

# Convexity constant of a domain and applications

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## Abstract

In the present paper we introduce a new characterization of the convexity of a planar domain, based on the convexity constant  $K(D)$  of a domain  $D \subset \mathbb{C}$ .

We show that in the class of simply connected planar domains,  $K(D) = 1$  characterizes the convexity of the domain  $D$ , and we derive the value of the convexity constant for some classes of doubly connected domains of the form  $D_\Omega = D - \bar{\Omega}$ , for certain choices of the domains  $D$  and  $\Omega$ .

Using the convexity constant of a domain, we derive an extension of the well-known Ozaki-Nunokawa-Krzyz univalence criteria for the case of non-convex domains, and we present some examples, which show that our condition is sharp.

*Keywords:* convex set, convexity constant of a domain, univalent function, univalence criteria.

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## 1. Introduction

Convexity of a planar domain plays an important notion in many areas of mathematics. With respect to this notion, the class of planar domains can be divided into two classes: the class of convex domains and the class of non-convex domains, but there is no continuous way of passing between the

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two, even though some domains are very close to being convex (for example the case of the punctured disk).

In the present paper we introduce a new characterization of the convexity of a domain by using the convexity constant  $K(D)$  of a domain (Definition 1). In Theorem 5 we show that in the class of simply connected domains,  $K(D) = 1$  characterizes uniquely the convexity of the domain  $D$ .

In Proposition 7 and Theorems 11, 14 and 18 we derive the value of the convexity constant for certain classes of doubly connected planar domains of the form  $D_\Omega = D - \overline{\Omega}$ , and we also propose a conjecture for the convexity constant  $K(D_\Omega)$  in the general case.

As applications, in Section 3, in Theorem 25 we obtain a general univalence criteria for analytic functions defined in non-convex domains  $D \subset U$  contained in the unit disk. We extend the result to the case of analytic functions defined in bounded non-convex domains  $D \subset \mathbb{C}$  (Theorem 26), and for the particular choice  $D = U$  we obtain as corollaries the Ozaki-Nunokawa-Krzyz univalence criterion (Theorem 22 and Theorem 24).

We conclude with an example, which shows that the condition in Theorem 25 and Theorem 26 is sharp.

## 2. The convexity constant of a domain

We will denote by  $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$  the open disk centered at  $z_0 \in \mathbb{C}$  of radius  $r > 0$ , and in particular  $U = B(0, 1)$  the unit disk.

For points  $z_1, z_2 \in \mathbb{C}$ , we will denote by  $z_1 z_2$  the line passing through  $z_1$  and  $z_2$  and by  $[z_1, z_2]$  the line segment with endpoints  $z_1$  and  $z_2$ .

We begin by characterizing the convexity of a domain by an appropriate number between 0 and 1, which is a measure of “how far off” is a domain from being convex, definition which might be of independent interest. The formal definition is the following:

**Definition 1.** For a simply connected domain  $D \subset \mathbb{C}$ , we define the convexity constant of the domain  $D$  by

$$K(D) = \inf_{\substack{a, b \in D \\ a \neq b}} \sup_{\gamma \in \Gamma(a, b; D)} \frac{|a - b|}{l(\gamma)},$$

where  $\Gamma(a, b; D)$  is the family of all rectifiable arcs  $\gamma \subset D$  with distinct endpoints  $a$  and  $b$ , and  $l(\gamma)$  denotes the length of  $\gamma$ .

**Remark 2.** Denoting by  $S(a, b; D) = \sup_{\gamma \in \Gamma(a, b; D)} \frac{|a - b|}{l(\gamma)}$ , since  $l(\gamma) \geq |a - b|$  for any  $a, b \in D$  and  $\gamma \in \Gamma(a, b; D)$ , we have

$$S(a, b; D) \in (0, 1], \quad a, b \in D, \quad a \neq b,$$

and

$$K(D) = \inf_{\substack{a, b \in D \\ a \neq b}} S(a, b; D) \in [0, 1].$$

**Remark 3.** If  $D$  is a convex domain, then two arbitrary distinct points  $a, b \in D$  can be joined by the line segment  $\gamma = [a, b]$  contained in  $D$ , and therefore

$$S(a, b; D) = \sup_{\gamma \in \Gamma(a, b)} \frac{|a - b|}{l(\gamma)} = 1,$$

for any distinct points  $a, b \in D$ .

Taking the infimum over all distinct points  $a, b \in D$  we obtain

$$K(D) = \inf_{a, b \in D} S(a, b; D) = \inf_{a, b \in D} 1 = 1,$$

and therefore the convexity constant of a convex domain  $D$  is  $K(D) = 1$ .

In order to show that  $K(D) = 1$  characterizes the class of convex domains, we need the following:

**Lemma 4.** If  $D \subset \mathbb{C}$  is a simply connected domain,  $a, b \in D$  are distinct points in  $D$  and there exists  $c \in [a, b]$  and  $\varepsilon > 0$  such that  $B(c, \varepsilon) \subset \mathbb{C} - \bar{D}$ , then

$$S(a, b; D) = \sup_{\gamma \in \Gamma(a, b; D)} \frac{|b - a|}{l(\gamma)} \leq \frac{|b - a|}{|a - c_\varepsilon| + |b - c_\varepsilon|} < 1,$$

where  $B(c, \varepsilon)$  is the open disk centered at  $c$  of radius  $\varepsilon > 0$  and  $c_\varepsilon$  is one of the points of intersection of  $\partial B_{c, \varepsilon}$  with the line passing through  $c$  and perpendicular to the line determined by  $a$  and  $b$  (i.e.  $c_\varepsilon = c \pm i\varepsilon \frac{b-a}{|b-a|}$ ).

