J. Math. Anal. Appl. 450 (2017) 1434–1458

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Dichotomies and asymptotic equivalence in alternately advanced and delayed differential systems

Aníbal Coronel [a,*], Christopher Maulén [b], Manuel Pinto [b], Daniel Sepúlveda [c]

[a] *GMA, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Chillán, Chile*
[b] *Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Chile*
[c] *Escuela de Matemática y Estadística, Facultad de Ciencias de la Educación, Universidad Central de Chile, Chile*

ARTICLE INFO

Article history:
Received 12 December 2016
Available online 3 February 2017
Submitted by W. Sarlet

Keywords:
Dichotomies
Asymptotic equivalence
Piecewise constant arguments
Cauchy and Green matrices
Hybrid equations
Stability of solutions

ABSTRACT

In this paper, ordinary and exponential dichotomies are defined in differential equations with equations with piecewise constant argument of general type. We prove the asymptotic equivalence between the bounded solutions of a linear system and a perturbed system with integrable and bounded perturbations.

© 2017 Elsevier Inc. All rights reserved.

1. Introduction

The study of differential equations with piecewise constant argument is motivated by several applications coming from different fields of science and by their own mathematical definition as hybrid dynamical systems [2,9,29]. For a longer discussion on applications consult the references [5,10,21,22,27,36,37]. Here the meaning of hybrid is given in the sense that they combine the behavior of differential and difference equations. In general, the typical form of this kind of equations is given by the following functional equation

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t), \mathbf{x}(\gamma(t))), \tag{1}$$

where $\mathbf{x} : \mathbb{R} \to \mathbb{C}^p$ is the unknown function, $t \in \mathbb{R}$ usually denotes the time, $\mathbf{F}$ is a given function from $\mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p$ to $\mathbb{C}^p$, and γ is a given general step function in the sense that

* Corresponding author.
E-mail addresses: acoronel@ubiobio.cl (A. Coronel), christoph.maulen.math@gmail.cl (C. Maulén), pintoj.uchile@gmail.cl (M. Pinto), daniel.sep.oe@gmail.com (D. Sepúlveda).

http://dx.doi.org/10.1016/j.jmaa.2017.01.087
0022-247X/© 2017 Elsevier Inc. All rights reserved.

$$\left.\begin{array}{l}\gamma:\mathbb{R}\to\mathbb{R} \text{ is defined by } \gamma(t)=\zeta_i \text{ for } t\in I_i=[t_i,t_{i+1}), \text{ where } \{t_i\}_{i\in\mathbb{Z}}\\ \text{and } \{\zeta_i\}_{i\in\mathbb{Z}} \text{ are two given (fix) sequences such that } t_i\le\zeta_i\le t_{i+1}\\ \text{with } t_i<t_{i+1} \text{ for each } i\in\mathbb{Z} \text{ and } t_i\to\pm\infty \text{ when } i\to\pm\infty.\end{array}\right\} \tag{2}$$

The genesis of the study of this kind of functional equations goes back to the work of Myshkis [28], who proposed an equation of type (1)–(2) with the particular step functions $\gamma(t)=[t]$ and $\gamma(t)=2\,[(t+1)/2]$. Here $[\cdot]$ denotes the greatest integer function. By simplicity of the presentation, we use hereinafter the terms DEPCA and DEPCAG to refer the differential equations with piecewise constant argument when the step function is based on the greatest integer function and when the step function is of the general type given by (2), respectively. In particular, note that the equations studied by Myshkis are DEPCAs. Later, in the early 80's, a systemic analysis of (1)–(2) was introduced by Wiener and collaborators, see [1,17,39,40] and references therein. Afterwards, the contribution to the development of the theory was given by many authors see for instance [3–5,7,10,18,21–23,25,29,33,36,39–44]. Nowadays, there exist an intense and increasing interest to understand the qualitative behavior, to get novel applications and to solve numerically the equations, since a general theory for (1)–(2) is far to be closed, see [5,6,11–13,15,14,16,35,38].

An important point to observe is the notion of the solution or more generally the types of approach to analyze (1)–(2). Actually, generally speaking, the notion of solution for functional differential equations is one of the most important tasks. Now, we recall that the original notion of the integration (or solution) of a DEPCA was introduced in [1,17,40] and is based on the reduction to discrete equations. This approach presents some disadvantages when we require a generalization to analyze DEPCAGs. In particular, for instance to solve the Cauchy problem requires that the initial moments must be integers, see [40] for details. Another approach to study a general quasilinear DEPCAG was introduced by Akhmet [3,4], and is based on the construction of an equivalent integral equation and remarking the clear influence of the discrete part. The methodology of Akhmet permits to overpass the difficulties of the methodology of Wiener and collaborators. Moreover, Akhmet adapts the notion of solution given by Wiener and used previously by Papaschinopoulos to study a particular type of DEPCA, see [29]. The notion of Akhmet solution is given in terms of continuity of the solution on each t_i, the existence of the derivatives on each t with possible exception of some t_i and the local satisfaction of the equation, see Definition 2.1 below. Then, in spite of its functional character a quasilinear DEPCAG has similar properties to ordinary differential equations. For further details, consult for instance [8,20,21,23,24,35]. Here, in this paper, we use the approach of Akhmet. Thus, we do not need to impose any restrictions on the discrete equations and we assume more easily verifiable conditions on the coefficients, similar to those for ordinary differential equations.

In this paper, we are interested in the asymptotic equivalence of some DEPCAGs. Now, in order to precise the different type of systems which will be used in the paper, we introduce a particular notation of each case. Indeed, throughout the paper, we consider that x, z, u, y, w, v satisfy the following particular cases of (1)–(2):

$$x'(t)=A(t)x(t), \tag{3}$$

$$z'(t)=A(t)z(t)+B(t)z(\gamma(t)), \tag{4}$$

$$u'(t)=B(t)u(\gamma(t)), \tag{5}$$

$$y'(t)=A(t)y(t)+B(t)y(\gamma(t))+g(t), \tag{6}$$

$$w'(t)=A(t)w(t)+B(t)w(\gamma(t))+f(t,w(t),w(\gamma(t))), \tag{7}$$

$$v'(t)=A(t)v(t)+B(t)v(\gamma(t))+g(t)+f(t,v(t),v(\gamma(t))). \tag{8}$$

Note that (3)–(6) are linear and (7)–(8) are nonlinear. The specific hypotheses about the different functions given in (3)–(8) are summarized in subsection 2.1.

We have found some previous results on the literature and specifically focused on the analysis DEPCAs of DEPCAGs of types (4)–(7). Particularly, here we comment the works of Akhmet [3,4] and Pinto [34], since they are more close to our contributions. Indeed, Akhmet [3,4], by applying, his approach has obtained fundamental results about the variation of constants formula and the stability of the perturbed system (7). Now, Pinto in [34], by applying a combined methodology based on the Green matrix and Akhmet approaches, has proven some important results related to the analysis of the DEPCAGs (4)–(7). In particular, he defines an appropriate Green matrix associated to (4), then formulated the solution of (6) in terms of this Green matrix and subsequently characterize the solution of (7) by an integral equation of the first kind. Using the integral equation and a Gronwall type inequality for DEPCAGs, he deduces an existence and uniqueness of solutions for (6). Moreover, using this approach he guarantees that the zero solution of (4) is exponentially asymptotically stable. Finally, he gives some equivalence results on stability and compares his results with the corresponding ones of Akhmet [3,4]. Probably, the major advantages of the approach introduced in [34] are two. Firstly, permits the introduction of more general hypotheses on the coefficients. Second, the Green matrix type taking account of the decomposition of any interval, $I_i = [t_i, t_{i+1})$ in the advanced intervals $I_i^+ = [t_i, \zeta_i]$ and the delayed intervals $I_i^- = [\zeta_i, t_{i+1})$, hence permits to analyze the alternately advanced and delayed differential systems in a unified way.

The paper is organized as follows. In section 2, we introduce several preliminary concepts and results like the hypotheses, the Cauchy and Green matrices type, the notion of solution for DEPCAGs, the Gronwall type inequality for DEPCAGs and the notion of dichotomies and stability. Moreover, in section 2 we give a detailed example for (4) with A and B constants. In sections 3–5, we present the main results of the paper which are summarized as follows

(1) *Stability of solutions for* (7). In Theorem 3.1 we prove that the σ-exponentially stability of the zero solution for the linear DEPCAG (4) implies σ_0-exponentially stability of the zero solution for (7), where σ_0 is defined in terms of $\sigma, A, f, \{t_i\}_{i\in\mathbb{Z}}$ and $\{\zeta_i\}_{i\in\mathbb{Z}}$.
(2) *Bounded solutions for* (6). In Proposition 4.1 we prove that σ-exponentially stability of (6) and a convergence of a series given in terms of the fundamental solutions for (3) and (4) implies that equation (6) has a unique bounded solution on $\mathbb{R}$. Moreover, in Theorem 5.3 we prove that there exists a unique bounded solution of the non-homogeneous linear DEPCAG (6) by assuming the linear DEPCAG (4) has a σ-exponential dichotomy and two series in terms of the fundamental solutions for (3) and (4) and the associated projection to the dichotomy converges, see [6] for other results. In [6], periodic and almost periodic solutions are also studied.
(3) *Asymptotic equivalence of* (6) *and* (8). In Theorem 5.1, we prove that if the linear system (4) has an ordinary dichotomy and in (8) f is integrable, then there exists a homeomorphism between the bounded solutions of (6) and the bounded solutions of (8).
(4) *Bounded solutions for* (7). In Theorem 5.5 we prove that: if (4) has a σ-exponential dichotomy satisfying the requirements of Theorem 5.3, then for any $\xi \in P\mathbb{C}^p$ the nonlinear equation (7) has a unique bounded solution w on $[t_0, \infty)$, with $Pw(t_0) = \xi$, the correspondence $\xi \to w$ is continuous and the bounded solution of equation (7) converges exponentially to zero as $t \to \infty$.

Moreover, we prove introduce four corollaries with particular importance in Corollary 5.2 we deduce that here exists a homeomorphism between $\mathbb{C}^p$ and the bounded solutions of (5).

2. Preliminaries

2.1. General and stability assumptions and notation

In this section we summarize several hypotheses used thorough of the paper. We organize these assumptions in two big groups (H1)–(H4) and (S1)–(S3). The distinction obeys fundamentally to the fact that (S1)–(S3) are frequently needed by the exponential stability results.

The first group is given as follows

(H1) Let us denote by $\mathbb{C}^{n\times m}$ the vectorial space of complex matrices of size $n\times m$. We assume that the coefficients of (3)–(7) defined by the functions $A, B : \mathbb{R} \to \mathbb{C}^{p\times p}$ and $g : \mathbb{R} \to \mathbb{C}^p$ are locally integrable in $\mathbb{R}$.

