{Y_u : u \in T\}$. □

In Proposition 12 we refer to the sequence $(i_n)_n = (1,1,2,1,2,3,\ldots)$, obtained with the first components of $\mathbb{N}^2$ ordered by the diagonal order, i.e., $i_n = n - 2^{-1}h(h+1)$, if $n \in \left]\, 2^{-1}h(h+1), 2^{-1}(h+1)(h+2)\,\right]$ and $h = 0,1,2,\ldots$. Let us note that $i_n \leqslant n$, for each $n \in \mathbb{N}$.

Proposition 12. *Let $\{\mathcal{B}_{m_1,m_2,\ldots,m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in a σ-algebra $\mathcal{S}$ of subsets of Ω with the property that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $h \in \mathbb{N}$ such that $\mathcal{B}_{m_1,m_2,\ldots,m_h}$ does not have sN-property and let $(i_n)_n = (1,1,2,1,2,3,\ldots)$. Then there exist a strictly increasing sequence $(j_n)_n$ in $\mathbb{N}$, a sequence $(B_{i_n j_n})_n$ of pairwise disjoints elements of $\mathcal{S}$, a sequence $(\mu_{i_n j_n})_n$ in $ba(\mathcal{S})$ and a covering $(\mathcal{C}_r)_r$ of $\mathcal{S}$ such that for each $n \in \mathbb{N}$*

$$\Sigma_s\{\left|\mu_{i_{n+1}j_{n+1}}(B_{i_sj_s})\right| : 1 \leqslant s \leqslant n\} < 1, \tag{5}$$

$$|\mu_{i_nj_n}(B_{i_nj_n})| > j_n, \tag{6}$$

$$|\mu_{i_nj_n}(\cup_s\{B_{i_sj_s} : n < s\})| < 1, \tag{7}$$

and for each $r \in \mathbb{N}$ and each strictly increasing sequence $(n_p)_p$ such that $i_{n_p} = r$, for each $p \in \mathbb{N}$, the set $\{\mu_{i_{n_p}j_{n_p}} : p \in \mathbb{N}\}$ is $\mathcal{C}_r$-pointwise bounded, i.e., for each $H \in \mathcal{C}_r$ we have that

$$\sup\{\left|\mu_{i_{n_p}j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty. \tag{8}$$

Proof. Let $\{\mathcal{C}_t : t \in \cup_s\mathbb{N}^s\}$ and T be the increasing web in $\mathcal{S}$ and the increasing tree determined in Example 1 such that for each $t \in T$ there exists a deep Ω-unbounded $\tau_s(\mathcal{S})$-closed absolutely convex subset M_t of $\mathrm{ba}(\mathcal{S})$ which is $\mathcal{C}_t$-pointwise bounded, i.e.,

$$\sup\{|\mu(H)| : \mu \in M_t\} < \infty \tag{9}$$

for each $H \in \mathcal{C}_t$.

Then, by induction, we prove that there exist a countable increasing tree $\{t^i : i \in \mathbb{N}\}$ contained in T, a strictly increasing sequence of natural numbers $(k_j)_j$, a set $\{B_{ij} : (i,j) \in \mathbb{N}^2, i \leqslant k_j\}$ of pairwise disjoint elements of $\mathcal{S}$ and a set $\{\mu_{ij} \in M_{t^i} : (i,j) \in \mathbb{N}^2, i \leqslant k_j\}$ such that if $(i,j) \in \mathbb{N}^2$ and $i \leqslant k_j$ then

$$\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v,\ 1 \leqslant v < j\} < 1, \tag{10}$$

$$|\mu_{ij}(B_{ij})| > j, \tag{11}$$

and for each $i \in \mathbb{N}$ and each $H \in \mathcal{C}_{t^i}$ we have

$$\sup_j\{|\mu_{ij}(H)| : i \leqslant j\} < \infty. \tag{12}$$

Fix $t^1 \in T$. By Proposition 10 with $B := \Omega$, $\alpha = 1$, $\{Q_1, \dots, Q_r\} := \emptyset$ and $\{t^i : 1 \leqslant i \leqslant k\} := \{t^1\}$ there exist $B_{11} \in \mathcal{S}$, $\mu_{11} \in M_{t^1}$ and an increasing tree T_1 such that

1. $|\mu_{11}(B_{11})| > 1$, $\{M_t : t \in T_1\}$ is a family of deep $\Omega\backslash B_{11}$-unbounded subsets and
2. $t^1 \in T_1 \subset T$.