*Proof.* Follows by simple geometric considerations, by noticing that any curve  $\gamma \in \Gamma(a, b; D)$  must intersect the line passing through  $c$  and perpendicular to the line determined by  $a$  and  $b$ , and that the two parts of  $\gamma$  thus formed have lengths greater than  $|a - c_\varepsilon|$ , respectively  $|b - c_\varepsilon|$  (see Figure 1 below).  $\square$

With this preamble, we can show that in the class of simply connected domains, the convexity constant  $K(D)$  introduced in Definition 1 characterizes the convexity of the domain  $D$ , as follows:

*Proof.* The direct implication follows from Remark 3.

$$S(a, b; D) \leq \frac{|b - a|}{|a - c_\varepsilon| + |b - c_\varepsilon|} < 1,$$
$$K(D) = \inf_{\substack{a, b \in D \\ a \neq b}} S(a, b; D) \leq \frac{|b - a|}{|a - c_\varepsilon| + |b - c_\varepsilon|} < 1,$$

**Remark 6.** In the previous theorem, the requirement that  $D$  is a simply connected domain is essential, as it can be seen by considering the punctured disk  $U_{\{0\}} = U - \{0\}$ , which is not convex but has convexity constant  $K(U_{\{0\}}) = 1$  ( $U_{\{0\}}$  is not a simply connected domain).

We conclude this section by determining explicitly the convexity constant of some particular classes of non-convex domains.

Given two domains  $\Omega \subset D \subset \mathbb{C}$ , we will denote  $D_\Omega$  the domain

$$D_\Omega = D - \overline{\Omega} = \{z \in \mathbb{C} : z \in D, z \notin \overline{\Omega}\} \quad (1)$$

The next result shows that introducing a cut in a domain (convex or not) produces a domain with the smallest possible convexity constant (farthest from being convex, in the sense of Definition 1):

**Proposition 7.** *If  $D$  is a domain in  $\mathbb{C}$  and  $\gamma$  is a Jordan arc which joins two distinct points  $z_0 \in \Delta$  and  $w_0 \in \partial\Delta$ , such that  $\gamma - \{w_0\} \subset \Delta$ , then  $K(D_\gamma) = 0$ .*

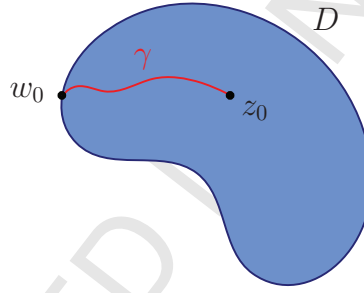


Figure 2: The convexity constant of the domain  $D_\gamma = D - \gamma$  is  $K(D_\gamma) = 0$ .

*Proof.* Consider  $c \in \gamma - \{z_0, w_0\}$  and  $r > 0$  such that  $B(c, r) \subset D$  and  $B(c, r) - \gamma$  is not a connected domain.

Consider the sequences  $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \subset B(c, r) - \gamma$  such that  $a_n$  and  $b_n$  lie in different connected components of  $B(c, r) - \gamma$  for each  $n \geq 1$ , and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = c$ . For an arbitrary curve  $\gamma_1 \in \Gamma(a_n, b_n, D_\gamma)$ , since  $a_n$  and  $b_n$  lie in different connected components of  $B(c, r) - \gamma$ ,  $\gamma_1$  must intersect  $\partial B(c, r)$ , and therefore we have

$$l(\gamma_1) \geq |a_n - \alpha_n| + |b_n - \beta_n| \geq 2r - |c - a_n| - |c - b_n|,$$

where  $\alpha_n = \alpha_n(\gamma_1), \beta_n = \beta_n(\gamma_1) \in \partial B(c, r)$  are the points of intersection between  $\gamma_1$  and  $\partial B(c, r)$ . It follows that for any  $n \geq 1$  we have

$$S(a_n, b_n; D_\gamma) = \sup_{\gamma_1 \in \Gamma(a_n, b_n; D)} \frac{|a_n - b_n|}{l(\gamma_1)} \leq \frac{|a_n - b_n|}{2r - |c - a_n| - |c - b_n|}.$$

We obtain

$$\begin{aligned}
 K(D_\gamma) &= \inf_{a,b \in D_\gamma} S(a,b; D_\gamma) \\
 &\leq \inf_{n \geq 1} S(a_n, b_n; D_\gamma) \\
 &= \inf_{n \geq 1} \frac{|a_n - b_n|}{2r - |c - a_n| - |c - b_n|} \\
 &= 0,
 \end{aligned}$$

since by construction  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = c$ , concluding the proof.  $\square$

The next result shows that the convexity constant of a convex domain with a closed disk removed is  $K(D_{B(z_0, r)}) = \frac{2}{\pi}$ . To prove this, we need the following

**Lemma 8.** *If  $D \subset \mathbb{C}$  is a convex domain and  $z_0 \in D$  and  $r > 0$  are chosen such that  $\overline{B(z_0, r)} \subset D$ , then:*

$$S(a, b; D_{B(z_0, r)}) = \begin{cases} 1, & \text{if } [a, b] \cap B(z_0, r) = \emptyset \\ \max_{i=1,2} \frac{|a-b|}{|a-a_i| + r|\arg(b_i - z_0) - \arg(a_i - z_0)| + |b-b_i|}, & \text{if } [a, b] \cap B(z_0, r) \neq \emptyset \end{cases},$$

where  $a_{1,2}$  and  $b_{1,2}$  denote the points of intersection of the circle  $\partial B(z_0, r)$  with the tangent lines from  $a$ , respectively  $b$ , to this circle (see Figure 3).

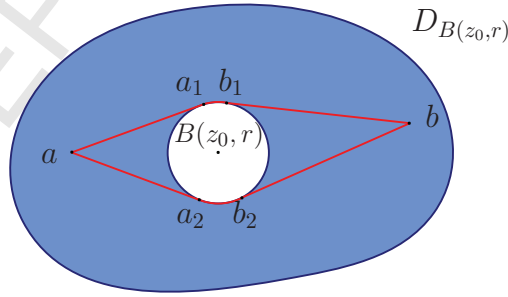


Figure 3: The domain  $D_{B(z_0, r)}$ .

*Proof.* Let  $a, b$  be arbitrary fixed distinct points in  $D_{B(z_0, r)}$ .