(H2) For an arbitrary matrix valued function $M \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{C}^{p\times p})$ and for each $i \in \mathbb{Z}$ consider the following notation $\rho_i(M) = \rho_i^+(M)\rho_i^-(M)$ with $\rho_i^{\pm}(M) = \exp\left(\int_{I_i^{\pm}} |M(s)|ds\right)$, where $\{I_i^+, I_i^-\}$ is a partition of I_i defined as follows

$$I_i^+ = [t_i, \gamma(t_i)] = [t_i, \zeta_i] \text{ and } I_i^- = (\gamma(t_i), t_{i+1}) = (\zeta_i, t_{i+1}). \tag{9}$$

The sets I_i^+ and I_i^- are so called the advanced and delayed intervals, respectively. We suppose that the functions A and B in equations (3)–(7) satisfy the following relations

$$\rho(A) = \sup_{i\in\mathbb{Z}} \rho_i(A) < \infty, \quad \nu_i^+(B) \le \nu^+ < 1, \ \nu_i^-(B) \le \nu^- < 1, \tag{10}$$

where $\nu_i^{\pm} = \rho_i^{\pm}(A)\ln(\rho_i^{\pm}(B))$.

(H3) We assume that the function f given as the third term on the right hand of the equation (7) satisfies the following three requirements: (i) $f : \mathbb{R}\times\mathbb{C}^p\times\mathbb{C}^p \to \mathbb{C}^p$ is a continuous function, i.e. $f \in C(\mathbb{R}\times\mathbb{C}^p\times\mathbb{C}^p, \mathbb{C}^p)$; (ii) $f(t, 0, 0) = 0$ for each $t \in \mathbb{R}$; and (iii) there exists $\eta \in L^1_{\text{loc}}(\mathbb{R}; [0,\infty))$ such for all $t \in \mathbb{R}$ and all $x_i, y_i \in \mathbb{C}^p, i = 1, 2$, the inequality

$$\left|f(t, x_1, y_1) - f(t, x_2, y_2)\right| \le \eta(t)\Big(|x_1 - x_2| + |y_1 - y_2|\Big) \tag{11}$$

holds.

(H4) Let $X(t)$ a fundamental matrix of solutions for (3) and denote by Φ the also called fundamental matrix of (3) and defined by $\Phi(t, s) = X(t)X^{-1}(s)$ for $(t, s) \in \mathbb{R}^2$. Consider that $J : \mathbb{R}^2 \to \mathbb{C}^{p\times p}$ is defined as follows

$$J(t,\tau) = I_p + \int_\tau^t \Phi(\tau, s)B(s)ds, \tag{12}$$

where I_p is the $p\times p$ identity matrix. For each $i \in \mathbb{Z}$, we assume that the matrix $J(t,\tau)$ is non-singular for all $t, \tau \in [t_i, t_{i+1})$.

Here we introduce three comments about (H1)–(H4): *(a)* note that when A and B are constant, the relation (10) is strongly simplified and naturally a simpler condition can be obtained; *(b)* the matrix Φ defined in (H4) satisfies the following properties:

$$\Phi(\tau,\tau) = I_p, \ \Phi(t, s)\Phi(s,\tau) = \Phi(t,\tau), \ \Phi(t, s) = \Phi^{-1}(s, t), \ \forall (s, t,\tau) \in \mathbb{R}^3; \tag{13}$$

and *(c)* (H2) implies (H4) (see Lemma 2.1).

Now, we introduce the following second group of hypotheses

(S1) For a given $t \in \mathbb{R}$ denote by $i(t)$ the unique integer such that $t \in I_{i(t)}$. We assume that estimate

$$\sup_{i(t)\in\mathbb{Z}} \sup_{t\in I_{i(t)}} |Z(t, t_{i(t)})| < \infty$$

is satisfied by $Z(t,s)$ the fundamental solution of (4) (see subsection 2.3).

(S2) $\inf_{i\in\mathbb{Z}}\{t_{i+1} - t_i\} > 0$.

(S3) The lengths of the $I_i^{\pm}$ satisfies the following bound

$$\overline{t} = \sup_{i\in\mathbb{Z}} \max\left\{\zeta_i - t_i, t_{i+1} - \zeta_i\right\} < \infty. \tag{14}$$

We note four facts: *(a)* (S1) is more general than (H2), since (H2) implies (S1) and condition (S2); *(a)* (S2) implies (S3); *(c)* for A and B constant matrices, the conditions (S2) and (S3) imply (S1); and *(d)* nowadays the most studied DEPCAGs satisfy (S3), see for instance [1,3,7,17,18,40].

2.2. Notion of solution for DEPCAGs

We state the notion of solutions for DEPCAG by following the given [1,15,17,18,40]. More precisely we have the following definition.

Definition 2.1. Consider a continuous function $F : I \times \mathbb{C}^p \times \mathbb{C}^p \to \times\mathbb{C}^p$ where $I \subset \mathbb{R}$ and γ is a general step function in the sense of (2). Then a function $\mathbf{v} : I \to \mathbb{C}^p$ such that $t \mapsto \mathbf{v}(t)$ is called a solution of the following DEPCAG

$$\mathbf{v}'(t) = F(t, \mathbf{v}(t), \mathbf{v}(\gamma(t))) \tag{15}$$

on an interval I if:

(a) $\mathbf{v}$ is continuous on I;

(b) the derivative $\mathbf{v}'(t)$ exists at each point $t \in I$ with the possible exception of the points $t_i \in I, i \in \mathbb{Z}$, where the one sided derivatives exist; and

(c) $\mathbf{v}$ satisfies pointwise the equation (15) on each interval $(t_i, t_{i+1}) \subset I, i \in \mathbb{Z}$, and (15) holds for the right derivative of $\mathbf{v}$ at the points $t_i \in I, i \in \mathbb{Z}$.

Naturally, the Definition 2.1 can be applied to precise the notion of solution for the DEPCAG of types (4)–(7) by considering that F has a particular definition in each case.

2.3. The fundamental matrix of solutions of (4)

In several parts of the paper we need the fundamental matrix of solutions of (4). For instance, to define the Green matrices the fundamental matrix of solutions of DEPCAG (4) is a central concept. Now, we recall that, under the assumption (H4), there exists $Z(t,s) \in \mathbb{C}^{p\times p}$ a fundamental matrix of solutions of DEPCAG (4), see for instance [4,36]. Indeed, to construct Z we proceed in two steps. Firstly, we define $Z(t,\tau)$ for $t \geq \tau$ and then for $t \leq \tau$. Let us consider that $(t,\tau) \in [t_j, t_{j+1}) \times [t_i, t_{i+1})$ with $t > \tau$ and $j \geq i$. By induction we can deduce that

$$
\begin{aligned}
Z(t,\tau) &= E(t,\gamma(t_j))E(t_j,\gamma(t_j))^{-1} \\
&\quad \times \prod_{k=j}^{i+2} \Big(E(t_k,\gamma(t_{k-1}))E(t_{k-1},\gamma(t_{k-1}))^{-1}\Big)\, E(t_{i+1},\gamma(\tau))E(\tau,\gamma(\tau))^{-1}, \\
&= E(t,\zeta_j)E(t_j,\zeta_j)^{-1} \\
&\quad \times \prod_{k=j}^{i+2} \Big(E(t_k,\zeta_{k-1})E(t_{k-1},\zeta_{k-1})^{-1}\Big)E(t_{i+1},\zeta_i)E(\tau,\zeta_i)^{-1},\ t \geq \tau,
\end{aligned} \tag{16}
$$

where $\prod$ denotes the backward product $\prod_{k=m}^{n} C_k = C_n C_{n-1} \cdots C_m$ and E is the matrix defined as follows

$$
E(t,\tau) = \Phi(t,\tau)J(t,\tau) \quad \text{with } \Phi \text{ and } J \text{ defined in (H4).} \tag{17}
$$

Note that the property of no-singularity of the matrix $J(\cdot,\zeta_i)$ on the interval I_i is the minimal requirement for the well definition of Z given in (16). Now, to define $Z(t,\tau)$ for $t < \tau$ we note that $Z(t,\tau)$ defined by (16) satisfies similar properties to (13), since we can easily deduce that

$$
Z(\tau,\tau) = I_p, \qquad \text{for } \tau \in \mathbb{R}, \tag{18}
$$

$$
Z(t,s)Z(s,\tau) = Z(t,\tau), \qquad \text{for } \tau \geq s \geq t, \tag{19}
$$

$$
Z(t,\tau) = [Z(\tau,t)]^{-1}, \qquad \text{for } t \leq \tau. \tag{20}
$$

In particular, the property (20) is important for our purpose since it allows to define $Z(t,\tau)$ for $t < \tau$ using (16). Hence $Z(t,s)$ is completely defined on $\mathbb{R}^2$ by (16) and (20) and naturally

$$
z(t) = Z(t,\tau)z(\tau) \text{ for } t \geq \tau \text{ or } z(t) = [Z(t,\tau)]^{-1}z(\tau) \text{ for } t < \tau \tag{21}
$$

defines the solution of (4) with initial condition $(\tau, z(\tau))$.

From (16) and (20) and for notational convenience, we introduce the matrices $Z^{\pm}$ as follows

$$
Z^{+}(t,s) = Z(t,s) \text{ for } t \geq s \text{ and } Z^{-}(t,s) = [Z(s,t)]^{-1} \text{ for } t < s, \tag{22}
$$

where Z is evaluated by (16).

The results of this subsection are formalized below in Lemma 2.1 and Corollary 2.3.

2.4. Green matrices type

In this subsection, we recall the concepts and notation of Green matrices G_k and G.

Definition 2.2 *(Green matrix $G_k(t,s)$ for $(t,s) \in [t_k,t_{k+1}]^2$).* Consider the notation defined in (2). For a given $k \in \mathbb{Z}$, the Green matrix type G_k is defined from $[t_k,t_{k+1}] \times [t_k,t_{k+1}]$ to $\mathbb{C}^{p\times p}$ by the following relation

$$
G_k(t,s) = \begin{cases} G_k^{+}(t,s), & \text{for } (t,s) \in [t_k,t_{k+1}] \times [t_k,\gamma(s)], \\ G_k^{-}(t,s), & \text{for } (t,s) \in [t_k,t_{k+1}] \times (\gamma(s),t_{k+1}], \end{cases} \tag{23}
$$

where

$$
\begin{aligned}
G_k^{+}(t,s) &= Z^{+}(t,\tau)\Phi(\tau,s), \quad \text{for } \tau \leq s \leq \gamma(s),\ \ t_k \leq \tau \leq t, \\
G_k^{-}(t,s) &= Z^{-}(t,\tau)\Phi(\tau,s), \quad \text{for } \gamma(s) < s \leq \tau,\ \ t \leq \tau \leq t_{k+1}.
\end{aligned}
$$

Here the notation $Z^{\pm}$ is defined in (22).

