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On local solvability of a class of abstract underdetermined systems

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ABSTRACT

In this work we present a necessary and sufficient condition for a class of abstract underdetermined systems to be solvable. We develop J. F. Trèves' ideas, presenting the so called condition (ψ) and its connection with the study of the solvability in consideration. We also prove the existence of finite order regularity solutions.

1. Introduction

In this paper we study the local solvability in top degree of the differential complex defined by the operators

$$\mathrm{L}_j = \partial_{t_j} - (\partial_{t_j}\phi)(t, A)A, \quad j = 1, \dots, n,$$

where A is a linear operator, densely defined in a Hilbert space H. We shall assume that A is unbounded, but it is self-adjoint, *positive definite* and it has a bounded inverse A^{-1}; and where $\phi(t, A)$ are power series with respect to A^{-1}, with coefficients in $C^\infty(\Omega)$, for some open set $\Omega \subset \mathbb{R}^n$, that is,

$$\phi(t, A) = \sum_{k \geq 0} \phi_k(t) A^{-k}.$$

These power series are assumed to be convergent in $L(H, H)$, as well as each of their t-derivatives, uniformly with respect to t on compact subsets of Ω.

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Our analysis will focus on a neighborhood Ω of the origin. If $\Omega \subset \mathbb{R}$ is an open interval containing the origin, [8] shows us necessary and sufficient conditions for the local solvability and hypoellipticity of the operator $\mathrm{L} = \partial_t - \phi(t, A)A$ at $t = 0$.

This work concerns the following problem:

For each $k \in \mathbb{Z}_+$ such that $N > \frac{n+k}{2}$, $N \in \mathbb{Z}_+$, find open neighborhoods of 0, $\omega_N \subset \Omega_N$, such that

$$\forall f \in C^\infty_{(n)}(\Omega_N, H^\infty),\ \exists v^{(N)} \in C^k_{(n-1)}(\omega_N, H^k) \text{ such that } \mathbb{L}v^{(N)} = f \text{ in } w_N, \tag{1.1}$$

where

$$\mathbb{L}v^{(N)} = \Big(\sum_{j=1}^{n} \mathrm{L}_j v_j^{(N)}\Big) dt_1 \wedge \ldots \wedge dt_n.$$

We denote by $C^\omega(\Omega)$ the space of analytic functions in Ω and assume $\phi_0 \in C^\omega(\Omega)$. In the text, $\Re\phi_0$ and $\Im\phi_0$ denote the real part and the imaginary part of ϕ_0, respectively.

Let B be the ball $\{t \in \mathbb{R}^n : |t| < R\} \subset\subset \Omega$.

Definition 1.1. We say that condition (ψ_1) holds on B if, for every real number a, the set

$$\{t \in B : \Re\phi_0 \leq a\} \text{ has no compact connected components.}$$

Definition 1.2. We say that condition (ψ_2) holds on B if, for every real number a, the set

$$\{t \in B : \Re\phi_0 \geq a\} \text{ has no compact connected components.}$$

Definition 1.3. We say that conditions (ψ_1) and (ψ_2) hold at 0 if, for any open ball B centered at the origin, there exists an open subset $\Omega' \subset B$ containing 0 such that both (ψ_1) and (ψ_2) hold on Ω'.

The main result states:

Theorem 1.4. *Condition (ψ_1) at 0 is necessary and sufficient to solve (1.1).*

The case where A is a linear operator, densely defined in H, unbounded and self-adjoint is also considered. That is, we study the problem:

For each $k \in \mathbb{Z}_+$ such that $N > \frac{n+k}{2}$, $N \in \mathbb{Z}_+$, find open neighborhoods of 0, $\omega_N \subset \Omega_N$, such that

$$\forall f \in C^\infty_{(n)}(\Omega_N, H^\infty),\ \exists v^{(N)} \in C^k_{(n-1)}(\omega_N, H^k) \text{ such that } \mathbb{L}_0 v^{(N)} = f \text{ in } w_N, \tag{1.2}$$

where

$$\mathbb{L}_0 v^{(N)} = \Big(\sum_{j=1}^{n} \mathrm{L}_{j,0} v_j^{(N)}\Big) dt_1 \wedge \ldots \wedge dt_n, \qquad \mathrm{L}_{j,0} = \partial_{t_j} - (\partial_{t_j}\Re\phi_0)(t)A, \quad j = 1, \ldots, n.$$

The result proved is the following:

Theorem 1.5. *Conditions (ψ_1) and (ψ_2) at 0 are necessary and sufficient to solve (1.2).*

Remark. In problems (1.1) and (1.2) a category argument shows that ω_N can be assumed to depend only on Ω_N and not on f.

2. Notations

Let H be a Hilbert space and let A be a linear operator, densely defined in H, unbounded but self-adjoint, *positive definite* and has a bounded inverse A^{-1}.

If Ω is an open set in $\mathbb{R}^n$, with variable $t = (t_1, \dots, t_n)$, we denote by $\mathcal{Q}_A(\Omega)$ the ring of the power series of the form

$$\phi(t, A) = \sum_{k \geq 0} \phi_k(t) A^{-k},$$

where $\phi_k \in C^\infty(\Omega)$, and the series of all t-derivatives converge in $L(H, H)$, uniformly on compact subsets of Ω.

We will use the scale of "Sobolev" spaces (for $s \in \mathbb{R}$) defined by A (see also [1,3,8]): if $s \geq 0$, H^s is the space of elements u of H such that $A^s u \in H$, equipped with the norm $\|u\|_s = \|A^s u\|_0$, where $\| \ \|_0$ denotes the norm in $H = H^0$; if $s < 0$, H^s is the completion of H for the norm $\|u\|_s = \|A^s u\|_0$. The inner product in H^s will be denoted by $(\ , \ )_s$. Whatever $s \in \mathbb{R}$, $m \in \mathbb{R}$, A^m is an isomorphism (for the Hilbert space structures) of H^s onto H^{s-m}. A good example of this construction is obtained when $A = (1 - \Delta_x)^{1/2}$ and $H = L^2(\mathbb{R}^\nu)$: then H^s is the "true" Sobolev space in $\mathbb{R}^\nu$, of degree s.

By H^∞ we denote the *intersection* of the spaces H^s, equipped with the projective limit topology, and by $H^{-\infty}$ their *union*, with the inductive limit topology. Since, for each $s \in \mathbb{R}$, H^s and H^{-s} can be regarded as the dual of each other, so can H^∞ and $H^{-\infty}$: given their topologies, they are the strong dual of each other.

We denote by $C^\infty(\Omega, H^\infty)$ the space of C^∞ *functions in* Ω *valued in* H^∞. It is the intersection of the spaces $C^j(\Omega, H^k)$ (of the j*-continuously differentiable functions defined in* Ω *and valued in* H^k) as the non-negative integers j, k tend to $+\infty$. We equip $C^\infty(\Omega, H^\infty)$ with its natural C^∞ topology. If K is any compact subset of Ω, we denote by $C_c^\infty(K, H^\infty)$ the subspace of $C^\infty(\Omega, H^\infty)$ consisting of the functions which vanish identically outside K. It is a closed linear subspace of $C^\infty(\Omega, H^\infty)$, hence a Fréchet space, and we denote by $C_c^\infty(\Omega, H^\infty)$ the inductive limit of $C_c^\infty(K, H^\infty)$ as K ranges over all compact subsets of Ω.

We will denote by $\mathcal{D}'(\Omega, H^{-\infty})$ the dual of $C_c^\infty(\Omega, H^\infty)$, and refer to it as the space of *distributions in* Ω *valued in* $H^{-\infty}$. By $D'^0(\Omega, H^{-\infty})$ we denote the dual of $C_c(\Omega, H^\infty)$, and refer to it as the space of *distributions of order* 0 *in* Ω *valued in* $H^{-\infty}$.

3. The spaces $\mathscr{H}^s$; $\mathscr{H}^s(K)$; $\mathscr{H}^s_{loc}(\Omega)$; $\mathscr{H}^M(\Omega), \mathscr{H}^{-M}(\Omega)$, $M \in \mathbb{Z}_+$; and $\mathscr{A}(\Omega, H^w)$

We will denote by $\mathscr{S}(\mathbb{R}^n, H^\infty)$ the space of all functions $u \in C^\infty(\mathbb{R}^n, H^\infty)$ such that, for all pairs of polynomials P and Q in the variable t, and with complex coefficients, $P(t)Q(\partial_t)u(t)$ remains in a bounded subset of H^∞ as t varies over $\mathbb{R}^n$, i.e., such that

$$\forall s \in \mathbb{R}, \quad \sup_{t \in \mathbb{R}^n} ||P(t)Q(\partial_t)u(t)||_s < \infty. \tag{3.1}$$

We equip $\mathscr{S}(\mathbb{R}^n, H^\infty)$ with its natural topology (i.e., we take as a basis of continuous seminorms the expressions in (3.1)).

We define the integral of a continuous function valued in a locally convex vector space as the limit of Riemann sums. Then, if $u \in \mathscr{S}(\mathbb{R}^n, H^\infty)$, we may form its Fourier transform $\mathscr{F}(u) = \hat{u}$ by

$$\hat{u}(\tau) = \int_{\mathbb{R}^n} e^{-it\tau} u(t) dt, \qquad \forall \tau \in \mathbb{R}^n.$$

It can be checked at once that $\hat{u}(\tau) \in H^\infty$ for every $\tau \in \mathbb{R}^n$; and that $\hat{u} \in \mathscr{S}(\mathbb{R}^n, H^\infty)$. Moreover, the Fourier transform is a continuous linear map from $\mathscr{S}(\mathbb{R}^n, H^\infty)$ into itself, and it can be verified that its inverse is given by the usual formula:

$$u(t) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{it\tau} \hat{u}(\tau) d\tau, \qquad \forall t \in \mathbb{R}^n,$$

which shows that the Fourier transform is an isomorphism from $\mathscr{S}(\mathbb{R}^n, H^\infty)$ onto itself.

As usually, except for a multiplicative constant, the Fourier transform can be extended as an isometry of $L^2(\mathbb{R}^n, H)$ onto itself. We have precisely:

$$\int ||\hat{u}(\tau)||_0^2 d\tau = (2\pi)^n \int ||u(t)||_0^2 dt.$$

We denote by $\mathscr{S}'(\mathbb{R}^n, H^{-\infty})$ the dual of $\mathscr{S}(\mathbb{R}^n, H^\infty)$, and we refer to it as the space of *tempered distributions on* $\mathbb{R}^n$*, valued in* $H^{-\infty}$. Since $C_c^\infty(\mathbb{R}^n, H^\infty)$ is dense in $\mathscr{S}(\mathbb{R}^n, H^\infty)$, we can identify $\mathscr{S}'(\mathbb{R}^n, H^{-\infty})$ (as a set) with a subspace of $\mathcal{D}'(\mathbb{R}^n, H^{-\infty})$. The transposition of the Fourier transform gives an isomorphism from $\mathscr{S}'(\mathbb{R}^n, H^{-\infty})$ onto itself, which extends the initial one, and will also be referred to as a Fourier transform.

We define the operator $\Lambda^s : \mathscr{S}'(\mathbb{R}^n, H^{-\infty}) \to \mathscr{S}'(\mathbb{R}^n, H^{-\infty})$ by

$$\Lambda^s(u(t)) = \mathscr{F}^{-1}\big\{(1 + |\tau|^2 + A^2)^{s/2}\hat{u}(\tau)\big\}.$$

Definition 3.1. $\mathscr{H}^s$, $s \in \mathbb{R}$, is the space of tempered distributions u on $\mathbb{R}^n$, valued in $H^{-\infty}$, such that its Fourier transform $\hat{u}$ is a measurable function and

$$(1 + |\tau|^2 + A^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n, H) \ .$$

The norm in $\mathscr{H}^s$ is given by

$$|||u|||_s^2 = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} ||(1 + |\tau|^2 + A^2)^{s/2}\hat{u}(\tau)||_0^2 d\tau \ .$$

If $u \in \mathscr{H}^s$ and $p \in \mathbb{R}$, then we have $|||\Lambda^s u|||_p^2 = |||u|||_{s+p}^2$, i.e., $\Lambda^s : \mathscr{H}^{s+p} \to \mathscr{H}^p$ is an isometry.

Definition 3.2. Let K be a compact subset of $\mathbb{R}^n$. Then we call

$$\mathscr{H}^s(K) = \{u \in \mathscr{H}^s \mid \text{supp } u \text{ is a subset of } K\},$$

with the topology induced by $\mathscr{H}^s$.

By supp u we will always mean the support of u.

Let Ω be an open subset of $\mathbb{R}^n$.

Definition 3.3. We call

$$\mathscr{H}^s_{loc}(\Omega) = \{u \in \mathcal{D}'(\Omega, H^{-\infty}) \mid \forall \phi \in C_c^\infty(\Omega), \text{ we have } \phi u \in \mathscr{H}^s\},$$

with the coarsest locally convex topology which renders all the maps $u \to \phi u$ from $\mathscr{H}^s_{loc}(\Omega)$ into $\mathscr{H}^s$ continuous.

