Doctopic: Partial Differential Equations ARTICLE IN PRESS YJMAA:20828

J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Liouville theorems for supersolutions of semilinear elliptic equations with drift terms in exterior domains

Takanobu Hara

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji-shi, Tokyo 192-0397, Japan

A R T I C L E I N F O

Article history:
Received 5 August 2016
Available online xxxx
Submitted by Y. Du

Keywords:
Liouville theorem
Drift terms
Semilinear equations
Potential theory
Fundamental solution

A B S T R A C T

In this paper, we prove nonexistence of positive supersolutions of a semilinear equation $-\mathrm{div}\,(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u = f(u)$ in exterior domains in $\mathbb{R}^n$ $(n \geq 3)$, where $A(x)$ is bounded and uniformly elliptic, $\mathbf{b}(x) = O(|x|^{-1})$, $\mathrm{div}\,\mathbf{b} = 0$ and f is a continuous and positive function in $(0, \infty)$ satisfying $f(u) \sim u^q$ as $u \to 0$ with $q \leq n/(n-2)$. Furthermore, we investigate general conditions on $\mathbf{b}$ and f for nonexistence of positive supersolutions.

1. Introduction

We consider nonexistence of positive (weak) solutions to differential inequalities

$$\mathcal{L}u := -\mathrm{div}\,(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u \geq f(x, u) \quad \text{in } \mathbb{R}^n \setminus \overline{B_R}, \tag{1}$$

where $n \geq 3$ and B_R is a ball of radius $R > 0$ centered at the origin and $A = A(x)$ is a bounded measurable matrix-valued function which satisfies

$$\|A\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad (A(x)\xi) \cdot \xi \geq |\xi|, \quad \forall x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n.$$

Throughout the paper, we also assume that the vector-valued function $\mathbf{b} = \mathbf{b}(x)$ belongs to $(L^2_{\mathrm{loc}}(\mathbb{R}^n))^n$ and $f : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is a continuous function. Specific conditions on $\mathbf{b}$ and f will be described later.

Gidas [6] and Gidas and Spruck [7] proved the nonexistence of positive C^2 supersolutions of

$$-\triangle u + \frac{\beta x}{|x|^2} \cdot \nabla u = u^q \quad \text{in } \mathbb{R}^n \setminus \overline{B_R} \tag{2}$$

E-mail address: takanobuhara526@gmail.com.

http://dx.doi.org/10.1016/j.jmaa.2016.10.053

for $1 < q \le (n-\beta)/(n-\beta-2)$. When $\beta = 0$, the range of the exponent is

$$1 < q \le \frac{n}{(n-2)}. \tag{3}$$

Note that if $\beta < 0$, then $1 < (n-\beta)/(n-\beta-2) < n/(n-2)$. Therefore, when $\mathbf{b} = O(|x|^{-1})$ and $\mathbf{b} \neq 0$, the nonexistence range (3) changes, in general. The exponent $(n-\beta)/(n-\beta-2)$ is sharp. Indeed, for $-\infty < \beta < n-2$ and $q > (n-\beta)/(n-\beta-2)$, the equation (2) has a positive solution $u(x) = c(n,q)|x|^{-2/(q-1)}$.

This type of nonexistence theorem has been extended for more general supersolutions of linear and nonlinear equations by many authors, see e.g. [22,19,17,18,5,12–14,4,1,2].

Recently, Armstrong and Sirakov [4] treated a wide class of second order (nonlinear) elliptic differential operators and nonlinearities. They developed a new method to show nonexistence of supersolutions of the equation

$$-Q[u] = f(x,u) \quad \text{in } \mathbb{R}^n \setminus \overline{B_R}, \tag{4}$$

where $Q[u]$ is several homogeneous elliptic differential operators with general nonlinearity $f(x,u)$. In particular, they proved that if $Q[u] = \triangle u$ and if $f(x,u) = f(u)$ satisfying $\liminf_{s\to 0} s^{-n/(n-2)} f(s) > 0$, then the equation (4) has no positive supersolutions.

On the other hand, Kondratiev et al. [14] gave a sufficient condition on $\mathbf{b}$ to assure the nonexistence of positive supersolutions of (1) for $f(x,u) = u^q$ with $1 < q \le n/(n-2)$. In [14], it was assumed that $A(x)$ is Hölder continuous and periodic with the same period, $\mathbf{b}$ satisfies some Kato type conditions, moreover,

$$|||\mathbf{b}||| = \left\{ C > 0;\ \frac{\int_{\mathbb{R}^n} |\mathbf{b}|^2 \phi^2\, dx}{\int_{\mathbb{R}^n} |\nabla \phi|^2\, dx} \le C^2 \quad \forall \phi \in C_c^\infty(\mathbb{R}^n) \right\} \tag{5}$$

is sufficiently small in some sense. Under these conditions, it was proved that (1) has no positive weak supersolutions if and only if $q \le n/(n-2)$.

In this paper, we give new sufficient conditions on $\mathbf{b}$ and f for nonexistence of positive supersolutions, using methods in [4] and techniques of (nonlinear) potential theory (see e.g. [11,21,15,10]). We shall prove the following:

Theorem 1. *Suppose that vector field* $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_1$ *satisfies*

$$|||\mathbf{b}_0||| < \infty \quad \textit{and} \quad \operatorname{div} \mathbf{b}_0 = 0 \quad \textit{in } \mathbb{R}^n \tag{6}$$

and

$$\mathbf{b}_1 \in (L^{n,1}(\mathbb{R}^n \setminus \overline{B_R}))^n \quad \textit{for some } R \ge 0. \tag{7}$$

Assume that $f(x,u) = |x|^{-\gamma} g(u)$, $\gamma < 2$ *and*

$$\liminf_{s\to 0} s^{-q} g(s) > 0$$

for $q = 1 + (2-\gamma)/(n-2)$. *Then* (1) *has no positive weak solutions.*

Here, $L^{p,\sigma}(\Omega)$ is a Lorentz space (see Section 2 for details). When $A(x) = I$, $\mathbf{b}_1 = 0$ and $f(x,u) = u^q$, Theorem 1 becomes as follows:

Corollary 2. *Suppose that vector field* $\mathbf{b}_0$ *satisfies* (6), *and that* $q \leq n/(n-2)$. *Then the differential inequality*

$$-\triangle u + \mathbf{b}_0 \cdot \nabla u \geq u^q \quad \text{in } \mathbb{R}^n \setminus \overline{B_R} \tag{8}$$

has no positive weak solutions.