In particular, the Green matrix (23), for the advanced situation $t_k \leq s \leq \gamma(s) = t_{k+1}$

$$G_k^+(t,s) = \begin{cases} Z^+(t,t_k)\Phi(t_k,s), & \text{for } t_k \leq s \leq \gamma(t_k), \ \ s < t, \\ \Phi(t,s), & \text{for } t \leq s \leq \gamma(t_k), \end{cases}$$

and for the delayed situations $t_k = \gamma(s) < s$

$$G_k^-(t,s) = \begin{cases} Z^-(t,t_{k+1})\Phi(t_{k+1},s), & \text{for } \gamma(s) < s \leq t_{k+1}, \ \ t > s, \\ \Phi(t,s), & \text{for } \gamma(s) < s \leq t < t_{k+1}. \end{cases}$$

Definition 2.3 *(Green matrix $G(t,s)$ for $(t,s) \in \mathbb{R}^2$).* Consider the notation $i(t)$ given in (S1). The Green matrix type $G : \mathbb{R}^2 \to \mathbb{C}^{p\times p}$ for $s > t$ is defined as follows

$$G(t,s) = \begin{cases} G_{i(t)}(t,s), & i(s) = i(t), \\ G_{i(t)}(t_{i(t)+1},s) + G_{i(t)}(t,t_{i(t)+1}), & i(s) = i(t)+1, \\ G_{i(t)}(t_{i(t)+1},t) + \displaystyle\sum_{k=i(t)+1}^{i(s)-1} G_k(t_{k+1},t_k) & \\ \qquad + G_{i(s)}(s,t_{i(s)-1}), & i(s) > i(t)+1, \end{cases} \tag{24}$$

and for $s < t$ by the following relation

$$G(t,s) = \begin{cases} G_{i(s)}(t,s), & i(s) = i(t), \\ G_{i(s)}(t_{i(s)+1},s) + G_{i(s)}(t,t_{i(s)+1}), & i(s) = i(t)+1, \\ G_{i(s)}(t_{i(s)+1},s) + \displaystyle\sum_{k=i(s)+1}^{i(t)-1} G_k(t_{k+1},t_k) & \\ \qquad + G_{i(t)}(t,t_{i(t)-1}), & i(t) > i(s)+1, \end{cases} \tag{25}$$

where $G_k(\cdot,\cdot)$ is the matrix introduced in Definition 2.2.

We note that

$$G(t,s) = \begin{cases} G^+(t,s), & \text{for } s \leq \gamma(s), \\ G^-(t,s), & \text{for } \gamma(s) < s, \end{cases}$$

where $G^{\pm}(t,s) = \sum_{k=i(s)}^{i(t)} G_k^{\pm}(t,s)$. This fact justifies the name of 'Green matrix type' for G. Moreover, $G_k(t_{k+1},t_k) = G_k^+(t_{k+1},t_k) = Z(t_{k+1},t_k)$, which gives the recurrence relation:

$$x(t_{i+1}) = Z(t_{i+1},t_i)x(t_i), i \in \mathbb{Z}. \tag{26}$$

Note that from (26), we have

$$X(t_{j+1}) = \prod_{k=i}^{j} Z_k(t_{k+1},t_k)x(t_i), \quad i \leq j, \tag{27}$$

which gives another way to obtain formula (16). It allows also to solve the linear non-homogeneous DEPCAG (6).

Here we recall an important result given in [34].

Lemma 2.1. *([34]) Assume that the condition (H2) is fulfilled, then the condition (H4) holds and the matrices $Z(t,s)$ and $Z(t,s)^{-1}$ are well defined for any $(t,s)\in\mathbb{R}^2$ with $t\geq s$. Moreover, there exists a positive constant number α such that*

$$|\Phi(t,s)|\leq\rho(A),\quad |Z(t,s)|\leq\alpha\quad \text{and}\quad |G_i(t,s)|\leq\alpha\rho(A) \tag{28}$$

for $(t,s)\in[t_i,t_{i+1}]^2$ and for each $i\in\mathbb{Z}$.

2.5. Variation of parameters formulas

A variation of parameters formula to (6) can be deduced by assuming that (H1) and (H4) hold. Indeed, in [34] was proven that the solution of the equation of (6) is given by

$$\begin{aligned} y(t) = Z(t,\tau)y(\tau) + Z(t,\tau)\int_{\tau}^{\zeta_i}\Phi(\tau,s)g(s)ds + \sum_{k=i+1}^{j} Z(t,t_k)\\ \times\int_{t_k}^{\zeta_k}\Phi(t_k,s)g(s)ds + \sum_{k=i}^{j-1} Z(t,t_{k+1})\int_{\zeta_k}^{t_{k+1}}\Phi(t_{k+1},s)g(s)ds\\ +\int_{\zeta_j}^{t}\Phi(t,s)g(s)ds,\quad \text{for } (\tau,t)\in I_i\times I_j. \end{aligned} \tag{29}$$

In particular, we have that

$$\begin{aligned} y(t) = Z(t,\tau)y(\tau) + \int_{\tau}^{\zeta_i} G_i^+(t,s)g(s)ds\\ + \sum_{k=i+1}^{j}\int_{t_k}^{\zeta_k} G_k^+(t,s)g(s)ds + \sum_{k=i}^{j-1}\int_{\zeta_k}^{t_{k+1}} G_k^-(t,s)g(s)ds\\ +\int_{\zeta_j}^{t} G_j^-(t,s)g(s)ds,\quad \text{for } (\tau,t)\in I_i^+\times I_j^-, \end{aligned} \tag{30}$$

where the notation $I_i^{\pm}$ is defined in (9). Note that, using Definition 2.1, a similar formula can be obtained for $(\tau,t)\in I_i^-\times I_j^-$. To be more precise, the following theorem can be stated.

Theorem 2.2. *([34]) Assume that the hypotheses (H1) and (H4) (or (H1) and (H2)) are fulfilled. Then, for any $(\tau,\xi)\in\mathbb{R}\times\mathbb{C}^p$ the solution of (6) such that $y(\tau)=\xi$ is defined on $\mathbb{R}$ and is given by*

$$y(t) = Z(t,\tau)\xi + \int_{\tau}^{t} G(t,s)g(s)ds. \tag{31}$$

In particular the formula (31) is reduced to (29) or (30) depending if $(\tau,t)\in I_i\times I_j$ or $(\tau,t)\in I_i^+\times I_j^-$, respectively.

By application of Theorem 2.2 to the DEPCAGs (4), (5), and (7) we can deduce some useful results which are formalized in the following three corollaries.

Corollary 2.3. *([34]) Assume that $A(t)$ and $B(t)$ satisfy the requirements of hypothesis (H1). Moreover, assume that (H4) or (H2) are fulfilled. Then, the following assertions are valid*

(i) *there exists $Z : \mathbb{R}^2 \to \mathbb{C}^{p\times p}$ the fundamental solution matrix of the linear system (4) and is given by (16) and (20),*
(ii) *for every $(\tau, \xi) \in \mathbb{R}\times\mathbb{C}^p$, there exists on all of $\mathbb{R}$ a unique solution (4) such that $z(\tau) = \xi$. This solution is given by (21).*

Furthermore, conversely, the existence of a solution $z(t) = z(t, \tau, \xi)$ of (4) defined on all of $\mathbb{R}$ implies that the condition (H4) must be true.

Corollary 2.4. *([34]) Assume that $B(t)$ satisfies the requirement of hypothesis (H1). Then, for every $(\tau, \xi) \in \mathbb{R} \times \mathbb{C}^p$ the solution of DEPCAG (5) such that $u(\tau) = \xi$ is given by the formulae (29) and (31) with $u(t) = z(t)$, $g = 0$, $\Phi(t, s) \equiv I_p$, or equivalently $u(t) = z(t)$, $g = 0$, $E(t, s) = J(t, s) = I_p + \int_s^t B(r)dr$,*

$$Z(t,\tau) = J(t,\gamma(t_j))J(t_j,\gamma(t_j))^{-1} \prod_{k=i+2}^{j}\left(J(t_k,\gamma(t_{k-1}))J(t_{k-1},\gamma(t_{k-1}))^{-1}\right) J(t_{i+1},\gamma(\tau))J(\tau,\gamma(\tau))^{-1},$$

for $(t, \tau) \in [t_j, t_{j+1}) \times [t_i, t_{i+1})$ with $t \geq \tau$.

Corollary 2.5. *([34]) Assume that the hypotheses (H1) and (H4) (or (H1) and (H2)) are fulfilled, then for any $(\tau, \xi) \in \mathbb{R} \times \mathbb{C}^p$, every $w(t) = w(t, \tau, \xi)$ solution of the quasilinear DEPCAG (7) such that $w(\tau) = \xi$, satisfies the integral equation (31) with $g(s) = f(s, z(s), z(\gamma(s)))$ or*

$$w(t) = Z(t,\tau)\xi + \int_\tau^t G(t,s)f(s,w(s),w(\gamma(s)))ds. \tag{32}$$

Conversely, any solution of the integral equation (32), is a solution in the sense of Definition 2.1, of the quasilinear DEPCAG (7).

To close the subsection we remark that, using the classical method of Wiener [40] (pp. 8, 18, 52, 88), several authors have obtained variation of parameters formula for the particular cases of the steps functions $\gamma(t)$ given by $[t], [t + 1/2], [t + 1]$. They have obtained the compact formula:

$$\begin{aligned} y(t) = Z(t,\tau)y(\tau) &+ Z(t,\tau)\int_\tau^{\gamma(t_i)} \Phi(\tau,s)g(s)ds \\ &+ \sum_{k=i}^{j-1} Z(t,t_{k+1})\int_{\gamma(t_k)}^{\gamma(t_{k+1})} \Phi(t_{k+1},s)g(s)ds \\ &+ \int_{\gamma(t_j)}^{t} \Phi(t,s)g(s)ds, \quad \text{for } (\tau,t) \in [t_i,t_{i+1})\times[t_j,t_{j+1}), \end{aligned} \tag{33}$$

which follows from (31), (24)–(25) and (23) since $G(t,s) = Z(t,t_{k+1})\Phi(t_{k+1},s)$ for $s \in [\gamma(t_k),\gamma(t_{k+1})] = [\zeta_k,\zeta_{k+1}]$. Under other conditions, this formula was extended to any DEPCAG by Akhmet [4]. This formula takes account of the intervals $[\zeta_k,\zeta_{k+1}]$ instead of the advanced intervals I_k^+ and the delayed intervals I_k^-. This representation allows not to see the Green matrix type.

2.6. Gronwall type inequalities for DEPCAGs

The Gronwall type inequality for DEPCAG introduced and proved in [33] is formulated in the following lemma.