We define $k_1 := 1$, $S^1 := \{t^1\}$ and $B^1 := B_{11}$.

Suppose that in the following $n-1$ steps of the inductive process we have obtained the finite sequence $k_2 < k_3 < \cdots < k_n$ in $\mathbb{N}\backslash\{1\}$, the increasing trees $T_2 \supset T_3 \supset \cdots \supset T_n$ contained in T_1, the subset $\{t^1, t^2, \dots, t^{k_n}\}$ of T_n, the set $\{B_{ij} : i \leqslant k_j,\ j \leqslant n\}$ formed by pairwise disjoint elements of $\mathcal{S}$ and the set $\{\mu_{ij} \in M_{t^i} : i \leqslant k_j,\ j \leqslant n\}$ such that, for each $1 < j \leqslant n$ and each $i \leqslant k_j$:

1. $|\mu_{ij}(B_{ij})| > j$, $\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v, 1 \leqslant v < j\} < 1$, the union $B^j := \cup\{B_{sv} : s \leqslant k_v,\ 1 \leqslant v \leqslant j\}$ verifies that $\{M_t : t \in T_j\}$ is a family of deep $\Omega\backslash B^j$-unbounded subsets,
2. $S^j := \{t^i : i \leqslant k_j\} \subset T_j$ and S^j has the increasing property respect to S^{j-1}.

To finish the induction procedure let $\{t^{k_n+1}, \dots, t^{k_{n+1}}\}$ be a subset of $T_n\backslash\{t^i : i \leqslant k_n\}$ that verifies the increasing property with respect to S^n. Then applying Proposition 10 to $\Omega\backslash B^n$, $\{B_{sv} : s \leqslant k_v, 1 \leqslant v \leqslant n\}$, T_n, the finite subset $S^{n+1} := \{t^i : i \leqslant k_{n+1}\}$ of T_n and $n+1$ we obtain a family $\{B_{in+1} : i \leqslant k_{n+1}\}$ of pairwise disjoint elements of $\mathcal{S}$ contained in $\Omega\backslash B^n$, a subset $\{\mu_{in+1} \in M_{t^i} : i \leqslant k_{n+1}\}$ of $\mathrm{ba}(\mathcal{S})$ and an increasing tree T_{n+1} contained in T_n such that for each $i \leqslant k_{n+1}$,

1. $|\mu_{in+1}(B_{in+1})| > n+1$, $\Sigma_{s,v}\{|\mu_{in+1}(B_{sv})| : s \leqslant k_v,\ 1 \leqslant v \leqslant n\} < 1$, the union $B^{n+1} := \cup\{B_{sv} : s \leqslant k_s,\ 1 \leqslant v \leqslant n+1\}$ has the property that $\{M_t : t \in T_{n+1}\}$ is a family of deep $\Omega \backslash B^{n+1}$-unbounded subsets,
2. $S^{n+1} \subset T_{n+1}$ and S^{n+1} has the increasing property respect to S^n.

By Claim 6, $\cup_n S_n = \{t^i : i \in \mathbb{N}\}$ is an increasing tree, whence, by Proposition 11, the sequence $(\mathcal{C}_{t^i})_i$ is a countable covering of the σ-algebra $\mathcal{S}$. As $(k_j)_j$ is increasing then $(i,j) \in \mathbb{N}^2$ and $i \leqslant j$ imply that $i \leqslant k_j$, whence $\{\mu_{ij} : j \in \mathbb{N}\backslash\{1,2,\ldots,i-1\}\} \subset M_{t^i}$ and from this inclusion and (9) with $t = t^i$ it follows (12), i.e., $\sup_j\{|\mu_{ij}(H)| : i \leqslant j\} < \infty$, for each $i \in \mathbb{N}$ and each $H \in \mathcal{C}_{t^i}$.

With a new induction procedure we determine the increasing sequence $(j_n)_n$ such that together with the sequence $(i_n)_n = (1,1,2,1,2,3,\ldots)$ verifies (5), (6), (7) and (8).