Remark 3. The nonexistence range $q \leq n/(n-2)$ is sharp. Let η be a smooth bump function, and let $\varphi = (1-\eta(x))\log(|x|)$. Then

$$\mathbf{b}_0(x) = \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}, 0, \dots, 0\right)^{\mathrm{T}}$$

satisfies (6) and $\mathbf{b}_0(x) \cdot x = 0$ outside of $\operatorname{supp} \eta$. Therefore, for this $\mathbf{b}_0$ and $q > n/(n-2)$, (8) has a positive solution $u(x) = |x|^{-2/(q-1)}$ with sufficiently large $R > 0$. Note that $\mathbf{b}(x) = \beta |x|^{-2} x$ does not satisfy (6) for any $\beta \neq 0$, because this vector field has a scalar potential $\beta \log(|x|)$.

Remark 4. From a similar cutoff argument, if $\mathbf{b}(x) = (b_i(x)) \in (L^{n,\infty}(\mathbb{R}^n \setminus \overline{B_R}))^n$ has a vector potential i.e. there exists a skew symmetric matrix valued function $V(x) = (v_{ij}(x))$ $(i, j = 1, \dots n)$ such that $\mathbf{b}_i(x) = \sum_{j=1}^n \partial_j v_{ij}(x)$, then we can apply Corollary 2. Here, we emphasize that we have considered the equation in exterior domains.

Remark 5. The assumption on $\mathbf{b}_1$ closely relate to Kato-type conditions. For example, by a simple calculation, if $\mathbf{b}_0 = 0$,

$$|\mathbf{b}_1(x)| \leq \frac{\beta(|x|)}{|x|} \quad \text{and} \quad \int_R^\infty \beta(s) \frac{ds}{s} < \infty \quad \text{for some } R \geq 0, \tag{9}$$

then $\mathbf{b}$ satisfies our condition (7) and the assumptions in [14].

Kondratiev et al. [14] proved an Aronson-type estimate of a fundamental solution of $(\partial_t + \mathcal{L})$. From this estimate, it follows that if u is a positive supersolution of $\mathcal{L}u = 0$ in $\mathbb{R}^n \setminus \overline{B_{1/2}}$, then there is a positive constant $c > 0$ such that

$$\inf_{\partial B_R} u \geq c R^{2-n} \quad \text{in } \mathbb{R}^n \setminus \overline{B_1}. \tag{10}$$

The lower bound plays an important role in the proof of nonexistence theorem. They used the assumptions on Hölder continuity and periodicity of $A(x)$ to prove this. Semenov [20] proved an Aronson-type estimate for bounded $A(x)$ and $\mathbf{b}_0$ satisfying the condition (6) (see also [16] for $A(x) = I$). Also, he studied somewhat general conditions on $\mathbf{b}$. Unfortunately, we can not use his result under the assumption (7).

We prove (10) more directly using a method in [21,15,10]. We do not need the Hölder continuity and periodicity of $A(x)$.

Next, we give another condition on $\mathbf{b}$ and f for nonexistence of supersolutions.

Theorem 6. *Instead of the assumptions on* $\mathbf{b}_1$ *and* f *as in* Theorem 1, *suppose that* $\mathbf{b}_1$ *satisfies*

$$|||\mathbf{b}_1 \mathbf{1}_{\mathbb{R}^n \setminus \overline{B_R}}||| \to 0 \quad \text{as } R \to \infty. \tag{11}$$

Assume that $f(x, u) = |x|^{-\gamma} g(u)$, $\gamma < 2$ *and*

$$\liminf_{s\to 0} s^{-q_0} g(s) > 0$$

for some $1 < q_0 < 1 + (2-\gamma)/(n-2)$. *Then* (1) *has no positive weak solutions.*

If $\mathbf{b} \in (L^n(\mathbb{R}^n \setminus \overline{B_R}))^n$ for some $R > 0$, then (11) holds since

$$|||\mathbf{b}\mathbf{1}_{\mathbb{R}^n\setminus\overline{B_R}}||| \le C(n,\sigma)\|\mathbf{b}\|_{L^n(\mathbb{R}^n\setminus\overline{B_R})} \to 0 \quad \text{as } R \to \infty.$$

Hence, we also have the following:

Corollary 7. *If* $\mathbf{b} \in (L^n(\mathbb{R}^n \setminus \overline{B_R}))^n$ *for some* $R > 0$, *and* $f(u) = u^q$ *with* $q < n/(n-2)$, *then* (1) *has no positive weak solutions.*

Remark 8. The nonexistence range $q < n/(n-2)$ is sharp. Indeed, as Example 3.12 in [14], the equation

$$-\triangle u + \frac{\beta x}{|x|^2 \log|x|} \cdot \nabla u = u^{n/(n-2)} \quad \text{in } \mathbb{R}^n \setminus \overline{B_R} \tag{12}$$

has a positive supersolution $u(x) = c(|x|\log|x|)^{2-n}$ for $\beta < 2-n$. Since

$$\frac{1}{|x|\log|x|} \in L^n(\mathbb{R}^n \setminus \overline{B_2}) \setminus L^{n,1}(\mathbb{R}^n \setminus \overline{B_2}),$$

the nonexistence theorem of supersolutions does not hold for the critical exponent $q = n/(n-2)$. Here, we also note that (10) cannot hold for $\mathbf{b} \in (L^n(\mathbb{R}^n \setminus \overline{B_R}))^n$ in general.

Organization of the paper In Section 2, we recall some properties of Lorentz spaces. In Section 3, we prove several quantitative properties of weak (super-, sub-)solutions to $\mathcal{L}u = f(x)$. In particular, we establish Lemma 15. In Section 4, we derive (10) using results of Section 3. In Section 5, we investigate behavior of

$$m(r) := \inf_{A(r)} u$$

by using results in Section 3 and 4, and we prove Theorem 1. In Section 6, we give a proof of Theorem 6 modifying the proof of Theorem 1.

Notation We use the following notation in this paper. Let Ω be a domain of $\mathbb{R}^n$.

- $B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$, $B_R := B_R(0)$.
- $A(R) := B_{2R} \setminus \overline{B_R}$.
- $|A|$:= the Lebesgue measure of a measurable set A.
- $\mathbf{1}_A(x)$:= the indicator function of A.
- $⨍_A f\, dx := \frac{1}{|A|}\int_A f\, dx$.
- $f_+ := \max\{f, 0\}$, $f_- := \max\{-f, 0\}$.

