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Asymptotic expansions of the Helmholtz equation solutions using approximations of the Dirichlet to Neumann operator

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ABSTRACT

This paper is concerned with the asymptotic expansions of the amplitude of the solution of the Helmholtz equation. The original expansions were obtained using a pseudo-differential decomposition of the Dirichlet to Neumann operator. This work uses first and second order approximations of this operator to derive new asymptotic expressions of the normal derivative of the total field. The resulting expansions can be used to appropriately choose the ansatz in the design of high-frequency numerical solvers, such as those based on integral equations, in order to produce more accurate approximation of the solutions around the shadow and the deep shadow regions than the ones based on the usual ansatz.

1. Introduction

Studying the Helmholtz equation at the high-frequency regime is fundamental in both the theoretical understanding of the corresponding solutions and the derivation of appropriate numerical schemes. The well-know asymptotic expansions developed by Melrose and Taylor [40] have significantly contributed in this matter and were the key in the design of several high-frequency integral equation methods. Indeed, integral equation methods are very efficient and widely used in the solution of acoustic scattering problems (see e.g. [23,21] and the references therein). However, the resulting linear systems are dense, ill-conditioned and with large size in particular when the frequency increases. Several effective strategies have been proposed to overcome these difficulties [23,21,3,15,20,7–9,36,42,17,5,31,43,45,16,13,14,12]. Despite this significant progress, integral formulations are limited at higher frequencies since the numerical resolution of field oscillations can easily lead to impractical computational times. This is why hybrid numerical methods based on a combination of integral equations and asymptotic methods have found an increasing interest for the solution of high-frequency scattering problems. Indeed, the methodologies developed in this connection that specifically

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concern scattering from a smooth convex obstacle were first introduced in [1,2]. Several other works followed these [19,28,4,18,25,35,30,27,26] and mainly consist of improving and analyzing this kind of numerical algorithms in single and multiple scattering configurations. All these methods are mainly based on construction of an appropriate ansatz for the solution of integral equations in the form of a highly oscillatory function of known phase modulated by an unknown slowly varying envelope, which is expected to generate linear systems quasi-independent of the frequency.

The high-frequency integral equation methods mentioned above use the asymptotic expansions developed in the well-known paper by Melrose and Taylor [40] in the context of convex obstacles. From these expansions, an ansatz is derived and incorporated into integral equations to eliminate the highly oscillatory part of the unknown which usually corresponds to the physical density, normal derivative of the total field, computed on the surface of the obstacles. This surface is decomposed into three regions, the illuminated and shadows regions in addition to the deep shadow one. Each region is then numerically treated differently and the ansatz is set in general on the illuminated one. Although carefully designed, the aforementioned high-frequency integral equation formulations result in ill-conditioned matrices that limit the numerical accuracy of the approximate solutions. One explanation lies in the fact that the rapidly decaying behavior of the unknown density in the deep shadow regions is not incorporated into the approximation spaces as it is not intrinsic to the chosen ansatz. Generally speaking, it is not clear how to extract all the information needed from the leading term in the expansion given in [40], which restricts the construction of the ansatz.

In this paper, we derive new expansions of the normal derivative of the total field using approximations of the Dirichlet to Neumann (DtN) operator. The original expansions employed a pseudo-differential decomposition of the DtN operator, and the related analysis focuses on the behavior of this field around the shadow boundary which leads to a corrected formula for the Kirchhoff approximation around this region [40]. However, it has been shown that these expansions are valid in the entire surface of the obstacles [25,28]. Here, we choose first and second order approximations of the DtN operator of the Bayliss–Turkel type [11,38]. These conditions were designed to deal with the infinite aspect of the computational domain for scattering problems. They were also employed in the design of the On-Surface Radiation Conditions [6]. Other approximations such as those developed in [29] can also be adapted to our analysis without specific difficulties. To obtain these new expansions, we follow a similar procedure to the one given in [40]. Briefly, it consists of first finding the kernel of a certain operator, which allows the computation of its amplitude, and then use the stationary phase method to get the final expansions around the shadow boundary. In this case, we can use some of the results derived by Melrose and Taylor [40] in our analysis. The resulting expansions can then be used to appropriately build an ansatz that contains the expected behavior of the solution in the three regions, namely, the illuminated and the deep shadow regions in addition to the shadow boundaries. This provides an improvement over the usual ansatz that behaves like Kirchhoff approximations, meaning that the corresponding solutions are accurate mostly in the illuminated regions.

This paper is organized as follows. After reviewing the functional setting needed for this analysis, we state the problem and explain our choice, regarding the approximation of the DtN operator, in the second section. The two following sections are, respectively, devoted to the derivation of asymptotic expansions in the context of first and second order approximations of the DtN operator. The last section is reserved for some conclusions.

In this work, we will use the following functional spaces (for more details, see for instance [44,22]). Let U be an open bounded set of $\mathbb{R}^n$.

- $D(U)$: space of smooth test functions with compact support, from U to $\mathbb{R}^n$.
- $D'(U)$: space of distributions.
- $S(\mathbb{R}^n)$: Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$.
- $S'(\mathbb{R}^n)$: space of tempered distributions, which is the dual space of $S(\mathbb{R}^n)$.
- $\mathcal{E}'$: space of compactly supported distributions.

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- OPS^m: space of pseudo-differential operators of order m.
- I^m: space of Fourier integral operators of order m.

