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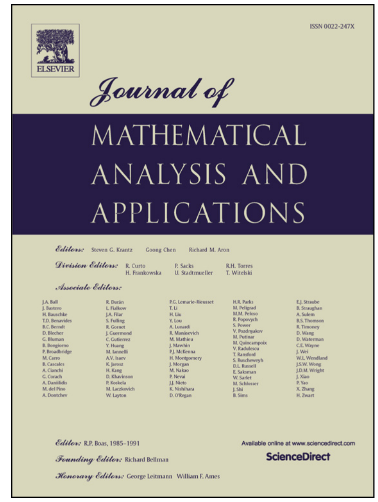

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Approximation by multivariate Kantorovich–Kotelnikov operators *

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Abstract

Approximation properties of multivariate Kantorovich-Kotelnikov type operators generated by different band-limited functions are studied. In particular, a wide class of functions with discontinuous Fourier transform is considered. The L_p-rate of convergence for these operators is given in terms of the classical moduli of smoothness. Several examples of the Kantorovich-Kotelnikov operators generated by the sinc-function and its linear combinations are provided.

Keywords Kantorovich–Kotelnikov operator, band-limited function, approximation order, modulus of smoothness, matrix dilation.

AMS Subject Classification: 41A58, 41A25, 41A63

1 Introduction

The Kantorovich–Kotelnikov operator is an operator of the form

$$K_w(f,\varphi;x)=\sum_{k\in\mathbb{Z}}\left(w\int_{\frac{k}{w}}^{\frac{k+1}{w}}f(u)du\right)\varphi(wx-k),\quad x\in\mathbb{R},\quad w>0, \tag{1}$$

where $f:\mathbb{R}\to\mathbb{C}$ is a locally integrable function and φ is an appropriate kernel satisfying certain "good" properties, as a rule φ is a band-limited function or a function with a compact support, e.g., B-spline. The operator K_w was introduced in [3], although in other forms, it was known previously, see, e.g., [12, 13, 20]. During the last years, in view of some important applications, this operator has drawn attention by many mathematicians and has been especially actively studied [4, 7, 8, 9, 10, 13, 15, 16, 17, 18, 23, 27, 28].

The operator K_w has several advantages over the generalized sampling operators

$$S_w(f,\varphi;x)=\sum_{k\in\mathbb{Z}}f\left(\frac{k}{w}\right)\varphi(wx-k),\quad x\in\mathbb{R},\quad w>0. \tag{2}$$

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First of all, using the averages $w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du$ instead of the sampled values $f(k/w)$ allows to deal with discontinues signals and reduce the so-called time-jitter errors. Note that the latter property is very useful in the theory of Signal and Image Processing. Moreover, unlike to the generalized sampling operators S_w, the operator (1) is continuous in $L_p(\mathbb{R})$ and, therefore, provides better approximation order than S_w in most cases.

In the literature, there are several generalizations and refinements of the Kantorovich–Kotelnikov operator K_w (see, e.g., [1, 11, 13, 16, 17, 18, 20, 23, 27, 28]). In this paper, we study approximation properties of the following multivariate analogue of (1)

$$Q_j(f,\varphi,\widetilde{\varphi};x)=\sum_{k\in\mathbb{Z}^d}\left(m^j\int_{\mathbb{R}^d} f(u)\widetilde{\varphi}(M^ju+k)du\right)\varphi(M^jx+k),\quad x\in\mathbb{R}^d,\quad j\in\mathbb{Z},\tag{3}$$

where M is a dilation matrix, $m=|\det M|$, and $\widetilde{\varphi}$ and φ are appropriate functions. Note that if $d=1$ and $\widetilde{\varphi}(x)=\chi_{[0,1]}(x)$ (the characteristic function of $[0,1]$), then (3) represents the standard Kantorovich–Kotelnikov operator K_{m^j}.

Convergence and approximation properties of the operator (3) have been actively studied by many authors (see [1, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 20, 23, 27, 28]). The most general results on estimates of the error of approximation by Q_j have been obtained in the case of compactly supported φ and $\widetilde{\varphi}$. In particular in [13] (see also [12]) it was proved the following result: *if φ and $\widetilde{\varphi}$ are compactly supported, $\varphi\in L_p(\mathbb{R}^d)$, $\widetilde{\varphi}\in L_q(\mathbb{R}^d)$, $1/p+1/q=1$, M is an isotropic dilation matrix, and Q_0 reproduces polynomials of degree $n-1$, then for any $f\in L_p(\mathbb{R})$, $1\le p\le\infty$, and $n\in\mathbb{N}$ we have*

$$\|f-Q_j(f,\varphi,\widetilde{\varphi})\|_{L_p(\mathbb{R}^d)}\le C(p,d,n,\varphi,\widetilde{\varphi})\omega_n(f,\|M^{-j}\|)_p,\tag{4}$$

where $\omega_n(f,h)_p$ is the modulus of smoothness of order n.

Concerning band-limited functions φ, it turns out that approximation properties of the operators Q_j have been studied mainly in the case where $\widetilde{\varphi}$ is some characteristic function (see, e.g., [3, 7, 8, 9, 10, 16, 23, 27, 28]) and, unlike to compactly supported functions φ, there are several limitations and drawbacks in the available results. First of all, the methods previously used are essentially restricted to the case of integrable functions φ, which do not allow to consider the functions of type $\operatorname{sinc}(x)=(\sin\pi x)/(\pi x)$. Secondly, the conditions imposed on the kernel φ cannot provide high rate of convergence of $Q_j(f)$ even for sufficiently smooth functions f. At that, the corresponding estimates are given in the terms of Lipschitz classes. For example, it follows from [8] that *for any $f\in L_p(\mathbb{R})\cap\operatorname{Lip}(\nu)$, $1\le p\le\infty$, $0<\nu\le 1$, we have*

$$\left\|f-\frac12\sum_{k\in\mathbb{Z}}\left(w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du\right)\operatorname{sinc}^2\left(\frac{wx-k}{2}\right)\right\|_{L_p(\mathbb{R})}=\mathcal{O}(w^{-\nu}),\quad w\to+\infty.\tag{5}$$

In this paper, we improve the mentioned drawbacks and study the rate of convergence of the operator (3) for a wide class of band-limited functions φ including non-integrable ones. In particular, estimation (4) is proved for a large class of functions $\widetilde{\varphi}$ including both compactly supported and band-limited functions, provided that $D^\beta(1-\widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}})(\mathbf{0})=0$ for all $\beta\in\mathbb{Z}_+^d$, $\|\beta\|_{\ell_1}<n$ (see Theorems 17 and 17$'$). In the partial cases, this gives an answer to the question posed in [3] about approximation properties of the following sampling series (see Section 6):

$$\sum_{k\in\mathbb{Z}}\left(w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du\right)\operatorname{sinc}(wx-k),\quad x\in\mathbb{R},\quad w>0.$$

The paper is organized as follows: in Section 2 we introduce notation and give some basic facts. In Section 3 we consider approximation properties of some generalized sampling operators of type Q_j.

These operators are defined similarly to the operators Q_j but with appropriate tempered distribution in place of the function $\widetilde{\varphi}$ that, in particular, allows to include in the consideration operators of type S_w. The L_p-rate of convergence of such generalized sampling operators is given in terms of Fourier transform of f and has several drawbacks, which we improve in the next sections. Section 4 is devoted to auxiliary results. In Section 5 we prove two main results that provide estimates of the error of approximation by the operator Q_j in terms of the classical moduli of smoothness. The results of this section can be considered as counterparts of the corresponding results from Section 3. Finally, in Section 6 we consider some special cases and provide a number of examples.

2 Notation and basic facts

$\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers. $\mathbb{R}^d$ is the d-dimensional Euclidean space, $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ are its elements (vectors), $(x, y) = x_1y_1 + \dots + x_dy_d$, $|x| = \sqrt{(x,x)}$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$; $B_r = \{x \in \mathbb{R}^d : |x| \le r\}$, $\mathbb{T}^d = [-\frac{1}{2}, \frac{1}{2}]^d$; $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$, $\mathbb{Z}_+^d := \{x \in \mathbb{Z}^d : x \ge \mathbf{0}\}$. If $\alpha, \beta \in \mathbb{Z}_+^d$, $a, b \in \mathbb{R}^d$, we set $[\alpha] = \sum\limits_{j=1}^d \alpha_j$, $\alpha! = \prod\limits_{j=1}^d (\alpha_j!)$,

$$\binom{\beta}{\alpha} = \frac{\beta!}{\alpha!(\beta-\alpha)!}, \quad D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial x^\alpha} = \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d},$$

δ_{ab} is the Kronecker delta.