Let $j_1 := 1$ and suppose that $|\mu_{i_1 j_1}|(\Omega) < s_1$, with $s_1 \in \mathbb{N}$. Let $\{N_u^1, 1 \leqslant u \leqslant s_1\}$ be a partition of $\{m \in \mathbb{N} : m > j_1\}$ in s_1 infinite subsets and define $B_u^1 := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N_u^1,\ s \leqslant k_t\}$, $1 \leqslant u \leqslant s_1$. From $\Sigma\{|\mu_{i_1 j_1}|(B_u^1) : 1 \leqslant u \leqslant s_1\} < s_1$ it follows that there exists u', with $1 \leqslant u' \leqslant s_1$, such that $|\mu_{i_1 j_1}|(B_{u'}^1) < 1$, whence the sets $N^{(1)} := N_{u'}^1$ and $B^1 := B_{u'}^1$ verify that $N^{(1)} \subset \{m \in \mathbb{N} : m > j_1\}$ and

$$|\mu_{i_1 j_1}|(B^1) < 1.$$

Assume that in the first l steps of this induction we have obtained a finite sequence $j_1 < j_2 < \cdots < j_l$ in $\mathbb{N}$ and a decreasing finite sequence $N^{(1)} \supset N^{(2)} \supset \cdots \supset N^{(l)}$ of infinite subsets of $\mathbb{N}$ such that for each $w \in \mathbb{N}$, $1 \leqslant w \leqslant l$, $N^{(w)} \subset \{n \in \mathbb{N} : n > j_w\}$ and the variation of the measure $\mu_{i_w j_w}$ in the set $B^w := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N^{(w)},\ s \leqslant k_t\}$ verifies the inequality

$$|\mu_{i_w j_w}|(B^w) < 1.$$

Let j_{l+1} be the first element in $N^{(l)}$ and suppose that $|\mu_{i_{l+1} j_{l+1}}|(\Omega) < s_{l+1}$, with $s_{l+1} \in \mathbb{N}$. Then $j_l < j_{l+1}$ and if $\{N_r^{l+1}, 1 \leqslant r \leqslant s_{l+1}\}$ is a partition of $\{m \in \mathbb{N}^{(l)} : m > j_{l+1}\}$ in s_{l+1} infinite disjoint subfamilies then the subsets $B_r^{l+1} := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N_r^{l+1}, s \leqslant k_t\}$, $1 \leqslant r \leqslant s_{l+1}$, verify that $\Sigma\{|\mu_{i_{l+1} j_{l+1}}|(B_r^{l+1}) : 1 \leqslant r \leqslant s_{l+1}\} < s_{l+1}$, whence it follows that there exists r', with $1 \leqslant r' \leqslant s_{l+1}$, such that the set $B^{l+1} := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N_{r'}^{l+1},\ s \leqslant k_t\}$ verifies that

$$|\mu_{i_{l+1} j_{l+1}}|(B^{l+1}) < 1.$$

Set $N^{(l+1)} := N_{r'}^{l+1}$. Then, by induction, we get a strictly increasing sequence $(j_n)_n$ in $\mathbb{N}$ and a decreasing sequence $(N^{(n)})_n$ of infinite subsets of $\mathbb{N}$, with $j_2 \in N^{(1)} \subset \{m \in \mathbb{N} : m > j_1\}$ and $j_{n+1} \in N^{(n)} \subset \{m \in N^{(n-1)} : m > j_n\}$, for each $n > 1$, such that the measurable sets $B^n := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N^{(n)},\ s \leqslant k_t\}$, $n \in \mathbb{N}$, verify that

$$|\mu_{i_n j_n}|(B^n) < 1. \tag{13}$$

The inclusion $j_s \in N^{(s-1)} \subset N^{(n)}$ when $n < s$ and the trivial inequalities $i_s \leqslant s \leqslant k_s \leqslant k_{j_s}$ imply that $\cup\{B_{i_s j_s} : s \in \mathbb{N},\ n < s\} \subset B^n$, hence from (13) it follows that

$$|\mu_{i_n j_n}|(\cup_s\{B_{i_s j_s} : n < s\}) < 1,$$

for each $n \in \mathbb{N}$, and this inequality imply (7) because the variation $|\mu|(B)$ of μ in a set $B \in \mathcal{S}$ verifies that $|\mu(B)| \leqslant |\mu|(B)$.