We will also use symbols of Hörmander's classes [34,33], we say $p(x,\xi) \in S^m_{\rho,\delta}$ if and only if

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}. \tag{1}$$

In particular we say $p(x,\xi) \in S^m$ if $p(x,\xi) \in S^m_{1,0}$. Note that each $p(x,\xi)$ admits an asymptotic expansion of the form

$$p(x,\xi) \sim \sum_{j\geq 0} p_j(x,\xi) \tag{2}$$

for $|\xi|$ large where the $p_j(x,\xi)$ are homogeneous functions of degree $m-j$ in ξ. Finally, if $p(x,\xi) \in S^m$, we say $p(x,D) \in OPS^m$, where D is the corresponding pseudo-differential operator.

2. Model problem

Consider a convex obstacle $O \subset \mathbb{R}^{n+1}$ such that $B = \partial O \subset \mathbb{R}^{n+1}$ is a hypersurface and let Ω be the exterior domain given by $\Omega = \mathbb{R}^{n+1} \setminus O$. We are interested in solutions of the following wave equation

$$\begin{cases} (\partial_{tt} - \Delta)u(x,t) = 0 \quad \text{in } \Omega \times \mathbb{R}, \\ u(x,t) = -\delta(t - x\cdot\omega) = -u^i(x,t) \quad \text{on } B \times \mathbb{R}, \end{cases} \tag{3}$$

where u^i is the incident wave, ω denotes the incidence direction, δ is the Dirac function, and $u \in D'(B\times\mathbb{R})$ [40]. Defining the function w by

$$w(x,k) = \int e^{ikt} u(x,t)dt, \tag{4}$$

leads to the well-posed problem [46]

$$\begin{cases} (\Delta + k^2)w(x,k) = 0 \quad \text{in } \Omega \times \mathbb{R}, \\ w(x,k) = -e^{ikx\cdot\omega} \quad \text{on } B \times \mathbb{R}, \\ w(x,k) = \mathcal{O}(|x|^{-n/2}) \text{ and } (\partial_{|x|} - ik)w(x,k) = o(|x|^{-n/2}) \text{ for } |x| \to \infty. \end{cases} \tag{5}$$

For each $x \in \partial O = B$, $\mathbf{n}$ denotes the outgoing normal vector. In what follows, we use the notation $w^t = w^s + w^i$ indicating the total field, where w^s is the scattered field solution of the problem (5) and $w^i = e^{ikx\cdot\omega}$. The normal derivative of w^t is written in the form

$$\begin{aligned} \partial_{\mathbf{n}} w^t &= \partial_{\mathbf{n}} w^s + \partial_{\mathbf{n}} w^i \\ &= \mathcal{K}(\omega,x,k)e^{ikx\cdot\omega}. \end{aligned} \tag{6}$$

In addition, we define the Kirchhoff operator (as given in [40]) by

$$Q = (\mathrm{DtN} + (\omega\cdot\mathbf{n})\partial_t)\, F : \ \mathcal{E}'(\mathcal{S}^n \times \mathbb{R}) \to \mathcal{D}'(B\times\mathbb{R}) \tag{7}$$

with $\omega \in \mathcal{S}^n$. Here, $\mathcal{S}^n$ indicates the unit sphere of dimension n and $\mathrm{DtN} \in OPS^1$ [40,32] stands for the Dirichlet to Neumann operator (called the forward Neumann operator in [40])

$$\mathrm{DtN} : \mathcal{E}'(B \times \mathbb{R}) \to D'(B \times \mathbb{R}),\ \mathrm{DtN}\ u^i = -\partial_{\mathbf{n}} u|_{B\times\mathbb{R}}, \tag{8}$$

and F is a Fourier integral operator defined by [40,24]

$$Fu(x,t) = \int\limits_{\mathcal{S}^n \times \mathbb{R}} \delta(t - s - x\cdot\omega)u(\omega, s)d\omega ds. \tag{9}$$

Using a pseudo-differential decomposition of this operator DtN, Melrose and Taylor derived the well-known expansion

$$\partial_{\mathbf{n}} w^t \sim \sum_{p,l=0}^{\infty} k^{2/3-p-2l/3} a_{p,l}(\omega, x)\Psi^{(l)}(k^{1/3} Z(\omega, x))e^{ikx\cdot\omega}, \tag{10}$$

where $\Psi(\tau) \sim \sum_{j=0}^{\infty} c_j \tau^{1-3j}$ as $\tau \to +\infty$, and it is rapidly decreasing in the sense of Schwartz as $\tau \to -\infty$. The real-valued function Z is positive on the illuminated region, negative on the shadow region and vanishes precisely to first order on the shadow boundary [40]. Here, $a_{p,l}$ result from the application of the stationary phase method and the expansion of the symbol of the operator Q [40].

Remark 1. As is mentioned in [40], the first term of the expansion (10) represents the classical Kirchhoff approximation. Indeed, if $\Psi(k^{1/3}Z(\omega, x))$ is replaced by the leading term in its asymptotic expansion

$$\Psi(\tau) \simeq -2i\tau, \text{ for } \tau \to +\infty, \tag{11}$$

and taking $a_{0,0}(\omega, x) = (\mathbf{n}\cdot\omega)/Z(\omega, x)$ in the illuminated region $Z(\omega, x) > 0$ ($\mathbf{n}\cdot\omega < 0$), we obtain

$$\partial_{\mathbf{n}} w^t \simeq k^{2/3} \frac{\mathbf{n}\cdot\omega}{Z(\omega, x)}(-2ik^{1/3}Z(\omega, x))e^{ikx\cdot\omega} = 2ik\mathbf{n}\cdot\omega e^{ikx\cdot\omega}. \tag{12}$$