A real $d \times d$ matrix M whose eigenvalues are bigger than 1 in modulus is called a dilation matrix. Throughout the paper we consider that such a matrix M is fixed and $m = |\det M|$, M^* denotes the conjugate matrix to M. Since the spectrum of the operator M^{-1} is located in B_r, where $r = r(M^{-1}) := \lim_{j\to+\infty} \|M^{-j}\|^{1/j}$ is the spectral radius of M^{-1}, and there exists at least one point of the spectrum on the boundary of B_r, we have

$$\|M^{-j}\| \le C_{M,\vartheta}\, \vartheta^{-j}, \quad j \ge 0, \tag{6}$$

for every positive number ϑ which is smaller in modulus than any eigenvalue of M. In particular, we can take $\vartheta > 1$, then

$$\lim_{j\to+\infty} \|M^{-j}\| = 0.$$

L_p denotes $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, with the usual norm $\|f\|_p = \|f\|_{L_p(\mathbb{R}^d)}$. We say that $\varphi \in L_p^0$ if $\varphi \in L_p$ and φ has a compact support. We use W_p^n, $1 \le p \le \infty$, $n \in \mathbb{N}$, to denote the Sobolev space on $\mathbb{R}^d$, i.e. the set of functions whose derivatives up to order n are in L_p, with the usual Sobolev semi-norm given by

$$\|f\|_{\dot{W}_p^n} = \sum_{[\nu]=n} \|D^\nu f\|_p.$$

If f, g are functions defined on $\mathbb{R}^d$ and $f\overline{g} \in L_1$, then $\langle f, g\rangle := \int_{\mathbb{R}^d} f\overline{g}$. If $f \in L_1$, then its Fourier transform is $\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x,\xi)}\, dx$.

If φ is a function defined on $\mathbb{R}^d$, we set

$$\varphi_{jk}(x) := m^{j/2} \varphi(M^j x + k), \quad j \in \mathbb{Z},\ k \in \mathbb{R}^d.$$

Denote by $\mathcal{S}$ the Schwartz class of functions defined on $\mathbb{R}^d$. The dual space of $\mathcal{S}$ is $\mathcal{S}'$, i.e. $\mathcal{S}'$ is the space of tempered distributions. The basic facts from distribution theory can be found, e.g., in [29]. Suppose $f \in \mathcal{S}$, $\varphi \in \mathcal{S}'$, then $\langle \varphi, f\rangle := \overline{\langle f, \varphi\rangle} := \varphi(f)$. If $\varphi \in \mathcal{S}'$, then $\widehat{\varphi}$ denotes its Fourier transform defined by $\langle \widehat{f}, \widehat{\varphi}\rangle = \langle f, \varphi\rangle$, $f \in \mathcal{S}$. If $\varphi \in \mathcal{S}'$, $j \in \mathbb{Z}, k \in \mathbb{Z}^d$, then we define φ_{jk} by $\langle f, \varphi_{jk}\rangle = \langle f_{-j,-M^{-j}k}, \varphi\rangle$ for all $f \in \mathcal{S}$.

Denote by $\mathcal{S}'_N$ the set of all tempered distribution whose Fourier transform $\widehat{\varphi}$ is a function on $\mathbb{R}^d$ such that $|\widehat{\varphi}(\xi)| \le C_\varphi |\xi|^N$ for almost all $\xi \notin \mathbb{T}^d$, $N = N(\varphi) \ge 0$, and $|\widehat{\varphi}(\xi)| \le C'_\varphi$ for almost all $\xi \in \mathbb{T}^d$.

Denote by $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ the class of functions φ given by

$$\varphi(x) = \int\limits_{\mathbb{R}^d} \theta(\xi) e^{2\pi i(x,\xi)}\, d\xi,$$

where θ is supported in a parallelepiped $\Pi := [a_1, b_1] \times \cdots \times [a_d, b_d]$ and such that $\theta\big|_\Pi \in C^d(\Pi)$.

Let $1 \le p \le \infty$. Denote by $\mathcal{L}_p$ the set

$$\mathcal{L}_p := \left\{ \varphi \in L_p \,:\, \|\varphi\|_{\mathcal{L}_p} := \Big\| \sum_{k\in\mathbb{Z}^d} |\varphi(\cdot + k)| \Big\|_{L_p(\mathbb{T}^d)} < \infty \right\}.$$

With the norm $\|\cdot\|_{\mathcal{L}_p}$, $\mathcal{L}_p$ is a Banach space. The simple properties are: $\mathcal{L}_1 = L_1$, $\|\varphi\|_p \le \|\varphi\|_{\mathcal{L}_p}$, $\|\varphi\|_{\mathcal{L}_q} \le \|\varphi\|_{\mathcal{L}_p}$ for $1 \le q \le p \le \infty$. Therefore, $\mathcal{L}_p \subset L_p$ and $\mathcal{L}_p \subset \mathcal{L}_q$ for $1 \le q \le p \le \infty$. If $\varphi \in L_p$ and compactly supported, then $\varphi \in \mathcal{L}_p$ for $p \ge 1$. If φ decays fast enough, i.e. there exist constants $C > 0$ and $\varepsilon > 0$ such that $|\varphi(x)| \le C(1 + |x|)^{-d-\varepsilon}$ for all $x \in \mathbb{R}^d$, then $\varphi \in \mathcal{L}_\infty$.

The modulus of smoothness $\omega_n(f, \cdot)_p$ of order $n \in \mathbb{N}$ for a function $f \in L_p$ is defined by

$$\omega_n(f, h)_p = \sup_{|\delta|<h,\, \delta\in\mathbb{R}^d} \|\Delta_\delta^n f\|_p,$$

where

$$\Delta_\delta^n f(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} f(x + \delta\nu).$$

3 Preliminary results

Scaling operator $\sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}$ is a good tool of approximation for many appropriate pairs of functions $\varphi, \widetilde{\varphi}$. Let us consider the case, where $\widetilde{\varphi}$ is a tempered distribution, e.g., the delta-function or a linear combination of its derivatives. In this case the inner product $\langle f, \widetilde{\varphi}_{jk}\rangle$ has meaning only for functions f in $\mathcal{S}$. To extend the class of functions f one can replace $\langle f, \widetilde{\varphi}_{jk}\rangle$ by $\langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle$. Approximation properties of such operators for certain classes of distributions $\widetilde{\varphi}$ and functions φ were studied, e.g., in [16] and [18].

Repeating step-by-step the proof of Theorem 4 in [18] and using Corollary 10 in [16], it is easy to prove the following result.

Proposition 1 *Let $N \in \mathbb{Z}_+$, $\widetilde{\varphi} \in \mathcal{S}'_N$, $\varphi \in \mathcal{B}$. Suppose that there exist $n \in \mathbb{N}$ and $\delta \in (0, 1/2)$ such that $\widehat{\varphi}\widehat{\widetilde{\varphi}}$ is boundedly differentiable up to order n on $\{|\xi| < \delta\}$; $\operatorname{supp}\widehat{\varphi} \subset B_{1-\delta}$; $D^\beta(1 - \widehat{\varphi}\widehat{\widetilde{\varphi}})(0) = 0$ for $[\beta] < n$. If $2 \le p < \infty$, $1/p + 1/q = 1$, $\gamma \in (N + d/p, N + d/p + \varepsilon)$, $\varepsilon > 0$, and*

$$f \in L_p, \quad \widehat{f} \in L_q, \quad \widehat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon}) \text{ as } |\xi| \to \infty, \tag{7}$$

then

$$\Big\| f - \sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle \varphi_{jk} \Big\|_p^q \le C_1 \|M^{*-j}\|^{\gamma q} \int\limits_{|M^{*-j}\xi| \ge \delta} |\xi|^{q\gamma} |\widehat{f}(\xi)|^q d\xi + C_2 \|M^{*-j}\|^{nq} \int\limits_{|M^{*-j}\xi| < \delta} |\xi|^{qn} |\widehat{f}(\xi)|^q d\xi,$$

where C_1 and C_2 do not depend on j and f.