From the proved relation $i_s \leqslant k_{j_s}$ and the trivial fact that $s \leqslant n$ implies that $j_s \leqslant j_n < j_{n+1}$ it follows that (10) implies (5). The inequality (6) is a particular case of (11). Finally from (12) with $i = r$ we get (8) because each (i_{n_p}, j_{n_p}) verifies that $r = i_{n_p} \leqslant n_p \leqslant j_{n_p}$.

To finish the proposition define $\mathcal{C}_r := \mathcal{C}_{t^r}$, for each $r \in \mathbb{N}$. □

We are at the position to present the proof of Theorem 2. Recall again that $(i_n)_n = (1, 1, 2, 1, 2, 3, \ldots)$.

Proof of Theorem 2. Assume Theorem 2 fails. Then by Proposition 12 there exist a strictly increasing sequence $(j_n)_n$ in $\mathbb{N}$, a sequence $(B_{i_n j_n})_n$ of pairwise disjoints elements of the σ-algebra $\mathcal{S}$, a sequence $(\mu_{i_n j_n})_n$ in ba$(\mathcal{S})$ and a covering $(\mathcal{C}_r)_r$ of $\mathcal{S}$ such that for each $n \in \mathbb{N}$

$$\Sigma_s\{|\mu_{i_n j_n}(B_{i_s j_s})| : s < n\}) < 1, \tag{14}$$

$$|\mu_{i_n j_n}(B_{i_n j_n})| > j_n, \tag{15}$$

$$|\mu_{i_n j_n}(\cup_s\{B_{i_s j_s} : n < s\})| < 1, \tag{16}$$

and for each strictly increasing sequence $(n_p)_p$ such that $i_{n_p} = r$ for each $p \in \mathbb{N}$ we have that the sequence $(\mu_{i_{n_p} j_{n_p}})_p = (\mu_{r j_{n_p}})_p$ is pointwise bounded in $\mathcal{C}_r$, i.e., for each $H \in \mathcal{C}_r$ we have that

$$\sup\{\left|\mu_{i_{n_p} j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty. \tag{17}$$

As $H_0 := \cup\{B_{i_s j_s} : s = 1, 2, \ldots\} \in \mathcal{S}$ and $(\mathcal{C}_r)_r$ is a covering of the σ-algebra $\mathcal{S}$ there exists $r' \in \mathbb{N}$ such that $H_0 \in \mathcal{C}_{r'}$. Fix a strictly increasing sequence $(n_q)_q$ in $\mathbb{N}\backslash\{1\}$ such that $i_{n_q} = r'$, for each $q \in \mathbb{N}$. Then, by (17),

$$\sup\{\left|\mu_{i_{n_q} j_{n_q}}(H_0)\right| : q \in \mathbb{N}\} < \infty. \tag{18}$$

The sets $C_q := \cup_s\{B_{i_s j_s} : s < n_q\}$, $B_{i_{n_q} j_{n_q}}$ and $D_q := \cup_s\{B_{i_s j_s} : n_q < s\}$ are a partition of the set H_0. By (14), (15) and (16), $\left|\mu_{i_{n_q} j_{n_q}}(C)\right| < 1$, $\mu_{i_{n_q} j_{n_q}}(B_{i_{n_q} j_{n_q}}) > j_{n_q} > n_q$ and $\left|\mu_{i_{n_q} j_{n_q}}(D)\right| < 1$, for each $q \in \mathbb{N}\backslash\{1\}$. Therefore the inequality

$$\left|\mu_{i_{n_q} j_{n_q}}(H_0)\right| > -\left|\mu_{i_{n_q} j_{n_q}}(C)\right| + \mu_{i_{n_q} j_{n_q}}(B_{i_{n_q} j_{n_q}}) - \left|\mu_{i_{n_q} j_{n_q}}\right|(D) > n_q - 2,$$

implies that

$$\lim_p \left|\mu_{i_{n_p} j_{n_p}}(H_0)\right| = \infty,$$

contradicting (18). □

The following corollary extends Theorems 2 and 3 in [14]. Again following [7, 7 Chapter 7, 35.1] a family $\{B_{m_1 m_2 \ldots m_i} : i, m_j \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant p\}$ of subsets of A is an *increasing p-web in A* if $(B_{m_1})_{m_1}$ is an increasing covering of A and $(B_{m_1 m_2 \ldots m_{i+1}})_{m_{i+1}}$ is an increasing covering of $B_{m_1 m_2 \ldots m_i}$, for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i < p$.