Generally speaking, the aforementioned high-frequency integral equation methods, based on an ansatz of the form $\partial_{\mathbf{n}} w^t = \eta(x)e^{ikx\cdot\omega}$, although delivering better accuracy than the Kirchhoff approach in the illuminated region, were designed replicating its behavior, and this explains why the decay in the deep shadow region is not observed but somehow forced. This is partly due to the fact that an explicit form of the leading term in the asymptotic expansion (10) is not available. We propose in this work to use approximations of the DtN map to derive new asymptotic expressions of $\partial_{\mathbf{n}} w^t$. This can allow construction of a new ansatz in order to improve the behavior and the accuracy of the solution in the shadow and the deep shadow regions. We use first and second order approximations of the DtN operator given by Bayliss–Turkel [11]

$$\partial_{\mathbf{n}} w^s(x,k) = -ikw^i(x,k) + \frac{c(x)}{2}w^i(x,k), \tag{13}$$

$$\partial_{\mathbf{n}} w^s(x,k) = -ikw^i(x,k) + \frac{c(x)}{2}w^i(x,k) - \frac{c(x)^2}{8(c(x)-ik)}w^i(x,k) - \frac{1}{2(c(x)-ik)}\partial_x^2 w^i(x,k), \tag{14}$$

where $c(x)$ represents the interface curvature. Although these conditions are approximations of the DtN operator, we use the sign "=" for the sake of the presentation.

The motivation behind this choice of conditions (13) and (14) is summarized in the next example. Suppose that Ω is a circle, in this case the exact solution of problem (5) as well as the plane wave are given by Bessel functions. We then compute the quantity

$$\partial_{\mathbf{n}} w^t = \partial_{\mathbf{n}} w^s + \partial_{\mathbf{n}} e^{ikx\cdot\omega} \tag{15}$$

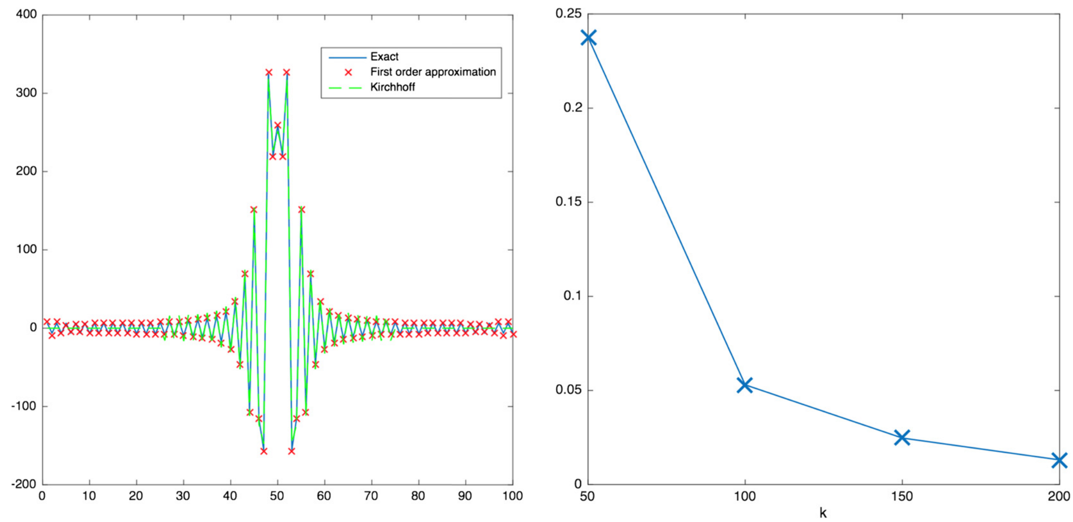

Fig. 1. Left: comparison of the exact solution of the problem (5) with the classical Kirchhoff approximation and a first order Bayliss–Turkel type approximation (13) for the unit circle illuminated by a plane wave incidence with $k = 150$. Right: relative error, computed for $k = 50, 100, 150, 200$, of the solution on the shadow and the deep shadow regions for the unit circle, obtained by first order Bayliss–Turkel type approximation (13).

using, (a) the exact solution, (b) the first order approximation (13) approximating $\partial_{\mathbf{n}} w^s$, and (c) the Kirchhoff approximation which uses the formula (12) in the illuminated region. The resulting calculations are exhibited in Fig. 1. We can observe that the Kirchhoff approach produces a reasonable approximation only in the illuminated region. In contrast, the solution, based on condition (13), generates a satisfactory approximation over the entire boundary, see the left of Fig. 1. The figure in the right validates this observation. Indeed, this figure exhibits the relative error associated with the solution based on (12) on the shadow and the deep shadow regions. In this case, the one related to the Kirchhoff approximation is equal to 1 since the solution is taken as zero in those regions. This explains why this procedure leads to inaccurate solutions in the high-frequency integral formulations. As is shown in [10,38], the second order approximation (14) of the DtN operator improves further the accuracy of the solution.

Remark 2. The condition (13) was derived in the case of two and three dimensions while the condition (14) was derived only in the two dimensional case [39]. Our computations do not distinguish between these two cases. However, at the end of this analysis, we give an example of an expansion derived for a three dimensional second order approximation of the DtN operator.