2. Lorentz spaces

First, we recall some properties of Lorentz spaces. For $1 < p < \infty$ and $1 \le \sigma \le \infty$, we take

$$\|f\|_{L^{p,\sigma}(\Omega)} := \begin{cases} \left(p\int_0^\infty \left(t|\{x \in \Omega;\ |f(x)| \ge t\}|^{1/p}\right)^\sigma \frac{dt}{t}\right)^{1/\sigma} & \text{if } \sigma < \infty, \\ \sup_{t>0} t|\{x \in \Omega;\ |f(x)| \ge t\}|^{1/p} & \text{if } \sigma = \infty, \end{cases}$$

and

$$L^{p,\sigma}(\Omega) := \left\{ f;\ \Omega \to \mathbb{R} \text{ measurable};\ \|f\|_{L^{p,\sigma}(\Omega)} < \infty \right\}.$$

Note that $L^{p,p}(\Omega) = L^p(\Omega)$ and $L^{p,\sigma}(\Omega) \subsetneq L^{p,\tau}(\Omega)$ for $1 \le \sigma < \tau \le \infty$. In particular,

$$\|f\|_{L^{p,\tau}(\Omega)} \le C(p,\sigma,\tau)\|f\|_{L^{p,\sigma}(\Omega)}.$$

For $1 \le \sigma < \infty$, $\|\cdot\|_{L^{p,\sigma}(\Omega)}$ is defined by an integral. Hence, for any $f \in L^{p,\sigma}(\mathbb{R}^n)$,

$$\|f\mathbf{1}_{\mathbb{R}^n\setminus\overline{B_R}}\|_{L^{p,\sigma}(\mathbb{R}^n)} \to 0 \quad \text{as } R \to \infty. \tag{13}$$

On the other hand, $\|f\mathbf{1}_{\mathbb{R}^n\setminus\overline{B_R}}\|_{L^{p,\infty}(\mathbb{R}^n)}$ does not generally go to 0 as $R \to \infty$, for example, $|x|^{-1} \in L^{n,\infty}(\mathbb{R}^n)\setminus L^{n,\sigma}(\mathbb{R}^n)$. The following Hölder type inequality is standard:

Lemma 9 *([8, p. 52]). Assume that $1 < p, p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then,*

$$\left|\int_\Omega fg\ dx\right| \le \|f\|_{L^{p,1}(\Omega)}\|g\|_{L^{p',\infty}(\Omega)}.$$

According to a sharp form of Sobolev inequality (see e.g. [3]), for any $\phi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathbf{b}|^2\phi^2\ dx &\le \||\mathbf{b}|^2\|_{L^{n/2,\infty}(\mathbb{R}^n)}\|\phi^2\|_{L^{n/(n-2),1}(\mathbb{R}^n)}\\ &= \|\mathbf{b}\|^2_{L^{n,\infty}(\mathbb{R}^n)}\|\phi\|^2_{L^{2n/(n-2),2}(\mathbb{R}^n)} \le S_2^2\|\mathbf{b}\|^2_{L^{n,\infty}(\mathbb{R}^n)}\|\nabla\phi\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

In particular, for any $1 \le \sigma \le \infty$,

$$|||\mathbf{b}||| \le C(n,\sigma)\|\mathbf{b}\|_{L^{n,\sigma}(\mathbb{R}^n)}. \tag{14}$$

3. Regularity of solutions to $\mathcal{L}u = f(x)$

In this section, we review the properties of weak (super-, sub-)solutions of $\mathcal{L}u = f(x)$. We say that $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak (super-, sub-)solution of the equation $\mathcal{L}u = f(x,u)$ in Ω if $f(x,u) \in L^1_{\mathrm{loc}} \cap H^{-1}(\Omega)$ and

$$\int_\Omega A\nabla u\cdot\nabla\phi + \mathbf{b}\cdot\nabla u\phi\, dx = (\ge,\le) \int_\Omega f(x,u)\phi\, dx, \tag{15}$$

for any nonnegative $\phi \in C_c^\infty(\Omega)$. By the uniform ellipticity of A, we have the following:

Lemma 10. *Let u be a weak subsolution to the equation $\mathcal{L}u = 0$ in Ω. Then* $\max\{u,k\}$ *is a weak subsolution to the same equation. Let u be a weak supersolution to the equation $\mathcal{L}u = 0$ in Ω. Then* $\min\{u,k\}$ *is a weak supersolution to the same equation.*

Since A is uniformly elliptic and $\operatorname{div}\mathbf{b}_0 = 0$,

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$$\begin{aligned}\int_\Omega & A\nabla u\cdot\nabla u + \mathbf{b}\cdot\nabla u u\,dx \\ &= \int_\Omega A\nabla u\cdot\nabla u\,dx + \frac{1}{2}\int_\Omega \mathbf{b}_0\cdot\nabla(u^2)\,dx + \int_\Omega \mathbf{b}_1\cdot\nabla u u\,dx \\ &\geq \int_\Omega |\nabla u|^2\,dx - |||\mathbf{b}_1|||\int_\Omega |\nabla u|^2\,dx\end{aligned} \tag{16}$$

for any $u \in C_c^\infty(\mathbb{R}^n)$. Therefore, when $|||\mathbf{b}_1||| < 1$, the operator $\mathcal{L}$ is coercive. Hereafter, for simplicity, we always assume that

$$|||\mathbf{b}_1||| \leq \frac{1}{2}. \tag{17}$$

Note that from (14), there is a constant $\mathcal{B}_1 = \mathcal{B}_1(n)$ such that

$$\|\mathbf{b}_1\|_{L^{n,1}(\mathbb{R}^n)} \leq \mathcal{B}_1 \Longrightarrow |||\mathbf{b}_1||| \leq \frac{1}{2}. \tag{18}$$

The following comparison principle is proved by standard methods:

Lemma 11 *(Comparison principle). Assume (17). Let $u \in H^1(\Omega)$ be a weak supersolution to $\mathcal{L}u = 0$ in Ω, and let $v \in H^1(\Omega)$ be a weak subsolution to $\mathcal{L}u = 0$ in Ω. If $(u - v)_- \in H_0^1(\Omega)$, then $u \geq v$ in Ω.*

Moreover, from De Giorgi or Moser's iteration technique and the John–Nirenberg lemma, we can get the following estimates (see e.g. [9]):

Lemma 12. *Assume (17). Let u be a weak subsolution to $\mathcal{L}u = 0$ in $B_R(x_0)$. Then for any $p > 0$ there exists a constant C_B depending only on n, $\|A\|_{L^\infty(\mathbb{R}^n)}$, $|||\mathbf{b}_0|||$ and p such that*

$$\sup_{B_{R/2}(x_0)} u_+ \leq C_B \left(\fint_{B_R(x_0)} u_+^p\,dx \right)^{1/p}. \tag{19}$$

Lemma 13. *Assume (17). Let u be a nonnegative weak supersolution to $\mathcal{L}u = 0$ in $B_{2R}(x_0)$. Then there exist constants $\sigma > 0$ and C_W depending only on n, $\|A\|_{L^\infty(\mathbb{R}^n)}$ and $|||\mathbf{b}_0|||$ such that*

$$\left(\fint_{B_R(x_0)} u^\sigma\,dx \right)^{1/\sigma} \leq C_W \inf_{B_R(x_0)} u. \tag{20}$$

Next, we assume the smallness of $\|\mathbf{b}_1\|_{L^{n,1}(\Omega)}$. Let

$$\|\mathbf{b}_1\|_{L^{n,1}(\Omega)} \leq \mathcal{B}_2 := \min\{\mathcal{B}_1, \frac{1}{2S_\infty}\}, \tag{21}$$

where $S_\infty = (n(n-2))^{-1/2}|B_1|^{-1/n}$.