Proposition 1 does not provide approximation order of $\sum_{k\in\mathbb{Z}^d}\langle\widehat{f},\widehat{\widetilde{\varphi}_{jk}}\rangle\varphi_{jk}$ better than $\|M^{*-j}\|^n$ even for very smooth functions f. This can be fixed under stronger restrictions on φ given in the following definition.

Definition 2 *A tempered distribution $\widetilde{\varphi}$ and a function φ is said to be* strictly compatible *if there exists $\delta\in(0,1/2)$ such that $\overline{\widehat{\varphi}}(\xi)\widehat{\widetilde{\varphi}}(\xi)=1$ a.e. on $\{|\xi|<\delta\}$ and $\widehat{\varphi}(\xi)=0$ a.e. on $\{|l-\xi|<\delta\}$ for all $l\in\mathbb{Z}^d\setminus\{0\}$.*

Remark 3 It is well known that the shift-invariant space generated by a function φ has approximation order n if and only if the Strang–Fix condition of order n is satisfied for φ (that is $D^\beta\widehat{\varphi}(l)=0$ whenever $l\in\mathbb{Z}^d$, $l\neq\mathbf{0}$, $[\beta]<n$). This fact appeared in the literature in different situations many times (see, e.g., [1], [2], [11], and [22, Ch. 3]). The condition $D^\beta(1-\widehat{\varphi}\widehat{\widetilde{\varphi}})(0)=0$, $[\beta]<n$, is also a natural requirement for providing approximation order n of scaling operators. This assumption often appears (especially in wavelet theory) in other terms, in particular, in terms of polynomial reproducing property (see [12, Lemma 3.2]). It is clear that to provide an infinitely large approximation order, these conditions should be satisfied for any n. Clearly, the latter holds for strictly compatible functions φ and $\widetilde{\varphi}$.

Supposing that $\widehat{\widetilde{\varphi}}(\xi)=1$ a.e. on $\{|\xi|<\delta\}$, it is easy to see that the simplest example of φ satisfying Definition 2 is the tensor product of the sinc functions.

Proposition 4 [16, Theorem 11] *Let $N\in\mathbb{Z}_+$, $\widetilde{\varphi}\in\mathcal{S}'_N$, $\varphi\in\mathcal{B}$, $\widetilde{\varphi}$ and φ are strictly compatible. If $2\le p<\infty$, $1/p+1/q=1$, $\gamma\in(N+d/p,N+d/p+\varepsilon)$, $\varepsilon>0$, and a function f satisfies (7), then*

$$\Big\|f-\sum_{k\in\mathbb{Z}^d}\langle\widehat{f},\widehat{\widetilde{\varphi}_{jk}}\rangle\varphi_{jk}\Big\|_p^q\le C\|M^{*-j}\|^{\gamma q}\int\limits_{|M^{*-j}\xi|\ge\delta}|\xi|^{q\gamma}|\widehat{f}(\xi)|^q d\xi, \tag{8}$$

where C does not depend on j and f.

Note that Propositions 4 is an analog of the following Brown's result [5]:

$$\Big|f(x)-\sum_{k\in\mathbb{Z}}f(-2^{-j}k)\,\mathrm{sinc}(2^jx+k)\Big|\le C\int\limits_{|\xi|>2^{j-1}}|\widehat{f}(\xi)|d\xi,\quad x\in\mathbb{R},$$

whenever the Fourier transform of f is summable on $\mathbb{R}$.

There are several drawbacks in Propositions 1 and 4. First, they are proved only in the case $p\ge2$. Second, there are additional restrictions on the function f. Even in the case $\widetilde{\varphi}\in L_q^0$, where $\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_{jk}\rangle\varphi_{jk}$ has meaning for every $f\in L_p$, the error estimate is obtained only for functions f satisfying (7) with $N=0$. Third, the error estimate is given in the terms of decreasing of Fourier transforms unlike to common estimates, which usually are given in the terms of moduli of smoothness.

Below, under more restrictive conditions on $\widetilde{\varphi}$, we obtain analogues of the above propositions for all $f\in L_p$, $1\le p\le\infty$, and give the estimates of the error of approximation in terms of the classical moduli of smoothness.

4 Auxiliary results

The following auxiliary statements will be useful for us.

Lemma 5 *Let either $1<p<\infty$, $\varphi\in\mathcal{B}$ or $1\le p\le\infty$, $\varphi\in\mathcal{L}_p$, and $a=\{a_k\}_{k\in\mathbb{Z}^d}\in\ell_p$. Then*

$$\Big\|\sum_{k\in\mathbb{Z}^d}a_k\varphi_{0k}\Big\|_p\le C_{\varphi,p}\|a\|_{\ell_p}. \tag{9}$$

Proof. In the case $\varphi \in \mathcal{B}$, the proof of (9) follows from [16, Proposition 9]. Concerning the case $\varphi \in \mathcal{L}_p$ see [11, Theorem 2.1]. $\diamond$

Lemma 6 *Let $f \in L_p$. Suppose that either $1 < p < \infty$, $\varphi \in \mathcal{B}$ or $1 \le p \le \infty$, $\varphi \in L_q^0$, $1/p+1/q=1$. Then*

$$\left(\sum_{k\in\mathbb{Z}^d} |\langle f, \varphi_{0k}\rangle|^p \right)^{\frac{1}{p}} \le C'_{\varphi,p}\|f\|_p. \tag{10}$$

Proof. In the case $\varphi \in \mathcal{B}$, the proof of (10) see in [16, Proposition 6]. The case $\varphi \in L_q^0$ follows from [17, Lemmas 4 and 5]. $\diamond$

Combining the above lemmas, we obtain the following statement.

Lemma 7 *Let $f \in L_p$, $1 < p < \infty$, and $j \in \mathbb{N}$. Suppose $\varphi \in \mathcal{B}$ and $\widetilde{\varphi} \in \mathcal{B} \cup \mathcal{L}_q$, $1/p + 1/q = 1$. Then*

$$\Big\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_p \le C\|f\|_p, \tag{11}$$

where C does not depend on f and j.

Proof. If $\widetilde{\varphi} \in \mathcal{B}$, then the proof of (11) directly follows from Lemmas 5 and 6. In the case $\widetilde{\varphi} \in \mathcal{L}_q$, we find $g \in L_q$, $\|g\|_q \le 1$, such that

$$\Big\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_p = \Big| \Big\langle \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}, g \Big\rangle \Big| = \Big| \Big\langle f, \sum_{k\in\mathbb{Z}^d} \langle \varphi_{jk}, g\rangle \widetilde{\varphi}_{jk} \Big\rangle \Big|. \tag{12}$$

It follows from the Hölder inequality and Lemmas 5 and 6 that

$$\begin{aligned} \Big| \Big\langle f, \sum_{k\in\mathbb{Z}^d} \langle \varphi_{jk}, g\rangle \widetilde{\varphi}_{jk} \Big\rangle \Big| &\le \|f\|_p \Big\| \sum_{k\in\mathbb{Z}^d} \langle \varphi_{jk}, g\rangle \widetilde{\varphi}_{jk} \Big\|_q \\ &\le C_{\widetilde{\varphi},q}\|f\|_p \left(\sum_{k\in\mathbb{Z}^d} |\langle \varphi_{jk}, g\rangle|^q \right)^{1/q} \le C_{\widetilde{\varphi},q} C'_{\varphi,q}\|f\|_p. \end{aligned} \tag{13}$$

Thus, combining (12) and (13), we obtain (11). $\diamond$

Similarly one can prove the following generalization of Lemma 7 to the limiting cases $p = 1, \infty$.

Lemma 7′ *Let $f \in L_p$, $p = 1$ or $p = \infty$, $j \in \mathbb{N}$. Suppose that*
(i) *$\varphi \in L_1$ and $\widetilde{\varphi} \in \mathcal{L}_\infty$ in the case $p = 1$;*
(ii) *$\varphi \in \mathcal{L}_\infty$ and $\widetilde{\varphi} \in L_1$ in the case $p = \infty$.*
Then

$$\Big\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_p \le C\|f\|_p,$$

where C does not depend on f and j.