Corollary 13. *Let $\mathcal{S}$ be a σ-algebra of subsets of Ω and let $\{\mathcal{B}_{m_1 m_2 \ldots m_i} : i, m_j \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant p\}$ be an increasing p-web in $\mathcal{S}$. Then there exists $\mathcal{B}_{n_1, n_2, \ldots, n_p}$ such that if $(\mathcal{B}_{n_1, n_2, \ldots, n_p s_{p+1}})_{s_{p+1}}$ is an increasing covering of $\mathcal{B}_{n_1, n_2, \ldots, n_p}$ there exists $n_{p+1} \in \mathbb{N}$ such that each $\tau_s(\mathcal{B}_{n_1, n_2, \ldots, n_p n_{p+1}})$-Cauchy sequence $(\mu_n)_n$ in ba$(\mathcal{S})$ is $\tau_s(\mathcal{S})$-convergent.*

Proof. By Theorem 2 there exists $\mathcal{B}_{n_1 n_2 \ldots n_p}$ which has sN-property. Hence there exists $\mathcal{B}_{n_1,n_2,\ldots,n_p n_{p+1}}$ which has N-property. Then a $\tau_s(\mathcal{B}_{n_1,n_2,\ldots,n_p n_{p+1}})$-Cauchy sequence $(\mu_n)_n$ is $\tau_s(\mathcal{A})$-relatively compact. As $\overline{L(\mathcal{B}_{n_1,n_2,\ldots,n_p n_{p+1}})} = L(\mathcal{S})$ the sequence $(\mu_n)_n$ has no more than one $\tau_s(\mathcal{A})$-adherent point, whence $(\mu_n)_n$ is $\tau_s(\mathcal{A})$-convergent. □

4. Applications

We present some applications of Theorem 2 concerning localizations of bounded finitely additive vector measures.

A *finitely additive vector measure*, or simply a *vector measure*, μ defined in an algebra $\mathcal{A}$ of subsets of Ω with values in a topological vector space E is a map $\mu\colon \mathcal{A} \to E$ such that $\mu(B \cup C) = \mu(B) + \mu(C)$, for each pairwise disjoint subsets $B, C \in \mathcal{A}$. The vector measure μ is *bounded* if $\mu(\mathcal{A})$ is a bounded subset of E, or, equivalently, if the E-valued linear map $\mu\colon L(\mathcal{A}) \to E$ defined by $\mu(\chi_B) := \mu(B)$, for each $B \in \mathcal{A}$, is continuous.

A locally convex space $E(\tau)$ is an (LF)- or (LB)-*space* if it is, respectively, the inductive limit of an increasing sequence $(E_m(\tau_m))_m$ of Fréchet or Banach spaces where the relative topology $\tau_{m+1}|_{E_m}$ induced on E_m is coarser than τ_m, for each $m \in \mathbb{N}$. $(E_m(\tau_m))_m$ is a *defining* sequence for $E(\tau)$ with *steps* $E_m(\tau_m)$, $m \in \mathbb{N}$, and we write $E(\tau) = \Sigma_m E_m(\tau_m)$. If $\tau_{m+1}|_{E_m} = \tau_m$, for each $m \in \mathbb{N}$, then $E(\tau)$ is a *strict* (LF)-, or (LB)-space. From [7, 19.4(4)] it follows that if $\mu\colon \mathcal{A} \to E(\tau)$ is a vector bounded measure with values in a strict (LF)-space $E(\tau) = \Sigma_m E_m(\tau_m)$ then there exists $n \in \mathbb{N}$ such that $\mu(\mathcal{A})$ is a bounded subset of the step $E_n(\tau_n)$. For σ-algebras the following extension of this result is contained in [14, Theorem 4].

Theorem 14. *Let μ be a bounded vector measure defined in a σ-algebra $\mathcal{S}$ of subsets of Ω with values in an (LF)-space $E(\tau) = \Sigma_m E_m(\tau_m)$. Then there exists $n \in \mathbb{N}$ such that $\mu(\mathcal{S})$ is a bounded subset of $E_n(\tau_n)$.*

Theorem 2 provides the following proposition that contains Theorem 14 as a particular case.