Lemma 14. *Assume (21). Let Ω be a bounded domain. Let $u \in H_0^1(\Omega)$ be a weak solution to $\mathcal{L}u = f$ in Ω. Then there exists a positive constant C depending only on n such that*

$$\|u\|_{L^{n/(n-2),\infty}(\Omega)} \le C\|f\|_{L^1(\Omega)}.$$

Proof. By the result in [10], if $u \in H_0^1(\Omega)$ satisfies the equation $-\mathrm{div}\,(A\nabla u) + \mathbf{b}_0 \cdot \nabla u = g$, then

$$\|u\|_{L^{n/(n-2),\infty}(\Omega)} \le S_\infty^2 \|g\|_{L^1(\Omega)}$$

and

$$\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} \le S_\infty \|g\|_{L^1(\Omega)}.$$

Hence, applying the second inequality for $g = -\mathbf{b}_1 \cdot \nabla u + f$, we get

$$\begin{aligned}\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} &\le S_\infty \left(\|\mathbf{b}_1 \cdot \nabla u\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}\right) \\ &\le S_\infty \|\mathbf{b}_1\|_{L^{n,1}(\Omega)} \|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} + S_\infty \|f\|_{L^1(\Omega)}.\end{aligned}$$

By the assumption on $\mathbf{b}_1$, this implies that

$$\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} \le C\|f\|_{L^1(\Omega)}.$$

Consequently, we get

$$\begin{aligned}\|u\|_{L^{n/(n-2),\infty}(\Omega)} &\le S_\infty^2 \left(\|\mathbf{b}_1 \cdot \nabla u\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}\right) \\ &\le S_\infty^2 \|\mathbf{b}_1\|_{L^{n,1}(\Omega)} \|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} + S_\infty^2 \|f\|_{L^1(\Omega)} \\ &\le C\|f\|_{L^1(\Omega)}.\end{aligned}$$

This completes the proof. □

Below, we also assume that

$$\|\mathbf{b}_1\|_{L^{n,1}(\Omega)} \le \mathcal{B}_3 := \min\{\mathcal{B}_2, \frac{C(n)}{C_L C_W}\}, \tag{22}$$

where, $C(n)$ is a sufficiently small constant depending only on n to be determined later (see (28)). Under this condition, we get the following potential lower bound:

Lemma 15. *Assume (22). Let $f \in L^1(B_{2R}(x_0)) \cap H^{-1}(B_{2R}(x_0))$ and $f \ge 0$. Let u be a nonnegative weak supersolution to $\mathcal{L}u \ge f(x)$ in $B_{2R}(x_0)$. Let x_0 be a Lebesgue point of u. Then there exists a constant C_L depending only on n, $\|A\|_{L^\infty(\mathbb{R}^n)}$ and $|||\mathbf{b}_0|||$ such that*

$$u(x_0) \ge \frac{1}{C_L} \mathbf{I}_2^f(x_0, R),$$

where

$$\mathbf{I}_2^f(x_0, R) = \int_0^R \left(s^{2-n} \int_{B_s(x_0)} f\,dx \right) \frac{ds}{s}.$$

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Proof. Without loss of generality, we may assume that $x_0 = 0$ and 0 is a Lebesgue point of u. For $k = 0, 1, \ldots$, we take $R_k = 2^{1-k}R$. Fix k. Let $\eta(x) = (1-(8/R_k)\mathrm{dist}(x, \overline{B_{R_{k+1}}}))_+$ and let $w \in H_0^1(B_{R_k})$ be a weak solution to

$$\mathcal{L}w = \eta f \quad \text{in } B_{R_k}.$$

Then $w \geq 0$ in B_{R_k} by the minimum principle. Let

$$M := \sup_{\partial B_{\frac{3}{4}R_k}} w, \qquad m := \inf_{\partial B_{\frac{3}{4}R_k}} w.$$

According to Lemma 12 and 13, we have

$$M \leq C(n)C_L C_W m. \tag{23}$$

Let $\tilde{w} := \min\{w, M\}$. Since $\tilde{w}$ is a weak supersolution in $B_{\frac{3}{4}R_k}$, the minimum principle yields $\tilde{w} \geq m$ in $B_{\frac{3}{4}R_k}$. Since $\tilde{w} \geq m$ in $B_{\frac{3}{4}R_k}$, we have

$$m \int\limits_{B_{R_{k+1}}} f\, dx \leq \int\limits_{B_{R_k}} f\tilde{w}\, dx \leq \int\limits_{B_{R_k}} A\nabla w \cdot \nabla \tilde{w} + \mathbf{b} \cdot \nabla w \tilde{w}\, dx. \tag{24}$$

Note that if $\nabla \tilde{w}(x) \neq 0$ then $\tilde{w}(x) = w(x)$. Therefore,

$$\int\limits_{B_{R_k}} A\nabla w \cdot \nabla \tilde{w}\, dx = \int\limits_{B_{R_k}} A\nabla \tilde{w} \cdot \nabla \tilde{w}\, dx \leq \|A\|_{L^\infty(\Omega)} \int\limits_{B_{R_k}} |\nabla \tilde{w}|^2\, dx.$$

Moreover, since $\operatorname{div} \mathbf{b}_0 = 0$, we have

$$\int\limits_{B_{R_k}} \mathbf{b}_0 \cdot \nabla w \tilde{w}\, dx = -\int\limits_{B_{R_k}} \mathbf{b}_0 \cdot \tilde{w} \nabla \tilde{w}\, dx = \frac{-1}{2} \int\limits_{B_{R_k}} \mathbf{b}_0 \cdot (\nabla \tilde{w}^2)\, dx = 0.$$