3. Expansion of the Kirchhoff amplitude using the first order approximation

The analysis produced in this paper is based on some results derived in the paper [40] and it consists of determining the asymptotic behavior of the Kirchhoff amplitude

$$a_Q(\omega, x, k) = \mathcal{K}(\omega, x, k)e^{ikx\cdot\omega}. \tag{16}$$

It begins by determining the kernel associated with the operator (7) in the case where the DtN operator is approximated by (13). For the sake of the presentation, we use the notation $\mathcal{C}(x) = c(x)/2$.

Theorem 3. *Let $O \subset \mathbb{R}^{n+1}$ be a strictly convex bounded obstacle such that $\partial O = B$, where B is C^∞ hypersurface with positive curvature in $\mathbb{R}^{n+1}$. Suppose that Ω is an open set of $\mathbb{R}^{n+1}$ such that $\Omega = \mathbb{R}^{n+1}\backslash O$, and let w^s be a solution of (5). Using the approximation (13), the operator Q (7) can be written as*

$$Q = ((1 - \mathbf{n}\cdot\omega)\partial_t + \mathcal{C}(x))F : \ \mathcal{E}'(\mathcal{S}^n \times \mathbb{R}) \to \mathcal{D}'(B \times \mathbb{R}), \tag{17}$$

where $\kappa_Q(\omega, x, t)$ *is its kernel given by*

$$\kappa_Q(\omega, x, t) = ((1 - \mathbf{n}\cdot\omega)\partial_t + \mathcal{C}(x))\kappa_F(\omega, x, t), \tag{18}$$

and $\kappa_F(\omega, x, t) = \delta(t - \omega\cdot x)$ *is the kernel of the Fourier integral operator* F *(9).*

Proof. Using the approximation (13) and the definition of the total field, we can write

$$\partial_{\mathbf{n}} w^t(x, k) = (-ik(1 - \mathbf{n}\cdot\omega) + \mathcal{C}(x))e^{ikx\cdot\omega}. \tag{19}$$

To obtain the kernel of the operator Q, we compute the Fourier transform with respect to the variable t. Let $\varphi_x(k) = \varphi(x, k) \in S(\mathbb{R})$, we have then

$$\begin{aligned}
\langle \widehat{\partial_{\mathbf{n}} w^t}(x, k), \varphi(x, k)\rangle_{S',S} &= \left\langle \partial_{\mathbf{n}} w^t(x, k), \widehat{\varphi}(x, k)\right\rangle_{S',S} \\
&= \int\limits_{\mathbb{R}} \partial_{\mathbf{n}} w^t(x, k)\widehat{\varphi}(x, k)dk \\
&= \int\limits_{\mathbb{R}\times\mathbb{R}} \partial_{\mathbf{n}} w^t(x, k)\varphi(x, t)e^{-ikt}dkdt \\
&= \int\limits_{\mathbb{R}\times\mathbb{R}} \left[-ik(1 - \mathbf{n}\cdot\omega)e^{ikx\cdot\omega} + \mathcal{C}(x)e^{ikx\cdot\omega}\right] \varphi(x, t)e^{-ikt}dkdt \\
&= (1 - \mathbf{n}\cdot\omega) \int\limits_{\mathbb{R}\times\mathbb{R}} -ike^{ikx\cdot\omega}\varphi(x, t)e^{-ikt}dkdt \\
&\quad + \mathcal{C}(x) \int\limits_{\mathbb{R}\times\mathbb{R}} e^{ikx\cdot\omega}\varphi(x, t)e^{-ikt}dkdt \\
&= \langle((1 - \mathbf{n}\cdot\omega)\partial_t + \mathcal{C}(x))\delta(t - \omega\cdot x), \varphi(x, t)\rangle_{S',S} \\
&= \left\langle \kappa_Q(\omega, x, t), \varphi(x, t)\right\rangle_{S',S},
\end{aligned} \tag{20}$$

where $\langle \widehat{ikf(k)}, \varphi\rangle_{S',S} = \langle -\partial_t \widehat{f}(t), \varphi\rangle_{S',S}$ and $\langle \widehat{e^{ik\omega\cdot x}}, \varphi\rangle_{S',S} = \langle \delta(t - \omega\cdot x), \varphi\rangle_{S',S}$. Therefore

$$\kappa_Q(\omega, x, t) = ((1 - \mathbf{n}\cdot\omega)\partial_t + \mathcal{C}(x))\kappa_F(\omega, x, t), \tag{21}$$

and $Q = ((1 - \mathbf{n}\cdot\omega)\partial_t + \mathcal{C}(x))F$. □

In the following, we use the same decomposition of the operator F given in [40] (equation (5.9)), that is,

$$F = J(E_1\mathcal{A} + E_2\mathcal{A}')K, \tag{22}$$

with $E_1 \in OPS^{-n/2+1/6}$, $E_2 \in OPS^{-n/2-1/6}$, and J and K are elliptic Fourier integral operators of order 0. The operators $\mathcal{A}$ and $\mathcal{A}'$ are Fourier Airy integral operators and are defined by

$$\mathcal{A}^{(l)}u(x, t) = \int e^{itk+ix\xi} Ai^{(l)}(\xi_1 k^{-1/3})\widehat{u}(\xi, k)d\xi dk, \tag{23}$$

$$\widehat{(\mathcal{A}u)}(\xi, k) = Ai(k^{-1/3}\xi_1)\widehat{u}(\xi, k), \tag{24}$$

$$\widehat{(\mathcal{A}'u)}(\xi, k) = Ai'(k^{-1/3}\xi_1)\widehat{u}(\xi, k), \tag{25}$$

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where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $k \in \mathbb{R}$, l is an integer indicating the order of the derivative, Ai is the Airy function

$$Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(\frac{t^3}{3}+st)} dt, \tag{26}$$

with

$$A_{\pm}(s) = Ai(e^{\pm 2\pi i/3} s), \tag{27}$$

and Ai' is its derivative. Finally, we define the operator $\mathcal{A}^{-1}$ [40] as follows

$$\widehat{(\mathcal{A}^{-1}u)}(\xi, k) = \frac{1}{A_+(k^{-1/3}\xi_1)} \widehat{u}(\xi, k). \tag{28}$$

The next theorem is concerned with the computation of the amplitude of the operator (17).