Proposition 15. *Let μ be a bounded vector measure defined in a σ-algebra $\mathcal{S}$ of subsets of Ω with values in a topological vector space $E(\tau)$. Suppose that $\{E_{m_1,m_2,\cdots,m_i} : m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant p\}$ is an increasing p-web in E. Then there exists $E_{n_1,n_2,\cdots,n_p}$ such that if $E_{n_1,n_2,\cdots,n_p}(\tau_{n_1,n_2,\cdots,n_p})$ is an (LF)-space, the topology $\tau_{n_1,n_2,\cdots,n_p}$ is finer than the relative topology $\tau|_{E_{n_1,n_2,\cdots,n_p}}$ and if $(E_{n_1,n_2,\cdots,n_p,s_{p+1}}(\tau_{n_1,n_2,\cdots,n_p,s_{p+1}}))_{s_{p+1}}$ is a defining sequence for $E_{n_1,n_2,\cdots,n_p}(\tau_{n_1,n_2,\cdots,n_p})$ there exists $n_{p+1} \in \mathbb{N}$ such that $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2,\cdots,n_p,n_{p+1}}(\tau_{n_1,n_2,\cdots,n_p,n_{p+1}})$.*

Proof. Let $\mathcal{B}_{m_1,m_2,\ldots,m_i} := \mu^{-1}(E_{m_1,m_2,\ldots,m_i})$ for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i \leqslant p$. By Theorem 2 there exists $(n_1, n_2, \ldots, n_p) \in \mathbb{N}^p$ such that $\mathcal{B}_{n_1,n_2,\ldots,n_p}$ has sN-property. Let $(E_{n_1,n_2,\ldots,n_p,s_{p+1}}(\tau_{n_1 n_2 \ldots n_p s_{p+1}}))_{s_{p+1}}$ be a defining sequence for $E_{n_1,n_2,\ldots,n_p}(\tau_{n_1,n_2,\ldots,n_p})$ and let $\mathcal{B}_{n_1,n_2,\ldots,n_p,s_{p+1}} := \mu^{-1}(E_{n_1,n_2,\ldots,n_p,s_{p+1}})$.

As $(\mathcal{B}_{n_1,n_2,\ldots,n_p,s_{p+1}})_{s_{p+1}}$ is an increasing covering of $\mathcal{B}_{n_1,n_2,\ldots,n_p}$ there exists n_{p+1} such that $\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}}$ has N-property, whence $L(\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}})$ is a dense subspace of $L(\mathcal{S})$ and then the map with closed graph

$$\mu|_{L(\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}})}\colon L(\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}}) \to E_{n_1,n_2,\ldots,n_p,n_{p+1}}(\tau_{n_1,n_2,\ldots n_p,n_{p+1}})$$

has a continuous extension $\upsilon\colon L(\mathcal{S}) \to E_{n_1,n_2,\ldots,n_p,n_{p+1}}(\tau_{n_1,n_2,\ldots,n_p,n_{p+1}})$ (by [12, 2.4 Definition and (N_2)] and [13, Theorems 1 and 14]). The continuity of $\mu\colon L(\mathcal{S}) \to E(\tau)$ implies that $\upsilon(A) = \mu(A)$, for each $A \in \mathcal{S}$. Whence $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2,\ldots,n_p,n_{p+1}}(\tau_{n_1,n_2,\ldots,n_p,n_{p+1}})$. □

Proposition 15 also holds if we replace (LF)-space by an inductive limit of Γ_r-spaces (see [13, Definition 1] and, taking into account [12, Property (N_2) after 2.4 Definition], apply again [13, Theorems 1 and 14]).

A particular case of this proposition is the next corollary, which it is also a concrete generalization of Theorem 14.

Corollary 16. *Let μ be a bounded vector measure defined in a σ-algebra $\mathcal{S}$ of subsets of Ω with values in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of (LF)-spaces. There exists $n_1 \in \mathbb{N}$ such that for each defining sequence $(E_{n_1,m_2}(\tau_{n_1,m_2}))_{m_2}$ of $E_{n_1}(\tau_{n_1})$ there exists $n_2 \in \mathbb{N}$ which verifies that $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$.*

A sequence $(x_k)_k$ in a locally convex space E is *subseries convergent* if for every infinite subset J of $\mathbb{N}$ the series $\Sigma\{x_k : k \in J\}$ converges and $(x_k)_k$ *is bounded multiplier* if for every bounded sequence of scalars $(\lambda_k)_k$ the series $\Sigma_k \lambda_k x_k$ converges.