Using Lemma 9 and Lemma 14, we have

$$\begin{aligned} \left| \int\limits_{B_{R_k}} \mathbf{b}_1 \cdot \nabla w \tilde{w}\, dx \right| &\leq M \|\mathbf{b}_1\|_{L^{n,1}(\Omega)} \|\nabla w\|_{L^{n/(n-1),\infty}(B_{R_k})} \\ &\leq C(n) M \|\mathbf{b}_1\|_{L^{n,1}(\Omega)} \|f\|_{L^1(B_{R_{k+1}})}. \end{aligned} \tag{25}$$

Take $v \in H_0^1(B_{R_k})$ such that

$$\begin{cases} \mathcal{L}v = 0 & \text{in } B_{R_k} \setminus \overline{B_{\frac{3}{4}R_k}}, \\ v = M & \text{on } \overline{B_{\frac{3}{4}R_k}}. \end{cases}$$

Then, by Lemma 11, $v - \tilde{w} \geq 0$ in B_{R_k}. Since $\tilde{w}$ is a weak supersolution to $\mathcal{L}\tilde{w} = 0$ in B_{R_k}, we have

$$\langle \mathcal{L}\tilde{w}, (v - \tilde{w}) \rangle \geq 0.$$

This implies that

$$\int_{B_{R_k}} |\nabla \tilde{w}|^2\, dx \le C \int_{B_{R_k}} |\nabla v|^2\, dx = C \int_{B_{R_k} \setminus \overline{B_{\frac{3}{4}R_k}}} |\nabla v|^2\, dx. \tag{26}$$

Next, we choose $\phi \in C_c^\infty(B_{R_k})$ such that $\phi \equiv 1$ on $B_{R_{k+1}}$ and $|\nabla \phi| \le \frac{C}{R_k}$. Since $\mathcal{L}v = 0$ in $B_{R_k} \setminus \overline{B_{\frac{3}{4}R_k}}$ and $v - M\phi \in H_0^1(B_{R_k} \setminus \overline{B_{\frac{3}{4}R_k}})$,

$$\langle \mathcal{L}v, (v - M\phi)\rangle = 0.$$

Hence,

$$\int_{B_{R_k} \setminus \overline{B_{\frac{3}{4}R_k}}} |\nabla v|^2\, dx \le C \int_{B_{R_k} \setminus \overline{B_{\frac{3}{4}R_k}}} |\nabla M\phi|^2\, dx \le CM^2 R_k^{n-2}. \tag{27}$$

Combining these inequalities (23)–(27), we obtain

$$m \int_{B_{R_{k+1}}} f\, dx \le (C(n)(C_L C_W)^2) m^2 R_k^{n-2} + (C(n) C_L C_W \|\mathbf{b}_1\|_{L^{n,1}(\Omega)}) m \int_{B_{R_{k+1}}} f\, dx.$$

Thus, if

$$\|\mathbf{b}_1\|_{L^{n,1}(\Omega)} \le \frac{1}{2} \frac{1}{C(n) C_L C_W}, \tag{28}$$

then, we also get

$$m \int_{B_{R_{k+1}}} f\, dx \le C m^2 R_k^{n-2}.$$

If $m = 0$, then we have $f|_{B_{R_{k+1}}} = 0$ by Lemma 13. Thus,

$$R_k^{2-n} \int_{B_{R_{k+1}}} f\, dx \le Cm.$$

On the other hand, by Lemma 11, we have

$$w \le u - \inf_{\partial B_{R_k}} u \quad \text{in } B_{R_k}.$$

Therefore,

$$m \le \inf_{\partial B_{\frac{3}{4}R_k}} (u - \inf_{\partial B_{R_k}} u) \le \inf_{\partial B_{R_{k+1}}} (u - \inf_{\partial B_{R_k}} u).$$

Consequently, we have

$$R_k^{2-n} \int_{B_{R_{k+1}}} f\, dx \le C \left(\inf_{\partial B_{R_{k+1}}} u - \inf_{\partial B_{R_k}} u \right), \tag{29}$$

for $k = 0, 1, \ldots$. Summing over all $k = 0, 1, \ldots$, we arrive at

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$$\sum_{k=0}^{\infty} R_k^{2-n} \int_{B_{R_{k+1}}} f\,dx \le C \lim_{k\to\infty} \inf_{\partial B_{R_{k+1}}} u.$$

By Lemma 11, we have

$$\lim_{k\to\infty} \inf_{\partial B_{R_k}} u = \lim_{k\to\infty} \inf_{B_{R_k}} u \le \lim_{k\to\infty} \fint_{B_{R_k}} u\,dx = u(0).$$

Since

$$C(n) \sum_{k=0}^{\infty} R_k^{2-n} \int_{B_{R_{k+1}}} f\,dx \ge \int_0^R s^{1-n} \int_{B_s} f\,dx\,ds = \mathbf{I}_2^f(0,R),$$

we arrived at the desired lower bound. □

4. Global behavior of supersolutions of $\mathcal{L}u = 0$ in exterior domains

In this section, we investigate the asymptotic behavior of supersolutions to $\mathcal{L}u = 0$ in $\mathbb{R}^n \setminus \overline{B_R}$. First, we show the existence of a global weak supersolution which approaches zero near infinity.

Lemma 16. *Assume (17). Then there exists a positive function U satisfying $\|\nabla U\|_{L^2(\mathbb{R}^n)} < \infty$ and*

$$\begin{cases} \mathcal{L}U = \mathbf{1}_{B_{1/2}} & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{30}$$

Moreover, if (22) holds, then there exists a positive constant C such that

$$\frac{1}{C}|x|^{2-n} \le U(x) \le C|x|^{2-n} \quad \text{in } \mathbb{R}^n \setminus \overline{B_1}. \tag{31}$$

Proof. Let $R > 1/2$ and let $U_R \in H_0^1(B_R)$ be a weak solution to $\mathcal{L}U_R = \mathbf{1}_{B_{1/2}}$. By the coercivity of $\mathcal{L}$ and Sobolev's inequality, we have

$$\begin{aligned} \frac{1}{2}\|\nabla U_R\|^2_{L^2(B_R)} &\le \int_{B_R} \mathbf{1}_{B_{1/2}} U_R\,dx \\ &\le \|\mathbf{1}\|_{L^{2n/(n+2)}(B_{1/2})} \|U_R\|_{L^{2n/(n-2)}(B_R)} \\ &\le C(n) \cdot S \|\nabla U_R\|_{L^2(B_R)}. \end{aligned}$$