If  $[a, b] \cap B(z_0, r) = \emptyset$ , since  $[a, b] \subset D$  (since  $D$  is assumed convex), it follows that  $[a, b] \subset D_{B(z_0, r)}$  (or  $[a, b] \subset D_{B(z_0, r)} \cup \{t\}$ , where  $t$  denotes the point of intersection of  $[a, b]$  with  $\partial B(z_0, r)$ , in the case when  $[a, b]$  is tangent to the circle  $\partial B(z_0, r)$ ), and therefore  $S(a, b; D_{B(z_0, r)}) = 1$  in this case.

If  $[a, b] \cap B(z_0, r) \neq \emptyset$ , for an arbitrarily chosen curve  $\gamma \in \Gamma(a, b; D_{B(z_0, r)})$ , it follows that  $\gamma$  either intersects the half-lines  $(z_0 a_1$  and  $(z_0 b_1$ , or  $(z_0 a_2$  and  $(z_0 b_2$  (here  $(z_0 a_1$  denotes the half-lines with endpoint  $z_0$  and passing through  $a_1$ ).

In the first case, denoting by  $c \in \gamma \cap (z_0 a_1$  and  $d \in \gamma \cap (z_0 b_1$  the points of intersection, we have:

$$\begin{aligned} l(\gamma) &= l(\gamma_{ac}) + l(\gamma_{cd}) + l(\gamma_{db}) \\ &\geq |a - c| + l(\gamma_{cd}) + |b - d| \\ &\geq |a - a_1| + l(\gamma_{cd}) + |b - b_1| \end{aligned}$$

(the last inequality follows from the fact that the pairs of lines  $aa_1$ ,  $z_0 a_1$ , respectively  $bb_1$ ,  $z_0 b_1$  are orthogonal).

Denoting by  $\gamma_{cd}$  the part of  $\gamma$  between  $c$  and  $d$ , if  $\gamma_{cd}(t) = z_0 + R(t)e^{i\theta(t)}$ ,  $0 \leq t \leq 1$ , is a differentiable parametrization of the arc  $\gamma_{cd}$ , we obtain:

$$\begin{aligned} l(\gamma_{cd}) &= \int_{\gamma_{cd}} |dz| = \int_0^1 \left| R'(t)e^{i\theta(t)} + i\theta'(t)R(t)e^{i\theta(t)} \right| dt \\ &= \int_0^1 \sqrt{(R'(t))^2 + (\theta'(t)R(t))^2} dt \\ &\geq \int_0^1 |\theta'(t)| |R(t)| dt \\ &\geq r \int_0^1 |\theta'(t)| dt \\ &\geq r |\theta(1) - \theta(0)| \\ &= r |\arg(d - z_0) - \arg(c - z_0)| \\ &= r |\arg(b_1 - z_0) - \arg(a_1 - z_0)|. \end{aligned}$$

Combining the last two inequalities above, we obtain:

$$\frac{|a - b|}{l(\gamma)} \leq \frac{|a - b|}{|a - a_1| + r |\arg(b_1 - z_0) - \arg(a_1 - z_0)| + |b - b_1|},$$



and therefore:

$$\frac{|a-b|}{l(\gamma)} \leq \max_{i=1,2} \frac{|a-b|}{|a-a_i| + r |\arg(b_i - z_0) - \arg(a_i - z_0)| + |b-b_i|}.$$

A similar argument shows that this inequality is also valid in the second case (the case when  $\gamma \cap (z_0 a_2 \neq \emptyset$  and  $\gamma \cap (z_0 b_2 \neq \emptyset)$ ), and therefore it is valid for any  $\gamma \in \Gamma(a, b; D_{B(z_0, r)})$ . From the definition of  $S(a, b; D_{B(z_0, r)})$  we obtain therefore:

$$S(a, b; D_{B(z_0, r)}) \leq \max_{i=1,2} \frac{|a-b|}{|a-a_i| + r |\arg(b_i - z_0) - \arg(a_i - z_0)| + |b-b_i|}.$$

Since for an arbitrarily chosen  $\varepsilon > 0$  we can approximate the arcs  $[a, a_i] \cup \widehat{a_i b_i} \cup [b_i, b]$  ( $i = 1, 2$ ) by Jordan arcs  $\gamma_\varepsilon \in \Gamma(a, b; D_{B(z_0, r)})$  contained in  $D$  such that

$$\frac{|a-b|}{l(\gamma_\varepsilon)} > \max_{i=1,2} \frac{|a-b|}{|a-a_i| + r |\arg(b_i - z_0) - \arg(a_i - z_0)| + |b-b_i|} - \varepsilon,$$

combining with the above inequality we obtain

$$S(a, b; D_{B(z_0, r)}) = \max_{i=1,2} \frac{|a-b|}{|a-a_i| + r |\arg(b_i - z_0) - \arg(a_i - z_0)| + |b-b_i|},$$

concluding the proof.  $\square$

**Remark 9.** The value of  $S(a, b; D_{B(z_0, r)})$  given in the previous lemma (in the case when  $[a, b] \cap B(z_0, r) \neq \emptyset$ ) represents the length of the longer curve  $\gamma_i = [a, a_i] \cup \widehat{a_i b_i} \cup [b_i, b]$  ( $i = 1, 2$ ), where  $\widehat{a_i b_i}$  denotes the smaller of the two circular arcs of  $\partial B(z_0, r)$  with endpoints  $a_i$  and  $b_i$ .

**Remark 10.** Denoting by  $x = |a - z_0|$ ,  $y = |b - z_0|$  and  $d = \text{dist}(z_0, ab)$  - the distance from  $z_0$  to the line determined by  $a$  and  $b$ , simple geometric considerations show that the value of  $S(a, b; D_{B(z_0, r)})$  obtained in the previous lemma can be written as follows:

$$S(a, b; D_{B(z_0, r)}) = \frac{\sqrt{y^2 - d^2} + \sqrt{x^2 - d^2}}{\sqrt{y^2 - r^2} + \sqrt{x^2 - r^2} + r \left( \arcsin \frac{r}{x} - \arcsin \frac{d}{x} + \arcsin \frac{r}{y} - \arcsin \frac{d}{y} \right)},$$

for any  $a, b \in D_{B(z_0, r)}$  with  $[a, b] \cap B(z_0, r) \neq \emptyset$ .