We give below two important properties of the spaces $\mathscr{H}^s$ and $\mathscr{H}^s_{loc}(\Omega)$:

$\forall s, r \in \mathbb{R}, r \geq 0$, we have continuous injections

$$\mathscr{H}^{s+r} \to \mathscr{H}^s, \quad \mathscr{H}^{s+r}_{loc}(\Omega) \to \mathscr{H}^s_{loc}(\Omega) .$$

$\forall s, r \in \mathbb{R}, r \geq 0, \forall \alpha \in \mathbb{Z}^n_+$, $A^r \partial_t^\alpha$ is a continuous operator

$$\text{from } \mathscr{H}^s \text{ into } \mathscr{H}^{s-r-|\alpha|}, \qquad \text{from } \mathscr{H}^s_{loc}(\Omega) \text{ into } \mathscr{H}^{s-r-|\alpha|}_{loc}(\Omega).$$

Definition 3.4. Let $M, k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}^n_+$. We call

$$\mathscr{H}^M(\Omega) = \{u \in L^2(\Omega, H) \mid A^k \partial_t^\alpha u \in L^2(\Omega, H),\ |\alpha| + k \leq M\},$$

equipped with the norm

$$|||u|||_M = \left(\sum_{|\alpha|+k \leq M} \|A^k \partial_t^\alpha u\|^2_{L^2(\Omega,H)} \right)^{1/2} .$$

Definition 3.5. Let $M, k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}^n_+$. We call

$$\mathscr{H}^{-M}(\Omega) = \left\{ T \in \mathcal{D}'(\Omega, H) \mid T = \sum_{|\alpha|+k \leq M} A^k \partial_t^\alpha u_{\alpha,k},\ u_{\alpha,k} \in L^2(\Omega, H) \right\},$$

equipped with the norm

$$|||T|||_{-M} = inf \left\{ \left(\sum_{|\alpha|+k \leq M} \|u_{\alpha,k}\|^2_{L^2(\Omega,H)} \right)^{1/2} \right\}.$$

We also give a version of Gagliardo's inequality in the abstract set-up:

Proposition 3.6 *(Gagliardo's inequality). If s, s_1, s_2 are real numbers such that $s > s_1 \geq s_2$, then for each $\epsilon > 0$, there exists $M > 0$ depending on ϵ such that*

$$|||u|||_{s_1} \leq \epsilon |||u|||_s + M |||u|||_{s_2}, \quad \forall u \in \mathscr{H}^s(K).$$

We denote by $\mathscr{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$. As in the classical theory we have:

Theorem 3.7. *If $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $u \in \mathscr{H}^s$, $s \in \mathbb{R}$, then $\phi u \in \mathscr{H}^s$ and*

$$|||\phi u|||_s \leq C |||u|||_s, \quad C = C(\phi, s).$$

Proposition 3.8. *If $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $s \in \mathbb{R}$, then*

$$[\Lambda^s, \phi] : \mathscr{H}^p \to \mathscr{H}^{p-s+1} \text{ is continuous}, \quad \forall p \in \mathbb{R}.$$

Corollary 3.9. *Let $M \in \mathbb{Z}_+$, $\gamma \in \mathbb{Z}^n_+$ and $s \in \mathbb{R}$. If $Q(t, \partial_t) = \sum_{|\gamma| \leq M} a_\gamma \partial_t^\gamma$, $a_\gamma \in \mathscr{S}(\mathbb{R}^n)$, then*

$$[\Lambda^s, Q] : \mathscr{H}^p \to \mathscr{H}^{p-s-M+1} \text{ is continuous}, \quad \forall p \in \mathbb{R}.$$

Another important class of functions is the following (see [7]):

Definition 3.10. We denote by $\mathscr{A}(\Omega, H^\omega)$ the subspace of $C^\infty(\Omega, H^\infty)$ given by the set of functions $u \in C^\infty(\Omega, H^\infty)$ such that, for every $t_0 \in \Omega$, there exist a relatively compact open neighborhood U of t_0 contained in Ω and $C > 0$ such that, for every $k \in \mathbb{Z}_+$ and every $\alpha \in \mathbb{Z}_+^n$,

$$\sup_{t\in U} \|\partial_t^\alpha A^k u(t)\|_0 \ \leq \ C^{|\alpha|+k+1}(|\alpha|+k)!\ . \tag{3.2}$$

Alternatively, we have (see [6]):

Definition 3.11. We denote by $C^\infty(\Omega, E^\sigma)$ the subspace of $C^\infty(\Omega, H^\infty)$ given by the set of functions $u \in C^\infty(\Omega, H^\infty)$ such that, for every $t_0 \in \Omega$, there exists a relatively compact open neighborhood U of t_0 contained in Ω such that

$$E^\sigma = \{u(t)|\ e^{\sigma A}u(t) \in H,\ \ t \in U\} \ \ \text{for some } \sigma > 0,$$

i.e.,

$$\sum_{k=0}^{\infty} \frac{||A^k u(t)||_0}{k!}\sigma^k < \infty \ \ ,\ t \in U,\ \ \text{for some } \sigma > 0.$$

Indeed, it suffices to set $|\alpha| = 0$ in (3.2) and take σ such that $C\sigma < 1$. Conversely, $u(t) \in E^\sigma \subset E^{\sigma'}$, for some $0 < \sigma' < 1$, $\sigma' < \sigma$, $t \in U$. Thus, there exists $k_0 \in \mathbb{Z}_+$ such that, if $k > k_0$, we have

$$\frac{||A^k u(t)||_0}{k!}\sigma'^k < 1.$$

By induction on $|\alpha|$, we conclude the proof.

4. The differential complex

From now on we consider Ω an open subset of $\mathbb{R}^n$, $0 \in \Omega$, and define, for $p = 0, ..., n$, the spaces

$$C^\infty_{(p)}(\Omega, H^\infty) \doteq \left\{ f = \sum_{|J|=p} f_J(t)dt_J,\ \ f_J \in C^\infty(\Omega, H^\infty) \right\} \ \ \text{and}$$

$$\mathcal{D}'_{(p)}(\Omega, H^{-\infty}) \doteq \left\{ f = \sum_{|J|=p} f_J\ dt_J,\ \ f_J \in \mathcal{D}'(\Omega, H^{-\infty}) \right\},$$

where J is an ordered multi-index $(j_1, ..., j_p)$ of integers such that $1 \leq j_1 < j_2 < ... < j_p \leq n$, $|J| = p$ its length and $dt_J = dt_{j_1} \wedge ... \wedge dt_{j_p}$. Given $\phi \in \mathcal{Q}_A(\Omega)$, we define the operators

$$\mathrm{L}_j = \partial_{t_j} - (\partial_{t_j}\phi)(t, A)A, \quad j = 1, \ldots, n, \qquad (\partial_{t_j}\phi)(t, A) = \sum_{k=0}^{\infty}(\partial_{t_j}\phi_k)A^{-k},$$

and introduce, for $p = 0, ..., n-1$, the differential complex

$$\mathbb{L}\ :\ C^\infty_{(p)}(\Omega, H^\infty) \longrightarrow C^\infty_{(p+1)}(\Omega, H^\infty) \ \ \text{or}$$
$$\mathbb{L}\ :\ \mathcal{D}'_{(p)}(\Omega, H^{-\infty}) \longrightarrow \mathcal{D}'_{(p+1)}(\Omega, H^{-\infty})$$

given by

$$\mathbb{L}f = \sum_{j=1}^{n} \sum_{|J|=p} \mathrm{L}_j f_J dt_j \wedge dt_J.$$

It is easily seen that $[\mathrm{L}_j, \mathrm{L}_k] = \mathrm{L}_j(\mathrm{L}_k) - \mathrm{L}_k(\mathrm{L}_j) = 0$ and so $\mathbb{L}^2 = 0$, that is, $\mathbb{L}$ is a complex; and that $\mathbb{L}f = 0$ if $p = n$. In the same way, we also define the differential complex

$$\mathbb{L}_0 f = \sum_{j=1}^{n} \sum_{|J|=p} \mathrm{L}_{j,0} f_J dt_j \wedge dt_J,$$

where $\mathrm{L}_{j,0} = \partial_{t_j} - (\partial_{t_j} \Re\phi_0)(t)A,\ j = 1, \ldots, n$.

Definition 4.1. $\mathbb{L}$ is locally solvable at the origin in degree p, $1 \le p \le n$, if, given a neighborhood $\Omega' \subset \Omega$ of the origin, there exists a neighborhood $\Omega'' \subset \Omega'$, $0 \in \Omega''$, such that

$$\forall f \in C^\infty_{(p)}(\Omega', H^\infty),\ \mathbb{L}f = 0,\ \exists u \in \mathcal{D}'_{(p-1)}(\Omega'', H^{-\infty}) \text{ such that } \mathbb{L}u = f \text{ in } \Omega''.$$

Lemma 4.2. $\mathbb{L}$ *is locally solvable at the origin in degree* p *if and only if this is true of* $\mathbb{L}_0$*.*

Proof. Set

$$\alpha_1(t, A) = i\ \Im\phi_0(t)I + \phi(t, A) - \phi_0(t)I\ , \qquad t \in \Omega'.$$

It is immediately checked that $u \mapsto U(t)u$, where

$$U(t) = e^{\alpha_1(t,A)A},$$

defines an automorphism of $\mathcal{D}'_{(p)}(\Omega', H^{-\infty})$ or of $C^\infty_{(p)}(\Omega', H^\infty)$, $0 \le p \le n$.

For each $f \in \mathcal{D}'_{(p)}(\Omega', H^{-\infty})$, $0 \le p \le n$, we have

$$\mathbb{L}\ Uf = U\mathbb{L}_0 f$$

and

$$\mathbb{L}_0 U^{-1} f = U^{-1}\mathbb{L}f.$$

Then, for $1 \le p \le n$, the end of the proof is an easy consequence of both equalities above. □

In virtue of Lemma 4.2, we get

Lemma 4.3. *Definition 4.1 for* $p = n$ *is equivalent to:*

∃ an open set $U \subset \Omega'$, $0 \in U$, *such that* $\forall g \in C^\infty_c(U, H^\infty)$, $\exists\ v_j \in \mathcal{D}'(U, H^{-\infty})$, $j = 1, \ldots, n$, *satisfying*

$$\mathrm{L}_{1,0}v_1 + \ldots + \mathrm{L}_{n,0}v_n = g \qquad \textit{in}\ \ U.$$

5. Solvability

From now on we assume $\phi_0 \in C^\omega(\Omega)$ *real-valued*. We recall that $B = \{t \in \mathbb{R}^n : |t| < R\} \subset\subset \Omega$ and initially we study the following equation (local solvability):

$[S_0]^1$ For every $f \in C_c^\infty(B, H^\infty)$, there exist $u_j \in \mathcal{D}'(B, H^{-\infty})$, $j = 1, \ldots, n$, satisfying

$$\mathrm{L}_{1,0}u_1 + \ldots + \mathrm{L}_{n,0}u_n = f \qquad \text{in} \quad B. \tag{5.1}$$

Theorem 5.1. *Condition* (ψ_1) *on* B *is necessary and sufficient for the local solvability of the equation (5.1).*

6. Conditions (ψ_1) and (ψ_2)

Proposition 6.1. *Conditions* (ψ_1) *and* (ψ_2) *on* B *are equivalent to*

$$\min_K \phi_0 = \min_{\partial K} \phi_0, \qquad \textit{for all compact subset } K \textit{ of } B$$

and

$$\max_K \phi_0 = \max_{\partial K} \phi_0, \qquad \textit{for all compact subset } K \textit{ of } B,$$

respectively, where ∂K *is the boundary of* K*.*

Proof. See [4], Proposition 2.2. □

Proposition 6.2. *(Case* $n = 1$*) Let* J *be the set* $\{t \in \mathbb{R} : |t| < a\}$ *and let* ϕ_0 *be a real analytic function in* J*. Then condition* (ψ_1) *holding on* J *is equivalent to*

$$(\psi_1') \qquad \textit{if } \phi_0'(\bar{t}) < 0 \textit{ for some } \bar{t} \in J, \textit{then } \phi_0'(t) \le 0, \quad \forall t \in J, \quad t > \bar{t}.$$

Proof. The equivalence is trivial if ϕ_0 is a constant. If (ψ_1') is not fulfilled, then there exist points $t_2 > t_1$ in J such that $\phi_0'(t_1) < 0 < \phi_0'(t_2)$. By intermediate value theorem, the set Z containing points where ϕ_0' is null intersects the interval $]t_1, t_2[$. We may write $Z \cap]t_1, t_2[= \{s_1, ..., s_N\}$, which has a finite number of points, since ϕ_0 is analytic and it is not a constant. Necessarily there exists j such that ϕ_0' changes the sign, from minus to plus, at s_j. Thus, s_j is a strict local minimum point of ϕ_0 and, therefore, (ψ_1) does not hold when K is a small compact interval centered at s_j.