Therefore we have

$$\|\nabla U_R\|_{L^2(B_R)} \le C(n). \tag{32}$$

The right-hand side does not depend on R. By Lemma 10 U_R, is a weak subsolution to $\mathcal{L}u = 0$ in $\mathbb{R}^n \setminus \overline{B_{1/2}}$. Using Lemma 12, Lemma 9 and Sobolev's inequality, for any $x \in \mathbb{R}^n \setminus \overline{B_1}$ we have

$$\begin{aligned} U_R(x) &\le C_B \fint_{B_{|x|/2}(x)} U_R \, dy \\ &\le C_B \frac{1}{|B_{|x|/2}|} \|\mathbf{1}\|_{L^{2n/(n+2)}(B_{|x|/2}(x))} \|U_R\|_{L^{2n/(n-2)}(B_{|x|/2}(x))} \\ &\le C|x|^{1-n/2} \|\nabla U_R\|_{L^2(B_R)} \le C|x|^{1-n/2}. \end{aligned}$$

By the maximum principle, U_R increases with respect to R. Let $U = \lim_{R\to\infty} U_R$; then U has the upper bound

$$U(x) \le C|x|^{1-n/2} \quad \text{in } \mathbb{R}^n \setminus \overline{B_1}.$$

Moreover, by (32), ∇U_R converges to ∇U weakly in $L^2(\mathbb{R}^n)$. Therefore, U is a weak solution to (30).

Next, we shall prove the second part of the theorem. Using Lemma 12, Hölder's inequality, and Lemma 14, we get

$$\begin{aligned} U_R(x) &\le C_B \fint_{B_{|x|/2}(x)} U_R \, dy \\ &\le C_B \frac{1}{|B_{|x|/2}|} \|\mathbf{1}\|_{L^{n/2,1}(B_{|x|/2}(x))} \|U_R\|_{L^{n/(n-2),\infty}(B_{|x|/2}(x))} \\ &\le C|x|^{2-n} \|U_R\|_{L^{n/(n-2),\infty}(B_R)} \le C|x|^{2-n}. \end{aligned}$$

Therefore, U has the desired upper bound. Finally, we derive the lower bound of U. For $|x| > 1$ and $s \ge 3|x|/2$, $B_{1/2} \subset B_s(x)$. Thus, applying Lemma 15 for $\mathcal{L}u = \mathbf{1}_{B_{1/2}}$ in $B_{4|x|}(x)$, we obtain

$$\begin{aligned} U(x) \ge \frac{1}{C_L} \mathbf{I}_2^{\mathbf{1}_{B_{1/2}}}(x, 2|x|) &= \frac{1}{C_L} \int_0^{2|x|} s^{2-n} |B_{1/2} \cap B_s(x)| \frac{ds}{s} \\ &\ge \frac{1}{C_L} \int_{3|x|/2}^{2|x|} s^{2-n} |B_{1/2} \cap B_s(x)| \frac{ds}{s} \\ &= \frac{1}{C_L C(n)} |x|^{2-n}. \end{aligned}$$

This completes the proof. □

Using the function U and an argument in [4], we can show the following Hadamard-type properties of positive supersolutions of $\mathcal{L}u = 0$.

Lemma 17. *Assume (17). Let $u \not\equiv 0$ be a nonnegative weak supersolution to $\mathcal{L}u = 0$ in $\mathbb{R}^n \setminus \overline{B_1}$.*

1. *Let*

$$m(r) := \inf_{A(r)} u$$

for $r \ge 1$. Then $m(r)$ is bounded from above.

2. *Let U be the function in Lemma 16 and let*

$$\rho(r) := \inf_{A(r)} \frac{u}{U}$$

for $r \geq 1$. Then

$$\rho(r) = \inf_{\mathbb{R}^n \setminus \overline{B_r}} \frac{u}{U}.$$

Moreover, if (22) holds, then we have the lower bound

$$m(r) \geq \frac{1}{C} r^{2-n}.$$

5. Proof of Theorem 1

In this section, we consider semilinear equations $\mathcal{L}u \geq f(x,u)$ and prove Theorem 1. Below, we normalize $\mathcal{L}$ for the simplification of arguments. By the definition of the Lorentz norm $\|\cdot\|_{L^{n,1}(\Omega)}$, taking sufficiently large R_0, we may assume that $\|\mathbf{b}_1\|_{L^{n,1}(\mathbb{R}^n \setminus \overline{B_{R_0}})}$ is sufficiently small. Moreover, rescaling by

$$A_r(x) = A(rx), \quad (\mathbf{b})_r(x) = r\mathbf{b}(rx), \quad f_r(x,s) = r^2 f(rx,s) \quad (r = 2R_0),$$

we may assume that $R = \frac{1}{2}$. Note that $\|A_r\|_{L^\infty(\mathbb{R}^n)} = \|A\|_{L^\infty(\mathbb{R}^n)}$ and $|||(\mathbf{b}_i)_r||| = |||\mathbf{b}_i|||$ $(i = 0, 1)$. Therefore, we may assume that $R = \frac{1}{2}$ and that condition (22) holds, without loss of generality.

Proof of Theorem 1. Suppose existence of a positive weak solution u. For $r \geq 1$, we take

$$m(r) := \inf_{A(r)} u, \quad \rho(r) := \inf_{A(r)} \frac{u}{U},$$

where U is the function in Lemma 16. We derive a contradiction by investigating the behavior of $m(r)$. We divide the proof into several steps.

Step 1. Fix $r \geq 1$. Since u is a nonnegative weak supersolution to $\mathcal{L}u = 0$ in $B_{4r} \setminus B_{r/2}$, by Lemma 13 and a simple covering argument, we can show that

$$\left(\fint_{A(r)} u^\sigma \, dx \right)^{1/\sigma} \leq C_* \inf_{A(r)} u, \tag{33}$$

where C_* is a constant depending only on n and C_W. Let $\alpha(n) = \inf\{|A(1) \cap B_{1/8}(x_0)|;\ x_0 \in A(1)\}$ and let $C_1 = C_*(2|A_1|/\alpha(n))^{1/\sigma}$. Also, let

$$Q(r) = \{x \in A(r);\ m(r) \leq u(x) \leq C_1 m(r)\}.$$

Then, by Chebyshev's inequality,

$$C_1 m(r) \left(\frac{|A(r) \setminus Q(r)|}{|A(r)|} \right)^{1/\sigma} \leq \left(\fint_{A(r)} u^\sigma \, dx \right)^{1/\sigma} \leq C(n) C_W \cdot m(r).$$