A Fréchet space E is Fréchet Montel if each bounded subset of E is relatively compact. Important classes of Montel and Fréchet Montel spaces are considered and studied while Schwartz Theory of Distributions is described, for instance, in [6, Chapter 3, Examples 3, 4, 5 and 6.].

The following corollary is a generalization of [14, Corollary 1.4] and it follows partially from Corollary 16.

Corollary 17. *Let $(x_k)_k$ be a subseries convergent sequence in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of (LF)-spaces. Then there exists $n_1 \in \mathbb{N}$ such that for each defining sequence $(E_{n_1,m_2}(\tau_{n_1,m_2}))_{m_2}$ for $E_{n_1}(\tau_{n_1})$ there exists $n_2 \in \mathbb{N}$ such that $\{x_k : k \in \mathbb{N}\}$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$. If, additionally, $E_{n_1,n_2}(\tau_{n_1,n_2})$ is a Fréchet Montel space then the sequence $(x_k)_k$ is bounded multiplier in $E_{n_1,n_2}(\tau_{n_1,n_2})$.*

Proof. As the sequence $(x_k)_k$ is subseries convergent then the additive vector measure $\mu\colon 2^{\mathbb{N}} \to E(\tau)$ defined by $\mu(J) := \Sigma_{k\in J} x_k$, for each $J \in 2^{\mathbb{N}}$, is bounded, because as $(f(x_k))_k$ is subseries convergent for each $f \in E'$ we get that $\Sigma_{k=1}^{\infty} |f(x_k)| < \infty$.

By Corollary 16 there exists $n_1 \in \mathbb{N}$ such that for each defining sequence $(E_{n_1,m_2}(\tau_{n_1,m_2}))_{m_2}$ for $E_{n_1}(\tau_{n_1})$ there exists $n_2 \in \mathbb{N}$ with the property that $\mu(2^{\mathbb{N}}) = \{\Sigma_{k\in J} x_k : J \in 2^{\mathbb{N}}\}$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$. Then $\Sigma_k |\lambda_k f(x_k)| < \infty$ for each continuous linear form f defined on $E_{n_1,n_2}(\tau_{n_1,n_2})$ and each bounded sequence $(\lambda_k)_k$ of scalars, whence $(\Sigma_{j=1}^{k} \lambda_j x_j)_k$ is a bounded sequence in $E_{n_1,n_2}(\tau_{n_1,n_2})$ which has at most one adherent point, because $\Sigma_k \lambda_k f(x_k)$ converges for each $f \in (E_{n_1,n_2}(\tau_{n_1,n_2}))'$. If $E_{n_1,n_2}(\tau_{n_1,n_2})$ is a Montel space then the bounded subset $\{\Sigma_{j=1}^{k} \lambda_j x_j : k \in \mathbb{N}\}$ is relatively compact and then the series $\Sigma_k \lambda_k x_k$ converges in $E_{n_1,n_2}(\tau_{n_1,n_2})$. $\square$

Recall that a vector measure μ defined in an algebra $\mathcal{A}$ of subsets of Ω with values in a Banach space E is *strongly additive* whenever given a sequence $(B_n)_n$ of pairwise disjoint elements of $\mathcal{A}$ the series $\Sigma_n \mu(B_n)$ converges in norm [2, I.1. Definition 14]. Each strongly additive vector measure μ is bounded [2, I.1. Corollary 19].

Corollary 18. *Let μ be a bounded vector measure defined in a σ-algebra $\mathcal{S}$ of subsets of Ω with values in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of (LB)-spaces such that each $E_m(\tau_m)$ admit a defining sequence $(E_{m,m_2}(\tau_{m,m_2}))_{m_2}$ of Banach spaces which does not contain a copy of l^∞. If H is a dense subset of $E'(\tau_s(E))$ such that $f\mu$ is countably additive for each $f \in H$, then there exists $(n_1, n_2) \in \mathbb{N}^2$ such that μ is a $E_{n_1,n_2}(\tau_{n_1,n_2})$-valued countably additive vector measure.*