With this preparation we can now prove the following:

**Theorem 11.** *If  $D \subset \mathbb{C}$  is a convex domain,  $z_0 \in D$  and  $r > 0$  are chosen such that  $B(z_0, r) \subset D$ , then the convexity constant of the domain  $D_{B(z_0, r)}$  is given by*

$$K(D_{B(z_0, r)}) = \frac{2}{\pi}.$$

*Proof.* Using the previous remark, it suffices to show that the minimum value of

$$\frac{\sqrt{y^2 - d^2} + \sqrt{x^2 - d^2}}{\sqrt{y^2 - r^2} + \sqrt{x^2 - r^2} + r \left( \arcsin \frac{r}{x} - \arcsin \frac{d}{x} + \arcsin \frac{r}{y} - \arcsin \frac{d}{y} \right)}, \quad (2)$$

defined for  $x = |a - z_0|$ ,  $y = |b - z_0|$  and  $d = \text{dist}(z_0, ab)$  such that  $a, b \in D_{B(z_0, r)}$  and  $[a, b] \cap B(z_0, r) \neq \emptyset$ , equals  $\frac{2}{\pi}$ .

For arbitrarily chosen distinct points  $a, b \in D_{B(z_0, r)}$  with  $[a, b] \cap B(z_0, r) \neq \emptyset$ , denoting by  $\sin \alpha = \frac{r}{x}$ ,  $\sin \beta = \frac{d}{x}$ ,  $\sin \gamma = \frac{r}{y}$  si  $\sin \delta = \frac{d}{y}$ ,  $0 \leq \beta \leq \alpha < \frac{\pi}{2}$  si  $0 \leq \delta \leq \gamma < \frac{\pi}{2}$ , we can rewrite (2) as follows:

$$\frac{y \cos \gamma + x \cos \beta}{y \cos \delta + x \cos \alpha + r(\alpha - \beta) + r(\gamma - \delta)}.$$

Using the inequality

$$\begin{aligned} & \frac{x \cos \beta + y \cos \gamma}{x \cos \alpha + r(\alpha - \beta) + y \cos \delta + r(\gamma - \delta)} \\ & \geq \min \left\{ \frac{x \cos \beta}{x \cos \alpha + r(\alpha - \beta)}, \frac{y \cos \gamma}{y \cos \delta + r(\gamma - \delta)} \right\} \\ & = \min \left\{ \frac{\cos \beta}{\cos \alpha + (\alpha - \beta) \sin \alpha}, \frac{\cos \gamma}{\cos \delta + (\gamma - \delta) \sin \gamma} \right\}, \end{aligned}$$

it follows that a lower bound for  $K(D_{B(z_0, r)})$  is given by the minimum value of the expression

$$\frac{\cos \beta}{\cos \alpha + (\alpha - \beta) \sin \alpha},$$

defined for  $0 \leq \beta \leq \alpha < \frac{\pi}{2}$ .

We will show that the minimum value is  $\frac{2}{\pi}$ , that is:

$$\frac{\cos \beta}{\cos \alpha + (\alpha - \beta) \sin \alpha} \geq \frac{2}{\pi}, \quad (3)$$

for any  $0 \leq \beta \leq \alpha < \frac{\pi}{2}$ .

For this, notice first that for  $\beta \in [0, \frac{\pi}{2})$  fixed, we have

$$\frac{\cos \beta}{\cos \alpha + (\alpha - \beta) \sin \alpha} \geq \frac{\cos \beta}{\frac{\pi}{2} - \beta}, \quad (4)$$

for any  $\alpha \in [\beta, \frac{\pi}{2})$ , since the expression on the left is a decreasing function of  $\alpha \in [\beta, \frac{\pi}{2}]$ , and therefore the minimum is attained for  $\alpha = \frac{\pi}{2}$ .

We have:

$$\frac{d}{d\beta} \left( \frac{\cos \beta}{\frac{\pi}{2} - \beta} \right) = \sin \beta \frac{\tan(\frac{\pi}{2} - \beta) - (\frac{\pi}{2} - \beta)}{(\frac{\pi}{2} - \beta)^2} > 0$$

for any  $\beta \in (0, \frac{\pi}{2})$ , which shows that the expression in the right side of (4) is an increasing function of  $\beta \in [0, \frac{\pi}{2})$ , and therefore:

$$\frac{\cos \beta}{\cos \alpha + (\alpha - \beta) \sin \alpha} \geq \frac{\cos \beta}{\frac{\pi}{2} - \beta} \geq \frac{\cos 0}{\frac{\pi}{2} - 0} = \frac{2}{\pi}$$

for any  $0 \leq \beta \leq \alpha < \frac{\pi}{2}$ .

It follows that a lower bound for the convexity constant of the domain  $D_{B(z_0, r)}$  is given by:

$$K(D_{B(z_0, r)}) \geq \frac{2}{\pi}. \quad (5)$$

To prove the reverse inequality, consider the sequences of points  $(a_n)_{n \geq 1}$ ,  $(b_n)_{n \geq 1} \subset D_{B(z_0, r)}$ , such that  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ , where  $a, b \in \partial B(z_0, r)$  are chosen such that  $[a, b]$  is a diameter of the circle  $\partial B(z_0, r)$ . We have:

$$K(D_{B(z_0, r)}) \leq \inf_{n \geq 1} S(a_n, b_n; D_{B(z_0, r)}) = \frac{|a - b|}{l(\widehat{ab})} = \frac{2r}{\pi r} = \frac{2}{\pi},$$

which together (5) concludes the proof of the theorem.  $\square$

**Remark 12.** It is interesting to note that the value  $\frac{2}{\pi}$  of the convexity constant of the domain  $D_{B(z_0, r)}$  is independent of the choice of the convex domain  $D$  (with the property  $\overline{B(z_0, r)} \subset D$ ).