Theorem 4. *Let K and J be elliptic Fourier integral operators of order 0. Then the operator Q and its kernel κ_Q can be respectively written as*

$$Q = J\mathcal{A}^{-1}P_1K + J\mathcal{A}^{-1}P_2K, \tag{29}$$

$$\kappa_Q(\omega, x, t) = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt} \left((1-\mathbf{n}\cdot\omega)a(\omega, x, \xi, k) + \mathcal{C}(x)b(\omega, x, \xi, k)\right) \frac{1}{A_+(k^{-1/3}\xi_1)} d\xi dk, \tag{30}$$

such that $P_1 \in OPS^{-n/2+5/6}$ (resp. $P_2 \in OPS^{-n/2-1/6}$) with a symbol of the form $(1-\mathbf{n}\cdot\omega)p_1$ (resp. $\mathcal{C}(x)p_2$), where $a(\omega, x, \xi, k)$ and $b(\omega, x, \xi, k)$ are defined in (48), and $\mathcal{A}^{-1}$ is a pseudo-differential operator defined by (28) [40]. In addition, its amplitude is given by

$$a_Q(\omega, x, k) = \int e^{ik\psi_2(\omega,x,\zeta)} \left((1-\mathbf{n}\cdot\omega)a_1(\omega, x, \zeta, k) + \mathcal{C}(x)b_1(\omega, x, \zeta, k)\right) \frac{1}{A_+(k^{2/3}\zeta_1)} d\zeta, \tag{31}$$

with $\xi = k\zeta$, $a_1(\omega, x, \zeta, k) = k^n a(\omega, x, k\xi, k)$, $b_1(\omega, x, \zeta, k) = k^n b(\omega, x, k\xi, k)$ and $\psi_2(\omega, x, \zeta) = \psi_1(x, \zeta, 1) - \omega\cdot\zeta$.

Proof. Using (22) and equation (17) we obtain

$$Q = J(E_3\mathcal{A} + E_4\mathcal{A}')K + \mathcal{C}(x)J(E_1\mathcal{A} + E_2\mathcal{A}')K, \tag{32}$$

where $E_1 \in OPS^{-n/2+1/6}$, $E_2 \in OPS^{-n/2-1/6}$, $E_3 \in OPS^{-n/2+1/6+1}$, and $E_4 \in OPS^{-n/2-1/6+1}$. The operators $\mathcal{A}$ and $\mathcal{A}'$ are Airy operators given by (23). Using Theorem 6.5 in [40], we can write

$$Q = J\mathcal{A}^{-1}P_1K + J\mathcal{A}^{-1}P_2K, \tag{33}$$

with $P_1 \in OPS^{-n/2-1/6+1}$ and $P_2 \in OPS^{-n/2-1/6}$. To compute the oscillatory integral related to Q, consider $\varphi(x,t) \in S(\mathbb{R}^n \times \mathbb{R})$ and the Dirac delta function $\delta_{(\omega,t)} \in \mathcal{E}'(\mathcal{S}^n \times \mathbb{R})$ used here to find the kernel of Q at the base point [40,41], we have then

$$\begin{aligned}
\langle Q\delta_{(\omega,0)}, \varphi(x,t)\rangle &= \langle J\mathcal{A}^{-1}P_1K\delta_{(\omega,0)} + J\mathcal{A}^{-1}P_2K\delta_{(\omega,0)}, \varphi(x,t)\rangle \\
&= \langle J\mathcal{A}^{-1}P_3\delta_0, \varphi(x,t)\rangle + \langle J\mathcal{A}^{-1}P_4\delta_0, \varphi(x,t)\rangle \\
&= \langle JP_3\mathcal{A}^{-1}\delta_0, \varphi(x,t)\rangle + \langle JP_4\mathcal{A}^{-1}\delta_0, \varphi(x,t)\rangle \\
&= \langle Q_1\delta_{(\omega,0)}, \varphi(x,t)\rangle + \langle Q_2\delta_{(\omega,0)}, \varphi(x,t)\rangle,
\end{aligned}$$

where

$$Q_1\delta_{(\omega,0)} = JP_3\mathcal{A}^{-1}\delta_0, \quad Q_2\delta_{(\omega,0)} = JP_4\mathcal{A}^{-1}\delta_0, \tag{34}$$

and

$$Ju(x,t) = \int e^{i\psi_1(x,\eta,\tau)+i\tau(t-t_1)-iy\eta} a_J(x,y,t_1\eta,\tau)u(y,t_1)dydt_1d\eta d\tau, \tag{35}$$