Conversely, if (ψ_1) does not hold on J, then there exists an interval $I \subset\subset J$ such that

$$\min_I \phi_0 < \min_{\partial I} \phi_0.$$

Let

$$\phi_0(t_1) \doteq \min_I \phi_0.$$

Since t_1 is a local minimum point of ϕ_0, it follows that $\phi_0'(t_1) = 0$ and, as ϕ_0' is analytic in J, there exists an open interval $I' \subset I$, $t_1 \in I'$, such that t_1 is an isolated zero of ϕ_0' in I'. Again, t_1 being a local minimum point of ϕ_0 in I' implies that

$$\phi_0'(t) < 0,\, t < t_1 \ t \in I',$$
$$\phi_0'(t) > 0,\, t > t_1 \ t \in I',$$

which is the contradiction of (ψ_1'). □

Proposition 6.3. (ψ_1) *implies the existence of a constant* $C > 0$ *satisfying the following property:*

- $\forall\ t \in B$, $\exists$ *a curve* γ_t, *analytic by parts, which connects* t *to a point on the border of* B, *satisfying* $\phi_0(s) \le \phi_0(t)$ *for every* s *on* γ_t, *and the length of* $\gamma_t \le C$.

Proof. See [4], Proposition 2.6. □

7. Sufficiency of Theorem 5.1

The formal adjoint of $\mathrm{L}_{j,0}$ is equal to $\mathrm{L}_{j,0}^* = -\big(\partial_{t_j} + \partial_{t_j}\phi_0(t)A\big)$.

If ϕ_0 is a constant, then

$$u_1(t) = \int_{T_0}^{t_1} f(s_1, t_2, ..., t_n)ds_1, \quad T_0 \in [-R, R], \quad u_2 = ... = u_n = 0,$$

is a $C^\infty(B, H^\infty)$ solution of the equation.

Let us suppose that ϕ_0 is not a constant. We take $t \in B$. By Proposition 6.3, there exists a curve γ_t and a constant $C > 0$ such that γ_t connects $t \in B$ to a point $\bar{t} \in \partial B$, satisfying $\phi_0(s) \le \phi_0(t)$ for every s on γ_t, and the length of $\gamma_t \le C$. We have

$$\begin{aligned} -u(t) &= \int_{\gamma_t} d_s\Big(e^{(\phi_0(s)-\phi_0(t))A}u(s)\Big) \\ &= -\int_0^1 \sum_{j=1}^n \Big[e^{(\phi_0(\gamma_t(\nu))-\phi_0(t))A}\mathrm{L}_{j,0}^* u(\gamma_t(\nu))\Big] s_j'(\nu)d\nu, \qquad \forall\ u \in C_c^\infty(B, H^\infty). \end{aligned}$$

Hence,

$$\begin{aligned} \|u(t)\|_0 &\le \int_0^1 \Big(\sum_{j=1}^n \|\ e^{(\phi_0(\gamma_t(\nu))-\phi_0(t))A}\mathrm{L}_{j,0}^* u(\gamma_t(\nu))\ \|_0^2\Big)^{1/2} \Big(\sum_{j=1}^n [s_j'(\nu)]^2\Big)^{1/2} d\nu \\ &\le C_1 \sum_{j=1}^n \Big(\sup_{\nu\in[0,1]} \|\mathrm{L}_{j,0}^* u(\gamma_t(\nu))\|_0\Big) \underbrace{\int_0^1 \|\gamma_t'(\nu)\|d\nu}_{\le C} \\ &\le C_2 \sum_{j=1}^n \sup_B \|\mathrm{L}_{j,0}^* u\|_0, \end{aligned}$$

which implies

$$\sup_{t\in B} \|u(t)\|_0 \le C_2 \sum_{j=1}^n \sup_{t\in B} \|\mathrm{L}_{j,0}^* u(t)\|_0, \qquad \forall u \in C_c^\infty(B, H^\infty). \tag{7.1}$$

Define, with the topology induced by $(C_c(B, H^\infty))^n$, the subspace

$$E = \{(\mathrm{L}^*_{1,0}\psi, ..., \mathrm{L}^*_{n,0}\psi) : \psi \in C^\infty_c(B, H^\infty)\} \quad \hookrightarrow \quad (C_c(B, H^\infty))^n.$$

For each $f \in C^\infty(\overline{B}, H^\infty) \supset C^\infty_c(B, H^\infty)$, consider the mapping

$$\begin{aligned} E &\xrightarrow{T} \mathbb{C} \\ (\mathrm{L}^*_{1,0}\psi, ..., \mathrm{L}^*_{n,0}\psi) &\longmapsto \int_B (\psi(t), f(t))_0 dt \ . \end{aligned}$$

Due to (7.1), T is well defined and it is continuous, that is,

$$|T(\mathrm{L}^*_{1,0}\psi, ..., \mathrm{L}^*_{n,0}\psi)| \leq C_3 \sum_{j=1}^{n} \sup_{t\in B} \|\mathrm{L}^*_{j,0}\psi(t)\|_0.$$

By Hahn–Banach theorem, there exists a continuous linear functional $\tilde{T} : (C_c(B, H^\infty))^n \longrightarrow \mathbb{C}$ which is an extension of T, that is, $\tilde{T} = (\tilde{T}_1, ..., \tilde{T}_n) \in \left(\mathcal{D}'^0(B, H^{-\infty})\right)^n \hookrightarrow \left(\mathcal{D}'(B, H^{-\infty})\right)^n$.

Therefore,

$$\begin{aligned} < \mathrm{L}_{1,0}\tilde{T}_1 + ... + \mathrm{L}_{n,0}\tilde{T}_n \ , \ \psi > &= \sum_{j=1}^{n} < \tilde{T}_j \ , \mathrm{L}^*_{j,0} \ \psi > \\ &= \tilde{T}(\mathrm{L}^*_{1,0}\psi, ..., \mathrm{L}^*_{n,0}\psi) \\ &= T(\mathrm{L}^*_{1,0}\psi, ..., \mathrm{L}^*_{n,0}\psi) \\ &= \int_B (\psi(t), f(t))_0 dt \\ &= < f, \psi >, \qquad \forall \psi \in C^\infty_c(B, H^\infty) \ . \end{aligned}$$

We now prove the sufficiency when $n = 1$. Indeed, conditions (ψ_1) and (ψ'_1) on J are equivalent to the following property (see [8], page 208):

- (ψ''_1) There is a point T_0 in the closure of J such that, for every $t \in J$ and for every $s \in J$ belonging to the interval joining t to T_0, we have

$$\phi_0(t) - \phi_0(s) \leq 0.$$

Suppose that (ψ''_1) holds. Then, a solution of the equation

$$\partial_t u - (\partial_t \phi_0)(t) A u = f \quad \text{in } J,$$

with an arbitrary $f \in C^\infty_c(J, H^\infty)$, is given by

$$u(t) = \int_{T_0}^{t} e^{(\phi_0(t)-\phi_0(s))A} f(s) ds \quad \in \quad C^\infty(J, H^\infty).$$

8. Necessity of Theorem 5.1

Lemma 8.1. *If $[S_0]^1$ is satisfied, then for any compact $K \subset B$, there exists an integer $M \geq 0$ and a constant $C > 0$ such that, $\forall u, f \in C_c^\infty(K, H^\infty)$,*

$$\left| \int (f(t), u(t))_0 dt \right| \leq C \left(\sup_{t \in K} \sum_{l+|\beta| \leq M} \| A^l \partial_t^\beta f(t) \|_0 \right) \left(\sum_{j=1}^{n} \sup_{t \in K} \sum_{l+|\beta| \leq M} \| A^l \partial_t^\beta \mathrm{L}_{j,0}^* u(t) \|_0 \right). \tag{8.1}$$

Proof. Consider the bilinear mapping

$$\begin{aligned} C_c^\infty(K, H^\infty) \times C_c^\infty(K, H^\infty) &\xrightarrow{J} \mathbb{C} \\ (f, u) &\longmapsto \int \big(f(t), u(t)\big)_0 dt \ . \end{aligned}$$

We know that $C_c^\infty(K, H^\infty)$, equipped with the seminorms

$$\sup_{t \in K} \sum_{l+|\beta| \leq M} \| A^l \partial_t^\beta f(t) \|_0,$$

is a Fréchet space and that it is a metrizable space under the seminorms

$$\sum_{j=1}^{n} \sup_{t \in K} \sum_{l+|\beta| \leq M} \| A^l \partial_t^\beta \mathrm{L}_{j,0}^* u(t) \|_0 \ .$$

For fixed u, $f \longmapsto J(f, u)$ is continuous.

For fixed f, by hypothesis there exist $v_j \in \mathcal{D}'(B, H^{-\infty})$, $j = 1, ..., n$, such that $\sum \mathrm{L}_{j,0} v_j = f$. This yields

$$\left| \int \big(f(t), u(t)\big)_0 dt \right| \leq C(K) \sum_{j=1}^{n} \sup_{t \in K} \sum_{l+|\beta| \leq M} \| A^l \partial_t^\beta \mathrm{L}_{j,0}^* u(t) \|_0 \ .$$

So $u \longmapsto J(f, u)$ is continuous. From the classical functional analysis theory, J is continuous in both variables. This proves (8.1). □

We now prove the necessity of condition (ψ_1) on B.

We are going to show that, if (ψ_1) does not hold on B, then (8.1) cannot hold either, for some compact $K \subset B$, whatever the integer $M \geq 0$ and the constant $C > 0$.

Assuming that (ψ_1) does not hold on B, we can find a compact K_1 in B, $\overset{\circ}{K_1} \neq \emptyset$, such that

$$\underbrace{\min_{K_1} \phi_0}_{\doteq \phi_0(t_0)} < \min_{\partial K_1} \phi_0.$$

Let us set

$$\alpha_0(t_0, t) \doteq \phi_0(t) - \min_{K_1} \phi_0, \qquad t \in K_1.$$

Then we have

$$\alpha_0(t_0, t) > 0, \, t \in \partial K_1,$$
$$\alpha_0(t_0, t) \geq 0, \;\; t \in K_1.$$

Due to the continuity of α_0 in B we can find a compact K, $K_1 \subset K \subset B$, $K \neq K_1$, such that $\alpha_0(t_0, t) \geq 0$, $\forall t \in K$. We select a function $g \in C^\infty$, $g(t) \geq 0$, with compact support in K, equal to one in K_1 and such that, for a suitable constant $c > 0$,

$$\alpha_0(t_0, t) \geq c > 0 \quad \text{whenever} \quad \nabla g(t) \neq 0. \tag{8.2}$$

Observe that, given any $u_0 \in H^\infty$, the function

$$h(t) = e^{-\alpha_0(t_0,t)A} u_0$$

is a solution of the homogeneous equation $-\mathrm{L}^*_{j,0} h = 0$. Consequently,

$$-\mathrm{L}^*_{j,0}\big(gh\big) = \big(\partial_{t_j} g\big) h. \tag{8.3}$$

We use the spectral resolution dE_λ of the operator A and consider a point, which we denote by τ^2, towards $+\infty$ in the spectrum of A. We shall denote by Π_τ the spectral projector of A corresponding to the interval

$$J_\tau = \big\{\lambda \in \mathbb{R}_+ : |\lambda - \tau^2| \leq \tau\big\},$$

that is to say,

$$\Pi_\tau = \int_{J_\tau} dE(\lambda) \; .$$

We choose $u_{0,\tau} \in H^\infty$ such that

$$\|u_{0,\tau}\|_0 = 1, \qquad \Pi_\tau u_{0,\tau} \;=\; u_{0,\tau} \; .$$

Note that such an element $u_{0,\tau}$ always exists. Indeed, we know that there exists $u_1(\tau) \in H$ such that $u_1(\tau) = \Pi_\tau u_1(\tau) \neq 0$. We may then take, for some $\epsilon > 0$,

$$u_{0,\tau} = e^{-\epsilon A} u_1(\tau) / \|e^{-\epsilon A} u_1(\tau)\|_0.$$

Note that

$$h(t) = e^{-\alpha_0(t_0,t)A} u_0 = \int_{J_\tau} e^{-\alpha_0(t_0,t)\lambda} dE(\lambda) u_0 \; . \tag{8.4}$$

We apply (8.1) with $u = gh$ and $f = F\big(\tau(t - t_0)\big) u_{0,\tau}$, where F is a non-negative C^∞ function on $\mathbb{R}^n$, vanishing for $|t| > 1$ and equal to 1 for $t = 0$. Note that supp$F \subset K$, for large τ.