Lemma 2.6. *([33]) Let $u,\eta : \mathbb{R} \to [0,\infty)$ be two functions such that u is continuous and η is locally integrable satisfying*

$$\theta = \sup_{i\in\mathbb{Z}} \left\{ \theta_i \ : \ \theta_i := 2\int_{I_i} \eta(s)ds \right\} < 1. \tag{34}$$

Suppose that for $\tau \le t$ or $t \le \tau$, we have the inequality

$$u(t) \le u(\tau) + \left| \int_\tau^t \eta(s)[u(s) + u(\gamma(s))]ds \right|.$$

Then

$$u(t) \le u(\tau)\exp\left\{ \tilde{\theta} \int_\tau^t \eta(s)ds \right\} \quad with \quad \tilde{\theta} = \frac{2-\theta}{1-\theta}.$$

If we consider the forward and backward situation in a separated way, then in (34) instead of integration on the all I_i to define θ_i we need only an integration on I_i^+ or I_i^- respectively. More precisely, we have the following result.

Corollary 2.7. *([34]) The results in Lemma 2.6 are true*

$$for \quad t \ge \tau, \quad if \quad \theta = \sup_{i\in\mathbb{Z}} \left\{ \theta_i^+ \ : \ \theta_i^+ := 2\int_{I_i^+} \eta(s)ds \right\} < 1,$$

$$for \quad t \le \tau, \quad if \quad \theta = \sup_{i\in\mathbb{Z}} \left\{ \theta_i^- \ : \ \theta_i^- := 2\int_{I_i^-} \eta(s)ds \right\} < 1.$$

2.7. Existence and uniqueness of solution of the quasilinear system (7)

The existence, uniqueness, boundedness, stability and continuous dependences of the solutions $w(t) = w(t,\tau,\xi)$ of DEPCAG (7) are precisely stated as follows.

Proposition 2.8. *([34]) Assume that conditions (H1), (H3) and (H4) (or (H1)–(H3))are fulfilled. Moreover, assume that the given function A in (H1) and the existing function η in (H3) satisfies the estimate $\alpha\rho(A)\theta < 1$ with $\rho(A)$, α and θ given in (10), (28) and (34), respectively. Then, for every $(\tau,\xi) \in \mathbb{R}\times\mathbb{C}^p$, there exists $w(t) = w(t,\tau,\xi)$ solution of (7) in the sense of Definition 2.1 and satisfying the following properties:*

(i) $w(\tau) = \xi$,
(ii) w *is defined on all of* $\mathbb{R}$,
(iii) w *is solution of the integral equation* (32),
(iv) w *is unique and depends continuously on* τ *and* ξ.

Moreover, if there exists a constant $c \geq 1$ *such that*

$$|Z(t,s)| \leq c, \text{ for } t \geq s \tag{35}$$

and if $\eta \in L^1(\mathbb{R})$, *then* w *is bounded and is stable, namely*

$$\exists c_1 \in \mathbb{R}^+ \ : \ |w(t,\tau,\xi_1) - w(t,\tau,\xi_2)| \leq c_1|\xi_1 - \xi_2|, \ \forall t \geq \tau, \ \forall \xi_1, \xi_2 \in \mathbb{C}^p. \tag{36}$$

The estimate $\alpha\rho(A)\theta < 1$ required as one of the hypotheses of Proposition 2.8 is more frequently written in an explicit way as follows

$$\theta_i = 2\alpha\rho(A)\int_{I_i} \eta(s)ds \leq \theta = \sup_{i\in\mathbb{Z}} \theta_i < 1.$$

Moreover, we remark two facts. Firstly, if we are only interested in the forward (or backward) continuation, i.e. only for $t \geq \tau$ (or $t \leq \tau$), then instead of the condition $\alpha\rho(A)\theta < 1$ we need only an integration on I_i^+ (or I_i^-) i.e. the inequality

$$2\alpha\rho_i^+(A)\int_{I_i^+} \eta(s)ds \leq \theta^+ < 1 \qquad \Big(\text{ or } 2\alpha\rho_i^-(A)\int_{I_i^+} \eta(s)ds \leq \theta^+ < 1\Big),$$

is required. Second, we remark that the Proposition 2.8 generalizes the corresponding results obtained by Akhmet [3,4].

2.8. Exponential stability

The definitions of Lyapunov stability of the solutions of DEPCAG can be given in the same way as for ordinary differential equations. Let us formulate only one of them.

Definition 2.4. The zero solution of DEPCAG (7) is σ-exponentially stable if for an arbitrary positive ϵ, there exists a positive number $\delta = \delta(\tau,\epsilon)$ such that $|\xi| \leq \delta$ implies that $|x(t,\tau,\xi)| \leq c|\xi|e^{-\sigma(t-\tau)}$ for all $t \geq \tau \geq 0$.

Let $Z(t,s)$ be the fundamental matrix of the linear DEPCAG (4) (see (16) and (22)). By the representations (21) and (16), the stability of linear system (4) can be analogously expressed as theorems for ordinary differential equations [8,19,20,24]. An example of this fact is (35) and the following theorem (see [4]).

Theorem 2.9. *The zero solution of the linear DEPCAG* (4) *is* σ*-exponentially stable if and only if there exist two positive numbers* c *and* σ *such that*

$$|Z(t,s)| \leq ce^{-\sigma(t-s)}, \quad \text{for } t \geq s \geq 0. \tag{37}$$

By Lemma 2.1

$$|G(t,s)| \leq c\rho(A)e^{\sigma\bar{t}}e^{-\sigma(t-s)}, \text{ for } t \geq s \geq 0, \tag{38}$$

where $\bar{t}$ is the notation introduced in (S3) (see (14)).

From the respective stability of the difference system (26):

$$z(t_{i+1}) = Z(t_{i+1}, t_i)z(t_i), \ i \in \mathbb{Z}, \tag{39}$$

whose solutions are given by

$$z(t_{j+1}) = \prod_{k=i}^{j} Z(t_{k+1}, t_k)z(t_i) = Z(t_{j+1}, t_i)z(t_i), \quad i < j, \tag{40}$$

we can formulate several theorems which provide sufficient conditions for the stability of linear systems of DEPCAG (4). The stability of the solution $z = 0$ of the difference system (39) is deduced from boundedness or convergence of $Z(t_{j+1}, t_i)$ as $j \to \infty$. Taking into account formula (16), from the hypotheses (S1) and (S2), we have, e.g. the following theorem.

Theorem 2.10. *Assume that conditions (H1), (H4), (S1) and (S3) are fulfilled and the zero solution of DEPCAG (4) is exponentially asymptotically stable if* $0 < \rho < 1$*, we have*

$$|E(t_{k+1}, \zeta_k)E(t_k, \zeta_k)^{-1}| = |Z(t_{k+1}, t_k)| \leq \rho, \quad k \in \mathbb{N}. \tag{41}$$

We observe that the condition (41) is the natural ones. Moreover, we note that there exist other conditions which permit to get similar results. For instance, under other several additional assumptions, Akhmet [4] consider $|E(t_{k+1}, \zeta_{k+1})E(t_{k+1}, \zeta_k)^{-1}| \leq \rho$ instead of (41). This condition is equivalent to (41) if A and B are constants and scalars and (S3) holds. At present, the DEPCAG more studied satisfy (S3), see [1,3,7,17, 18,40] and also our example given below in subsection 2.9.

On the other hand, we note that some interesting results similar to Theorems 2.9 and 2.10 can be found in [4,17,39,40].

2.9. An example for (4) with A and B constants

Study the dichotomic character in the following linear DEPCAG:

$$x'(t) = Ax(t) + Bx(\gamma(t)) \tag{42}$$

where A and B are fixed real constant matrices such that A^{-1} exists and the function $\gamma(t)$ is defined by sequences t_i and ζ_i satisfying:

$$\zeta_i - t_i = \nu^+, \ t_{i+1} - \zeta_i = \nu^-, \quad i \in \mathbb{Z},$$

where $\nu^+, \nu^- > 0$ are fixed numbers. Calling,

$$\Lambda(s) = e^{sA} + A^{-1}(e^{sA} - I_p)B = e^{sA}[I_p + A^{-1}(I_p - e^{-sA})B],$$

we obtain $E(t, \tau) = \Lambda(t - \tau)$.

Then, to apply Theorem 2.10 we must study:

$$Z(t_{i+1},t_i)=E(t_{i+1},\zeta_i)E(t_i,\zeta_i)^{-1}=\Lambda(\nu^-)\Lambda(-\nu^+)^{-1}=\Lambda_1.$$

For $A=0$, we have $\Lambda(s)=I_p+sB$ and

$$\Lambda_1=(I_p+\nu^-B)(I_p-\nu^+B)^{-1}.$$

For $\gamma(t)=c[t+d/c], c>0,\ c>d$, we have $\zeta_i=c,\ t_i=ci-d, \nu^+=d,\ \nu^-=c-d, \Lambda_1=\Lambda(c-d)\Lambda(-d)^{-1}$. In particular, for the famous case of Cooke and Wiener [17], $\gamma(t)=2[t+1/2]$, we have $\Lambda_1=\Lambda(1)\Lambda(-1)^{-1}$. The zero solution is stable if $\rho(\Lambda_1)\leq 1$ and exponentially stable if $\rho(\Lambda_1)<1$. In these conditions, there exist $c\geq 1$ and κ constants such that:

$$|Z(t,\tau)|\leq ce^{\kappa\ln(t-\tau)},\quad t\geq\tau. \tag{43}$$

The situation for $t<\tau$ can be treated similarly.

In the case $A=a\neq 0$ and $B=b$ be scalars, i.e. the equation

$$x'(t)=ax(t)+bx(\gamma(t)), \tag{44}$$

we can find explicit conditions on the coefficients and the sequences for providing exponential stability for of zero solution of (44), see [5,17,39,40]. On the basis of the previous analysis, we must have $\rho=|\Lambda_1|=|\Lambda(\nu^-)\Lambda(-\nu^+)^{-1}|<1$, either of the inequalities

$$-b>a>0,\quad [e^{a\nu^-}+e^{-a\nu^+}]\left[1+\frac{b}{a}\right]>2\frac{b}{a}, \tag{45}$$

$$-b>a,\ a<0,\quad [e^{a\nu^-}+e^{-a\nu^+}]\left[1+\frac{b}{a}\right]>2\frac{b}{a} \tag{46}$$

is sufficient for the zero solution to be exponentially stable. For the completely delayed case $\nu^+=0$ and, hence, $t_{i+1}-t_i=\nu^-$, the conditions (45) and (46) are, respectively, transformed to

$$-b>a>0,\ e^{a\nu^-}<\frac{b-a}{b+a}\quad\text{and}\quad -b>a,\quad a<0,\quad e^{a\nu^-}<\frac{b-a}{b+a}\cdot$$

The stability for the case $a=0$ can be also studied.

3. Stability of solutions for (7)

Theorem 3.1. *Assume that the hypotheses (H1)–(H3), (S1) and (S3) are fulfilled and the zero solution of linear DEPCAG (4) is σ-exponentially stable. Moreover, assume the function η in (H3), $\rho(A)$ defined in (10) and σ satisfy the following requirements*

$$\theta:=\sup_{i\in\mathbb{Z}}\left\{2ce^{\sigma\bar{t}}\rho(A)\int_{t_i}^{\zeta_i}\eta(s)ds\right\}<1,\qquad \beta:=\limsup_{t\to\infty}\frac{\int_\tau^t\eta(s)ds}{t}<\infty. \tag{47}$$

Then, there exists $\sigma_0=\sigma-\beta\mu c\rho(A)e^{2\sigma\bar{t}}$ with $\mu=(2-\theta)(1-\theta)^{-1}$ such that the zero solution of (7) is σ_0-exponentially stable. In particular, if $\eta\in L^1([0,\infty))$, then the σ-exponential stability follows and the stability of the zero solution of (4) implies the stability of the solution zero of (7).