Also, since for  $r \searrow 0$ , the limiting domain is  $D_{\{z_0\}} = D - \{z_0\}$ , and since  $K(D_{\{z_0\}}) = K(D) = 1$  (since  $D$  is convex) and  $K(D_{B(z_0, r)}) = \frac{2}{\pi}$  for any  $r > 0$  sufficiently small, it follows that

$$\frac{2}{\pi} = \lim_{r \searrow 0} K(D_{B(z_0, r)}) \neq K(D_{\{z_0\}}) = 1,$$

which shows that the convexity constant  $K(D)$  is not a continuous function with respect to the domain  $D$ .

Also, it can be shown that without additional hypothesis,  $K(D)$  is not a monotone function with respect to  $D$ .

For  $z_0 \in \mathbb{C}$  and  $r > 0$ , we will denote by  $S(z_0, r)$  the interior of the square having  $z_0$  as center of symmetry and sides parallel to the coordinate axes, of length equal to  $r$ :

$$S(z_0, r) = \{z \in \mathbb{C} : |\operatorname{Re}(z - z_0)| < \frac{r}{2}, |\operatorname{Im}(z - z_0)| < \frac{r}{2}\}$$

We have the following:

**Lemma 13.** *If  $D \subset \mathbb{C}$  is a convex domain and  $z_0 \in D$  and  $r > 0$  are chosen such that  $\overline{S(z_0, r)} \subset D$ , then*

$$S(a, b; D_{S(z_0, r)}) = \begin{cases} 1, & \text{if } [a, b] \cap S(z_0, r) = \emptyset \\ \frac{|a-b|}{l(\gamma_{ab})}, & \text{if } [a, b] \cap S(z_0, r) \neq \emptyset \end{cases},$$

where  $\gamma_{ab}$  is the shortest polygonal arc from  $a$  to  $b$ , contained in  $D - S(z_0, r)$ , and having as intermediate points vertices of  $\partial S(z_0, r)$  (see Figure 4).

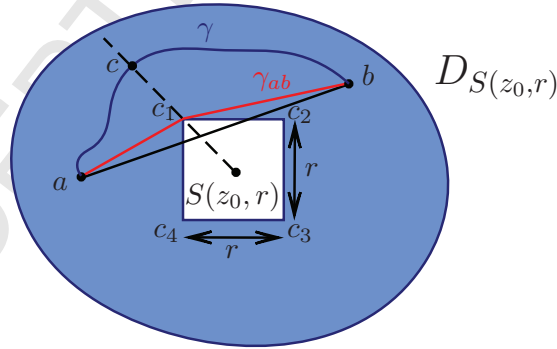


Figure 4: The domain  $D_{S(z_0, r)} = D - \overline{S(z_0, r)}$ .

*Proof.* The proof is similar to the proof of Lemma 8.

For example consider the case when  $[a, b] \cap [c_1, c_2] \neq \emptyset$  and  $[a, b] \cap [c_1, c_4] \neq \emptyset$  (see Figure 4).

An arbitrary curve  $\gamma \in \Gamma(a, b; D_{S(z_0, r)})$  must intersect the half-line  $(z_0 c_1)$  (or the three half-lines  $(z_0 c_2)$ ,  $(z_0 c_3)$  and  $(z_0 c_4)$ ). Denoting by  $c$  the point of intersection, we have

$$l(\gamma) = |a - c| + |c - b| \geq |a - c_1| + |b - c_1| = l(\gamma_{ab}),$$

where in this case  $\gamma_{ab} = [a, c_1] \cup [c_1, b]$ , and therefore

$$S(a, b; D_{S(z_0, r)}) \leq \frac{|a - b|}{l(\gamma_{ab})}.$$

To prove the reverse inequality, we can approximate the curve  $\gamma_{ab}$  by curves  $(\gamma_n)_{n \geq 1} \subset \Gamma(a, b; D_{S(z_0, r)})$  with  $\lim_{n \rightarrow \infty} l(\gamma_n) = l(\gamma_{ab})$ , and therefore we obtain in this case

$$S(a, b; D_{S(z_0, r)}) \leq \frac{|a - b|}{l(\gamma_{ab})},$$

where  $\gamma_{ab} = [a, c_1] \cup [c_1, b]$ .

There are other cases to consider, but the proof being similar, we omit it.  $\square$

Using the previous lemma we can show the following:

**Theorem 14.** *If  $D$  is a convex domain and  $z_0 \in D$  and  $r > 0$  are chosen such that  $\overline{S(z_0, r)} \subset D$ , then the convexity constant of the domain  $D_{S(z_0, r)}$  is given by*

$$K(D_{S(z_0, r)}) = \frac{1}{2}. \quad (6)$$

*Proof.* The proof is similar to the proof of Theorem 11. The key is to show that

$$K(D_{S(z_0, r)}) = \inf_{\substack{a, b \in \partial S(z_0, r) \\ a \neq b}} \frac{|a - b|}{l(\gamma_{ab})}$$

is attained on the boundary of  $S(z_0, r)$ , where as in the previous lemma,  $\gamma_{ab}$  denotes the shortest polygonal path in  $D - S(z_0, r)$  having as intermediate points vertices of  $\partial S(z_0, r)$ .

Next, using elementary calculus it can be shown that the above infimum equals  $\frac{1}{2}$ , being attained at points  $a = \frac{c_1 + c_2}{2}$  and  $b = \frac{c_3 + c_4}{2}$  (or  $a = \frac{c_1 + c_4}{2}$  and  $b = \frac{c_2 + c_3}{2}$ ).  $\square$

**Remark 15.** Note that the previous equality can be written

$$K(D_{S(z_0, r)}) = \frac{1}{2} = \min_{\substack{a, b \in \partial S(z_0, r) \\ a \neq b}} \frac{|a - b|}{l(\gamma_{ab})},$$

where  $\gamma_{\alpha\beta}$  denotes the shorter of the two arcs of the boundary of  $\partial S(z_0, r)$  with endpoints  $a$  and  $b$ .