From our choice of $u_{0,\tau}$ we derive at once (τ is large)

$$\big\|A^l u_{0,\tau}\big\|_0 = \big\|A^l \Pi_\tau u_{0,\tau}\big\|_0 = \left(\int_{J_\tau} \lambda^{2l} d\|E(\lambda) u_{0,\tau}\|_0^2 \right)^{1/2}$$
$$\leq (\tau^2 + \tau)^l \|\Pi_\tau u_{0,\tau}\|_0 \quad \leq \quad 2^l \tau^{2l},$$

whence

$$\sup_{t\in K}\sum_{l+|\beta|\le M}\left\|A^l\partial_t^\beta f(t)\right\|_0 = C'_M\ \tau^{2M}.$$

And if we combine (8.2), (8.3) and (8.4) (taking into account the definition of J_τ), we get

$$\sum_{j=1}^{n}\sup_{t\in K}\sum_{l+|\beta|\le M}\|A^l\partial_t^\beta \mathrm{L}^*_{j,0}u(t)\|_0 \le C''_M\tau^{2M}\ e^{-c(\tau^2-\tau)}\quad ,\quad \text{for large } \tau\ .$$

Thus, if (8.1) is valid, we should have (for large τ)

$$\left|\int(u(t),f(t))_0 dt\right| \le C_M\tau^{4M}e^{-c\tau^2/2}. \tag{8.5}$$

On the other hand,

$$\int(u(t),f(t))_0 dt = \frac{1}{\tau^n}\int\limits_{|s|\le 1}\int\limits_{J_\tau} e^{-\alpha_0(t_0,t_0+s/\tau)\lambda}d\|E(\lambda)u_{0,\tau}\|_0^2F(s)ds\ .$$

But, if $s\in\operatorname{supp}F$ (hence $|s|\le 1$),

$$0\le\alpha_0\Big(t_0,t_0+\frac{s}{\tau}\Big)\le C_1\tau^{-2}\ .$$

Therefore, for a suitable constant $c'>0$ and all sufficiently large τ,

$$\left|\int\big(u(t),f(t)\big)_0 dt\right|\ \ge\ \frac{1}{\tau^n}\ e^{-C_1(1+\frac{1}{\tau})}\underbrace{\int F(s)ds}_{>0}\underbrace{\int\limits_{J_\tau} d\|E(\lambda)u_{0,\tau}\|_0^2}_{\|\Pi_\tau u_{0,\tau}\|_0^2=1}\ \ge\ c'\tau^{-n}$$

and this contradicts (8.5).

9. A more precise result of the local solvability of the underdetermined system

In this section we write $||v||_{s,B'}=\sup\limits_{t\in B'}||v(t)||_s$, where B' is an open subset whose compact closure is contained in $B=\{t\in\mathbb{R}^n : |t|<R\}$. We recall the following well known lemma:

Lemma 9.1. *Let ω be an open subset of $\mathbb{R}^n$ and let Φ be a real C^∞ function in $\overline{\omega}$. Suppose that there exist $M>0$ and $0\le\theta<1$ such that*

$$|\nabla\Phi|\ge M|\Phi|^\theta\quad \mathit{in}\ \ \omega. \tag{9.1}$$

Define $\Sigma=\{t\in\omega\mid\nabla\Phi(t)=0\}$ and consider, for each $t\in\omega\setminus\Sigma$, the solution $\gamma_t(\tau)$ of

$$\begin{cases}\dot\gamma_t=-\frac{\nabla\Phi}{|\nabla\Phi|}(\gamma_t)\\ \gamma_t(0)=t\end{cases}$$

defined in $[0,\delta(t)[$. Then there exists a $C>0$ and $\sigma\ge 1$ such that

$$\Phi(t)-\Phi(\gamma_t(\tau))\ge C\tau^\sigma,\quad \forall\tau\in[0,\delta(t)[.$$

Remarks. The inequality above implies that $\delta(t)$ is uniformly bounded in ω. There exists the limit

$$\lim_{\tau \to \delta(t)_-} \gamma_t(\tau) \doteq l(t) \in \partial\omega \cup \Sigma$$

for each $t \in \omega \setminus \Sigma$. Furthermore, if $l(t) \in \Sigma$ then $\Phi(t) > 0$.

Proposition 9.2. *Let $\mathfrak{L}_0^* = -(d_t + d(\phi_0)A)$ be defined on B and suppose that the real analytic function ϕ_0 satisfies (ψ_1) and (9.1) on B. Then, for every $\epsilon > 0$ there exist $B' \subset\subset B$ and a constant $C' = C'(B') > 0$ such that*

$$||v||_{\epsilon,B'} \leq C'||\mathfrak{L}_0^* v||_{0,B'} \ , \qquad \forall v \in C_c^\infty(B', H^\infty).$$

Proof. Let $u \in C_c^\infty(B, H^\infty)$ and consider $B' \subset\subset B$ and $t \in B' \setminus \Sigma$. We study two cases:

First case: $l(t) \in \partial B$.

We know that

$$\begin{aligned} -e^{\phi_0(t)A}u(t) &= \int_{\gamma_t} d_s(e^{\phi_0(s)A}u(s)) \\ &= \int_{\gamma_t} e^{\phi_0(s)A}\mathfrak{L}_0^* u(s)ds. \end{aligned}$$

Hence,

$$A^\epsilon u(t) = -\int_{\gamma_t} A^\epsilon e^{(\phi_0(s)-\phi_0(t))A}\mathfrak{L}_0^* u(s)ds.$$

Taking Lemma 9.1 into account, we have

$$\begin{aligned} ||u(t)||_\epsilon &\leq \int_{\gamma_t} ||A^\epsilon e^{(\phi_0(s)-\phi_0(t))A}\mathfrak{L}_0^* u(s)||_0 |ds| \\ &\leq C_1 \sup_{s\in\gamma_t} ||\mathfrak{L}_0^* u(s)||_0 \ length(\gamma_t) \\ &\leq C_1 ||\mathfrak{L}_0^* u||_{0,B} \ length(\gamma_t) \ , \quad \forall t \in B' \setminus \Sigma. \end{aligned}$$

As ϕ_0 has no local minimum, Σ has empty interior and then

$$||u||_{\epsilon,B'} \leq C'||\mathfrak{L}_0^* u||_{0,B} \ .$$

Therefore,

$$||v||_{\epsilon,B'} \leq C'||\mathfrak{L}_0^* v||_{0,B'} \ , \quad \forall v \in C_c^\infty(B', H^\infty).$$

Second case: $l(t) \in \Sigma$ ($\phi_0(t) > 0$).

We choose t_0 arbitrarily close to $l(t)$ satisfying:

- $\phi_0(t_0) < 0$ (ϕ_0 has no local minimum).
- $t_0 \in V$, where V is an open convex neighborhood of $l(t)$ in which $\phi_0(s) < \phi_0(t)$, $\forall s \in V$.

Notice that

$$e^{\phi_0(t)A}u(t) = \{e^{\phi_0(t)A}u(t) - e^{\phi_0(l(t))A}u(l(t))\} + \{e^{\phi_0(l(t))A}u(l(t)) - e^{\phi_0(t_0)A}u(t_0)\} + e^{\phi_0(t_0)A}u(t_0).$$

Using

$$e^{\phi_0(t)A}u(t) - e^{\phi_0(l(t))A}u(l(t)) = -\int\limits_{\gamma_t} e^{\phi_0(s)A}\mathfrak{L}_0^* u(s)ds$$

we have

$$\begin{aligned} u(t) = -\int\limits_{\gamma_t} e^{(\phi_0(s)-\phi_0(t))A}\mathfrak{L}_0^* u(s)ds \ + \ e^{-\phi_0(t)A}\{e^{\phi_0(l(t))A}u(l(t)) - e^{\phi_0(t_0)A}u(t_0)\} \\ + e^{(\phi_0(t_0)-\phi_0(t))A}u(t_0). \end{aligned}$$

To estimate $||u(t)||_\epsilon$ we estimate the following terms:

1.

$$\left|\left|\int\limits_{\gamma_t} A^\epsilon e^{(\phi_0(s)-\phi_0(t))A}\mathfrak{L}_0^* u(s)ds\right|\right|_0 \le C_2 \sup_{s\in\gamma_t} ||\mathfrak{L}_0^* u(s)||_0 \ length(\gamma_t), \tag{9.2}$$

as in the first case.

2.

$$||A^\epsilon e^{(\phi_0(t_0)-\phi_0(t))A}u(t_0)||_0 \ .$$

As $l(t) \in \Sigma$, it follows that $\phi_0(t_0) < 0 < \phi_0(t)$. Thus,

$$||A^\epsilon e^{(\phi_0(t_0)-\phi_0(t))A}u(t_0)||_0 \le C_3||u(t_0)||_0 \ . \tag{9.3}$$

3.

$$||A^\epsilon e^{-\phi_0(t)A}\{e^{\phi_0(l(t))A}u(l(t)) - e^{\phi_0(t_0)A}u(t_0)\}||_0 \ .$$

By the choice of t_0 it follows that $\phi_0(s) < \phi_0(t)$, $\forall s \in [l(t), t_0]$. We use

$$e^{\phi_0(l(t))A}u(l(t)) - e^{\phi_0(t_0)A}u(t_0) = -\int\limits_{s\in[l(t),t_0]} e^{\phi_0(s)A}\mathfrak{L}_0^* u(s)ds.$$

Thus, we get

$$\begin{aligned} ||A^\epsilon e^{-\phi_0(t)A}\{e^{\phi_0(l(t))A}u(l(t)) - e^{\phi_0(t_0)A}u(t_0)\}||_0 &\le \int\limits_{s\in[l(t),t_0]} ||A^\epsilon e^{(\phi_0(s)-\phi_0(t))A}\mathfrak{L}_0^* u(s)||_0|ds| \\ &\le C_4 \sup_{s\in[l(t),t_0]} ||\mathfrak{L}_0^* u(s)||_0 \ |t_0 - l(t)|. \end{aligned} \tag{9.4}$$

Adding the estimates (9.2), (9.3) and (9.4), we obtain, $\forall t \in B' \setminus \Sigma$,

$$||u(t)||_\epsilon \leq C_2 \sup_{s\in B} ||\mathfrak{L}_0^* u(s)||_0 \ length(\gamma_t) + C_3 \sup_{s\in B} ||u(s)||_0 + C_4 \sup_{s\in B} ||\mathfrak{L}_0^* u(s)||_0 \ |t_0 - l(t)| \ .$$

Since t_0 is arbitrarily close to $l(t)$, it follows that

$$||u(t)||_\epsilon \leq C_2 \sup_{s\in B} ||\mathfrak{L}_0^* u(s)||_0 \ length(\gamma_t) + C_3 \sup_{s\in B} ||u(s)||_0, \quad \forall t \in B' \setminus \Sigma.$$

As ϕ_0 has no local minimum, Σ has empty interior and then

$$||u||_{\epsilon,B'} \leq C_5(||\mathfrak{L}_0^* u||_{0,B} + ||u||_{0,B}).$$

Hence,

$$||v||_{\epsilon,B'} \leq C_5(||\mathfrak{L}_0^* v||_{0,B'} + ||v||_{0,B'}) \ , \quad \forall v \in C_c^\infty(B', H^\infty).$$

If necessary, we can take a small enough B' such that

$$||v||_{0,B'} \leq \frac{1}{C_5+1} ||v||_{\epsilon,B'} \ .$$

Therefore,

$$||v||_{\epsilon,B'} \leq C' ||\mathfrak{L}_0^* v||_{0,B'} \ , \quad \forall v \in C_c^\infty(B', H^\infty). \qquad \square$$

Proposition 9.3. *Let $\mathfrak{L}_0^* = -(d_t + d(\phi_0)A)$ be defined on B. Suppose that for every $\epsilon > 0$ there exists a $B' \subset\subset B$ and $C' = C'(B') > 0$ such that*

$$||v||_{\epsilon,B'} \leq C' ||\mathfrak{L}_0^* v||_{0,B'} \ , \quad \forall v \in C_c^\infty(B', H^\infty).$$

Then, for every $s \in \mathbb{R}$ and every $\epsilon > 0$ we have

$$||v||_{s+\epsilon,B'} \leq C' ||\mathfrak{L}_0^* v||_{s,B'} \ , \quad \forall v \in C_c^\infty(B', H^\infty). \tag{9.5}$$

Proof.

$$\begin{aligned} ||v||_{s+\epsilon,B'} &= ||A^s v||_{\epsilon,B'} \\ &\leq C' ||\mathfrak{L}_0^* A^s v||_{0,B'} \\ &= C' ||\mathfrak{L}_0^* v||_{s,B'} \ , \quad \forall v \in C_c^\infty(B', H^\infty). \qquad \square \end{aligned}$$

Remark. (9.5) can be written as

$$\sup_{t\in B'} ||v(t)||_{s+\epsilon} \leq C' \sum_{j=1}^n \sup_{t\in B'} ||\mathrm{L}_{j,0}^* v(t)||_s. \tag{9.6}$$

Proposition 9.4. *Suppose that for every $s \in \mathbb{R}$ and every $\epsilon > 0$ there exist $B' \subset\subset B$ and $C' = C'(B') > 0$ such that*

$$||v||_{s+\epsilon,B'} \leq C' ||\mathfrak{L}_0^* v||_{s,B'} \ , \quad \forall v \in C_c^\infty(B', H^\infty).$$