Proof. If we consider that $w(t) = w(t,\tau,\xi)$ is a solution of (7) such that $w(\tau) = \xi$ and, without loss of generality, we assume that $t_i \le \tau < \zeta_i \le t_{i+1} < \cdots < t_j \le \zeta_j < t$. Then by Corollary 2.5 and Theorem 2.2 (see (32) and (29)), we have that

$$\begin{aligned} w(t) &= Z(t,\tau)w(\tau) + \int_\tau^t G(t,s)f(s,w(s),w(\gamma(s)))ds \\ &= Z(t,\tau)\xi + Z(t,\tau)\int_\tau^{\zeta_{i(\tau)}} \Phi(\tau,s)f(s,w(s),w(\gamma(s)))ds \\ &\quad + \sum_{k=i(\tau)+1}^{j(t)} Z(t,t_k)\int_{t_k}^{\zeta_k} \Phi(t_k,s)f(s,w(s),w(\gamma(s)))ds \\ &\quad + \sum_{k=i(\tau)}^{j(t)-1} Z(t,t_{k+1})\int_{\zeta_k}^{t_{k+1}} \Phi(t_{k+1},s)f(s,w(s),w(\gamma(s)))ds \\ &\quad + \int_{\zeta_{j(t)}}^{t} \Phi(t,s)f(s,w(s),w(\gamma(s)))ds. \end{aligned}$$

By the hypothesis of the σ-exponential stability of the solution for (4), the assumption (H3), the application of Theorem 2.9 and Lemma 2.1, and by (S3), we deduce the following bound for w

$$\begin{aligned} |w(t)| &\le ce^{-\sigma(t-\tau)}|\xi| + c\rho(A)e^{-\sigma(t-\tau)}\int_\tau^{\zeta_{i(\tau)}} \eta(s)\Big(|w(s)| + |w(\gamma(s))|\Big)ds \\ &\quad + \sum_{k=i(\tau)+1}^{j(t)} c\rho(A)e^{-\sigma(t-t_k)}\int_{t_k}^{\zeta_k} \eta(s)\Big(|w(s)| + |w(\gamma(s))|\Big)ds \\ &\quad + \sum_{k=i(\tau)}^{j(t)-1} c\rho(A)e^{-\sigma(t-t_{k+1})}\int_{\zeta_k}^{t_{k+1}} \eta(s)\Big(|w(s)| + |w(\gamma(s))|\Big)ds \\ &\quad + \rho(A)\int_{\zeta_{j(t)}}^{t} e^{-\sigma(t-s)}\eta(s)\Big(|w(s)| + |w(\gamma(s))|\Big)ds \\ &\le ce^{-\sigma(t-\tau)}|\xi| \\ &\quad + \int_\tau^t c\rho(A)e^{-\sigma(t-s-\overline{t})}\eta(s)\left(|w(s)| + e^{-\sigma\gamma(s)}|w(\gamma(s))|e^{\sigma\gamma(s)}\right)ds, \end{aligned}$$

which can be rewritten as follows

$$u(t) \le cu(\tau) + \int_\tau^t c\rho(A)e^{2\sigma\overline{t}}\eta(s)\left(u(s) + u(\gamma(s))\right)ds \quad \text{with } u(t) = |w(t)|e^{\sigma t}.$$

Now, by virtue of the DEPCAG Gronwall inequality given in Lemma 2.6, we obtain that

$$|w(t)| \le c\exp\left(-\sigma(t-\tau) + c\mu\rho(A)e^{2\sigma\overline{t}}\int_\tau^t \eta(s)ds\right) \quad \text{with} \quad \mu = \frac{2-\theta}{1-\theta}.$$

Hence, the last inequality combined with (47) proves that the zero solution is σ_0-exponentially stable. The other assertions follow similarly. The theorem is proved. □

If in (H3), the function η is constant we have an interesting result similar to the ones obtained previously by Akhmet in [4] under other conditions.

Corollary 3.2. *Assume that conditions (H1)–(H3), (S1) and (S3) are fulfilled. Moreover, in (H3) consider that η is constant, i.e. $\eta(t) \equiv \eta_0$. Suppose that the zero solution of the linear DEPCAG (4) is σ-exponentially stable and consider that*

$$\theta = 2c\bar{t}\eta_0\rho(A)e^{2\sigma\bar{t}} < 1, \quad \sigma - c\rho(A)\mu\eta_0 e^{2\sigma\bar{t}} = \sigma_0 > 0, \quad \mu = \frac{2-\theta}{1-\theta}.$$

Then, the solution zero of system (7) is σ_0-exponentially stable.

4. Bounded solutions for (6)

The bounded solutions on all of $\mathbb{R}$ of the linear nonhomogeneous equation (6) can be studied by considering the convergence of the series

$$\sum_{k=-\infty}^{-1} |Z(0,t_{k+1})| \int_{\gamma(t_k)}^{\gamma(t_{k+1})} |\Phi(t_{k+1},s)|ds < \infty, \tag{48}$$

$$\sum_{k=0}^{\infty} |Z(0,t_{k+1})| \int_{\gamma(t_k)}^{\gamma(t_{k+1})} |\Phi(t_{k+1},s)|ds < \infty. \tag{49}$$

As $|Z(0,t_{k+1})| < ce^{-\sigma|t_{k+1}|}$ estimations of the integrals in (48) and (49) allow give conditions for convergence of the above series. For example, for $t_k = k$ and in general with (H1) and (S2), the conditions (48) and (49) hold, see Lopez-Fenner and Pinto [26].

Proposition 4.1. *Let $g : \mathbb{R} \to \mathbb{C}^p$ be a bounded function. The following assertions with respect to the solution of the equation (6) are valid*

(a) *Assume that (48) holds and the solution of (4) is σ-exponentially stable on $\mathbb{R}$, i.e.*

$$|Z(t,s)| \le ce^{-\sigma(t-s)} \text{ for } t \ge s.$$

Then equation (6) has a unique bounded solution $y : \mathbb{R} \to \mathbb{C}^p$ defined by

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} G(t,s)g(s)ds = \sum_{k=-\infty}^{i(t)-1} \int_{t_k}^{t_{k+1}} G_k(t,s)g(s)ds + \int_{t_{i(t)}}^{t} \Phi(t,s)g(s)ds \\ &= \sum_{k=-\infty}^{i(t)-1} Z(t,t_k) \int_{t_k}^{\zeta_k} \Phi(t_k,s)g(s)ds + \sum_{k=-\infty}^{i(t)-1} Z(t,t_{k+1}) \\ &\quad \times \int_{\zeta_k}^{t_{k+1}} \Phi(t_{k+1},s)g(s)ds + \int_{i(t)}^{t} \Phi(t,s)g(s)ds. \end{aligned} \tag{50}$$

(b) *Assume that (49) holds and the following condition*

$$|Z(t,s)| \leq ce^{-\sigma(s-t)}, \quad s \geq t \tag{51}$$

is satisfied. Then the unique bounded solution of equation (6) on $\mathbb{R}$ is given by

$$\begin{aligned} y(t) &= -\int_t^{\infty} G(t,s)g(s)ds \\ &= -\int_t^{t_{i(t)+1}} \Phi(t,s)g(s)ds - \sum_{k=i(t)+1}^{\infty} \int_{t_k}^{t_{k+1}} G_k(t,s)g(s)ds \\ &= -\int_t^{t_{i(t)+1}} \Phi(t,s)g(s)ds - \sum_{k=i(t)+1}^{\infty} Z(t,t_k) \int_{t_k}^{\zeta_k} \Phi(t_k,s)g(s)ds \\ &\quad - \sum_{k=i(t)+1}^{\infty} Z(t,t_{k+1}) \int_{\zeta_k}^{t_{k+1}} \Phi(t_{k+1},s)g(s)ds. \end{aligned} \tag{52}$$

(c) *The map $g \to y_g$ is continuous and satisfies the estimate*

$$|y_g(t)| \leq \hat{c}|g|_{\infty} \quad \textit{with} \quad |g|_{\infty} = \sup_{t\in\mathbb{R}} |g(t)| \quad \textit{with} \quad \hat{c} = c\rho(A)e^{\sigma\overline{t}}, \tag{53}$$

for a positive constant $\hat{c}$ independent of g.

Proof. (a) In a standard way, it is not difficult to show that y given by (50) is a well defined bounded function and is a solution of (6). Moreover, for g fixed, the nonhomogeneous linear system (6) has a unique bounded solution on all of $\mathbb{R}$, because for $\omega \neq 0$ any solution $Z(t,\tau)\omega$ of the homogeneous linear system (3) is unbounded as $t \to -\infty$. Now, we deduce that the unique bounded solution of (6) is necessarily given by (50). Indeed, from (31) we have that any solution y of (31) is given by

$$y(t) = Z(t,0)\omega + \int_0^t G(t,s)g(s)ds, \quad \text{with} \quad \omega = y(0). \tag{54}$$

Note that $\int_0^t = \int_0^{\zeta_{i(0)}} + \sum_{k=i(0)}^{i(t)-1} \int_{\zeta_k}^{\zeta_{k+1}} + \int_{\zeta_{i(t)}}^t$. Supposing, by simplicity, that $\zeta_{i(0)} = 0$, and that the convergence of the series

$$\sum_{k=0}^{-\infty} Z(0,t_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \Phi(t_{k+1},s)g(s)ds = v_{-\infty}, \tag{55}$$

we get

$$\begin{aligned} &\int_0^t G(t,s)g(s)ds \\ &= \sum_{k=0}^{i(t)-1} \int_{\zeta_k}^{\zeta_{k+1}} G_k(t,s)g(s)ds + \int_{\zeta_{i(t)}}^t G_{i(t)}(t,s)g(s)ds \end{aligned}$$

$$= \sum_{k=0}^{i(t)-1} \int_{\zeta_k}^{\zeta_{k+1}} Z(t,t_{k+1})\Phi(t_{k+1},s)g(s)ds + \int_{\zeta_{i(t)}}^{t} \Phi(t,s)g(s)ds$$

$$= Z(t,0) \sum_{k=0}^{i(t)-1} Z(0,t_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \Phi(t_{k+1},s)g(s)ds + \int_{\zeta_{i(t)}}^{t} \Phi(t,s)g(s)ds$$

$$= Z(t,0) \left(\sum_{k=0}^{i(-\infty)} - \sum_{k=i(t)}^{i(-\infty)} \right) Z(0,t_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \Phi(t_{k+1},s)g(s)ds$$

$$+ \int_{\zeta_{i(t)}}^{t} \Phi(t,s)g(s)ds. \tag{56}$$

Then, introducing (55) and (56) in (54), we have

$$y(t) = Z(t,0)[\omega + v_{-\infty}]$$

$$+ \sum_{k=-\infty}^{i(t)-1} Z(t,t_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \Phi(t_{k+1},s)g(s)ds + \int_{\zeta_{i(t)}}^{t} \Phi(t,s)g(s)ds$$

$$= \int_{-\infty}^{t} G(t,s)g(s)ds,$$

by (33) and taking $\omega = -v_{-\infty}$ we avoid the homogenous unbounded solution in the first term and (50) is deduced.