It is interesting to note that the same representation formula holds also in the case of the domains  $D_{B(z_0, r)}$ :

$$K(D_{B(z_0, r)}) = \frac{2}{\pi} = \min_{\substack{a, b \in \partial B(z_0, r) \\ a \neq b}} \frac{|a - b|}{l(\gamma_{ab})},$$

where  $\gamma_{ab}$  denotes the shortest of the two arcs of the boundary of  $\partial B(z_0, r)$  with endpoints  $a$  and  $b$ .

We believe the above representation of the convexity constant of a doubly connected domain  $D_\Omega = D - \overline{\Omega}$  applies more generally. We propose the following conjecture:

**Conjecture 16.** If  $D$  and  $\Omega$  are convex domains with  $\overline{\Omega} \subset D$ , the convexity constant of the domain  $D_\Omega = D - \overline{\Omega}$  is given by

$$K(D_\Omega) = \min_{\substack{a, b \in \partial \Omega \\ a \neq b}} \frac{|a - b|}{l(\gamma_{ab})},$$

where  $\gamma_{ab}$  denotes the shorter of the two arcs of the boundary  $\partial \Omega$  with endpoints  $a$  and  $b$  (see Figure 5).

For  $z_0 \in (-1, 1)$  and  $0 < \alpha, \beta < \pi$ , we will denote by  $A(z_0, \alpha, \beta)$  the angular region

$$A(z_0, \alpha, \beta) = \{z_0 + re^{i\theta} : r > 0, -\arg(e^{i\beta} - z_0) < \theta < \arg(e^{i\alpha} - z_0)\}. \quad (7)$$

Simple geometric consideration show the following:

**Lemma 17.** For any  $\alpha, \beta \in (0, \pi)$ ,  $z_0 \in (-1, 1)$  and  $a, b \in U_{A(z_0, \alpha, \beta)}$  we have:

$$S(a, b; U_{A(z_0, \alpha, \beta)}) = \begin{cases} 1, & \text{if } [a, b] \cap (z_0, 1) = \emptyset \\ \frac{|a - b|}{|a - z_0| + |b - z_0|}, & \text{if } [a, b] \cap (z_0, 1) \neq \emptyset \end{cases}.$$

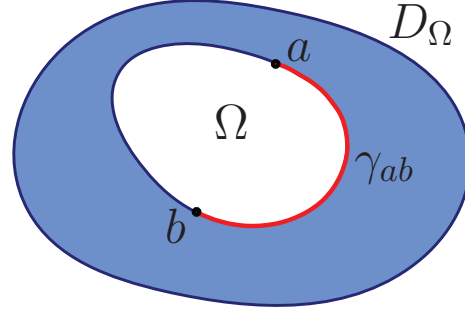


Figure 5: The convexity constant of the domain  $D_\Omega = D - \bar{\Omega}$  is given by  $\gamma_{ab}$ .

Using the above lemma and arguments similar to the proof of the Theorem 11 and Theorem 14, we obtain the following:

**Theorem 18.** *The convexity constant of the domain  $U_{A(z_0, \alpha, \beta)}$  is given by*

$$K(U_{A(z_0, \alpha, \beta)}) = \begin{cases} 1, & \text{if } z_0 \in \left[ \frac{\cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2}}, 1 \right) \\ \sin \frac{\arg(e^{i\alpha} - z_0) + \arg(e^{i\beta} - z_0)}{2}, & \text{if } z_0 \in \left( -1, \frac{\cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2}} \right) \end{cases}$$

*Proof.* If the condition  $z_0 \in \left[ \frac{\cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2}}, 1 \right)$  holds, it is easy to see that  $U_{A(z_0, \alpha, \beta)}$  is a convex domain, and therefore  $K(U_{A(z_0, \alpha, \beta)}) = 1$  in this case.

For  $z_0 \in \left( -1, \frac{\cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2}} \right)$ , using the above lemma it can be shown that if  $a \in [z_0, e^{i\alpha}]$  and  $b \in [z_0, e^{i\beta}]$  then

$$S(a, b; U_{A(z_0, \alpha, \beta)}) \leq S(a_1, b_1; U_{A(z_0, \alpha, \beta)})$$

for any points  $a_1, b_1 \in U_{A(z_0, \alpha, \beta)}$  of the line through  $a$  and  $b$ , such that  $[a_1, b_1] \cap (z_0, 1) \neq \emptyset$ . It follows that the value of the convexity constant of  $U_{A(z_0, \alpha, \beta)}$  is attained for points  $a \in [z_0, e^{i\alpha}]$  and  $b \in [z_0, e^{-i\beta}]$ , that is

$$K(U_{A(z_0, \alpha, \beta)}) = \min_{s, t \in [0, 1]} \frac{|sz_0 + (1-s)e^{i\alpha} - tz_0 - (1-t)e^{-i\beta}|}{(1-s)|e^{i\alpha} - z_0| + (1-t)|e^{-i\beta} - z_0|}.$$

It is not difficult to check that the above minimum is attained for  $s = t = 0$ , that is for points  $a = e^{i\alpha}$  and  $b = e^{-i\beta}$ , and using the law of sines in

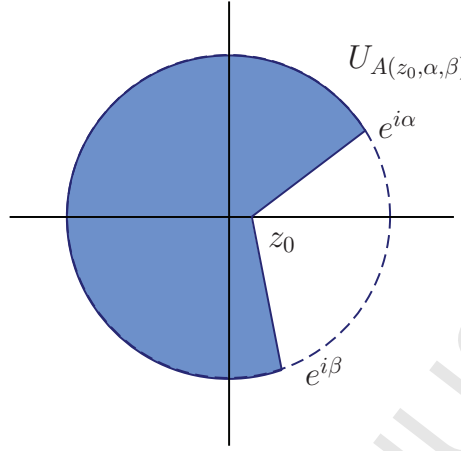


Figure 6: The domain  $U_{A(z_0, \alpha, \beta)} = U - \overline{A(z_0, \alpha, \beta)}$ .