Then, for every $s \in \mathbb{R}$, every $\epsilon > 0$ and every $f \in C_c^\infty(B', H^s)$, there exists a $u_j \in \mathcal{D}'^0(B', H^{s+\epsilon})$, $j = 1, ..., n$, such that

$$\sum_{j=1}^{n} \mathrm{L}_{j,0} u_j = f.$$

Proof. Define, with the topology induced by $(C_c(B', H^{-s-\epsilon}))^n$, the subspace

$$E = \{(\mathrm{L}_{1,0}^* \psi, ..., \mathrm{L}_{n,0}^* \psi) : \psi \in C_c^\infty(B', H^\infty)\} \quad \hookrightarrow \quad (C_c(B', H^{-s-\epsilon}))^n.$$

For each $f \in C_c^\infty(B', H^s)$, consider the mapping

$$\begin{aligned} E &\xrightarrow{T} \mathbb{C} \\ (\mathrm{L}_{1,0}^* \psi, ..., \mathrm{L}_{n,0}^* \psi) &\longmapsto \int_{B'} (A^{-s}\psi(t), A^s f(t))_0 dt \ . \end{aligned}$$

Due to (9.6), T is well defined and it is continuous, that is,

$$\begin{aligned} |T(\mathrm{L}_{1,0}^* \psi, ..., \mathrm{L}_{n,0}^* \psi)| &\leq \int_{B'} ||A^s f(t)||_0 dt \ \sup_{t \in B'} ||\psi(t)||_{-s} \\ &\leq CC' \sum_{j=1}^{n} \sup_{t \in B'} \|\mathrm{L}_{j,0}^* \psi(t)\|_{-s-\epsilon}. \end{aligned}$$

By Hahn–Banach theorem, T admits a continuous extension $u : (C_c(B', H^{-s-\epsilon}))^n \longrightarrow \mathbb{C}$, that is, it admits $u_j \in \mathcal{D}'^0(B', H^{s+\epsilon})$, $j = 1, ..., n$, such that

$$\begin{aligned} < \mathrm{L}_{1,0} u_1 + ... + \mathrm{L}_{n,0} u_n \ , \ \psi > &= \sum_{j=1}^{n} < u_j \ , \mathrm{L}_{j,0}^* \ \psi > \\ &= u(\mathrm{L}_{1,0}^* \psi, ..., \mathrm{L}_{n,0}^* \psi) \\ &= T(\mathrm{L}_{1,0}^* \psi, ..., \mathrm{L}_{n,0}^* \psi) \\ &= \int_{B'} (A^{-s}\psi(t), A^s f(t))_0 dt \\ &= < f, \psi >, \qquad \forall \psi \in C_c^\infty(B', H^\infty) \ . \qquad \square \end{aligned}$$

Remark. Propositions 9.2, 9.3 and 9.4 are valid if ϕ_0 is a real C^∞ function satisfying (ψ_1) and (9.1) on B.

10. Finite order regularity solutions of Theorem 5.1

Define $\Delta = \mathrm{L}_{1,0}^2 + ... + \mathrm{L}_{n,0}^2 - A^2$. Then $\Delta = \Delta^0 + Q$, where $\Delta^0 = \sum_{j=1}^n \partial_{t_j}^2 - A^2$ and

$$Q = \sum_{j=1}^{n} \Big\{ - \partial_{t_j}^2 \phi_0(t) A - 2\partial_{t_j} \phi_0(t) A \partial_{t_j} + (\partial_{t_j} \phi_0(t))^2 A^2 \Big\}.$$

Proposition 10.1. *There exists an open set $U \subset\subset B$, $0 \in U$, such that, for some $C > 0$, we have*

$$|||u|||_2 \leq C(|||\Delta u|||_0 + |||u|||_0), \qquad \forall u \in C_c^\infty(U, H^\infty).$$

Proof. We may assume $\nabla\phi_0(0) = 0$. If $u \in C_c^\infty(B, H^\infty)$, then

$$\begin{aligned}|||\Delta^0 u|||_0^2 &= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} ||\mathscr{F}(\Delta^0 u)(\tau)||_0^2 d\tau \\ &= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} ||(|\tau|^2 + A^2)\mathscr{F}(u)(\tau)||_0^2 d\tau \\ &\geq \frac{1}{2}\,|||u|||_2^2 - |||u|||_0^2\,.\end{aligned}$$

Thus,

$$|||u|||_2^2 \leq 4|||\Delta u|||_0^2 + 4|||Qu|||_0^2 + 2|||u|||_0^2\,. \tag{10.1}$$

Given $\epsilon > 0$ there exists an open neighborhood of the origin $U \subset\subset B$ such that $|\partial_{t_j}\phi_0(t)| \leq \epsilon$, $\forall t \in \overline{U}$, $\forall j = 1, ..., n$. It follows that there exists a constant $M > 0$ such that $|\partial_{t_j}^2\phi_0(t)| \leq M$, $\forall t \in \overline{U}$, $\forall j = 1, ..., n$. If $u \in C_c^\infty(U, H^\infty)$, then

$$\begin{aligned}|||Qu|||_0^2 &= \int\limits_{\mathbb{R}^n} ||Qu(t)||_0^2 dt \\ &\leq \sum_{j=1}^n \Big(4\int\limits_{\mathbb{R}^n} ||\partial_{t_j}^2\phi_0(t)Au(t)||_0^2 dt \;+\; 4\int\limits_{\mathbb{R}^n} ||2\partial_{t_j}\phi_0(t)\partial_{t_j}Au(t)||_0^2 dt \,+ \\ &\quad 2\int\limits_{\mathbb{R}^n} ||(\partial_{t_j}\phi_0(t))^2A^2u(t)||_0^2 dt\Big) \\ &\leq 4nM^2|||u|||_1^2 \;+\; 6n\epsilon^2|||u|||_2^2.\end{aligned}$$

Using this estimate in (10.1) we obtain

$$|||u|||_2^2 \leq 4|||\Delta u|||_0^2 + 16nM^2|||u|||_1^2 + 24n\epsilon^2|||u|||_2^2 + 2|||u|||_0^2. \tag{10.2}$$

By Gagliardo's inequality, there exists $M' > 0$, depending on ϵ, such that

$$|||u|||_1^2 \leq 2\epsilon^2|||u|||_2^2 + 2M'^2|||u|||_0^2\,.$$

Combining this estimate and (10.2) it follows that

$$|||u|||_2^2 \leq 4|||\Delta u|||_0^2 \;+\; (32nM^2\epsilon^2 + 24n\epsilon^2)|||u|||_2^2 \;+\; (32nM^2M'^2 + 2)|||u|||_0^2\,.$$

If we now choose ϵ such that $(32nM^2 + 24n)\epsilon^2 < 1$, we finally prove the proposition. □

Lemma 10.2. *For every open set $V \subset\subset U$, for every $s \in \mathbb{R}$, there exists $C = C(s, V) > 0$ such that*

$$|||v|||_{s+2} \leq C(|||\Delta v|||_s + |||v|||_s), \qquad \forall v \in C_c^\infty(V, H^\infty). \tag{10.3}$$

Proof. Let $V \subset\subset U$ be an open set and let also $\chi \in C_c^\infty(U)$ be identically equal to one in V. For a real number s and for $v \in C_c^\infty(V, H^\infty)$, we obtain

$$\begin{aligned}|||v|||_{s+2} &= |||\Lambda^s(\chi v)|||_2 \\ &\leq |||\chi\Lambda^s v|||_2 + |||[\Lambda^s, \chi]v|||_2 \\ &\leq |||\chi\Lambda^s v|||_2 + C_1|||v|||_{s+1}.\end{aligned}$$

If we apply Proposition 10.1, we obtain

$$|||v|||_{s+2} \leq C_0|||\Delta(\chi\Lambda^s v)|||_0 + C_2|||v|||_{s+1}.$$

Since $\Delta(\chi\Lambda^s) = \Lambda^s\Delta\chi + [\Delta\chi, \Lambda^s]$, where $[\Delta\chi, \Lambda^s]$ is an operator of order $s+1$, it follows that

$$\begin{aligned}|||v|||_{s+2} &\leq C_0|||\Lambda^s\Delta(\chi v)|||_0 + C_3|||v|||_{s+1} \\ &= C_0|||\Delta(\chi v)|||_s + C_3|||v|||_{s+1} \\ &\leq C_4(|||\Delta v|||_s + |||v|||_{s+1}).\end{aligned} \tag{10.4}$$

By Gagliardo's inequality, there exists $M > 0$, depending on C_4, such that

$$|||v|||_{s+1} \leq \frac{1}{C_4+1}|||v|||_{s+2} + M|||v|||_s.$$

Using this estimate in (10.4), we prove the assertion. □

Lemma 10.3. *Let V be an open set such that $V \subset\subset U$. If $u \in \mathscr{H}^{s+1}_{loc}(V)$ is such that $\Delta u \in \mathscr{H}^s_{loc}(V)$, then $u \in \mathscr{H}^{s+2}_{loc}(V)$.*

Proof. Let $W \subset\subset V$ be an open set and let $\theta \in C^\infty_c(V)$ be identically equal to one in W. It will suffice to show that $\theta u \in \mathscr{H}^{s+2}$.

Let $B_\epsilon = e^{-\epsilon A}\rho_\epsilon * \cdot$, where $\{\rho_\epsilon\}$ is the usual family of mollifiers in $\mathbb{R}^n$. As in the classical theory, we have $B_\epsilon(\theta u) \to \theta u$ in $\mathscr{H}^{s+1}$ as $\epsilon \to 0$ and also

$$\Delta B_\epsilon(\theta u) = B_\epsilon\Delta(\theta u) + [\Delta, B_\epsilon](\theta u) \xrightarrow{\epsilon\to 0} \Delta(\theta u) \quad \text{in} \quad \mathscr{H}^s$$

by Friedrich's lemma, since $\Delta(\theta u) \in \mathscr{H}^s$. Thus, if we take $\epsilon_n \to 0$ and if we apply (10.3) for $v = B_{\epsilon_m}(\theta u) - B_{\epsilon_n}(\theta u)$, we conclude that $\{B_{\epsilon_n}(\theta u)\}$ is a Cauchy sequence in $\mathscr{H}^{s+2}$. Hence, $\theta u \in \mathscr{H}^{s+2}$. □

Proposition 10.4. *Let $s \in \mathbb{R}$. If $u \in \mathcal{D}'(U, H^{-\infty})$ is such that $\Delta u \in \mathscr{H}^s_{loc}(U)$, then $u \in \mathscr{H}^{s+2}_{loc}(U)$.*

Proof. Let $u \in \mathcal{D}'(U, H^{-\infty})$ and let $s \in \mathbb{R}$. Let also $V \subset\subset U$ be an open set. As in the classical theory of distributions, we can find $l \in \mathbb{Z}_+$ such that $u\big|_V \in \mathscr{H}^{s-l+1}_{loc}(V)$. Since $\Delta u\big|_V \in \mathscr{H}^{s-l}_{loc}(V)$, if we apply Lemma 10.3 replacing u and s by $u\big|_V$ and $s-l$, respectively, we obtain $u\big|_V \in \mathscr{H}^{s-l+2}_{loc}(V)$. By an iteration process, it follows that $u\big|_V \in \mathscr{H}^{s+2}_{loc}(V)$. Therefore, $u \in \mathscr{H}^{s+2}_{loc}(U)$. □

Lemma 10.5. *For $N \geq 1$, Δ^N is hypoelliptic in U.*

Proof. Let $u \in \mathcal{D}'(U, H^{-\infty})$ such that $\Delta u \in C^\infty(U, H^\infty)$. By Proposition 10.4, $u \in \mathscr{H}^{s+2}_{loc}(U)$, for every s. Therefore, $u \in C^\infty(U, H^\infty)$. By induction on N, Δ^N is hypoelliptic in U. □

Lemma 10.6. *For every open set $V \subset\subset U$, for every $s \in \mathbb{R}$ and for every $M \geq 0$, there exists $C' = C'(s, V) > 0$ such that*

$$|||v|||_{s+2} \leq C'(|||\Delta v|||_s + |||v|||_{s-M}), \qquad \forall v \in C^\infty_c(V, H^\infty). \tag{10.5}$$

Proof. Let $\epsilon < 1/C$. By Gagliardo's inequality there exists $M' = M'(\epsilon) > 0$ such that

$$|||v|||_s \leq \epsilon |||v|||_{s+2} + M' |||v|||_{s-M}, \qquad \forall v \in C_c^\infty(V, H^\infty).$$

Combining this estimate and (10.3), we prove the lemma. □

Lemma 10.7. *For every open set $V \subset\subset U$, for every $s \in \mathbb{R}$ and for every $N \in \mathbb{Z}_+^*$, there exists $C = C(s, N, V) > 0$ such that*

$$|||v|||_{s+2N} \leq C(|||\Delta^N v|||_s + |||v|||_s), \qquad \forall v \in C_c^\infty(V, H^\infty). \tag{10.6}$$