Hence,

$$|A(r) \setminus Q(r)| \leq \frac{1}{2}\alpha(n)r^n. \tag{34}$$

Let $x_0 \in A(r)$. Then, since

$$\big(Q(r) \cap B_{r/8}(x_0)\big) \cup (A(r) \setminus Q(r)) \supset A(r) \cap B_{r/8}(x_0),$$

by the definition of $\alpha(n)$ and (34), we have

$$|Q(r) \cap B_{r/8}(x_0)| \geq \alpha(n)r^n - |A(r) \setminus Q(r)| \geq \frac{1}{2}\alpha(n)r^n.$$

Step 2. Let

$$F(r,s) := \inf_{\substack{r \leq |x| \leq 2r \\ s \leq t \leq C_1 s}} f(x,t).$$

Since u satisfies

$$\mathcal{L}u(x) \geq f(x,u(x)) \geq \inf_{Q(r)} f(x,u(x))\mathbf{1}_{Q(r)} \geq F(r,m(r))\mathbf{1}_{Q(r)},$$

using Lemma 15 in $B_{r/2}(x_0)$, we obtain

$$\begin{aligned} u(x_0) &\geq \frac{1}{C_L}\int_0^{r/4} s^{1-n} \int_{B_s(x_0)} F(r,m(r))\mathbf{1}_{Q(r)}\,dx\,ds \\ &\geq \frac{1}{C_L}\int_{r/8}^{r/4} s^{1-n}|Q(r) \cap B_{r/8}(x_0)|\,ds \cdot F(r,m(r)) \\ &\geq \frac{1}{C_L}\cdot\frac{1}{2}\alpha(n)r^n \int_{r/8}^{r/4} s^{1-n}\,ds \cdot F(r,m(r)) \\ &\geq \frac{1}{C(n)C_L} r^2 F(r,m(r)). \end{aligned} \tag{35}$$

Therefore, taking the infimum over $A(r)$, we get

$$m(r) \geq \frac{1}{C} r^2 F(r,m(r)).$$

By Lemma 17, $m(r)$ is bounded. Thus

$$r^\gamma F(r,m(r)) \leq Cr^{-2+\gamma} \to 0 \quad \text{as } r \to \infty.$$

By the assumption on $f(x,s)$, this implies $m(r) \to 0$ or $m(r) \to \infty$ as $r \to \infty$. The latter is impossible, since $m(r)$ is bounded, so we must have that

$$m(r) \to 0 \quad \text{as } r \to \infty.$$

By the assumption on f, for sufficiently large r, we have

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$$m(r) \leq Cr^{2-n}.$$

By Lemma 16, we have the lower bound $U(x) \geq \frac{1}{C}|x|^{2-n}$. Therefore, we have

$$\rho(r) \leq C.$$

Step 3. Next, we derive a lower bound of ρ. By Lemma 17, we have

$$u(x) \geq \rho(r/2)U(x) \quad \text{in } \mathbb{R}^n \setminus \overline{B_{r/2}}.$$

Thus, $u - \rho(r/2)U$ is nonnegative in $\mathbb{R}^n \setminus \overline{B_{r/2}}$. Fix $x \in A(r)$. By a similar calculation to (35), we get

$$(u - \rho(r/2)U)(x) \geq \frac{1}{C(n)C_L}\frac{1}{C}r^2 F(r, m(r)).$$

By Lemma 16, we have the upper bound $U(x) \leq C|x|^{2-n}$. Hence,

$$u(x) \geq \rho(r/2)U(x) + \frac{1}{C}r^2 F(r, m(r)) \geq \left(\rho(r/2) + \frac{1}{C}r^n F(r, m(r))\right) U(x),$$

for any $x \in A(r)$. Taking the infimum over $A(r)$, we get

$$\rho(r) \geq \rho(r/2) + \frac{1}{C}r^n F(r, m(r)),$$

where C is a constant independent of r. By the assumption on f, there is a positive constant $c > 0$ such that $\liminf_{r\to\infty} r^n F(r, m(r)) \geq c$. Repeating this estimate, we arrive at

$$\rho(2^k r) \to \infty \quad \text{as } k \to \infty,$$

which contradicts the upper bound on ρ. This completes the proof. □

6. Proof of Theorem 6

By the same argument as in Section 5, without loss of generality, we may assume that

$$|||\mathbf{b}_1||| \leq \mathcal{B}_4,$$

where $\mathcal{B}_4 = \mathcal{B}_4(n, \Lambda, q_0)$ is a sufficiently small constant to be determined later (see (39)).

In order to prove Theorem 6 we prepare two lemmas. First, we show the following sharp form the weak Harnack inequality

Lemma 18. *Assume*

$$|||\mathbf{b}_1||| \leq \frac{n}{(n-2)p_0} - 1 \tag{36}$$

for some $p_0 \in (0, \frac{n}{n-2})$. If $u \not\equiv 0$ is a nonnegative weak supersolution to $\mathcal{L}u = 0$ in $\mathbb{R}^n \setminus \overline{B_{1/2}}$, then

$$\inf_{\partial B_R} u \geq \frac{1}{C}R^{-n/p_0}. \tag{37}$$

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Proof. Let $L = \inf_{\partial B_1} u$. By the strong minimum principle, L is positive. By considering $\tilde{u} = \min\{u, L\}$, we may assume that u is a weak supersolution in $\mathbb{R}^n$. Set $p_k = p_0(\frac{n}{n-2})^{-k}$ for $k = 1, 2, \ldots$. Since $p_k \to 0$ as $k \to \infty$, by Lemma 13, there is a sufficiently large k_* such that

$$\left(\fint_{B_R} u^{p_{k_*}} \, dx \right)^{1/p_{k_*}} \leq C \inf_{B_{R/2}} u. \tag{38}$$

For $k = 0, 1, \ldots k_*$, we choose a nested ball sequence $B_{R/2} = B_0 \Subset B_1 \Subset \cdots \Subset B_{k_*} = B_R$. Fix $1 \leq k \leq k_0$. Set $p = p_k$, $w = u^{p/2}$ and take $\eta \in C_c^\infty(B_k)$ with $\eta \equiv 1$ in B_{k-1}. Choosing a test function $\phi = u^{p-1}\eta$, we get

$$\begin{aligned} 0 \leq (p-1) \int_\Omega A\nabla u \cdot \nabla u u^{p-2} \eta^2 \, dx + 2 \int_\Omega A\nabla u \cdot u^{p-1} \nabla\eta\eta \, dx \\ - \frac{2}{p} \int_\Omega \mathbf{b}_0 \cdot u^p \nabla\eta\eta \, dx + \int_\Omega \mathbf{b}_1 \cdot \nabla u u^{p-1} \eta^2 \, dx. \end{aligned}$$