the triangle with vertices  $z_0$ ,  $e^{i\alpha}$  and  $e^{-i\beta}$  we obtain

$$\begin{aligned} K(U_{A(z_0, \alpha, \beta)}) &= \frac{|e^{i\alpha} - e^{-i\beta}|}{|e^{i\alpha} - z_0| + |e^{-i\beta} - z_0|} \\ &= \frac{\sin(\alpha_1 + \beta_1)}{\sin\left(\frac{\pi}{2} - \beta_1 + \frac{\beta - \alpha}{2}\right) + \sin\left(\frac{\pi}{2} - \alpha_1 - \frac{\beta - \alpha}{2}\right)}, \end{aligned}$$

where  $\alpha_1 = \arg(e^{i\alpha} - z_0)$  and  $\beta_1 = \arg(e^{i\beta} - z_0)$ , and therefore we obtain

$$K(U_{A(z_0, \alpha, \beta)}) = \frac{\sin\left(\frac{\alpha_1 + \beta_1}{2}\right)}{\cos\left(\frac{\alpha_1 - \beta_1}{2} + \frac{\beta - \alpha}{2}\right)} = \sin\left(\frac{\alpha_1 + \beta_1}{2}\right),$$

as needed.  $\square$

In the particular case  $z_0 = 0$ , from the theorem above we obtain the following:

**Corollary 19.** *The convexity constant of the domain  $U_{A(0, \alpha, \beta)}$  is given by*

$$K(U_{A(0, \alpha, \beta)}) = \begin{cases} 1, & \text{if } \alpha + \beta \geq \pi \\ \sin \frac{\alpha + \beta}{2} & \text{if } 0 < \alpha + \beta < \pi \end{cases}$$



In the symmetric case  $\alpha = \beta \in (0, \pi)$ , from the theorem above we also obtain the following:

**Corollary 20.** *The convexity constant of the domain  $U_{A(z_0, \alpha, \alpha)}$  is given by*

$$K(U_{A(z_0, \alpha, \alpha)}) = \begin{cases} 1, & \text{if } z_0 \in [\cos \alpha, 1) \\ \sin \arg(e^{i\alpha} - z_0), & \text{if } z_0 \in (-1, \cos \alpha) \end{cases}.$$

In particular, for  $z_0 = 0$  we have:

$$K(D_{A(0, \alpha, \alpha)}) = \begin{cases} 1, & \text{if } \frac{\pi}{2} \leq \alpha < \pi \\ \sin \alpha, & \text{if } 0 < \alpha < \frac{\pi}{2} \end{cases}$$

Considering  $\alpha \searrow 0$ , from the previous corollary we obtain the following:

**Corollary 21.** *The convexity constant of the domain  $U_{[0,1]} = U - \{z \in \mathbb{R} : 0 \leq z \leq 1\}$  is  $K(U_{[0,1]}) = 0$ .*

### 3. Applications

We denote by  $\mathcal{A}$  the class of analytic functions  $f : U \rightarrow \mathbb{C}$  in  $U$ , normalized by the conditions  $f(0) = f'(0) - 1 = 0$  and  $f(z) \neq 0$  for  $z \in U - \{0\}$ .

S. Ozaki and M. Nunokawa obtained the following univalence criterion for functions belonging to the class  $\mathcal{A}$ :

**Theorem 22** ([2]). *If  $f \in \mathcal{A}$  and*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq |z|^2, \quad z \in U, \quad (8)$$

*then  $f$  is univalent in  $U$ .*

**Remark 23.** *For an arbitrary function  $f \in \mathcal{A}$  with Taylor series expansion given by  $f(z) = z + a_2 z^2 + \dots$ , consider the function  $g : U - \{0\} \rightarrow \mathbb{C}$ ,*

$$g(z) = \frac{z^2 f'(z)}{f^2(z)} - 1, \quad z \in U - \{0\}. \quad (9)$$

Since

$$\lim_{z \rightarrow 0} g(z) = \lim_{z \rightarrow 0} \frac{z^2(1 + 2a_2 z + \dots)}{z^2(1 + a_2 z + \dots)^2} - 1 = 0,$$

it follows that  $g$  can be extended to an analytic function in  $U$ , denoted for simplicity also by  $g$ .

A simple computations shows that the Taylor series expansion of  $g$  is given by

$$g(z) = (a_3 - a_2^2) z^2 + \dots, \quad z \in U.$$

Let us note that if the condition (8) holds, then we have  $|g(z)| \leq |z|^2 < 1$  for all  $z \in U$ .

Conversely, if  $|g(z)| < 1$  for all  $z \in U$ , then by Schwarz lemma it follows that

$$|g(z)| \leq |z|^2, \quad z \in U,$$

which shows that the condition (8) holds.

Therefore the condition (8) in the previous theorem is equivalent to the condition

$$\left| \left( \frac{1}{f(z)} - \frac{1}{z} \right)' \right| < 1, \quad z \in U. \quad (10)$$

The previous remark shows that the Ozaki-Nunokawa univalence criterion (Theorem 22) is equivalent to the following theorem, obtained independently by J. G. Krzyz:

**Theorem 24** ([1]). *If the function  $f : U \rightarrow \mathbb{C}$  belongs to the class  $\mathcal{A}$  and satisfies*

$$\left| \left( \frac{1}{f(z)} - \frac{1}{z} \right)' \right| < 1, \quad z \in U, \quad (11)$$

*then  $f$  is univalent in  $U$ .*

The convexity of the unit disk  $U$  plays an essential role in the proof of Theorem 22 and Theorem 24, and it is not immediate how one could extend the result for example in the case of analytic functions defined in non-convex domains.

Using the convexity constant of a domain  $D \subset \mathbb{C}$  defined in the previous section, we can obtain sufficient conditions for the univalence of an analytic function defined in non-convex domains. A first univalence criteria is the following:

**Theorem 25.** *Let  $D \subset U$  be a domain in  $\mathbb{C}$  containing the origin, and let  $f : D \rightarrow \mathbb{C}$  be analytic in  $D$  and normalized by  $f(0) = f'(0) - 1 = 0$  and*

$f(z) \neq 0$  for  $z \in D - \{0\}$ . If

$$\left| \left( \frac{1}{f(z)} - \frac{1}{z} \right)' \right| \leq K(D), \quad z \in D, \quad (12)$$

then the function  $f$  is univalent in  $D$ .