Proof. By an iteration process using (10.5) we obtain

$$|||v|||_{s+2N} \leq C'^N |||\Delta^N v|||_s + C'^N |||\Delta^{N-1} v|||_{s-M} + C'^{N-1} |||\Delta^{N-2} v|||_{s+2-M} + \\ C'^{N-2} |||\Delta^{N-3} v|||_{s+4-M} + \cdots + C'^2 |||\Delta v|||_{s+2(N-2)-M} + C' |||v|||_{s+2(N-1)-M}.$$

We now choose $M = 2(N-1)$. It follows that

$$|||v|||_{s+2N} \leq C'^N |||\Delta^N v|||_s + C'^N |||\Delta^{N-1} v|||_{s-2(N-1)} + C'^{N-1} |||\Delta^{N-2} v|||_{s-2(N-2)} + \\ C'^{N-2} |||\Delta^{N-3} v|||_{s-2(N-3)} + \cdots + C'^2 |||\Delta v|||_{s-2} + C' |||v|||_s.$$

But for every $k = 1, 2, 3, ..., N-1$, Δ^k is a continuous operator from $\mathscr{H}^s$ into $\mathscr{H}^{s-2k}$. The proof is complete. □

Lemma 10.8. *For every $s \in \mathbb{R}$, for every $N \in \mathbb{Z}_+^*$ and for every $t_0 \in U$, there exists a neighborhood $\omega_{s,N}$ of t_0 such that for some constant $C_1 = C_1(s, N) > 0$,*

$$|||v|||_{s+2N} \leq C_1 |||\Delta^N v|||_s, \qquad \forall v \in C_c^\infty(\omega_{s,N}, H^\infty).$$

Proof. Let us suppose that there exist $s \in \mathbb{R}$, $N \in \mathbb{Z}_+^*$ and $t_0 \in U$, which can be translated to the origin, such that for all sufficiently large $j \in \mathbb{N}$, there exists $v_j \in C_c^\infty(B_{1/j}(0), H^\infty)$ such that

$$|||v_j|||_{s+2N} > j\ |||\Delta^N v_j|||_s.$$

Writing $u_j = v_j / |||v_j|||_{s+2N}$, we obtain

$$u_j \in C_c^\infty(B_{1/j}(0), H^\infty), \qquad |||u_j|||_{s+2N} = 1, \qquad |||\Delta^N u_j|||_s < 1/j.$$

From the equality, we conclude that a subsequence of u_j converges weakly in $\mathscr{H}^{s+2N}(\overline{B_1(0)})$, hence, it converges in $\mathscr{H}^s(\overline{B_1(0)})$ to some $u \in \mathscr{H}^s(\overline{B_1(0)})$, which means that $\Delta^N u_{j_l} \longrightarrow \Delta^N u$ in the sense of distributions. From the inequality we obtain $\Delta^N u_{j_l} \longrightarrow 0$ in $\mathscr{H}^s(\overline{B_1(0)})$. Thus, $\Delta^N u = 0$ is such that $\operatorname{supp} u = \{0\}$. Therefore $u = 0$.

On the other hand, by (10.6), it follows that

$$|||u_{j_l}|||_{s+2N} = 1 \leq C(|||\Delta^N u_{j_l}|||_s + |||u_{j_l}|||_s),$$

such that $|||\Delta^N u_{j_l}|||_s \longrightarrow 0$ and $|||u_{j_l}|||_s \longrightarrow 0$, which is a contradiction. □

Proposition 10.9. *Given $M \in \mathbb{Z}$ and $N \in \mathbb{Z}_+^*$ there exists a neighborhood ω of the origin such that, given $f \in \mathscr{H}^M(\omega)$, there exists $u \in \mathscr{H}^{2N+M}(\omega)$ satisfying $\Delta^N u = f$ in ω.*

Proof. If Δ^* denotes the formal adjoint of Δ, then we may find an analogous estimate to that in Proposition 10.1. Thus, as obtained in Lemma 10.8, there exists a neighborhood $\omega = \omega_{M,N}$ of the origin such that

$$|||\phi|||_{-M} \le C'|||\Delta^{*N}\phi|||_{-2N-M}, \quad \forall \phi \in C_c^\infty(\omega, H^\infty). \tag{10.7}$$

Define, with the norm induced by $\mathscr{H}^{-2N-M}(\omega)$, the subspace

$$E = \Delta^{*N}(C_c^\infty(\omega, H^\infty)) \subset C_c^\infty(\omega, H^\infty) \subset \mathscr{H}^{-2N-M}(\omega).$$

We now consider the mapping

$$\begin{aligned} E &\xrightarrow{T} \mathscr{H}^{-M}(\omega) \\ \Delta^{*N}\phi &\longmapsto \phi. \end{aligned}$$

Due to (10.7), T is well defined and it is continuous. Therefore, T admits a continuous linear extension $F : \mathscr{H}^{-2N-M}(\omega) \to \mathscr{H}^{-M}(\omega)$, since E is a subspace of the Hilbert space $\mathscr{H}^{-2N-M}(\omega)$ and $\mathscr{H}^{-M}(\omega)$ is a Banach space.

Let the transpose of F be the map $G : \mathscr{H}^{M}(\omega) \to \mathscr{H}^{2N+M}(\omega)$. For $f \in \mathscr{H}^{M}(\omega)$ and $\phi \in C_c^\infty(\omega, H^\infty)$, we obtain

$$< \Delta^N Gf, \phi > = < f, F\Delta^{*N}\phi > \quad = \quad < f, \phi > . \qquad \square$$

Theorem 10.10. Δ *is analytic-hypoelliptic in* U*.*

Before the proof, we note that it follows from Lemma 10.5 that $u \in C^\infty(U, H^\infty)$. Since the statement of the theorem is local, it is sufficient to prove that every point in U has an open neighborhood ω where u is analytic. In view of Lemma 10.8, we may take $\omega \subset\subset U$ so small such that

$$|||\partial_t^\alpha A^l v|||_0 \le C'|||\Delta v|||_0, \quad |\alpha| + l \le 2, \quad \forall v \in C_c^\infty(\omega, H^\infty). \tag{10.8}$$

We denote by ω_ϵ the open set of points in ω at distance $> \epsilon$ from the complementary of ω, and introduce the notation

$$N_\epsilon(u) = \Big(\int\limits_{\omega_\epsilon} ||u(t)||_0^2 dt \Big)^{\frac{1}{2}} .$$

Lemma 10.11. *With a constant* C*, independent of* v, ϵ *and* ϵ_1*, we have*

$$\epsilon^{|\alpha|+l} N_{\epsilon+\epsilon_1}(\partial_t^\alpha A^l v) \le C\Big(\epsilon^{|\alpha|+l} N_{\epsilon_1}(\Delta v) + \sum_{|\beta|+l<2} \epsilon^{|\beta|+l} N_{\epsilon_1}(\partial_t^\beta A^l v)\Big) , \tag{10.9}$$

if $|\alpha| + l \le 2$ *and* $v \in C^\infty(\omega, H^\infty)$*.*

Proof. We can choose $\phi \in C_c^\infty(\omega_{\epsilon_1})$ such that $\phi = 1$ in $\omega_{\epsilon+\epsilon_1}$ and

$$|\partial_t^\alpha \phi| \le C_\alpha \epsilon^{-|\alpha|} \tag{10.10}$$

for suitable constants C_α independent of ϵ and ϵ_1.

Using (10.8) and (10.10) we obtain, if $|\alpha| + l \leq 2$,

$$N_{\epsilon+\epsilon_1}(\partial_t^\alpha A^l v) \leq |||\partial_t^\alpha A^l(\phi v)|||_0 \quad \leq \quad C'|||\Delta(\phi v)|||_0$$
$$\leq C\big(N_{\epsilon_1}(\Delta v) + \epsilon^{-2} N_{\epsilon_1}(v) + \epsilon^{-1}\sum_{j=1}^n N_{\epsilon_1}(\partial_{t_j} v) + \epsilon^{-1} N_{\epsilon_1}(Av)\big).$$

If we multiply by ϵ^2, the estimate (10.9) follows when $|\alpha| + l = 2$, and it is trivial if $|\alpha| + l < 2$. □

Proof of Theorem 10.10. Choose a small open set $\omega \subset\subset U$ such that Lemma 10.11 is valid and that $\int_\omega dt < 1$.

Writing $c_i = \sup_{t\in\omega} |\partial_t^\alpha \partial_{t_i}^2 \phi_0(t)| + 2\sup_{t\in\omega} |\partial_t^\alpha \partial_{t_i}\phi_0(t)| + \sup_{t\in\omega} |\partial_t^\alpha (\partial_{t_i}\phi_0(t))^2|$ we have, by hypothesis for some constant D,

$$\sum_{i=1}^n c_i \leq D^{|\alpha|+1}\ |\alpha|!\ .$$

This implies that

$$\epsilon^{|\alpha|}\sum_{i=1}^n c_i \leq D^{|\alpha|+1}\ |\alpha|!\ j^{-|\alpha|}, \tag{10.11}$$

where the supremum in the formula of c_i is now taken over $\omega_{j\epsilon}$ and such that $j\epsilon < 1$ is sufficiently small.

The analyticity of $f \ = \Delta u$ means that

$$\sup_\omega\ ||\partial_t^\alpha A^l f||_0 \leq D^{|\alpha|+l+1}(|\alpha|+l)^{|\alpha|+l}\ ,$$

for some constant D. And this implies that

$$\epsilon^{|\alpha|+l}\sup_{\omega_{|\alpha|\epsilon}}\ ||\partial_t^\alpha A^l f||_0 \leq D^{|\alpha|+l+1}\ , \tag{10.12}$$

$(|\alpha| + l)\epsilon < 1$.

We now claim that there exists a constant C_1 such that for every $\epsilon > 0$ and every integer $j > 0$, we have

$$\epsilon^{|\alpha|+l} N_{j\epsilon}(\partial_t^\alpha A^l u) \leq C_1^{|\alpha|+l+1} \quad \text{if} \quad |\alpha| < 2 + j. \tag{10.13}$$

It is easy to verify that this is true when $j = 1$. Assuming that (10.13) is proved for one value of j, we shall show that (10.13) follows with j replaced by $j+1$. To do so we only have to estimate the derivatives $\partial_t^\alpha A^l u$ with $|\alpha| = 2 + j$. We can write $\alpha = \alpha' + \alpha''$ where $|\alpha'| = j$ and $|\alpha''| = 2$. Applying $\partial_t^{\alpha'} A^l$ to $\Delta u = f$ gives

$$\Delta\partial_t^{\alpha'} A^l u = \partial_t^{\alpha'} A^l f + g\ , \tag{10.14}$$

where

$$g = \sum_{i=1}^n \sum_{0<\gamma\leq\alpha'} \binom{\alpha'}{\gamma}\Big(\partial_t^\gamma \partial_{t_i}^2 \phi_0(t)\partial_t^{\alpha'-\gamma} A^{l+1} u + 2\partial_t^\gamma \partial_{t_i}\phi_0(t)\partial_t^{\alpha'+e_i-\gamma} A^{l+1} u$$
$$- \partial_t^\gamma (\partial_{t_i}\phi_0(t))^2 \partial_t^{\alpha'-\gamma} A^{2+l} u\Big);$$

here $e_i = (0, ...0, 1, 0, ..., 0)$, where 1 is at the i-th position. We can estimate the right-hand side of g by means of (10.13). In view of (10.11) and the fact that

$$\sum_{|\gamma|=k, \gamma \leq \alpha'} \binom{\alpha'}{\gamma} = \binom{j}{k},$$

we obtain

$$\begin{aligned} \epsilon^{2+j+l} N_{j\epsilon}(g) &\leq \sum_{k=1}^{j} \binom{j}{k} D^{k+1} k! j^{-k} C_1^{j+2+l-k+1} \\ &\leq \sum_{k=1}^{j} D^{k+1} C_1^{j+2+l-k+1} \quad \leq \quad 2D^2 C_1^{j+2+l} \end{aligned}$$

if $C_1 > \sup\,(2D, 1)$. If we use this estimate and (10.12) in (10.14), it follows that

$$\epsilon^{2+j+l} N_{j\epsilon}(\Delta \partial_t^{\alpha'} A^l u) \leq D^{j+l+1} + 2D^2 C_1^{j+2+l} \tag{10.15}$$

if $C_1 > \sup\,(2D, 1)$.

We now apply Lemma 10.11 to $\partial_t^{\alpha'} u$ with $\epsilon_1 = j\epsilon$ and α replaced by α''. In view of (10.15) and (10.13), we then obtain

$$\epsilon^{|\alpha'+\alpha''|+l} N_{(j+1)\epsilon}(\partial_t^{\alpha'+\alpha''} A^l u) \leq C(D^{j+l+1} + 2D^2 C_1^{j+2+l} + C_2 C_1^{2+j})$$

where $C_2 = \sum_{|\beta|+l<2} 1$. Hence (10.13) follows with j replaced by $j+1$ provided that

$$C(D^{j+l+1} + 2D^2 C_1^{j+2+l} + C_2 C_1^{2+j}) \leq C_1^{2+j+l+1}.$$

This condition is fulfilled for every j if $C_1 > \sup(2D, 1, C(1 + 2D^2 + C_2))$. Thus the proof of (10.13) is completed.