(b) The proof of (52) when (51) holds, follows similarly.

(c) To prove that y satisfies (53), we apply the properties of G_k and Z in (50) and (52). □

We remark that estimations of the type $|Z(0,t_{k+1})| < ce^{-\sigma|t_{k+1}|}$ imply (48) and (49). Moreover, we note that these kind of estimates are valid for example for the particular case $t_k = k$ and in general when (H2) and (S2) hold, see [26] for details. Then we have the following corollary.

Corollary 4.2. *Let $g : \mathbb{R} \to \mathbb{C}^p$ be a bounded function. The results of Proposition 4.1 are valid if the hypotheses (H2) and (S3) are fulfilled.*

5. Asymptotic equivalence, ordinary and exponential dichotomies

In Proposition 2.8 we have studied a uniform stability given by (36). Its dichotomic extension carries us to the ordinary dichotomy which includes an unstability.

5.1. Ordinary dichotomy and Green matrix

Definition 5.1. The linear DEPCAG (4) has an ordinary dichotomy if there exists a projection P and a positive c such that $|Z_P(t,s)| \leq c$ for all $(t,s) \in \mathbb{R}^2$, where the Green function $Z_P : \mathbb{R}^2 \to \mathbb{C}^{p\times p}$ is defined by

$$Z_P(t,s) = \begin{cases} Z(t,0)PZ(0,s), & t \geq s, \\ -Z(t,0)(I-P)Z(0,s), & t < s, \end{cases} \tag{57}$$

for a given a projection matrix $P \in \mathbb{C}^{p\times p}$.

Definition 5.2. Consider that Z_P denotes the function defined in (57). Then, the Green matrix type $\tilde{G} : \mathbb{R}^2 \to \mathbb{C}^{p\times p}$ is defined as follows

$$\begin{aligned} \tilde{G}(t,s) &:= \langle Z_P(t,\cdot), \Phi(\cdot,s)\rangle \\ &:= Z_P(t,\tau)\cdot\tilde{\Phi}_-(\tau,s) + \sum_{k=i(\tau)+1}^{i(t)} Z_P(t,t_k)\tilde{\Phi}_+(t_k,s) \\ &\quad + \sum_{k=i(\tau)}^{i(t)-1} Z_P(t,t_{k+1})\tilde{\Phi}_-(t_{k+1},s) + \tilde{\Phi}(t,s), \end{aligned}$$

where $\tilde{\Phi}_\pm(t_k,s) = \Phi(t_k,s)1_{I_k^\pm}(s)$ and $\tilde{\Phi}(t,s) = \Phi(t,s)1_{[\xi_{i(t)},t]}$. Here $1_{\mathcal{A}}$ denotes the standard characteristic function of the set $\mathcal{A} \subset \mathcal{U}$, i.e. $1_{\mathcal{A}}(s) = 1$ for $s \in \mathcal{A}$ and $1_{\mathcal{A}}(s) = 0$ for $s \notin \mathcal{A}$.

By (10) we can deduce that

$$|\tilde{G}(t,s)| \leq \tilde{c} \text{ for all } (t,s) \in \mathbb{R}^2 \text{ with } \tilde{c} = c\rho(A). \tag{58}$$

Now, for an integrable function $g : [\tau,\infty) \to \mathbb{C}^n$, we have that the solution of (6) is given by

$$\begin{aligned} y(t) &= \int_\tau^\infty \tilde{G}(t,s)g(s)ds = \int_\tau^t \tilde{G}(t,s)g(s)ds + \int_t^\infty \tilde{G}(t,s)g(s)ds \\ &=: y_+(t) + y_-(t), \end{aligned}$$

where

$$\begin{aligned} y_+(t) &= \int_\tau^t < Z_P(t,\cdot);\Phi(\cdot,s) > g(s)ds \\ &= Z_P(t,\tau)\int_\tau^{\zeta_{i(\tau)}} \Phi(\tau,s)g(s)ds + \sum_{k=i(\tau)+1}^{i(t)} Z_P(t,t_k)\int_{t_k}^{\zeta_k}\Phi(t_k,s)g(s)ds \\ &\quad + \sum_{k=i(\tau)}^{i(t)-1} Z_P(t,t_{k+1})\int_{\zeta_k}^{t_{k+1}} \Phi(t_{k+1},s)g(s)ds + \int_{\zeta_{i(t)}}^t \Phi(t,s)g(s)ds, \\ y_-(t) &= -\int_t^\infty < Z_P(t,\cdot);\Phi(\cdot,s) > g(s)ds \\ &= -Z_P(t,t_{i(t)+1})\int_t^{t_{i(t)+1}} \Phi(t_{i(t)+1},s)g(s)ds - \sum_{k=i(t)+1}^{\infty} Z_P(t,t_k) \\ &\quad \times \int_{t_k}^{\zeta_k}\Phi(t_k,s)g(s)ds - \sum_{k=i(t)+1}^{\infty} Z_P(t,t_{k+1})\int_{\zeta_k}^{t_{k+1}}\Phi(t_{k+1},s)g(s)ds. \end{aligned}$$

In particular, if we have an ordinary stability with $P = I$, $|Z(t,s)| \leq c$, $c \geq 1$, for $t \geq s$, the special bounded solution on $\mathbb{R}$ of (6) is given by $y_g^+(t) = \int_{-\infty}^{t} \tilde{G}(t,s)g(s)ds$ and in the unstable situation $P = 0$, $|Z(t,s)| \leq c$, $c \geq 1$, for $t \leq s$, the special bounded solution on $\mathbb{R}$ of (6) is given by $y_g^-(t) = \int_{t}^{\infty} \tilde{G}(t,s)g(s)ds$ and the analogous of the bound (53) is stated as follows

$$\|y_g^{\pm}\|_\infty \leq \tilde{c}\|g\|_1$$

with $\tilde{c}$ given in (58).

Now, we prove the asymptotic equivalence between system (6) and (8) when the perturbation f is integrable, i.e. $\eta, f(t,0,0) \in L^1([t_0,\infty))$.

Theorem 5.1. *Assume that the linear system (4) has an ordinary dichotomy with projection P on $[t_0,\infty)$ and the hypotheses (H1)–(H4) are fulfilled. Moreover, assume that instead of (H3)–(ii) the condition $f(t,0,0) \in L^1([t_0,\infty))$ is satisfied and the function η in (H3)–(iii) is belonging $L^1([t_0,\infty))$. Then there exists a homeomorphism between the bounded solutions of the linear system (6) and the bounded solutions of the quasilinear system (8). Moreover, $|y(t) - v(t)| \to 0$ as $t \to \infty$ if $Z(t,0)P \to 0$ as $t \to \infty$.*

Proof. Consider that y is a bounded solution of (6) and introduce the notation $BC([t_0,\infty),\mathbb{C}^p)$ for the space of bounded continuous functions with the topology defined by the norm $\|y\| = \sup_{s \geq t_0^*} |y(s)|$ with $t_0^* = \min\{t_0, \gamma(t_0)\}$. Now, we consider the operator $\mathcal{A} : BC([t_0,\infty),\mathbb{C}^p) \to BC([t_0,\infty),\mathbb{C}^p)$ defined by

$$(\mathcal{A}v)(t) = y(t) + \int_{t_0}^{\infty} \tilde{G}(t,s)f(s,v(s),v(\gamma(s)))ds.$$

From (58), (H3)–(i) and (H3)–(iii), we can prove that $\mathcal{A}$ is a contraction for t_0 sufficiently large, since

$$\|\mathcal{A}v - y\| \leq \tilde{c} \int_{t_0}^{\infty} \Big(\eta(s)\Big[|v(s)| + |v(\gamma(s))|\Big] + |f(s,0,0)|\Big)ds,$$

$$\begin{aligned}\|\mathcal{A}v_1 - \mathcal{A}v_2\| &\leq \tilde{c} \int_{t_0}^{\infty} \eta(s)\Big[|v_1(s) - v_2(s)| + |v_1(\gamma(s)) - v_2(\gamma(s))|\Big]ds \\ &\leq \beta\|v_1 - v_2\|,\end{aligned}$$

with $\beta = 2\tilde{c}\int_{t_0}^{\infty} \eta(s)ds < 1$. Hence, the integral equation

$$v(t) = y(t) + \int_{t_0}^{\infty} \tilde{G}(t,s)f\big(s,v(s),v(\gamma(s))\big)ds \tag{59}$$

has a unique bounded solution and this solution is the unique bounded continuous solution of (8). Then, summarizing, we have that for any bounded continuous y of (6), the integral equation (59) has a unique bounded continuous solution v which is the solution of (8). Reciprocally, by the properties of f, $\tilde{G}$ and the integral equation is straightforward to deduce that if v is a bounded solution of (8) then y defined by (59) is a bounded solution of (6). Moreover, the correspondence $y \to v$ is bicontinuous, since the estimates

$$\|v_1 - v_2\| \leq \|y_1 - y_2\| + \beta\|v_1 - v_2\|, \qquad \|y_1 - y_2\| \leq \|v_1 - v_2\| + \beta\|v_1 - v_2\|,$$

gives

$$(1+\beta)^{-1}\|y_1 - y_2\| \leq \|v_1 - v_2\| \leq (1-\beta)^{-1}\|y_1 - y_2\|.$$

Finally, by (58) and the properties of f, we deduce that for any $\epsilon > 0$ there exists $T \geq t_0$ such that

$$\left| \int_T^\infty \tilde{G}(t,s) f(s, v(s), v(\gamma(s)))ds \right| \leq \tilde{c} \int_T^\infty (2\|v\|\eta(s) + |f(s,0,0)|)\, ds < \epsilon$$

and

$$|\mathcal{A}v(t) - y(t)| \leq |Z(t,0)P| \int_{t_0}^{T} \left|\tilde{G}(0,s) f(s, v(s), v(\gamma(s)))\right| ds + \epsilon.$$

Hence, the condition $|Z(t,0)P| \to 0$ as $t \to \infty$ implies that $|y(t) - v(t)| \to 0$ as $t \to \infty$. □

We note that under the conditions of the Theorem 5.1 the solutions of (6) and (8) are defined for all $t \geq t_0$. Moreover, the solutions of (6) and (8) are uniquely determined by their initial values and also depend continuously on these initial values over any bounded interval. Then, we have the continuity property on the original interval $[t_0, \infty)$ and not only for t_0 sufficiently large. Moreover, we note that f is linear, the correspondence $y \to v$ is linear and homogeneous. Meanwhile, related with the hypothesis 'the linear system (6) has an ordinary dichotomy', we have that this condition is satisfied, for instance, if $\|B\|$ is small enough and $A(t) = A$ is a constant matrix whose characteristic roots with zero real parts of simple type or if $A(t)$ is periodic, where the characteristic exponent with zero real part of simple type.