Lemma 8 *Let $N \in \mathbb{Z}_+$, $\widetilde{\varphi} \in \mathcal{S}'_N$, $\varphi \in \mathcal{B} \cup \mathcal{L}_2$, and $\widetilde{\varphi}$ and φ be strictly compatible. If a function $f \in L_2$ is such that its Fourier transform is supported in $\{\xi : \ |M^{*-j}\xi| < \delta\}$, where δ is from Definition 2, then*

$$f = \sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle \varphi_{jk} \quad a.e. \tag{14}$$

Proof. In the case $\varphi \in \mathcal{B}$, the proof of the lemma follows from (8). In the case $\varphi \in \mathcal{L}_2$, it follows from [18, eq. (4.15)] that (8) holds and, therefore, one has (14). $\Diamond$

Let $\mathcal{B}_p^\sigma$, $1 \le p \le \infty$, $\sigma > 0$, denote the set of all entire functions f in $\mathbb{C}^d$ which are of (radial) exponential type σ and being restricted to $\mathbb{R}^d$ belong to L_p.

Lemma 9 [30, Theorem 3] *Let $n \in \mathbb{N}$, $1 \le p \le \infty$, $0 < h < 2\pi/\sigma$, $|\xi| = 1$, and $P_\sigma \in \mathcal{B}_p^\sigma$. Then*

$$\|D^{n,\xi} P_\sigma\|_p \le \left(\frac{\sigma}{2\sin(\sigma h/2)}\right)^n \|\Delta_{h\xi}^n P_\sigma\|_p,$$

where

$$D^{n,\xi} f(x) = \mathcal{F}^{-1}((ix,\xi)^n \widehat{f}(x)).$$

Corollary 10 *In terms of Lemma 9, we have*

$$\|P_\sigma\|_{\dot{W}_p^n} \le C\sigma^n \omega_n(P_\sigma, \sigma^{-1})_p,$$

where C does not depend on P_σ.

Proof. By Lemma 9, for any $1 \le j \le d$, we have that

$$\left\|\frac{\partial^n}{\partial x_j^n} P_\sigma\right\|_p \le C\sigma^n \|\Delta_{\sigma^{-1}e_j}^n P_\sigma\|_p \le C\sigma^n \omega_n(P_\sigma, \sigma^{-1})_p,$$

which obviously implies (4). $\Diamond$

We need several basic properties of the modulus of smoothness (see, e.g., [21, Ch. 4]).

Lemma 11 *Let $f, g \in L_p$, $1 \le p \le \infty$, and $n \in \mathbb{N}$. Then for any $\delta > 0$, we have*

(i) $\omega_n(f+g, \delta)_p \le \omega_n(f,\delta)_p + \omega_n(g,\delta)_p$;

(ii) $\omega_n(f,\delta)_p \le 2^n \|f\|_p$;

(iii) $\omega_n(f, \lambda\delta)_p \le (1+\lambda)^n \omega_n(f,\delta)_p$, $\lambda > 0$.

Let us also recall the Jackson-type theorem in L_p (see, e.g., [21, Theorem 5.2.1 (7)] or [26, 5.3.2]).

Lemma 12 *Let $f \in L_p$, $1 \le p \le \infty$, $n \in \mathbb{N}$, and $\sigma > 0$. Then there exists $P_\sigma \in \mathcal{B}_p^\sigma$ such that*

$$\|f - P_\sigma\|_p \le C\omega_n(f, 1/\sigma)_p, \tag{15}$$

where C is a constant independent of f and P_σ.

5 Main results

Our main results are based on the following lemma.

Lemma 13 *Let $1 \le p \le \infty$, $n \in \mathbb{N}$, $\varphi \in (\mathcal{B} \cup \mathcal{L}_2) \cap L_p$, and $\widetilde{\varphi} \in L_q$, $1/p + 1/q = 1$. Suppose that the functions φ and $\widetilde{\varphi}$ are strictly compatible and there exists a constant $c = c(n,p,d,\varphi,\widetilde{\varphi})$ such that*

$$\left\|\sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}\right\|_p \le c\|f\|_p. \tag{16}$$

Then

$$\left\|f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}\right\|_p \le C\omega_n\left(f, \|M^{-j}\|\right)_p, \tag{17}$$

where C does not depend on f and j.

Proof. Let $g \in L_p \cap L_2$ be such that

$$\|f - g\|_p \le \omega_n \left(f, \|M^{*-j}\|\right)_p. \tag{18}$$

Using (16) and (18), we have

$$\begin{aligned} \left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p &\le \|f-g\|_p + \left\| g - \sum_{k\in\mathbb{Z}^d} \langle g, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p + \left\| \sum_{k\in\mathbb{Z}^d} \langle g - f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \\ &\le (1+c)\omega_n \left(f, \|M^{*-j}\|\right)_p + \left\| g - \sum_{k\in\mathbb{Z}^d} \langle g, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p. \end{aligned} \tag{19}$$

By Lemma 12, for any $n \in \mathbb{N}$ and $g \in L_p \cap L_2$ there exists a function $J_n(g) : \mathbb{R}^d \to \mathbb{C}$ such that $\operatorname{supp} \widehat{J_n(g)} \subset \{|\xi| < \delta^{-1}\|M^{*-j}\|\}$ and

$$\|g - J_n(g)\|_p \le C_1 \omega_n \left(g, \delta^{-1}\|M^{*-j}\|\right)_p, \tag{20}$$

where δ is from Definition 2 and C_1 does not depend on g and j.

By Lemma 8, taking into account that $\langle \widehat{J_n(g)}, \widehat{\widetilde{\varphi}_{jk}}\rangle = \langle J_n(g), \widetilde{\varphi}_{jk}\rangle$ and $\{|\xi| < \delta^{-1}\|M^{*-j}\|\} \subset \{|M^{*-j}\xi| < \delta\}$, we have

$$\sum_{k\in\mathbb{Z}^d} \langle J_n(g), \widetilde{\varphi}_{jk}\rangle \varphi_{jk} = J_n(g). \tag{21}$$

Thus, using (20), (21), and (16), we derive

$$\begin{aligned} \left\| g - \sum_{k\in\mathbb{Z}^d} \langle g, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p &\le \|g - J_n(g)\|_p + \left\| \sum_{k\in\mathbb{Z}^d} \langle g - J_n(g), \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \\ &\le (1+c)\|g - J_n(g)\|_p \le C_2 \omega_n \left(g, \delta^{-1}\|M^{*-j}\|\right)_p. \end{aligned} \tag{22}$$

Next, using Lemma 11 and (18), we get

$$\begin{aligned} \omega_n \left(g, \delta^{-1}\|M^{*-j}\|\right)_p &\le (1+\delta^{-1})^n \omega_n \left(g, \|M^{*-j}\|\right)_p \\ &\le C_3 \left(\|f-g\| + \omega_n \left(f, \|M^{*-j}\|\right)_p\right) \le 2C_3 \omega_n \left(f, \|M^{*-j}\|\right)_p \\ &= C_4 \omega_n \left(f, \|M^{-j}\|\right)_p. \end{aligned} \tag{23}$$

Finally, combining (19), (22), and (23), we obtain (17). $\diamond$

The following statement is a multivariate analogue of Theorem 7 in [25]. It is also a counterpart of Proposition 4 in some sense.

Theorem 14 *Let $f \in L_p$, $1 < p < \infty$, and $n \in \mathbb{N}$. Suppose $\varphi \in \mathcal{B}$, $\widetilde{\varphi} \in \mathcal{B} \cup \mathcal{L}_q$, $1/p + 1/q = 1$, and φ and $\widetilde{\varphi}$ are strictly compatible. Then*

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \le C \omega_n \left(f, \|M^{-j}\|\right)_p, \tag{24}$$

where C does not depend on f and j.

Proof. The proof follows from Lemma 13 and Lemma 7. $\diamond$

Remark 15 By (6), it is easy to see that in (24) and in further similar estimates, the modulus $\omega_n\left(f, \|M^{-j}\|\right)_p$ can be replaced by $\omega_n\left(f, \vartheta^{-j}\right)_p$, where ϑ is a positive number smaller in modulus than any eigenvalue of M. Note that $\|M^{-1}\|$ may be essentially bigger than ϑ^{-1}.