Now it follows from (10.13) that u is analytic in ω. In fact, let K be a compact subset of ω and choose $c > 0$ so that $K \subset \omega_c$. Setting $j = |\alpha|$ and $\epsilon = c/j$ in (10.13), we then obtain

$$N_c(\partial_t^{\alpha} A^l u) \leq C_1^{|\alpha|+l+1} (|\alpha|/c)^{|\alpha|+l}.$$

Application of

$$\sup_{t \in K} ||A^l u(t)||_0^2 \leq C' \sum_{|\beta| \leq n} \int_{\omega_c} ||\partial_t^{\beta} A^l u(t)||_0^2 dt \tag{10.16}$$

with u replaced by $\partial_t^{\alpha} u$ gives, with a constant C,

$$\sup_{t \in K} \; ||\partial_t^{\alpha} A^l u(t)||_0 \leq C(C_1/c)^{|\alpha|+l} (|\alpha| + l + n)^{(|\alpha|+l+n)}.$$

The right-hand side can be estimated by $C^{|\alpha|+l+1}(|\alpha|+l)!$ for some constant C, which proves the analyticity of u.

For a proof of (10.16), see [5], page 109. □

Lemma 10.12. *For $N \geq 1$, Δ^N is analytic-hypoelliptic in U.*

Proof. By induction on N. $\square$

11. Sufficiency of Theorem 1.4

By hypothesis, there exists an open ball B' centered at the origin such that ψ_1 holds on B'. Applying Theorem 5.1, for every $f \in C_c^\infty(B', H^\infty)$, there exist $u_j^{(N)} \in \mathcal{D}'^0(B', H^{-\infty})$, $j = 1, ..., n$, such that

$$\sum_{j=1}^{n} \mathrm{L}_{j,0}\ u_j^{(N)} = \Delta^N f \quad \text{in} \quad B'.$$

From the fact that the distributions $u_j^{(N)}$ are of order zero, if the integer M is such that

$$M > \frac{n}{2} \tag{11.1}$$

then:

- we can choose an open ball, centered at the origin, $\Omega_N \doteq B_N(0) \subset\subset B'$ satisfying Lemma 10.8. In the equation above we take the restrictions of $u_j^{(N)}$, $j = 1, ..., n$, and of f to $B_N(0)$. Relabeling them, we get

$$\sum_{j=1}^{n} \mathrm{L}_{j,0}\ u_j^{(N)} = \Delta^N f \quad \text{in} \quad B_N(0), \tag{11.2}$$

  where $u_j^{(N)} \in \mathcal{D}'^0(B_N(0), H^{-\infty})$, $j = 1, ..., n$, and $f \in C^\infty(B_N(0), H^\infty)$.
- $u_j^{(N)} \in \mathscr{H}^{-M}(B_N(0))$.

Due to Proposition 10.9, there exists, for each j, $v_j^{(N)} \in \mathscr{H}^{2N-M}(B_N(0))$ satisfying

$$\Delta^N v_j^{(N)} = u_j^{(N)}. \tag{11.3}$$

If we take

$$2N - M > k + \frac{n}{2}, \tag{11.4}$$

it follows that $\mathscr{H}^{2N-M}(B_N(0)) \subset C^k(B_N(0), H^k)$. From (11.1) and (11.4), if $N > \frac{n+k}{2}$ then, for each j, $v_j^{(N)} \in C^k(B_N(0), H^k)$. Combining (11.2) and (11.3), we obtain

$$\Delta^N \Big(\sum_{j=1}^{n} \mathrm{L}_{j,0}\ v_j^{(N)} - f \Big) = 0.$$

Applying Lemma 10.12, it follows that there exists $g_N \in \mathscr{A}(B_N(0), H^w)$ such that

$$\sum_{j=1}^{n} \mathrm{L}_{j,0}\ v_j^{(N)} - f = g_N. \tag{11.5}$$

But we can choose a small ball $\omega_N \doteq B_N^{\cdot}(0) \subset\subset B_N(0)$ such that the function

$$h_N(t) = \int_0^{t_1} e^{(\phi_0(t_1,t_2,\dots,t_n)-\phi_0(s_1,t_2,\dots,t_n))A} g_N(s_1,t_2,\dots,t_n)ds_1 \in \mathscr{A}(B_{\dot N}(0), H^w)$$

is a solution of the equation

$$\mathrm{L}_{1,0}\ h_N = g_N \quad \text{in} \quad B_{\dot N}(0). \tag{11.6}$$

In fact, by hypothesis, $g_N \in C^\infty(B_N(0), E^\sigma)$ for some $\sigma > 0$. We have to prove that $h_N \in C^\infty(B_{\dot N}(0), E^{\sigma'})$ for some $\sigma' > 0$, for some $B_{\dot N}(0) \subset\subset B_N(0)$. Let $0 < \sigma' < \sigma$. It suffices to choose $B_{\dot N}(0)$ in order to have

$$0 < \sigma' + \phi_0(t_1,t_2,\dots,t_n) - \phi_0(s_1,t_2,\dots,t_n) < \sigma.$$

Finally, from (11.5) and (11.6), we obtain

$$\mathrm{L}_{1,0}\ (v_1^{(N)} - h_N) + \mathrm{L}_{2,0} v_2^{(N)} + \dots + \mathrm{L}_{n,0} v_n^{(N)} = f \qquad \text{in} \quad B_{\dot N}(0).$$

12. Necessity of Theorem 1.4

By hypothesis, (1.1) is fulfilled. By Lemma 4.3, there exists an open ball $B_{\dot N}(0) \subset B_N(0)$ such that for every $g \in C_c^\infty(B_{\ddot N}(0), H^\infty)$ there exist $v_j \in \mathcal{D}'(B_{\ddot N}(0), H^\infty), j = 1, \dots, n$, such that

$$\mathrm{L}_{1,0} v_1 + \ldots + \mathrm{L}_{n,0} v_n = g \qquad \text{in} \quad B_{\ddot N}(0).$$

Applying Theorem 5.1, ψ_1 holds on $B_{\ddot N}(0)$. Therefore, ψ_1 holds at 0.

13. Corollary

As a consequence of Theorem 1.4, we have

Corollary 13.1. *Condition* (ψ_2) *at* 0 *is necessary and sufficient to solve the equation*

$$\mathrm{L}_{1,0}^* u_1^{(N)} + \ldots + \mathrm{L}_{n,0}^* u_n^{(N)} = f \qquad \textit{in} \quad \omega_N.$$

Proof. In fact, setting $\chi_0 = -\phi_0$, it follows that

$$-\mathrm{L}_{j,0}^* = \partial_{t_j} - \partial_{t_j}\chi_0(t)A \quad \text{and} \quad \min_K \chi_0 = \min_{\partial K} \chi_0, \quad \forall \text{ compact } K \text{ of } B. \qquad \square$$

14. General case

Let H be a Hilbert space and let A be a linear operator, densely defined in H, unbounded, but self-adjoint. Let (E_λ), $-\infty < \lambda < \infty$, be a spectral resolution of A.

For $\epsilon > 0$, we consider three orthogonal projections of H defined by the operators $E_{-\epsilon}$, $E_\epsilon - E_{-\epsilon}$ and $I - E_\epsilon$, and their correspondent spaces $H_- = E_{-\epsilon}H$, $H_0 = (E_\epsilon - E_{-\epsilon})H$ and $H_+ = (I - E_\epsilon)H$, that are Hilbert spaces, since they are closed in H. These spaces are two by two orthogonal and they define H by $H = H_- \oplus H_0 \oplus H_+$. Let A_- be the restriction of A to the elements of its domain that are in H_-; in H_-, the operator A_- is self-adjoint, negative definite and has a bounded inverse. Let A_0 be the restriction of A to the elements of H_0; in H_0, the operator A_0 is self-adjoint and bounded. At long last, let A_+ be the restriction of A to the elements of its domain that are in H_+; in H_+, the operator A_+ is self-adjoint, positive definite and has a bounded inverse. Those three operators determine the operator A by $A = A_- + A_0 + A_+$.

As $-A_-$ is a self-adjoint operator, positive definite and has a bounded inverse in H_-, we may define the family of "Sobolev" spaces H_-^s, for $s \in \mathbb{R}$. Once A_+ is a self-adjoint operator, positive definite and has a bounded inverse in H_+, we define the family of "Sobolev" spaces H_+^s, for $s \in \mathbb{R}$. Then we define $H^s = H_-^s \oplus H_0 \oplus H_+^s$, for $s \in \mathbb{R}$. In this general case we also define H^∞, $H^{-\infty}$, $C^\infty(\Omega, H^\infty)$, $C_c^\infty(\Omega, H^\infty)$, $\mathcal{D}'(\Omega, H^{-\infty})$ etc, where Ω is an open set in $\mathbb{R}^n$.

The spaces H_+^∞ and H_-^∞ defined, respectively, by A_+ and by $-A_-$ have duals denoted by $H_+^{-\infty}$ and $H_-^{-\infty}$.

We now prove Theorem 1.5:

Proof. We split equation (1.2) into three equations:

$$\begin{cases} \sum_{j=1}^{n} \left(\partial_{t_j} u_{j-}^{(N)} + (\partial_{t_j}\phi_0)(t)(-A_-)u_{j-}^{(N)}\right) = f^-, \\ \sum_{j=1}^{n} \left(\partial_{t_j} u_{j0} - (\partial_{t_j}\phi_0)(t)A_0 u_{j0}\right) = f^0, \\ \sum_{j=1}^{n} \left(\partial_{t_j} u_{j+}^{(N)} - (\partial_{t_j}\phi_0)(t)A_+ u_{j+}^{(N)}\right) = f^+, \end{cases}$$

where $u_{j-}^{(N)} \in C^k(\omega_N, H_-^k)$, $u_{j0} \in C^k(\omega_n, H_0)$, $u_{j+}^{(N)} \in C^k(\omega_N, H_+^k)$, $f^- \in C^\infty(\Omega_N, H_-^\infty)$, $f^0 \in C^\infty(\Omega_N, H_0)$, $f^+ \in C^\infty(\Omega_N, H_+^\infty)$.

By Corollary 13.1, we solve the first equation above if and only if condition (ψ_2) holds at 0. By Theorem 1.4, we solve the third equation above if and only if condition (ψ_1) holds at 0. Now, a $C^\infty(\omega_N, H_0)$ solution of the second equation is given by

$$u_{20} \equiv ... \equiv u_{n0} \equiv 0, \qquad u_{10}(t) = \int_0^{t_1} e^{(\phi_0(t_1,t_2,...,t_n)-\phi_0(s_1,t_2,...,t_n))A_0} f^0(s_1, t_2, ..., t_n)ds_1.$$

Indeed, the exponential in the integrand defines a bounded linear operator in H_0 and an automorphism (depending smoothly on t) on the spaces of distributions $\mathcal{D}'(\omega_N, H_0)$, $C^\infty(\omega_N, H_0)$, etc. $\square$

15. Examples

Example 15.1 *(A solvable system but no solvable* $\mathrm{L}_{j,0}$*).* Consider $B = \{t \in \mathbb{R}^2 : |t| < R\} \subset\subset \Omega$ and $\phi_0 : \Omega \longrightarrow \mathbb{R}$ given by $\phi_0(t_1, t_2) = t_1^2 + t_2^2 - 3t_1t_2$.

Proof. $(0, 0)$ is the only critical point of ϕ_0, and the hessian of ϕ_0 at $(0, 0)$, $H(0, 0)$, is equal to $-5 < 0$. Hence, ϕ_0 does not have local minimum value and, consequently, the underdetermined system is solvable by Theorem 1.4.

We state that $\mathrm{L}_{1,0}$ is not solvable in B. Indeed, the function $(\partial_{t_1}\phi_0)(t_1, 0) = 2t_1$ does not satisfy (ψ_1') on $I_1 = \{t_1 : (t_1, 0) \in B\}$. Hence, $\mathrm{L}_{1,0}$ is not solvable in I_1. Therefore, $\mathrm{L}_{1,0}$ is not solvable in B. The proof that $\mathrm{L}_{2,0}$ is not solvable in B is analogous. $\square$

Example 15.2. Let $H = L^2(\mathbb{R}^\nu)$ and let $A = Q(D_x)$ be a positive pseudodifferential operator, elliptic, where $Q \in C^\infty(\mathbb{R}^\nu, \mathbb{R})$, $Q(\lambda\xi) = \lambda^m Q(\xi)$ if $m > 0$, $\lambda \geq 1$, $|\xi| \geq 1$ and that Q never vanishes. We assume

$$\phi(t, Q(D_x)) = \sum_{k=0}^{\infty} \phi_k(t)Q^{-k}(D_x) \in \mathcal{Q}_{Q(D_x)}(\Omega) , \quad \phi_0 \in C^\omega(\Omega) .$$

Remark. $|Q(\xi_1)| \leq C|\xi_1|^m$, if $|\xi_1| \geq 1$.