*Proof.* Under the hypothesis of the theorem it follows that the function  $\frac{1}{f(z)} - \frac{1}{z}$  is well defined and analytic in  $D$  (it has a removable singularity at  $z = 0$ ).

First note that if  $K(D) = 0$ , then from the hypothesis it follows that for some constant  $c \in \mathbb{C}$  we have

$$f(z) = \frac{z}{1 + cz}, \quad z \in D,$$

and therefore  $f$  is univalent in this case.

Without loss of generality we can therefore assume  $K(D) \neq 0$ .

Assuming that  $f$  is not univalent in  $D$ , there exist distinct points  $a, b \in D$  such that  $f(a) = f(b)$ , and by the hypothesis it also follows that  $a, b \neq 0$ .

For an arbitrarily fixed rectifiable arc  $\gamma \subset \Gamma(a, b; D)$  joining  $a$  and  $b$  in  $D$ , integrating  $\left( \frac{1}{f(z)} - \frac{1}{z} \right)'$  along  $\gamma$  and using the hypothesis, we obtain

$$\begin{aligned} \left| \frac{1}{b} - \frac{1}{a} \right| &= \left| \frac{1}{f(b)} - \frac{1}{f(a)} - \left( \frac{1}{b} - \frac{1}{a} \right) \right| \\ &= \left| \int_{\gamma} \left( \frac{1}{f(z)} - \frac{1}{z} \right)' dz \right| \\ &\leq \int_{\gamma} \left| \frac{1}{f(z)} - \frac{1}{z} \right|' |dz| \\ &\leq K(D) l(\gamma), \end{aligned}$$

or equivalent

$$\frac{|a - b|}{l(\gamma)} \leq |ab| K(D).$$

Since  $\gamma \in \Gamma(a, b; D)$  was arbitrarily chosen, we obtain

$$S(a, b; D) = \sup_{\gamma \in \Gamma(a, b; D)} \frac{|a - b|}{l(\gamma)} \leq |ab| K(D),$$

and since  $|ab| < 1$  and  $K(D) > 0$ , we obtain

$$K(D) \leq S(a, b; D) \leq |ab| K(D) < K(D),$$

a contradiction.

It follows that the function  $f$  is univalent in  $D$ , concluding the proof.  $\square$

We can generalize the previous theorem to the case of bounded domains in  $\mathbb{C}$  as follows:

**Theorem 26.** *Let  $D \subset aU$  ( $a > 0$ ) be a domain in  $\mathbb{C}$  containing the origin and let  $f : D \rightarrow \mathbb{C}$  be analytic in  $D$  and normalized by  $f(0) = f'(0) - 1 = 0$  and  $f(z) \neq 0$  for  $z \in D - \{0\}$ . If*

$$\left| \left( \frac{1}{f(z)} - \frac{1}{z} \right)' \right| \leq \frac{K(D)}{a^2}, \quad z \in D, \quad (13)$$

*then the function  $f$  is univalent in  $D$ .*

*Proof.* Follows from the previous theorem by considering the function

$$f_a(z) = \frac{1}{a} f(az) : \frac{1}{a} D \subset U \rightarrow \mathbb{C},$$

and using the fact that the convexity constant of a domain is scaling invariant, that is  $K(aD) = K(D)$ , for any  $a > 0$ .  $\square$

Using Remark 3, from the previous theorem we obtain the following:

**Corollary 27.** *Let  $D \subset aU$  ( $a > 0$ ) be a convex domain containing the origin and let  $f : D \rightarrow \mathbb{C}$  be analytic in  $D$  and normalized by  $f(0) = f'(0) - 1 = 0$  and  $f(z) \neq 0$  for  $z \in D - \{0\}$ . If*

$$\left| \left( \frac{1}{f(z)} - \frac{1}{z} \right)' \right| \leq \frac{1}{a^2}, \quad z \in D, \quad (14)$$

*then the function  $f$  is univalent in  $D$ .*

**Remark 28.** *In the particular case  $D = U$  and  $a = 1$ , from the above corollary we obtain the Ozaki-Nunokawa-Krzyz univalence criterion (Theorem 22 and Theorem 24), so Theorem 25 and Theorem 26 above can be viewed as generalizations of it, in the case of analytic functions defined in arbitrary (not necessarily convex) domains.*

As an example, let us consider the function  $f_a : U_{\{1/a\}} = U - \{1/a\} \rightarrow \mathbb{C}$  given by

$$f_a(z) = \frac{z}{(1 - az)^2}, \quad z \in U_{\{1/a\}},$$

where  $a \in \mathbb{C}$  with is a parameter.

It is easy to see that  $f_a$  is analytic in  $U_{\{1/a\}}$ ,  $f_a(0) = f'_a(0) - 1 = 0$  and  $f_a(z) \neq 0$  for  $z \neq 0$ . Also, since the convexity constant of the punctured disk is 1 (see Remark 6), we see that the hypothesis (12) of Theorem 25 is satisfied only if

$$\left| \left( \frac{1}{f_a(z)} - \frac{1}{z} \right)' \right| = \left| (-2a + a^2 z)' \right| = |a|^2 \leq 1 = K(U_{\{1/a\}}),$$

that is only if  $|a| \leq 1$ . Therefore, by Theorem 25 it follows that  $f_a$  is univalent in  $U_a = U - \{1/a\} = U$  for any  $a \in \mathbb{C}$  with  $|a| \leq 1$ , and in particular for  $|a| = 1$  we obtain that any rotation of the Kőbe function is univalent in  $U$ .

It is easy to see that  $f_a$  is not univalent in  $U_{\{1/a\}}$  for any  $|a| > 1$ , which shows that the hypothesis of Theorem 25 is sharp, in the sense that the constant  $K(D)$  appearing on the right of inequality (12) cannot be replaced by a larger constant.

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