In the following result, we give an analogue of Theorem 14 for the limiting cases $p = 1$ and $p = \infty$.

Theorem 14′ *Let $f \in L_p$, $p = 1$ or $p = \infty$, and $n \in \mathbb{N}$. Suppose the functions φ and $\widetilde{\varphi}$ are strictly compatible and*

(i) *$\varphi \in \mathcal{B} \cap L_1$ and $\widetilde{\varphi} \in \mathcal{L}_\infty$ in the case $p = 1$;*

(ii) *$\varphi \in \mathcal{L}_\infty$ and $\widetilde{\varphi} \in L_1$ in the case $p = \infty$.*

Then

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \le C\omega_n\left(f, \|M^{-j}\|\right)_p,$$

where C does not depend on f and j.

Proof. The proof follows from Lemmas 13 and 7′. ◇

Remark 16 *It is easy to see that Theorem 14′ is valid if we replace the condition $\varphi \in \mathcal{B} \cap L_1$ by $\varphi \in \mathcal{L}_2$.*

The next theorem is the main result of the paper. This result essentially extends the classes of functions φ and $\widetilde{\varphi}$ from Theorem 14 and can be considered as a counterpart of Proposition 1.

Theorem 17 *Let $f \in L_p$, $1 < p < \infty$, and $n \in \mathbb{N}$. Suppose $\varphi \in \mathcal{B}$, $\operatorname{supp}\widehat{\varphi} \subset B_{1-\varepsilon}$ for some $\varepsilon \in (0,1)$, $\widehat{\varphi} \in C^{n+d+1}(B_\delta)$ for some $\delta > 0$; $\widetilde{\varphi} \in \mathcal{B} \cup \mathcal{L}_q$, $1/p + 1/q = 1$, $\widehat{\widetilde{\varphi}} \in C^{n+d+1}(B_\delta)$ and $D^\beta(1 - \widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}})(\mathbf{0}) = 0$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] < n$. Then*

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \le C\omega_n\left(f, \|M^{-j}\|\right)_p, \tag{25}$$

where C does not depend on f and j.

Proof. First, let us prove that for any $f \in W_p^n$

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \le C_0 \|f\|_{\dot{W}_p^n} \|M^{-j}\|^n, \tag{26}$$

where C_0 does not depend on f and j.

Choose $0 < \delta' < \delta''$ such that $\widehat{\varphi}(\xi) \ne 0$ on $\{|\xi| \le \delta'\}$ and $\delta' \le \delta$. Set

$$F(\xi) = \begin{cases} \dfrac{1 - \overline{\widehat{\varphi}(\xi)}\widehat{\widetilde{\varphi}}(\xi)}{\overline{\widehat{\varphi}(\xi)}} & \text{if } |\xi| \le \delta', \\ 0 & \text{if } |\xi| \ge \delta'' \end{cases}$$

and extend this function such that $F \in C^{n+d+1}(\mathbb{R}^d)$. Define $\widetilde{\psi}$ by $\widehat{\widetilde{\psi}} = F$. Obviously, the function $\widetilde{\psi}$ is continuous and $\widetilde{\psi}(x) = O(|x|^{-\gamma})$ as $|x| \to \infty$, where $\gamma > n + d$.

Since $\widetilde{\psi} \in \mathcal{B}$ and $\widetilde{\varphi} \in \mathcal{L}_q \cup \mathcal{B}$, by Lemma 7, we have

$$\Big\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk} + \widetilde{\psi}_{jk}\rangle \varphi_{jk} \Big\|_p \le \Big\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_p + \Big\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{jk}\rangle \varphi_{jk} \Big\|_p \le C_1 \|f\|_p.$$

Now, taking into account that $\overline{\widehat{\varphi}(\xi)}(\widehat{\widetilde{\varphi}}(\xi) + \widehat{\widetilde{\psi}}(\xi)) = 1$ whenever $|\xi| \le \delta'$, we obtain from Lemma 13 that for every $f \in W_p^n$

$$\Big\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{ik} + \widetilde{\psi}_{jk}\rangle \varphi_{jk} \Big\|_p \le C_2 \omega_n \left(f, \|M^{-j}\|\right)_p \le C_3 \|f\|_{\dot{W}_p^n} \|M^{-j}\|^n.$$

Thus, to prove (26), it remains to verify that for $f \in W_p^n$

$$\Big\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{jk}\rangle \varphi_{jk} \Big\|_p \le C_4 \|f\|_{\dot{W}_p^n} \|M^{-j}\|^n. \tag{27}$$

Let $k \in \mathbb{Z}^d$, $z \in [-1/2, 1/2]^d - k$, $y = M^{-j} z$. Since $D^\beta \widehat{\widetilde{\psi}}(0) = 0$ whenever $[\beta] < n$, we have

$$\int_{\mathbb{R}^d} y^\alpha \widetilde{\psi}_{jk}(y)\, dy = 0, \quad j \in \mathbb{Z}, \quad \alpha \in \mathbb{Z}_+^d, \quad [\alpha] < n.$$

Hence, due to Taylor's formula with the integral remainder,

$$\begin{aligned}
|\langle f, \widetilde{\psi}_{jk}\rangle| &= \Big| \int_{\mathbb{R}^d} f(x) \overline{\widetilde{\psi}_{jk}(x)}\, dx \Big| \\
&= \Big| \int_{\mathbb{R}^d} \overline{\widetilde{\psi}_{jk}(x)} \Big(\sum_{\nu=0}^{n-1} \frac{1}{\nu!} \left((x_1 - y_1)\partial_1 + \dots + (x_d - y_d)\partial_d\right)^\nu f(y) \\
&\qquad + \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!} \Big((x_1 - y_1)\partial_1 + \dots + (x_d - y_d)\partial_d\Big)^n f(y + t(x-y))\, dt \Big)\, dx \Big| \\
&\le \int_{\mathbb{R}^d} dx\, |x-y|^n |\widetilde{\psi}_{jk}(x)| \int_0^1 \sum_{[\beta]=n} |D^\beta f(y + t(x-y))|\, dt.
\end{aligned}$$

From this, using Hölder's inequality and taking into account that

$$|x - y|^n \le \|M^{-j}\|^n |M^j x - z|^n,$$

$$|\widetilde{\psi}_{jk}(x)| \le \frac{C_5 m^{j/2}}{(1 + |M^j x + k|)^\gamma} \le \frac{C_6 m^{j/2}}{(1 + |M^j x - z|)^\gamma},$$

we obtain

$$
\begin{aligned}
|\langle f,\widetilde{\psi}_{jk}\rangle| &\le C_6 m^{j/2}\|M^{-j}\|^n \int\limits_{\mathbb{R}^d} dx \frac{|M^j x - z|^n}{(1+|M^j x - z|)^\gamma} \int\limits_0^1 \sum_{[\beta]=n} |D^\beta f(y+t(x-y))|\,dt \\
&\le C_6 m^{j/2}\|M^{-j}\|^n \Bigg(\int\limits_{\mathbb{R}^d} \frac{|M^j x - z|^n}{(1+|M^j x - z|)^\gamma}\,dx\Bigg)^{1/q} \\
&\qquad \times \Bigg(\int\limits_{\mathbb{R}^d} \frac{|M^j x - z|^n dx}{(1+|M^j x - z|)^\gamma}\Bigg(\int\limits_0^1 \sum_{[\beta]=n} |D^\beta f(y+t(x-y))|\,dt\Bigg)^p\Bigg)^{\frac{1}{p}} \\
&= C_6 m^{\frac{j}{2}-\frac{j}{q}}\|M^{-j}\|^n \Bigg(\int\limits_{\mathbb{R}^d} \frac{|x - z|^n\,dx}{(1+|x - z|)^\gamma}\Bigg)^{1/q} \qquad (28) \\
&\qquad \times \Bigg(\int\limits_{\mathbb{R}^d} \frac{|M^j x - z|^n dx}{(1+|M^j x - z|)^\gamma}\Bigg(\int\limits_0^1 \sum_{[\beta]=n} |D^\beta f(y+t(x-y))|\,dt\Bigg)^p\Bigg)^{\frac{1}{p}} \\
&\le C_7 m^{\frac{j}{2}-\frac{j}{q}}\|M^{-j}\|^n \Bigg(\int\limits_{\mathbb{R}^d} \frac{|M^j (x-y)|^n dx}{(1+|M^j (x-y)|)^\gamma}\int\limits_0^1 \sum_{[\beta]=n} |D^\beta f(y+t(x-y))|^p\,dt\Bigg)^{\frac{1}{p}} \\
&= C_7 m^{\frac{j}{2}-\frac{j}{q}}\|M^{-j}\|^n \Bigg(\int\limits_{\mathbb{R}^d} \frac{|M^j u|^n du}{(1+|M^j u|)^\gamma}\int\limits_0^1 \sum_{[\beta]=n} |D^\beta f(y+tu)|^p\,dt\Bigg)^{\frac{1}{p}}.
\end{aligned}
$$