Statement 1: $Q(D_x)$ is densely defined in L^2.

Proof. The domain of $Q(D_x) : D\big(Q(D_x)\big) \subset L^2 \longrightarrow L^2$ is given by $D\big(Q(D_x)\big) = \{u \in L^2 : Q(D_x)u \in L^2\}$ and, by Parseval's formula, $D\big(Q(D_x)\big) = \{u \in L^2 : Q(\xi)\hat{u}(\xi) \in L^2\}$.

On the other hand, for every $u \in \mathscr{S}$ we have $\hat{u} \in \mathscr{S}$, which implies

$$\int\limits_{\mathbb{R}^\nu} |Q(\xi)\hat{u}(\xi)|^2 d\xi \leq C_1 + C^2 \int\limits_{|\xi_1|\geq 1} |\xi|^{2m} |\hat{u}(\xi)|^2 d\xi < \infty,$$

that is, $Q(\xi)\hat{u} \in L^2$. Thus, $\mathscr{S} \subset D\big(Q(D_x)\big)$ and, as $D\big(Q(D_x)\big) \subset L^2$, it follows from the density of $\mathscr{S}$ in L^2 that $D\big(Q(D_x)\big)$ is dense in L^2. $\square$

Statement 2: $Q(D_x)$ is unbounded, since it has order greater than 0.

Statement 3: $Q(D_x)$ is a self-adjoint pseudodifferential operator.

Proof. For every u and for every v in the domain of $Q(D_x)$ we have, using Parseval's formula,

$$\int\limits_{\mathbb{R}^\nu} \Big(Q(D_x)u(x)\Big)\overline{v}(x)dx = \int\limits_{\mathbb{R}^\nu} u(x)\overline{Q(D_x)v(x)}dx.$$

As $Q(D_x)$ is a positive definite symmetric operator, it has a Friedrichs' extension, that is, the extension is positive definite and self-adjoint. $\square$

Statement 4: $Q^{-1}(D_x) \in L(L^2(\mathbb{R}^\nu), L^2(\mathbb{R}^\nu))$.

Proof. We have the pseudodifferential operator

$$Q^{-1}(D_x)u(x) = \frac{1}{(2\pi)^\nu} \int\int e^{i(x-y).\xi} Q^{-1}(\xi)u(y)dyd\xi, \qquad u \in \mathscr{S}.$$

But, $Q^{-1} \in S^{-m}(\mathbb{R}^\nu) \subset S^0(\mathbb{R}^\nu)$, where S^{-m} and S^0 are the spaces of symbols of order $-m$ and 0, respectively. Therefore, $Q^{-1}(D_x) \in L(L^2(\mathbb{R}^\nu), L^2(\mathbb{R}^\nu))$. $\square$

We denote by $\mathfrak{A}^s(\mathbb{R}^\nu) = \mathfrak{A}^s$ the space of elements u in $L^2(\mathbb{R}^\nu)$ such that $Q^s(D_x)u \in L^2$, equipped with the norm

$$\|u\|^2_{\mathfrak{A}^s} \;=\; \|Q^s(D_x)u\|^2_0 \;=\; \frac{1}{(2\pi)^\nu} \int\limits_{\mathbb{R}^\nu} |Q^{2s}(\xi)||\hat{u}(\xi)|^2 d\xi.$$

Then consider the spaces

$$\mathfrak{H}^s(\mathbb{R}^\nu) = \left\{ u \in \mathscr{S}' : \frac{1}{(2\pi)^\nu} \int\limits_{\mathbb{R}^\nu} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\}$$

and

$$\mathfrak{A}^s(\mathbb{R}^\nu) = \left\{ u \in L^2 : \frac{1}{(2\pi)^\nu} \int\limits_{\mathbb{R}^\nu} |Q^{2s}(\xi)||\hat{u}(\xi)|^2 d\xi < \infty \right\}.$$

Statement 5: $\bigcap\limits_{s\geq 0} \mathfrak{A}^s = \bigcap\limits_{s\geq 0} \mathfrak{H}^s = \mathfrak{H}^\infty$.

Proof. Since $Q(D_x)$ is elliptic, there exists $C > 0$ such that $|Q(\xi)| \geq C|\xi|^m$, $\forall \xi \in \mathbb{R}^\nu$. Conversely, we already know that there exists $C > 0$ such that $|Q(\xi)| \leq C|\xi|^m$, if $|\xi| \geq 1$. □

As the 5 statements are fulfilled, $\mathbb{L}$, defined by the operators $\mathrm{L}_j = \partial_{t_j} - (\partial_{t_j}\phi)(t, Q(D_x))Q(D_x)$, $j = 1, ..., n$, is solvable in terms of Theorem 1.4.

Example 15.3. Let $H = L^2(\mathbb{R}^\nu)$ and let $A = Q(D_x)$ be a positive pseudodifferential operator, elliptic, where $Q \in C^\infty(\mathbb{R}^\nu, \mathbb{R})$, $Q(\lambda\xi) = \lambda Q(\xi)$, $\lambda \geq 1$, $|\xi| \geq 1$ and that Q never vanishes. We assume

$$\sum_{k=0}^{\infty} \sup_{t\in K} |\partial_t^\alpha \phi_k(t)| \; ||Q^{-k}||_{L^\infty(\mathbb{R}^\nu)} < \infty, \quad \forall \ \text{compact } K \subset \Omega, \quad \forall \alpha \in \mathbb{Z}_+^n.$$

This is a particular case of Example 15.2 taking $m = 1$. This example is an intersection with a result presented in [9]. Setting

$$B(t, D_x) = -\Big(\sum_{k=0}^{\infty} \phi_k(t) Q^{-k}(D_x) \Big) Q(D_x), \quad \phi_0 \in C^\omega(\Omega), \quad \phi_k \in C^\infty(\Omega), \quad k \geq 1,$$

the fundamental hypotheses in [9] are:

- there is a C^∞ function of t in Ω, valued in $L^\infty(\mathbb{R}^\nu)$, $R(t,\xi)$, such that $B^0(t,\xi) = B(t,\xi) - R(t,\xi)$ is positive homogeneous of degree one with respect to ξ.
- $B^0(t,\xi)$ is a C^∞ function of t in Ω, with values in the space of C^1 functions of ξ in $\mathbb{R}^\nu - \{0\}$.

But

$$B(t,\xi) = \underbrace{-\phi_0(t)Q(\xi)}_{B^0(t,\xi)} \; \underbrace{-\sum_{k=1}^{\infty} \phi_k(t) Q^{-k+1}(\xi)}_{R(t,\xi)}.$$

Then $B^0 \in C^\infty(\Omega, S^1(\mathbb{R}^\nu))$ is positive homogeneous of degree one in ξ for $|\xi| \geq 1$ and $R(t,\xi)$ converges in the space $C^\infty(\Omega, L^\infty(\mathbb{R}^\nu))$, where S^1 is the space of symbols of order 1.

We now write

$$B^0(t,\xi) \;=\; \Re B^0(t,\xi) + i\, \Im B^0(t,\xi) \;=\; \underbrace{-(\Re\phi_0)(t)Q(\xi)}_{B_1^0(t,\xi)} - i\, (\Im\phi_0)(t)Q(\xi)\;,$$

and the hypotheses become:

- $B_1^0(t,\xi)$ is *real-valued* and positive homogeneous of degree one with respect to ξ.
- $B_1^0(t,\xi)$ is a C^∞ function of t in Ω with values in $C^1(\mathbb{R}^\nu - \{0\})$.

Actually, in our case $B_1^0 \in C^\infty(\Omega, S^1(\mathbb{R}^\nu))$ is *real-valued* and satisfies $B_1^0(t,\lambda\xi) = \lambda B_1^0(t,\xi)$, if $\lambda \geq 1,\ |\xi| \geq 1$.

It remains to verify that $\phi(t, Q(D_x)) \in \mathcal{Q}_{Q(D_x)}(\Omega)$. Indeed, since

$$\begin{aligned} ||Q^{-1}(D_x)u||_{L^2} &= \frac{1}{(2\pi)^{\nu/2}}\ ||Q^{-1}(\xi)\hat{u}||_{L^2} \\ &\leq ||Q^{-1}||_{L^\infty(\mathbb{R}^\nu)}\ ||u||_{L^2} \quad , u \in L^2, \end{aligned}$$

it follows that, for every compact $K \subset \Omega$, for every $\alpha \in \mathbb{Z}_+^n$,

$$\sum_{k=0}^{\infty} \sup_{t\in K} |\partial_t^\alpha \phi_k(t)|\ ||Q^{-k}(D_x)||_{L(L^2(\mathbb{R}^\nu))} \leq \sum_{k=0}^{\infty} \sup_{t\in K} |\partial_t^\alpha \phi_k(t)|\ ||Q^{-k}||_{L^\infty(\mathbb{R}^\nu)}\ <\ \infty.$$

Thus, $\sum\limits_{k=0}^{\infty} \phi_k(t)Q^{-k}(D_x)$ converges in $C^\infty\big(\Omega; L(L^2(\mathbb{R}^\nu))\big)$, that is, $\sum\limits_{k=0}^{\infty} \phi_k(t)Q^{-k}(D_x) \in \mathcal{Q}_{Q(D_x)}(\Omega)$.

Setting, for every $\xi \in \mathbb{R}^\nu$ and for every real r,

$$B(\xi, r) = \left\{ t \in B\ :\ B_1^0(t,\xi) < r \right\},$$

in [9] the complex $\mathbb{L}$ has the condition (ψ_1) holding on B, in dimension $n-1$, if

$$\begin{aligned} B - B(\xi, r) &= \left\{ t \in B\ :\ B_1^0(t,\xi) = -(\Re\phi_0)(t)Q(\xi) \geq r \right\} \\ &= \left\{ t \in B\ :\ (\Re\phi_0)(t) \leq \frac{-r}{Q(\xi)} \right\} \end{aligned}$$

has no compact connected component. Since $Q(\xi)$ never vanishes, condition (ψ_1) on B can be written as: for every real r, the set

$$\left\{ t \in B\ :\ (\Re\phi_0)(t) \leq r \right\}$$

has no compact connected component.

Therefore, keeping our notation, theorem $II.1.2$ in [9] can be written as

Theorem 15.4. *Suppose that the complex $\mathbb{L}$ has the condition (ψ_1) holding on B, in dimension $n-1$. Then, given any open set $\mathscr{O}' \subset\subset B$, $0 \in \mathscr{O}'$, and any element f of $C^\infty_{(n)}(\Omega, \mathfrak{H}^\infty)$, there is an element u of $\mathcal{D}'_{(n-1)}(\mathscr{O}', \mathfrak{H}^{-\infty})$ solution of $\mathbb{L}u = f$ in $\mathscr{O}'$.*

But, of course, we also have the solvability of the complex $\mathbb{L}$ in terms of Theorem 1.4.

Example 15.5. Let $H = L^2(\mathbb{R})$ and let $A = D_x$. It is well known that D_x is densely defined in $L^2(\mathbb{R})$, unbounded and it is self-adjoint. Furthermore, D_x defines $\mathfrak{H}^\infty$.

This example performs the general case. We have the solvability of the complex $\mathbb{L}_0$ defined by the operators $\mathrm{L}_{j,0} = \partial_{t_j} - (\partial_{t_j}\Re\phi_0)(t)D_x$, $j = 1, \ldots, n$, in terms of Theorem 1.5.

Let us now make a link with [2] and show that it is a consequence to have local solvability at $0 \in \mathbb{R}^{n+1}$. In fact, given $f \in C_c^\infty(B \times \mathbb{R}) \subset C_c^\infty(B, \mathfrak{H}^\infty)$, there exist $u_j \in \mathcal{D}'(B, \mathfrak{H}^{-\infty}) \subset \mathcal{D}'(B \times \mathbb{R})$, $j = 1, ..., n$, such that

$$\mathrm{L}_{1,0} u_1 + \ldots + \mathrm{L}_{n,0} u_n = f \qquad \text{in} \qquad B \times \mathbb{R}.$$

The locally integrable structure is characterized by

$$\begin{cases} \mathrm{L}_{j,0} Z = 0, \quad j = 1, ..., n, \\ dZ \neq 0, \quad \text{in } B \times J, \end{cases}$$

where J is an open interval centered at the origin in $\mathbb{R}$ and $Z(t,x) = x - i\Re\phi_0(t)$. Then we recall the condition $\mathscr{P}_{n-1}$ at $0 \in \mathbb{R}^{n+1}$: “there is an open neighborhood of 0 over which every *regular fiber* of Z has no compact connected component”. To find a *fiber* of Z over $B \times J$ we write

$$\begin{cases} Z = x + i\phi(t), \\ Z = x_0 + iy_0, \quad x_0 \in J, \quad y_0 \in \mathbb{R}. \end{cases}$$

Thus, to say that over $B \times J$ every *regular fiber* of Z, $\{t \in B : \phi(t) = y_0\} \times \{x_0\}$, has no compact connected component coincides with the definition of (ψ_1) and (ψ_2) at 0.

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