with $P_3 \in OPS^{-n/2-1/6+1}$ and $P_4 \in OPS^{-n/2-1/6}$ such that $P_1K\delta_{(\omega,0)} = P_3\delta_{(\omega,0)} = P_3\delta_0$ and $P_2K\delta_{(\omega,0)} = P_4\delta_{(\omega,0)} = P_4\delta_0$, taking ω as a parameter, and knowing that $\mathcal{A}^{-1}$ commute with P_3 and P_4 [40]. Here, $(y,t_1) \in \mathbb{R}^n \times \mathbb{R}$, (η,τ) indicates the dual of (x,t), and the phase function ψ_1 is defined in the three regions of the obstacle [40]. In the illuminated region $\{x \in \partial O,\ \mathbf{n}(x)\cdot\omega < 0\}$, it is given by

$$\psi_1(x,\eta,\tau) = -\frac{|\eta'|^2}{2\tau} - \frac{|x'|^2}{2}\tau + \frac{2}{3}(-\eta_1\tau^{-1/3})^{3/2}, \tag{36}$$

while in the shadow region $\{x \in \partial O,\ \mathbf{n}(x)\cdot\omega > 0\}$, we have

$$\psi_1(x,\eta,\tau) = -\frac{|\eta|'^2}{2\tau} - \frac{|x'|^2}{2}\tau - \frac{2}{3}(-\eta_1\tau^{-1/3})^{3/2}. \tag{37}$$

Finally, on the shadow boundary $\{x \in \partial O,\ \mathbf{n}(x)\cdot\omega = 0\}$, the phase function is as follows

$$\psi_1(x,\eta,\tau) = -\frac{|\eta'|^2}{2\tau} - \frac{|x'|^2}{2}\tau \tag{38}$$

since $\eta_1 = 0$ [40]. Here, $x' = (x_2,...,x_n)$, $\eta' = (\eta_2,...,\eta_n)$ such that $x \in \partial O, \eta \in \mathbb{R}^n$, $t,\tau \in \mathbb{R}$, and $a_J(x,y,t_1,\eta,\tau) \in S^0_{1,0}$. The pseudo-differential operator P_3 is defined by

$$P_3u(y,t_1) = \int e^{i(y-\omega)\xi+it_1k} p_3(\omega,y,t_1\xi,k)\widehat{u}(\xi,k)d\xi dk, \tag{39}$$

where $(\xi,k) \in \mathbb{R}^n \times \mathbb{R}$ is the dual couple of (y,t_1). Our objective is to compute $Q_1\delta_{(\omega,0)} = JP_3\mathcal{A}^{-1}\delta_0$. First, using standard calculations on composition of operators [34,37], we have

$$J \circ P_3u(x,t) = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-itk} p_{J\circ P_3}(\omega,x,\xi,k)\widehat{u}(\xi,k)d\xi dk, \tag{40}$$

with

$$\begin{aligned} p_{J\circ P_3}(\omega,x,\xi,k) &= q_1(\omega,x,\xi,k) \\ &= \int e^{i\psi_1(x,\eta,\tau)-i\psi_1(x,\xi,k)+iy(\xi-\eta)+it_1(k-\tau)+it(k+\tau)} a_J(x,y,t_1,\eta,\tau)p_3(\omega,y,t_1,\xi,k)dydt_1d\eta d\tau \\ &= p_3(\omega,x,\xi,k)\# a_J(x,\xi,k), \end{aligned} \tag{41}$$

thus

$$J \circ P_3u(x,t) = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-itk} q_1(\omega,x,\xi,k)\widehat{u}(\xi,k)d\xi dk. \tag{42}$$

To find $Q_1\delta_{(\omega,0)}$ we need to replace $u(x,t)$ by $\mathcal{A}^{-1}\delta_0$ in (42)

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$$\mathcal{A}^{-1}\delta_0(x,t) = \int e^{ix\xi+ikt}\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk. \tag{43}$$

We get

$$\begin{aligned} Q_1\delta_{(\omega,0)} &= J \circ P_3 \circ \mathcal{A}^{-1}\delta_0 \\ &= \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt}q_1(\omega,x,\xi,k)\widehat{\mathcal{A}^{-1}\delta_0}(\xi,k)d\xi dk \\ &= \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt}q_1(\omega,x,\xi,k)\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk. \end{aligned} \tag{44}$$

A similar approach for Q_2 gives

$$Q_2\delta_{(\omega,0)} = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt}q_2(\omega,x,\xi,k)\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk. \tag{45}$$

The equation (33) becomes

$$Q\delta_{(\omega,0)} = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt}\left[q_1(\omega,x,\xi,k)+q_2(\omega,x,\xi,k)\right]\frac{1}{A_+(k^{-1/3}\xi_1)}dkd\xi, \tag{46}$$

where $q_2(\omega,x,\xi,k) \in S_{1,0}^{-n/2-1/6}$ and $q_1(\omega,x,\xi,k) \in S_{1,0}^{-n/2+5/6}$. This shows that the kernel κ_Q is as follows

$$\kappa_Q(\omega,x,t) = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt}\left[(1-\mathbf{n}\cdot\omega)a(\omega,x,\xi,k)+\mathcal{C}(x)b(\omega,x,\xi,k)\right]\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk, \tag{47}$$

where

$$q_1(\omega,x,\xi,k) = (1-\mathbf{n}\cdot\omega)a(\omega,x,\xi,k), \quad q_2(\omega,x,\xi,k) = \mathcal{C}(x)b(\omega,x,\xi,k). \tag{48}$$

Taking now the inverse Fourier transform of κ_Q, we obtain the following amplitude

$$a_Q(\omega,x,k) = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi}[(1-\mathbf{n}\cdot\omega)a(\omega,x,\xi,k)+\mathcal{C}(x)b(\omega,x,\xi,k)]\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi. \tag{49}$$