Next, it follows from (28) and Lemma 5 that

$$
\begin{aligned}
\Bigg\|\sum_{k\in\mathbb{Z}^d} \langle f,\widetilde{\psi}_{jk}\rangle\varphi_{jk}\Bigg\|_p^p &\le C_8 m^{p(\frac{j}{2}-\frac{j}{p})}\sum_{k\in\mathbb{Z}^d} |\langle f,\widetilde{\psi}_{jk}\rangle|^p \\
&= C_8 m^{p(\frac{j}{2}-\frac{j}{p})}\sum_{k\in\mathbb{Z}^d} \int\limits_{[-1/2,1/2]^d-k} dz |\langle f,\widetilde{\psi}_{jk}\rangle|^p \\
&\le C_9\|M^{-j}\|^{pn}\int\limits_{\mathbb{R}^d} dz \int\limits_{\mathbb{R}^d} \frac{|M^j u|^n du}{(1+|M^j u|)^\gamma}\int\limits_0^1 \sum_{[\beta]=n} |D^\beta f(M^{-j}z+tu)|^p\,dt \\
&= C_9\|M^{-j}\|^{pn}\int\limits_{\mathbb{R}^d} dz \int\limits_{\mathbb{R}^d} \frac{|u|^n du}{(1+|u|)^\gamma}\int\limits_0^1 \sum_{[\beta]=n} |D^\beta f(z+tM^{-j}u)|^p\,dt \\
&= C_9\|M^{-j}\|^{pn}\int\limits_{\mathbb{R}^d} \frac{|u|^n du}{(1+|u|)^\gamma}\int\limits_0^1 dt \int\limits_{\mathbb{R}^d} \sum_{[\beta]=n} |D^\beta f(z+tM^{-j}u)|^p\,dz \\
&= C_9\|M^{-j}\|^{pn}\int\limits_{\mathbb{R}^d} \frac{|u|^n du}{(1+|u|)^\gamma}\int\limits_0^1 dt \int\limits_{\mathbb{R}^d} \sum_{[\beta]=n} |D^\beta f(z)|^p\,dz \\
&\le C_{10}\|M^{-j}\|^{pn}\|f\|^p_{\dot W^n_p}.
\end{aligned}
$$

This implies (27) and, therefore, (26).

Now let us prove inequality (25). By Lemma 12, there exists $P_\sigma \in \mathcal{B}_p^\sigma$, $\sigma = 1/\|M^{-j}\|$, such that

$$\|f - P_\sigma\|_p \le C_{11}\omega_n(f, 1/\sigma)_p. \tag{29}$$

Using Lemma 7, we obtain

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \le C_{12}\|f - P_\sigma\|_p + \left\| P_\sigma - \sum_{k\in\mathbb{Z}^d} \langle P_\sigma, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p. \tag{30}$$

Next, by (26) and Corollary 10, we derive

$$\left\| P_\sigma - \sum_{k\in\mathbb{Z}^d} \langle P_\sigma, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \le C_0 \|P_\sigma\|_{\dot{W}_p^n} \|M^{-j}\|^n \le C_{13}\omega_n\left(P_\sigma, 1/\sigma\right)_p. \tag{31}$$

Finally, combining (29)–(31) and using Lemma 11, we get

$$\begin{aligned}\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p &\le C_{14}\left(\|f - P_\sigma\|_p + \omega_n(P_\sigma, 1/\sigma)_p\right) \\ &\le C_{15}\left(\|f - P_\sigma\|_p + \omega_n(f, 1/\sigma)_p\right) \le C\omega_n\left(f, \|M^{-j}\|\right)_p,\end{aligned}$$

which implies (25). $\diamond$

In the following result, we give an analogue of Theorem 17 for the limiting cases $p = 1$ and $p = \infty$.

Theorem 17′ *Let $f \in L_p$, $p = 1$ or $p = \infty$, and $n \in \mathbb{N}$. Suppose the functions φ and $\widetilde{\varphi}$ are such that $\widehat{\varphi}, \widehat{\widetilde{\varphi}} \in C^{n+d+1}(B_\delta)$ for some $\delta > 0$, $D^\beta(1 - \widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}})(\mathbf{0}) = 0$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] < n$, $\varphi \in \mathcal{B}$,* supp $\widehat{\varphi} \subset B_{1-\varepsilon}$ *for some $\varepsilon \in (0,1)$, and*

(i) $\varphi \in L_1$ *and* $\widetilde{\varphi} \in \mathcal{L}_\infty$ *in the case* $p = 1$*;*

(ii) $\varphi \in \mathcal{L}_\infty$ *and* $\widetilde{\varphi} \in L_1$ *in the case* $p = \infty$*.*

Then

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_p \le C\omega_n\left(f, \|M^{-j}\|\right)_p,$$

where C does not depend on f and j.

Proof. The proof is similar to the proof of Theorem 17. We only note that one needs to use Lemmas 7′ and 13′ instead of Lemmas 7 and 13. At that, in the case $p = \infty$, using the first inequality in (28) and Lemma 5, we get

$$\begin{aligned}\left\| \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{jk}\rangle \varphi_{jk} \right\|_\infty &\le C_1 m^{\frac{j}{2}} \sup_{k\in\mathbb{Z}^d} |\langle f, \widetilde{\psi}_{jk}\rangle| \\ &\le C_2 m^j \|M^{-j}\|^n \|f\|_{\dot{W}_\infty^n} \sup_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \frac{|M^j x - z|^n}{(1 + |M^j x - z|)^\gamma}\, dx \\ &\le C_3 \|M^{-j}\|^n \|f\|_{\dot{W}_\infty^n}.\end{aligned}$$

Theorem 17′ is proved. $\diamond$

Remark 18 *Note that any function $\widetilde{\varphi} \in L_p^0$, $1 \le p \le \infty$, satisfies assumptions $\widetilde{\varphi} \in \mathcal{L}_p \subset L_p$ and $\widehat{\widetilde{\varphi}} \in C^{n+d+1}(B_\delta)$, and hence can be used in Theorems 17 and 17′.*

6 Special Cases

I. Let us start from the classical multivariate Kotelnikov type decomposition that can be obtain by using the function

$$\operatorname{sinc}(x) := \prod_{\nu=1}^{d} \frac{\sin(\pi x_\nu)}{\pi x_\nu}, \quad x \in \mathbb{R}^d.$$

In what follows, we restrict ourselves by the case $1 < p < \infty$.

Proposition 19 *Let $f \in L_p$, $1 < p < \infty$, and let U be a bounded measured subset of $\mathbb{R}^d$. Then*

$$\Big\| f - \sum_{k\in\mathbb{Z}^d} \frac{m^j}{\operatorname{mes} U} \int\limits_{M^{-j}U} f(-M^{-j}k + t)\, dt \operatorname{sinc}(M^j \cdot + k) \Big\|_p \le C\omega_1(f, \|M^{-j}\|)_p, \tag{32}$$

where the constant C does not depend on f and j. If, in addition, U is symmetric with respect to the origin, then in (32) the modulus of continuity $\omega_1(f, \|M^{-j}\|)_p$ can be replaced by the second order modulus of smoothness $\omega_2(f, \|M^{-j}\|)_p$.