On the other hand, we note that the equation $x'(t) = 0$ with $P = 0$ satisfies the hypothesis of Theorem 5.1, then we can deduce a result for equation (5).

Corollary 5.2. *Consider the hypotheses of Theorem 5.1. Then, there exists a homeomorphism between $\mathbb{C}^p$ and the bounded solutions of (7). Moreover, any solution u is convergent to some $\hat{u} \in \mathbb{C}^p$ as $t \to \infty$ and for every $\hat{u} \in \mathbb{C}^p$ there exists a unique solution u of (5) such that $u(t) \to \hat{u}$ as $t \to \infty$.*

5.2. Exponential dichotomy and Green matrix

For the conditional asymptotic stability we have

Definition 5.3. Consider that Z_P denotes the function defined in (57). Then, the linear DEPCAG (4) has a σ-exponential dichotomy if there exists a projection matrix P and a positive constant $c > 0$ such that $|Z_P(t,s)| \leq ce^{-\sigma|t-s|}$.

Now, the Green matrix in Definition 5.3 satisfies the estimate

$$|\tilde{G}_p(t,s)| \leq \hat{c}e^{-\sigma|t-s|}, \quad t, s \in \mathbb{R}, \quad \hat{c} = c\rho(A)e^{\sigma\bar{t}}. \tag{60}$$

Remark 5.1. G. Papaschinopoulos [29,30,32,31] proposes to define an exponential dichotomy for linear DEPCAG (4) when the difference equation (26) has an exponential dichotomy. Definition 5.3 is rather a natural notion of exponential dichotomy. However, if we take $A(t) = 0$, $B(t) = \mathrm{diag}(\lambda_1(t), \lambda_2(t))$, $\lambda_1(t) = -\frac{2}{\pi} + \sin(2\pi t)$, $\lambda_2(t) = -\lambda_1(t)$, $t_n = n$ for all $n \in \mathbb{Z}$, $\int_n^{n+\delta} \lambda_1(\xi)d\xi = -\frac{1}{2\pi}(4\delta - 1 + \cos(2\pi\delta))$ for all

$\delta \in [0,1]$ then the difference equation (26) has an exponential dichotomy with projection $P = \mathrm{diag}(1,0)$ but there is no P such that the estimation for Z_P in Definition 5.3 is satisfied. Indeed, for $t - [t] < 1/2$, $\int_{[t]}^{t} \lambda_1(s)ds \geq 0$ and is negative for $t - [t] > 1/2$, while $\int_{[t]}^{t} \lambda_2(s)ds$ satisfies the same with contrary sign. However, $\int_{[t]}^{t} \lambda_1(s)ds = 0 = \int_{[t]}^{t} \lambda_2(s)ds$ for $t - [t] = 1/2$.

Notice that a dichotomy condition on the ordinary differential equation (3) implies an exponential dichotomy for the difference equation (26) when $|B(t)|$ is small enough [26, Proposition 2]. However, an exponential dichotomy for the difference equation in (26) is not a necessary condition for an exponential dichotomy for the ordinary differential system (3). In fact, let's consider $t_n = n$ $A(t) = 0$ and $B(t) = \mathrm{diag}(-\frac{3}{2}, \frac{1}{2})$. Then the exponential dichotomy for difference system (26) is satisfied, with no exponential dichotomy for the ordinary differential system (3).

Assume the convergence of the series

$$\sum_{k=-\infty}^{0} \left| PZ(0,t_{k+1}) \int_{\gamma(t_k)}^{\gamma(t_{k+1})} \Phi(t_{k+1},s)ds \right| < \infty, \tag{61}$$

$$\sum_{k=0}^{\infty} \left| (I-P)Z(0,t_{k+1}) \int_{\gamma(t_k)}^{\gamma(t_{k+1})} \Phi(t_{k+1},s)ds \right| < \infty. \tag{62}$$

Note that $|PZ(0,t_{k+1})|$, $|(I-P)Z(0,t_{k+1})| \leq ce^{-\sigma|t_{k+1}|}$ and estimations of the integrals in the above series establish conditions for its convergence.

For example, $t_k = rk, 0 < r < 1$ and in general (61) and (62) are true if $(H2)$ and $(S2)$ hold. See Lopez-Fenner–Pinto [26].

We have the fundamental result about bounded solution on $\mathbb{R}$ of the linear non-homogeneous DEPCAG.

Theorem 5.3. *Let $g : \mathbb{R} \to \mathbb{C}^p$ be a bounded function. Assume that the linear DEPCAG (4) has a σ-exponential dichotomy such that (61) and (62) hold. Then there exists $y : \mathbb{R} \to \mathbb{C}^p$ a unique bounded solution of the non-homogeneous linear DEPCAG (6) are defined by*

$$y_g(t) = \int_{-\infty}^{\infty} \tilde{G}(t,s)g(s)ds = \int_{-\infty}^{t} \tilde{G}(t,s)g(s)ds + \int_{t}^{\infty} \tilde{G}(t,s)g(s)ds.$$

Moreover the correspondence $g \to y_g$ defines a Lipschitz continuous operator on $\mathcal{B}(\mathbb{R},\mathbb{C}^p)$ and $|y_g|_\infty \leq \hat{c}|g|_\infty$ with $\hat{c}$ given by (60).

Proof. We proceed as in the proof of Proposition 4.1 by noticing that $Z(t,0)$ can be decomposed as follows $Z(t,0) = Z(t,0)P + Z(t,0)(I-P)$. Moreover, in this case, we get that

$$\omega = \sum_{k=-\infty}^{0} PZ(0,t_{k+1}) \int_{\gamma(t_k)}^{\gamma(t_{k+1})} \Phi(t_{k+1},s)ds + \sum_{k=0}^{\infty} (I-P)Z(0,t_{k+1}) \int_{\gamma(t_k)}^{\gamma(t_{k+1})} \Phi(t_{k+1},s)ds,$$

instead of $\omega = -v_{-\infty}$. □

We note that estimations of the type $|PZ(0,t_{k+1})|$, $|(I-P)Z(0,t_{k+1})| \leq ce^{-\sigma|t_{k+1}|}$ imply the convergence of the series defined in (61) and (62). Furthermore, we observe that these kind of estimates are valid for example for the particular case $t_k = rk$ with $r \in (0,1)$ and in general when (H2) and (S2) hold, see [26] for details. Then we have the following corollary.

Corollary 5.4. *Let $g : \mathbb{R} \to \mathbb{C}^p$ be a bounded function. The results of Theorem 5.3 are valid if the hypotheses (H2) and (S2) are fulfilled.*

Now we study bounded perturbations which cannot be studied with ordinary dichotomy.

Theorem 5.5. *Assume that the linear system (4) has a σ-exponential dichotomy such that series (61) and (62) hold and f satisfies the hypothesis (H3) with η such that $|\eta(t)| \leq \eta_0$ for all $t \in [t_0,\infty)$. Moreover, consider that (H2) and the inequality $\beta = 2\hat{c}\eta_0(\sigma)^{-1} < 1$, with $\hat{c}$ defined in (60), are satisfied. Then, for any $\xi \in P\mathbb{C}^p$ the nonlinear equation (7) has a unique bounded solution w on $[t_0,\infty)$ with $Pw(t_0) = \xi$. Furthermore, the correspondence $\xi \to w$ is continuous and any bounded solution w of the equation (7) for $t \geq 0$, satisfies*

$$|w(t)| \leq (1-\beta)^{-1}c|\xi|e^{-\sigma_0 t}, \quad t \geq 0, \tag{63}$$

where

$$\sigma_0 = \sigma - \mu(1-\beta)^{-1}\hat{c}\eta_0 e^{\sigma\bar{t}} > 0, \quad \mu = \frac{2-\theta}{1-\theta}, \quad \theta = 2c\bar{t}\eta_0\rho(A)e^{2\sigma\bar{t}} < 1, \tag{64}$$

with η_0 sufficiently small.

Proof. The analysis of the bounded solutions for equation (7) is related with the nonlinear operator $\mathcal{D} : BC([t_0,\infty),\mathbb{C}^p) \to BC([t_0,\infty),\mathbb{C}^p)$ defined as follows

$$(\mathcal{D}w)(t) = Z(t,t_0)\xi + \int_{t_0}^{\infty} \tilde{G}(t,s)f(s,w(s),w(\gamma(s)))ds.$$

Now, by the hypothesis $\xi \in P\mathbb{C}^p$ we deduce that the operator $\mathcal{D}$ is equivalent to the operator $\mathcal{A}$ defined in the proof of Theorem 5.1, since $(\mathcal{D}w)(t) = (\mathcal{A}w)(t)$ for all $t \in [t_0,\infty)$ by considering that $y(t) = Z(t,t_0)P\xi = Z(t,t_0)\xi$ and $v(t) = w(t)$. Then, to prove the properties of $\mathcal{D}$ we proceed as in the proof of Theorem 5.1. Indeed, consider the integral equation (59) with $y(t) = Z(t,t_0)P\xi = Z(t,t_0)\xi$ and $v(t) = w(t)$. Again $\mathcal{A}$ (or equivalently $\mathcal{D}$) is a contraction since

$$\|\mathcal{A}w_1 - \mathcal{A}w_2\| \leq \beta\|w_1 - w_2\| \quad \text{with} \quad \beta = \frac{2\hat{c}\eta_0}{\sigma} < 1,$$

and then there exists a unique bounded continuous solution w of (7). The correspondence $\xi \to w_\xi$ is continuous since as in Theorem 5.1

$$\|y_{\xi_1} - y_{\xi_2}\| \leq c|\xi_1 - \xi_2| + \beta\|y_{\xi_1} - y_{\xi_2}\| \quad \text{and}$$
$$\|y_{\xi_1} - y_{\xi_2}\| \leq c(1-\beta)^{-1}|\xi_1 - \xi_2|.$$

Now, to prove that any bounded solution of the equation (7) for $t \geq 0$ converges exponentially to 0 as $t \to \infty$, we denote by w a bounded solution of (7) and define the function

$$z(t) = w(t) - \mathcal{A}w(t).$$

Note that z is well defined, continuous and bounded on $[0,\infty)$. Moreover, z is the solution of the linear DEPCAG (4) satisfying $Pz(0) = 0$ and hence $z(t) = Z(t,0)(I-P)z(0)$ which is bounded only if $(I-P)z(0) = 0$. Then $z(t) \equiv 0$, which implies that

$$w(t) = Z(t,0)\xi + \int_0^\infty \tilde{G}(t,s) f(s, w(s), w(\gamma(s)))ds \tag{65}$$

and $w(t) \to 0$ as $t \to \infty$. Indeed, let $\theta \in (\beta, 1)$ and considering that $\overline{\lim}_{t\to\infty} |w(t)| = \ell > 0$, then $|w(t)| \le \theta^{-1}\ell$ for $t \ge T$ and by (65) we deduce that

$$|w(t)| \le |Z(t,0)||\xi| + |Z(t,0)P| \left| \int_0^T Z_P(0,s) f(s, w(s), w(\gamma(s)))ds \right| + \beta\theta^{-1}\ell$$

which letting $t \to \infty$, gives $\ell \le \beta\theta^{-1}\ell$ which is impossible; and hence $\ell = 0$.