By the Cauchy–Schwarz inequality and Young's inequality $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$, we have

$$2\left(\frac{1}{p} - 1\right)^2 \int_\Omega |\nabla w|^2 \eta^2 \, dx \leq C \int_\Omega w^2 |\nabla\eta|^2 \, dx + \int_\Omega |\mathbf{b}_1|^2 w^2 \eta^2 \, dx.$$

Since

$$|||\mathbf{b}_1|||^2 \leq \left(\frac{1}{p_1} - 1\right)^2 \leq \left(\frac{1}{p} - 1\right)^2,$$

we have

$$\int_\Omega |\nabla w|^2 \eta^2 \, dx \leq C \int_\Omega w^2 |\nabla\eta|^2 \, dx.$$

On the other hand, by Sobolev's inequality, we have

$$\begin{aligned} \left(\int_\Omega (w\eta)^{2n/(n-2)} \, dx \right)^{(n-2)/n} &\leq S^2 \int_\Omega |\nabla(w\eta)|^2 \, dx \\ &\leq 2S^2 \left(\int_\Omega |\nabla w|^2 \eta^2 + w^2 |\nabla\eta|^2 \, dx \right). \end{aligned}$$

Since $w = u^{p_k/2}$, by combining the two inequalities, we obtain

$$\begin{aligned} \left(\int_{B_{k-1}} u^{p_{k-1}} \, dx \right)^{p_k/p_{k-1}} &= \left(\int_{B_{k-1}} w^{2n/(n-2)} \, dx \right)^{(n-2)/n} \\ &\leq C \|\nabla\eta\|_{L^\infty(\Omega)}^2 \int_{B_k} w^2 \, dx = C \|\nabla\eta\|_{L^\infty(\Omega)}^2 \int_{B_k} u^{p_k} \, dx. \end{aligned}$$

Repeating this estimate, we arrive at

$$\left(\fint_{B_{R/2}} u^{p_0} \, dx \right)^{1/p_0} \leq C \left(\fint_{B_R} u^{p_{k_*}} \, dx \right)^{1/p_{k_*}}.$$

Combining this inequality and (38), we get

$$\left(\fint_{B_R} u^{p_0} \, dx \right)^{1/p_0} \leq C \inf_{B_{R/2}} u.$$

Therefore, we have

$$LR^{-n/p_0} \leq \left(\frac{1}{|B_1|} \int_{B_R} u^{p_0} \, dx \right)^{1/p_0} \cdot R^{-n/p_0} \leq C \inf_{B_{R/2}} u.$$

Since $\inf_{B_R} u = \inf_{\partial B_R} u$ by the minimum principle, we obtain the desired lower bound. □

Next, we prove the following modified potential lower bound:

Lemma 19. *Assume (17). Let $f \in L^\infty(B_{2R}(x_0))$ be a nonnegative function. Let u be a nonnegative weak supersolution to $\mathcal{L}u \geq f(x)$ in $B_{2R}(x_0)$. Then there exist constants C_{L1} and C_{L2} depending only on n, $\|A\|_{L^\infty(\mathbb{R}^n)}$ and $|||\mathbf{b}_0|||$ such that*

$$u(x_0) + C_{L2} R^2 |||\mathbf{b}_1||| \|f\|_{L^\infty(\Omega)} \geq \frac{1}{C_{L1}} \mathbf{I}_2^f(x_0, R).$$

Proof. We follow the proof of Lemma 15. By the assumption on $\mathbf{b}_1$, we get

$$\begin{aligned}\left| \int_{B_{R_k}} \mathbf{b}_1 \cdot \nabla w (\tilde{w} \phi^2) \, dx \right| &\leq M \left(\int_{B_{R_k}} |\mathbf{b}_1|^2 \phi^2 \, dx \right)^{1/2} \left(\int_{B_{R_k}} |\nabla w|^2 \, dx \right)^{1/2} \\ &\leq M |||\mathbf{b}_1||| \|\nabla \phi\|_{L^2(B_{R_k})} \|\nabla w\|_{L^2(B_{R_k})}.\end{aligned}$$

On the other hand, by the energy inequality, we have

$$\|\nabla w\|_{L^2(B_{R_k})} \leq C R_k^n \|f\|_{L^\infty(B_{R_k})}.$$

Thus, we have

$$\left| \int_{B_{R_k}} \mathbf{b}_1 \cdot \nabla w (\tilde{w} \phi^2) \, dx \right| \leq C M R_k^n |||\mathbf{b}_1||| \|f\|_{L^\infty(B_{R_k})},$$

instead of (25). Consequently, we have

$$R_k^{2-n} \int_{B_{R_{k+1}}} f \, dx \leq C \left(\inf_{\partial B_{R_{k+1}}} u - \inf_{\partial B_{R_k}} u \right) + C 2^{-2k} R^2 |||\mathbf{b}_1||| \|f\|_{L^\infty(\Omega)},$$

for $k = 0, 1, \ldots$, instead of (29). Since $\sum_{k=0}^{\infty} 2^{-2k} = \frac{4}{3}$, taking the sum for all $k = 0, 1, \ldots$, we get the desired inequality. □

Proof of Theorem 6. We follow the proof of Theorem 1. Replacing Lemma 15 by the above Lemma, we get

$$u(x_0) + C_{L2} r^2 |||\mathbf{b}_1||| F(r, m(r)) \geq \frac{1}{C(n) C_{L_1}} r^2 F(r, m(r)),$$

instead of (35). Thus, if $|||\mathbf{b}_1||| \leq (2C(n) C_{L1} C_{L2})^{-1}$, then we have

$$m(r) \geq \frac{1}{C} r^2 F(r, m(r)).$$

Therefore, by the assumption on f, we have

$$m(r) \leq C r^{-2/(q_0 - 1)}.$$

On the other hand, if

$$|||\mathbf{b}_1||| \leq \frac{n}{(n-2) q_1} - 1$$

for some $n(q_0 - 1)/2 < q_1 < n/(n-2)$, then by Lemma 18, we have

$$m(r) \geq \frac{1}{C} r^{-n/q_1}.$$

This contradicts to the upper bound on $m(r)$, so taking

$$\mathcal{B}_4 := \min\{\frac{1}{2}, \frac{1}{2C(n) C_{L1} C_{L2}}, \frac{n}{(n-2) q_1} - 1\}, \tag{39}$$

we conclude the proof. □

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Please cite this article in press as: T. Hara, Liouville theorems for supersolutions of semilinear elliptic equations with drift terms in exterior domains, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2016.10.053