Proof. We use Theorem 17 for

$$\varphi(x) = \operatorname{sinc}(x) \quad \text{and} \quad \widetilde{\varphi}(x) = \frac{1}{\operatorname{mes} U} \chi_U(x).$$

Then, taking into account that $\widehat{\widetilde{\varphi}}(\mathbf{0}) = 1$ and

$$\langle f, \widetilde{\varphi}_{jk} \rangle = \frac{m^{j/2}}{\operatorname{mes} U} \int\limits_{M^{-j}U} f(-M^{-j}k + t)\, dt,$$

we can verify that all assumptions of Theorem 17 are satisfied with $n = 1$, which provides inequality (32).

Now, let U be symmetric with respect to the origin. In this case,

$$\frac{\partial}{\partial x_j}(1 - \widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}})(\mathbf{0}) = -\frac{\partial \overline{\widehat{\widetilde{\varphi}}}}{\partial x_j}(\mathbf{0}) = \frac{2\pi i}{\operatorname{mes} U} \int_U x_j dx = 0, \quad 1 \le j \le d.$$

Therefore, all assumptions of Theorem 17 are satisfied with $n = 2$. $\diamond$

Remark 20 *Relation (32) gives a general answer to the question posed in [3] concerning approximation properties of the sampling series given by*

$$\sum_{k\in\mathbb{Z}^d} \frac{m^j}{\operatorname{mes} U} \int\limits_{M^{-j}U} f(-M^{-j}k + t)\, dt \operatorname{sinc}(M^j \cdot + k)$$

in the spaces L_p for $1 < p < \infty$.

Remark 21 *It follows from Theorem 17′ that Proposition 19 is valid for all $f \in L_p$, $1 \le p \le \infty$, if we replace $\operatorname{sinc}(x)$ by $\operatorname{sinc}^2(x)$ in (32). In particular, this gives an improvement of estimate (5). The same conclusion holds for all propositions presented below.*

II. Now let us show that using of an appropriate linear combination of the function $\operatorname{sinc}(x)$ rather than this function itself can provide better rates of the approximation by the corresponding sampling operator.

Proposition 22 *Let $f \in L_p$, $1 < p < \infty$, $n \in \mathbb{N}$, and let U be a bounded measured subset in $\mathbb{R}^d$. Then there exists a finite set of numbers $\{a_l\}_{l\in\mathbb{Z}^d} \subset \mathbb{C}$ depending only on d, n, and U such that for*

$$\varphi(x) = \sum_l a_l \operatorname{sinc}(x+l) \tag{33}$$

we have

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \frac{m^j}{\operatorname{mes} U} \int\limits_{M^{-j}U} f(-M^{-j}k+t)\,dt\,\varphi(M^j\cdot+k) \right\|_p \le C\omega_n(f, \|M^{-j}\|)_p, \tag{34}$$

where the constant C does not depend on f and j.

Proof. Let $\widetilde{\varphi}(x) = \frac{1}{\operatorname{mes} U}\chi_U(x)$. Find complex numbers c_α, $\alpha \in \mathbb{Z}^d_+$, $[\alpha] < n$, satisfying

$$c_{\mathbf{0}} = 1, \quad \sum_{\mathbf{0}\le\alpha\le\beta} \binom{\beta}{\alpha} \overline{D^{\beta-\alpha}\widehat{\widetilde{\varphi}}(\mathbf{0})} c_\alpha = 0 \quad \forall \beta \in \mathbb{Z}^d_+,\ \mathbf{0} < [\beta] < n$$

and set

$$T(\xi) = \sum_{\mathbf{0}\le[\alpha]\le n} c_\alpha \prod_{j=1}^d g_{\alpha_j}(\xi_j), \tag{35}$$

where g_k is a trigonometric polynomial such that $\frac{d^l g_k}{dt^l}(0) = \delta_{kl}$ for all $l = 0,\dots,k$. It is not difficult to deduce explicit recursive formulas for finding such polynomials (see, e.g., [19, Sec. 3.4] or [24]). Obviously, $D^\alpha T(\mathbf{0}) = c_\alpha$ and $D^\alpha(T\cdot\widehat{\varphi})(\mathbf{0}) = D^\alpha(T\cdot\chi_{[-1/2,1/2]^d})(\mathbf{0}) = c_\alpha$ for all $\alpha \in \mathbb{Z}^d_+$, $[\alpha] < n$. If now $T(\xi) = \sum_l a_l e^{2\pi i(l,\xi)}$, then setting $\varphi(x) = \sum_l a_l \operatorname{sinc}(x+l)$, we obtain that $D^\beta(1 - \widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}})(\mathbf{0}) = 0$ for all $\beta \in \mathbb{Z}^d_+$, $[\beta] < n$. Thus, due to Theorem 17, we have inequality (34). $\diamond$

Let us write explicit formulas for the function (33) and for the polynomial T given by (35) in the cases $d = 1, 2$ and $n = 4$.

Example 1. Let first $d = 1$, $U = [-1/2, 1/2]$, and $M = 2$. Then

$$\widehat{\widetilde{\varphi}}(0) = 1,\ \widehat{\widetilde{\varphi}}'(0) = 0,\ \widehat{\widetilde{\varphi}}''(0) = -\frac{\pi^2}{3},\ \widehat{\widetilde{\varphi}}'''(0) = 0,$$

which yields $c_0 = 1,\ c_1 = 0,\ c_2 = \frac{\pi^2}{3},\ c_3 = 0$, and

$$T(\xi) = 1 + \frac{\pi^2}{3} g_2(\xi),$$

where

$$g_2(u) = -\frac{1}{8\pi^2}(2 - 5e^{2\pi iu} + 4e^{4\pi iu} - e^{6\pi iu}).$$

Hence

$$\varphi(x) = \frac{11}{12}\operatorname{sinc}(x) + \frac{5}{24}\operatorname{sinc}(x+1) - \frac{1}{6}\operatorname{sinc}(x+2) + \frac{1}{24}\operatorname{sinc}(x+3)$$

and, by Proposition 22, we have

$$\left\| f - \sum_{k\in\mathbb{Z}} 2^j \int\limits_{[-2^{-j-1},2^{-j-1}]} f(-2^{-j}k+t)\,dt\,\varphi(2^j\cdot+k) \right\|_p \le C\omega_4(f, 2^{-j})_p.$$

Example 2. Now let $d = 2$ and $U = [-1/2, 1/2]^2$. Simple calculations show that in this case one has

$$T(\xi) = g_0(\xi_1)g_0(\xi_2) + \frac{\pi^2}{3}g_2(\xi_1)g_0(\xi_2) + \frac{\pi^2}{3}g_0(\xi_1)g_2(\xi_2) = 1 + \frac{\pi^2}{3}(g_2(\xi_1) + g_2(\xi_2)),$$

and, therefore,

$$\begin{aligned}\varphi(x_1, x_2) = \frac{5}{6}\operatorname{sinc} x_1 \operatorname{sinc} x_2 &+ \frac{\operatorname{sinc} x_2}{24}(5\operatorname{sinc}(x_1+1) - 4\operatorname{sinc}(x_1+2) + \operatorname{sinc}(x_1+3)) \\ &+ \frac{\operatorname{sinc} x_1}{24}(5\operatorname{sinc}(x_2+1) - 4\operatorname{sinc}(x_2+2) + \operatorname{sinc}(x_2+3)).\end{aligned}$$

It follows from (34) that

$$\left\| f - \sum_{k\in\mathbb{Z}^2} m^j \int\limits_{M^{-j}[-1/2,1/2]^2} f(-M^{-j}k + t)\, dt\, \varphi(M^j \cdot + k) \right\|_p \le C\omega_4(f, \|M^{-j}\|)_p.$$

Example 3. Similarly to Example 2, if $d = 2$ and $U = B_1$, taking into account that

$$\widetilde{\varphi}(x) = \frac{\Gamma(1 + d/2)}{\pi^{d/2}}\chi_{B_1}(x), \quad \widehat{\widetilde{\varphi}}(\xi) = \Gamma(1 + d/2)\frac{J_{d/2}(2\pi|\xi|)}{(\pi|\xi|)^{d/2}},$$

where J_λ is the Bessel function of the first kind of order λ, we obtain

$$T(\xi_1, \xi_2) = 1 - \pi^2(g_2(\xi_1) + g(\xi_2))$$

and

$$\begin{aligned}\varphi(x_1, x_2) = \frac{3}{2}\operatorname{sinc} x_1 \operatorname{sinc} x_2 &- \frac{\operatorname{sinc} x_2}{8}(5\operatorname{sinc}(x_1+1) - 4\operatorname{sinc}(x_1+2) + \operatorname{sinc}(x_1+3)) \\ &- \frac{\operatorname{sinc} x_1}{8}(5\operatorname{sinc}(x_2+1) - 4\operatorname{sinc}(x_2+2) + \operatorname{sinc}(x_2+3)).\end{aligned}$$