Finally, from the integral equation (65) we get

$$\begin{aligned} |w(t)| \le{}& ce^{-\sigma t}|\xi| + \hat{c}\eta_0 \int_0^t e^{-\sigma(t-s)}(|w(s)| + |w(\gamma(s))|)ds \\ &+ \hat{c}\eta_0 \int_t^\infty e^{-\sigma(s-t)}(|w(s)| + |w(\gamma(s))|)ds. \end{aligned} \tag{66}$$

Define

$$m(t) = \sup_{s\ge t} |w(s)|.$$

Since $w(t) \to 0$ as $t \to \infty$, $m(t)$ exists and is monotone nonincreasing. Moreover, for each t there exists $\tilde{t} \ge t$ such that

$$m(t) = m(\tilde{t}) \text{ and } m(s) = m(t) = m(\tilde{t}) \text{ for } s \in [t, \tilde{t}]. \tag{67}$$

Thus (66) with $t = \tilde{t}$ yields

$$\begin{aligned} m(\tilde{t}) \le{}& ce^{-\sigma\tilde{t}}|\xi| + \hat{c}\eta_0 \int_0^{\tilde{t}} e^{-\sigma(\tilde{t}-s)}(|m(s)| + |m(\gamma(s))|)ds \\ &+ m(\tilde{t})\hat{c}\eta_0 \int_{\tilde{t}}^\infty e^{-\sigma(s-\tilde{t})}ds \end{aligned}$$

or by (67)

$$\begin{aligned} m(t) \le{}& ce^{-\sigma t}|\xi| \\ &+ \hat{c}\eta_0 \int_0^t e^{-\sigma(t-s)}\Big(|m(s)| + e^{-\sigma\gamma(s)}e^{\sigma\gamma(s)}|m(\gamma(s))|\Big)ds + \beta m(t), \end{aligned}$$

since $\beta = 2\hat{c}\eta_0(\sigma)^{-1} < 1$, $M(t) = e^{\sigma t}m(t)$ satisfies

$$M(t) \leq (1-\beta)^{-1}c|\xi| + (1-\beta)^{-1}\hat{c}\eta_0 e^{\sigma\bar{t}} \int_0^t \Big(|M(s)| + |M(\gamma(s))|\Big)ds,$$

which by DEPCAG Gronwall inequality Lemma 2.6 gives

$$M(t) \leq (1-\beta)^{-1}c|\xi| \exp\Big(t\mu(1-\beta)^{-1}\hat{c}\eta_0 e^{\sigma\bar{t}}\Big)$$

or

$$m(t) \leq (1-\beta)^{-1}c|\xi| \exp\Big(-\Big[\sigma - \mu(1-\beta)^{-1}\hat{c}\eta_0 e^{\sigma\bar{t}}\Big]t\Big),$$

where μ and θ are given by (64). Thus (63) is proved. □

Acknowledgments

A. Coronel thanks for the support of research projects GI 153209/C and DIUBB GI 152920/EF at Universidad del Bío-Bío, Chile. M. Pinto thanks for the support of FONDECYT 1120709.

References

[1] A.R. Aftabizadeh, J. Wiener, J.M. Xu, Oscillatory and periodic solutions of delay differential equations with piecewise constant argument, Proc. Amer. Math. Soc. 99 (1987) 673–679.
[2] R. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.
[3] M.U. Akhmet, Integral manifolds of differential equations with piecewise constant argument of generalized type, Nonlinear Anal. TMA 66 (2007) 367–383.
[4] M.U. Akhmet, Stability of differential equations with piecewise constant arguments of generalized type, Nonlinear Anal. TMA 68 (2008) 794–803.
[5] M.U. Akhmet, Nonlinear hybrid continuous/discrete-time models, Atlantis Stud. Math. Eng. Sci. 8 (2011).
[6] M.U. Akhmet, E. Yilmaz, Neural Networks with Discontinuous/Impact Activations, Nonlinear Systems and Complexity, vol. 9, 2014.
[7] A. Alonso, J. Hong, R. Obaya, Almost periodic type solutions of differential equations with piecewise constant argument via almost periodic type sequences, Appl. Math. Lett. 13 (2000) 131–137.
[8] T.A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Academic Press, Inc., New York, 1985.
[9] S. Busenberg, K.L. Cooke, Vertically Transmitted Diseases, Springer-Verlag, Berlin, 1993.
[10] A. Cabada, J.B. Ferreiro, J.J. Nieto, Green's function and comparison principles for first order differential equations with piecewise constant arguments, J. Math. Anal. Appl. 291 (2004) 690–697.
[11] S. Castillo, M. Pinto, Existence and stability of almost periodic solutions to differential equations with piecewise constant arguments, Electron. J. Differential Equations 2015 (58) (2015) 1.
[12] A. Chávez, S. Castillo, M. Pinto, Discontinuous almost periodic type functions, almost automorphy of solutions of differential equations with discontinuous delay and applications, Electron. J. Qual. Theory Differ. Equ. 2014 (75) (2014) 1–17.
[13] A. Chávez, S. Castillo, M. Pinto, Discontinuous almost automorphic functions and almost automorphic solutions of differential equations with piecewise constant arguments, Electron. J. Differential Equations 2014 (56) (2014) 1.
[14] K.-Sh. Chiu, J.-Ch. Jeng, Stability of oscillatory solutions of differential equations with general piecewise constant arguments of mixed type, Math. Nachr. 288 (10) (2015) 1085–1097.
[15] K.S. Chiu, M. Pinto, Periodic solutions of differential equations with a general piecewise constant argument and applications, Electron. J. Qual. Theory Differ. Equ. 46 (2010), 19 pp.
[16] K.S. Chiu, M. Pinto, J.C. Jeng, Existence and global convergence of periodic solutions in recurrent neural network models with a general piecewise alternately advanced and retarded argument, Acta Appl. Math. 133 (1) (2014) 133–152.
[17] K.L. Cooke, J. Wiener, An equation alternately of retarded and advanced type, Proc. Amer. Math. Soc. 99 (1987) 726–732.
[18] K.L. Cooke, J. Wiener, Oscillations in systems of differential equations with piecewise constant delays, J. Math. Anal. Appl. 137 (1989) 221–239.
[19] W.A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath, Boston, MA, 1965.
[20] R.D. Driver, Ordinary and Delay Differential Equations, Springer-Verlag, New York, 1977.
[21] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations, Kluwer Academic Publishers Group, Dordrecht, 1992.

[22] K. Gopalsamy, M.R.S. Kulenovic, G. Ladas, On a logistic equation with piecewise constant arguments, Differential Integral Equations 4 (1991) 215–223.
[23] I. Gyori, G. Ladas, Oscillation Theory of Delay Differential Equations. With Applications, Oxford University Press, New York, 1991.
[24] J. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.
[25] K.N. Jayasree, S.G. Deo, Variation of parameters formula for the equation of Cooke and Wiener, Proc. Amer. Math. Soc. 112 (1991) 75–90.
[26] J. Lopez-Fenner, M. Pinto, On a Hartman linearization for a class of ODE with impulsive effect, Nonlinear Anal. 38 (1990) 307–325.
[27] Y. Muroya, Persistence, contractivity and global stability in logistics equations with piecewise constant delays, J. Math. Anal. Appl. 270 (2002) 602–635.
[28] A.D. Myshkis, On certain problems in the theory of differential equations with deviating arguments, Uspekhi Mat. Nauk 32 (1977) 173–202.
[29] G. Papaschinopoulos, Linearization near the integral manifold for a system of differential equations with piecewise constant argument, J. Math. Anal. Appl. 215 (1997) 317–333.
[30] G. Papaschinopoulos, Exponential dichotomy, topological equivalence and structural stability for differential equations with piecewise constant argument, Analysis 145 (1994) 239–247.
[31] Garyfalos Papaschinopoulos, Some results concerning a class of differential equations with piecewise constant argument, Math. Nachr. 166 (1994) 193–206.
[32] G. Papaschinopoulos, A linearization result for a differential equation with piecewise constant argument, Analysis 16 (1996) 161–170.
[33] M. Pinto, Asymptotic equivalence of nonlinear and quasilinear differential equations with piecewise constant argument, Math. Comput. Modelling 49 (2009) 1750–1758.
[34] M. Pinto, Cauchy and Green matrices type and stability in alternately advanced and delayed differential systems, J. Difference Equ. Appl. 17 (2) (2011) 235–254.
[35] M. Pinto, G. Robledo, Controllability and observability for a linear time varying system with piecewise constant delay, Acta Appl. Math. 136 (2015) 193–216.
[36] M. Pinto, G. Robledo, A Grobman–Hartman theorem for a differential equation with piecewise constant generalized argument, preprint on arXiv, http://arxiv.org/abs/1506.00533.
[37] G. Seifert, Almost periodic solutions of certain differential equations with piecewise constant delays and almost periodic time dependence, J. Differential Equations 164 (2000) 451–458.
[38] T. Veloz, M. Pinto, Existence, computability and stability for solutions of the diffusion equation with general piecewise constant argument, J. Math. Anal. Appl. 426 (1) (2015) 330–339.
[39] J. Wiener, Differential equations with piecewise constant delays, in: V. Lakshminkantham (Ed.), Trends in the Theory and Practice of Nonlinear Differential Equations, Marcel Dekker, New York, 1983, pp. 547–580.
[40] J. Wiener, Generalized Solutions of Functional Differential Equations, World Scientific, Singapore, 1993.
[41] J. Wiener, V. Lakshmikantham, Excitability of a second-order delay differential equations, Nonlinear Anal. 38 (1999) 1–11.
[42] Y.H. Xia, Z. Huang, M. Han, Existence of almost periodic solutions for forced perturbed systems with piecewise constant arguments, J. Math. Anal. Appl. 333 (2007) 798–816.
[43] P. Yang, Y. Liu, W. Ge, Green's function for second order differential equations with piecewise constant arguments, Nonlinear Anal. 64 (2006) 1812–1830.
[44] R. Yuan, The existence of almost periodic solutions of retarded differential equations with piecewise constant argument, Nonlinear Anal. 48 (2002) 1013–1032.