Applying the change of variable $\xi = k\zeta$ with $\xi \in \mathbb{R}^n$, we find

$$a_Q(\omega,x,k) = \int e^{ik\psi_2(\omega,x,\zeta)}\left[(1-\mathbf{n}\cdot\omega)a_1(\omega,x,\zeta,k)+\mathcal{C}(x)b_1(\omega,x,\zeta,k)\right]\frac{1}{A_+(k^{2/3}\zeta_1)}d\zeta, \tag{50}$$

such that $a_1(\omega,x,\zeta,k) = k^n a(\omega,x,k\zeta,k)$, $b_1(\omega,x,\zeta,k) = k^n b(\omega,x,k\zeta,k)$ and $\psi_2(\omega,x,\zeta) = \psi_1(x,\zeta,1) - \omega\cdot\zeta$. □

The remaining part of the computation of the asymptotic expression of a_Q (50) consists of applying the stationary phase method. First, we need the next lemma [40].

Lemma 5. *The function $\Psi \in S^1(\mathbb{R})$ defined as follows*

$$\Psi(\tau) = e^{-i\tau^3/3}\int \frac{1}{A_+(s)}e^{-is\tau}ds \tag{51}$$

is rapidly decreasing for $\tau \to -\infty$, where $A_+(s) = Ai(e^{\frac{2\pi i}{3}}s)$.

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Theorem 6. *The asymptotic expansion of the Kirchhoff amplitude a_Q is given by*

$$\begin{aligned} a_Q(\omega,x,k) = \sum_{p,l=0}^{P,L} k^{2/3-p-2l/3}\big((1-\mathbf{n}\cdot\omega)a_{p,l}(\omega,x) \\ + \mathcal{C}(x)b_{p,l}(\omega,x)\big)\psi^{(l)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega} + R_{P,L}(k), \end{aligned} \tag{52}$$

such that

$$|R_{P,L}(k)| \leq C_{PL}k^{-min(2L/3,P+1/3)}, \tag{53}$$

and where $p \in \{0,1..,P\}$, $l \in \{0,1..,L\}$, C_{PL} is a constant depending on L and P, ω is the incidence direction, and $Z(\omega,x)$ is a continuous real function that is positive on the illuminated region, negative on the shadow region, and vanishing on the shadow boundary. The functions $a_{p,l}$ and $b_{p,l}$ result from the expansion of the symbols a_1 and b_1 (see Theorem 4) and the application of the stationary phase method.

Proof. First, let us note that

$$\begin{aligned} \frac{1}{A_+}(k^{2/3}\zeta_1) &= \mathcal{F}^{-1}\left(\widehat{\frac{1}{A_+(k^{2/3}\zeta_1)}}\right) \\ &= \mathcal{F}^{-1}\left(\int e^{-ikt\zeta_1}\frac{k^{2/3}}{A_+(k^{2/3}\zeta_1)}d\zeta_1\right) \\ &= \mathcal{F}^{-1}\left(e^{ik\frac{t^3}{3}}\Psi(k^{1/3}t)\right) \\ &= k^{1/3}\int e^{ikt\zeta_1+ik\frac{t^3}{3}}\Psi(k^{1/3}t)dt. \end{aligned} \tag{54}$$

Using (54) and (50), a_Q becomes

$$a_Q(\omega,x,k) = k^{1/3}\int e^{ik\psi_2(\omega,x,\zeta)+ikt\zeta_1+ik\frac{t^3}{3}}\left[(1-\mathbf{n}\cdot\omega)a_1(\omega,x,\zeta,k)+\mathcal{C}(x)b_1(\omega,x,\zeta,k)\right]\Psi(k^{1/3}t)dtd\zeta, \tag{55}$$

where $a_1(\omega,x,\zeta,k) \in S^{n/2+7/6}$ and $b_1(\omega,x,\zeta,k) \in S^{n/2+1/6}$ are given in Theorem 4. Let us assume now that

$$(1-\mathbf{n}\cdot\omega)a_1(\omega,x,\zeta,k)+\mathcal{C}(x)b_1(\omega,x,\zeta,k) = d_1(\omega,x,\zeta,k) \in S^{n/2+7/6}. \tag{56}$$

Using properties of pseudo-differential operators [37], we can write

$$d_1(\omega,x,\zeta,k) = \sum_{p=0}^{P} k^{n/2+7/6-p}d_p(\omega,x,\zeta), \tag{57}$$

with $P = n/2+7/6$, and

$$d_p(\omega,x,\zeta) = (1-\mathbf{n}\cdot\omega)a_p(\omega,x,\zeta)+\mathcal{C}(x)b_p(\omega,x,\zeta). \tag{58}$$