Hence,

$$\left\| f - \sum_{k\in\mathbb{Z}^2} \frac{m^j}{\pi} \int\limits_{M^{-j}B_1} f(-M^{-j}k + t)\, dt\, \varphi(M^j \cdot + k) \right\|_p \le C\omega_4(f, \|M^{-j}\|)_p.$$

III. Another improvement of the estimate given in Proposition 19 can be obtained by using an appropriate linear combination of the averaging operator rather than linear combinations of the function $\varphi(x) = \operatorname{sinc}(x)$. Thus, the following estimate is a trivial consequence of Proposition 22:

$$\left\| f - \sum_{k\in\mathbb{Z}^d}\sum_{l} a_l \frac{m^j}{\operatorname{mes} U} \int\limits_{M^{-j}(U+l)} f(-M^{-j}k + t)\, dt\, \operatorname{sinc}(M^j \cdot + k) \right\|_p \le C\omega_n(f, \|M^{-j}\|)_p, \qquad (36)$$

where a_l is the l-th coefficient of the polynomial T defined by (35) and the constant C does not depend on f and j.

Now let us obtain an analog of (36) for other functions φ.

Proposition 23 *Let $f \in L_p$, $1 < p < \infty$, $n \in \mathbb{N}$, and let U be a bounded measured subset in $\mathbb{R}^d$. Suppose the function φ satisfies conditions of Theorem 17. Then there exists a finite set of numbers $\{b_l\}_{l\in\mathbb{Z}^d} \subset \mathbb{C}$ depending only on d, n, U, and φ such that*

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \sum_l b_l \frac{m^j}{\operatorname{mes} U} \int\limits_{M^{-j}(U-l)} f(-M^{-j}k + t)\, dt\, \varphi(M^j \cdot +k) \right\|_p \le C\omega_n(f, \|M^{-j}\|)_p, \tag{37}$$

where the constant C does not depend on f and j.

Proof. Find complex numbers c'_α, $\alpha \in \mathbb{Z}^d_+$, $[\alpha] < n$, satisfying

$$c'_{\mathbf{0}} = 1, \qquad \sum_{\mathbf{0}\le\alpha\le\beta} \binom{\beta}{\alpha} \overline{D^{\beta-\alpha}\widehat{\varphi}(\mathbf{0})} c'_\alpha = 0 \quad \forall \beta \in \mathbb{Z}^d_+,\ \mathbf{0} < [\beta] < n.$$

Next, for $\widetilde{\varphi}(x) = \frac{1}{\operatorname{mes} U}\chi_U(x)$, we find complex numbers c_α, $\alpha \in \mathbb{Z}^d$, $[\alpha] < n$, satisfying

$$c_{\mathbf{0}} = 1, \qquad \sum_{\mathbf{0}\le\alpha\le\beta} \binom{\beta}{\alpha} \overline{D^{\beta-\alpha}\widehat{\widetilde{\varphi}}(\mathbf{0})} c_\alpha = c'_\beta \quad \forall \beta \in \mathbb{Z}^d_+,\ \mathbf{0} < [\beta] < n,$$

and set

$$Q(\xi) = \sum_{\mathbf{0}\le[\alpha]\le n} c_\alpha \prod_{j=1}^d g_{\alpha_j}(\xi_j), \tag{38}$$

where g_α is as in (35). If now $Q(\xi) = \sum\limits_l b_l e^{2\pi i(l,\xi)}$, then setting

$$\widetilde{\psi}(x) = \sum_l b_l \widetilde{\varphi}(x+l),$$

we obtain that $D^\beta(1 - \overline{\widehat{\widetilde{\varphi}}}\widehat{\widetilde{\psi}})(\mathbf{0}) = 0$ for all $\beta \in \mathbb{Z}^d_+$, $[\beta] < n$. Thus, due to Theorem 17, we have (37). $\diamond$

Example 4. In this example, we consider radial functions $\varphi(x) = R_\delta(x)$ given by the Bochner-Riesz type kernel

$$R_\delta(x) := \frac{\Gamma(1+\delta)}{\pi^\delta} \frac{J_{d/2+\delta}(2\pi|x|)}{|x|^{d/2+\delta}}.$$

Some results on approximation properties of sampling expansions generated by radial functions with a diagonal matrix M can be found in [6]. Let also $d = 2$, $n = 4$, and $U = B_1$. By analogy with the above examples, we can compute that the polynomial $Q(\xi)$ given in (38) has the form

$$Q(\xi) = 1 + (2\delta - \pi^2)(g_2(\xi_1) + g_2(\xi_2))$$

and, therefore, by (37), we derive

$$\left\| f - \sum_{k\in\mathbb{Z}^2} \sum_{l_1=0}^3 \sum_{l_2=0}^3 \frac{b_{l_1,l_2} m^j}{\pi} \int\limits_{M^{-j}(B_1-(l_1,l_2))} f(-M^{-j}k + t)\, dt\, R_\delta(M^j \cdot +k) \right\|_p \le C\omega_4(f, \|M^{-j}\|)_p,$$

where

$$b_{0,0} = 1 - \frac{2\delta - \pi^2}{2\pi^2}, \quad b_{1,0} = b_{0,1} = \frac{5(2\delta - \pi^2)}{8\pi^2},$$

$$b_{2,0} = b_{0,2} = \frac{-(2\delta - \pi^2)}{2\pi^2}, \quad b_{3,0} = b_{0,3} = \frac{2\delta - \pi^2}{8\pi^2},$$

and

$$b_{1,1} = b_{1,2} = b_{2,1} = 0.$$

IV. Finally, from Theorem 14, we obtain the following result related to the classical Kotelnikov decomposition.

Proposition 24 *Let $f \in L_p$, $1 < p < \infty$, $\widehat{f}$ be locally summable, and $n \in \mathbb{N}$. Then*

$$\left\| f - m^{j/2} \sum_{k\in\mathbb{Z}^d} \int\limits_{[-1/2,1/2]^d} \widehat{f}(M^{*j}\xi) e^{-2\pi i(k,\xi)} d\xi \; \mathrm{sinc}(M^j \cdot +k) \right\|_p \le C\omega_n \left(f, \|M^{-j}\|\right)_p, \tag{39}$$

where the constant C does not depend on f and j.

Proof. We apply Theorem 14 for $\varphi(x) = \widetilde{\varphi}(x) = \mathrm{sinc}(x)$. Since $\widehat{\mathrm{sinc}}(\xi) = \chi_{[-1/2,1/2]^d}(\xi)$ and $\widehat{\mathrm{sinc}_{jk}}(\xi) = m^{-j/2} e^{2\pi i(k, M^{*-j}\xi)} \chi_{[-1/2,1/2]^d}(M^{*-j}\xi)$, we have

$$\langle f, \mathrm{sinc}_{jk}\rangle = \langle \widehat{f}, \widehat{\mathrm{sinc}_{jk}}\rangle = m^{j/2} \int\limits_{\mathbb{R}^d} \widehat{f}(M^{*j}\xi) \overline{\widehat{\widetilde{\varphi}}(M^{*j}\xi)} d\xi = \int\limits_{[-1/2,1/2]^d} \widehat{f}(M^{*j}\xi) e^{-2\pi i(k,\xi)} d\xi,$$

which, by Theorem 14, proves the proposition. $\diamond$

Finally, let us note that (39) can be also written in the following form

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \mathcal{F}^{-1}(\chi_{M^{*j}[-1/2,1/2]^d} \widehat{f})(M^{*-j}k) \; \mathrm{sinc}(M^j \cdot +k) \right\|_p \le C\omega_n \left(f, \|M^{-j}\|\right)_p. \tag{40}$$

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