Proof. By Corollary 16 there exists $(n_1, n_2) \in \mathbb{N}^2$ such that $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$. As $E_{n_1,n_2}(\tau_{n_1,n_2})$ does not contain a copy of l^∞ then, by ([2, I.4. Theorem 2]), the measure μ is strongly additive, hence if $(B_n : n \in \mathbb{N})$ is a sequence of pairwise disjoint subsets of $\mathcal{S}$ then $\Sigma_n \mu(B_n)$ converges to the vector x in $E_{n_1,n_2}(\tau_{n_1,n_2})$. Therefore $f(x) = \Sigma_n f\mu(B_n)$ for each $f \in E'$ and, by countably additivity of

$f\mu$ when $f \in H$, we have that $f(x) = \Sigma_n f\mu(B_n) = f\mu(\cup_n B_n)$ for each $f \in H$. By density $x = \mu(\cup_n B_n)$, whence $\Sigma_n \mu(B_n) = \mu(\cup_n B_n)$ in $E_{n_1,n_2}(\tau_{n_1,n_2})$. $\square$

Proposition 19. *Let μ be a bounded* vector measure *defined in a σ-algebra $\mathcal{S}$ of subsets of Ω with values in a topological vector space $E(\tau)$. Suppose that $\{E_{m_1,m_2,\ldots,m_i} : m_j \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant p\}$ is an* increasing p-web in E. *There exists $E_{n_1,n_2,\ldots,n_p}$ such that if $(E_{n_1,n_2,\ldots,n_p,m_{p+1}})_{m_{p+1}}$ is an increasing covering of $E_{n_1,n_2,\ldots,n_p}$ with the property that each relative topology $\tau|_{E_{n_1,n_2,\ldots,n_p,m_{p+1}}}$, $m_{p+1} \in \mathbb{N}$, is sequentially complete then there exists $n_{p+1} \in \mathbb{N}^p$ such that $\mu(\mathcal{S}) \subset E_{n_1,n_2,\cdots,n_p,n_{p+1}}$.*

Proof. Let $\mathcal{B}_{m_1,m_2,\ldots,m_i} := \mu^{-1}(E_{m_1,m_2,\ldots,m_i})$ for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i \leqslant p+1$. By Theorem 2 there exists $(n_1, n_2, \ldots, n_p) \in \mathbb{N}^p$ such that $\mathcal{B}_{n_1,n_2,\ldots,n_p}$ has sN-property, whence there exists $n_{p+1} \in \mathbb{N}^p$ such that $\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}}$ has N-property, therefore $E_{n_1,n_2,\ldots,n_p n_{p+1}}(\tau|_{E_{n_1,n_2,\ldots,n_p,n_{p+1}}})$ is a dense subspace of $E(\tau)$, hence density and sequential completeness imply that the continuous restriction of μ to $L(\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}})$ has a continuous extension v to $L(\mathcal{S})$ with values in the space $E_{n_1,n_2,\ldots,n_p,n_{p+1}}(\tau|_{E_{n_1,n_2,\ldots,n_p,n_{p+1}}})$. As $\mu\colon L(\mathcal{S}) \to E(\tau)$ is continuous then $v = \mu$ and we get that $\mu(\mathcal{S}) \subset E_{n_1,n_2,\ldots,n_p,n_{p+1}}$. $\square$

Corollary 20. *Let μ be a bounded additive vector measure defined in a σ-algebra $\mathcal{S}$ of subsets of Ω with values in an inductive limit $E(\tau) = \Sigma_{m_1} E_{m_1}(\tau_{m_1})$ of an increasing sequence $(E_{m_1}(\tau_{m_1}))_{m_1}$ of countable dimensional topological vector spaces. Then there exists n_1 such that span$\{\mu(\mathcal{S})\}$ is a finite dimensional subspace of $E_{n_1}(\tau_{n_1})$.*

Proof. For each $m_1 \in \mathbb{N}$ let $(E_{m_1,m_2})_{m_2}$ be an increasing covering of E_{m_1} by finite dimensional vector subspaces. $\{E_{m_1,m_2} : m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant 2\}$ is an *increasing* 2-*web in* E. As the relative topology $\tau|_{E_{m_1,m_2}}$ induced on E_{m_1,m_2} is complete then, by Proposition 19, there exists $(n_1, n_2) \in \mathbb{N}^2$ such that $\mu(\mathcal{S}) \subset E_{n_1,n_2}$. $\square$

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