This shows that the integral (55) can be rewritten as

$$k^{1/3}\sum_{p=0}^{P}k^{n/2+7/6-p}\int e^{ikf(\zeta,t)}d_p(\omega,x,\zeta)\Psi(k^{1/3}t)dtd\zeta, \tag{59}$$

such that

$$f(\zeta,t)=\psi_2(\omega,x,\zeta)+t\zeta_1+\frac{t^3}{3}, \tag{60}$$

see [40] for more details regarding the definition of f. To get the asymptotic expansion of a_Q, it remains to apply the stationary phase method to

$$\int e^{ikf(\zeta,t)}d_p(\omega,x,\zeta)\Psi(k^{1/3}t)dtd\zeta. \tag{61}$$

The conditions for this application are satisfied and the computation of the critical points is done in [40]. Using standard calculations regarding the stationary phase method [37,47], we obtain

$$\begin{aligned}&\int e^{ikf(\zeta,t)}d_p(\omega,x,\zeta)\Psi(k^{1/3}t)dtd\zeta\\&=k^{-1/3}\sum_{l=0}^{L}k^{-2l/3-(n+1)/2}d_{p,l}(\omega,x)\Psi^{(l)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega}+R_L(k),\end{aligned} \tag{62}$$

where $d_{p,l}(\omega,x)=\partial^l_{\zeta_1}d_p(\omega,x,\zeta)|_{\zeta=\zeta_c}$ with (ζ_c,t_c) is the critical point, and

$$|R_L(k)|\le C_Lk^{-n/2-2L/3-3/2}, \tag{63}$$

with C_L a constant. Using (62) in (59), we obtain

$$\begin{aligned}a_Q(\omega,x,k)=\sum_{p,l=0}^{P,L}&k^{2/3-p-2l/3}\big((1-\mathbf{n}\cdot\omega)a_{p,l}(\omega,x)\\&+\mathcal{C}(x)b_{p,l}(\omega,x)\big)\Psi^{(l)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega}+R_{P,L}(k),\end{aligned} \tag{64}$$

with

$$|R_{P,L}(k)|\le C_{PL}k^{-min(2L/3,P+1/3)}, \tag{65}$$

such that $l\in\{0,1..,L\}$ and P a real number. □

The next theorem establishes a relation between the functions $a_{p,l}$ and $b_{p,l}$ found in (52).

Theorem 7. *The functions $a_{p,l}$ and $b_{p,l}$ in (52), $p,l\ge 0$, satisfy the equation*

$$b_{p,l}(\omega,x)=-\frac{a_{p,l}(\omega,x)}{ik}. \tag{66}$$

Proof. From equation (17) and the operators F and J used in Theorem 4, P_4 and P_3 are two pseudo-differential operators with symbols $p_4\in S^{-n/2-1/6}$ and $p_3\in S^{-n/2-1/6+1}$ respectively, we have

$$Q=(1-\mathbf{n}\cdot\omega)\partial_tF+\mathcal{C}(x)F,\quad Fu(x,t)=JP_4\mathcal{A}^{-1}u(x,t). \tag{67}$$

Similar calculations to the equation (45) give

$$Fu(x,t) = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt} b(\omega,x,\xi,k) \frac{1}{A_+(k^{-1/3}\xi_1)} \widehat{u}(\xi,k) d\xi dk, \tag{68}$$

with $b(\omega,x,\xi,k) = p_4(\omega,x,\xi,k) \# a_J(x,\xi,k) \in S^{-n/2-1/6}$, where $a_J(x,\xi,k)$ is given in Theorem 4 and $\#$ is defined by (41). Knowing that $\partial_t Fu(x,t) = JP_3\mathcal{A}^{-1}u(x,t)$ and

$$\begin{aligned} \partial_t Fu(x,t) &= \partial_t \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt} b(\omega,x,\xi,k) \frac{1}{A_+(k^{-1/3}\xi_1)} \widehat{u}(\xi,k) d\xi dk, \\ &= \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt} (-ik) b(\omega,x,\xi,k) \frac{1}{A_+(k^{-1/3}\xi_1)} \widehat{u}(\xi,k) d\xi dk, \end{aligned} \tag{69}$$

implies that $a(\omega,x,\xi,k) = -ikb(\omega,x,\xi,k)$. We know that $b(\omega,x,\xi,k) \in S^{-n/2-1/6}$, and then usual pseudo-differential calculus results in $a(\omega,x,\xi,k) \in S^{-n/2+5/6}$. This allows us to conclude that $a_{p,l}(\omega,x) = -ikb_{p,l}(\omega,x)$. □

3.1. Some estimates of the asymptotic expansion (52)

Two estimates are established in this subsection. For the completion of the paper, we recall the next lemma [40].

Lemma 8. *The function Ψ given by (51) is rapidly decreasing for $\tau \to -\infty$ and*

$$\Psi(\tau) \sim \sum_{j=0}^{\infty} c_j \tau^{1-3j} \quad \textit{for } \tau \to +\infty. \tag{70}$$

The next result compares the asymptotic expansion (52) with $\partial_{\mathbf{n}} w^t$.

Proposition 9. *If a_Q is the amplitude given by (52), then*

$$|a_Q(\omega,x,k) - \partial_{\mathbf{n}} w^t(x,k)| \leq Ck^{-1} \textit{ for } k \to +\infty, \tag{71}$$

where C is a real constant and

$$a_Q(\omega,x,k) \sim \sum_{p,l=0}^{P,L} k^{2/3-p-2l/3} \left((1-\mathbf{n}\cdot\omega)a_{p,l}(\omega,x) + \mathcal{C}(x)b_{p,l}(\omega,x)\right) \psi^{(l)}(k^{1/3}Z(\omega,x)) e^{ikx\cdot\omega}. \tag{72}$$

Proof. We know that

$$\partial_{\mathbf{n}} w^t(x,k) = -ik(1-\mathbf{n}\cdot\omega)e^{ikx\cdot\omega} + \mathcal{C}(x)e^{ikx\cdot\omega}. \tag{73}$$

Using Theorem 7, the expansion (72) can be written as

$$\begin{aligned} a_Q(\omega,x,k) &\sim k^{2/3} \left((1-\mathbf{n}\cdot\omega) - \frac{\mathcal{C}(x)}{ik}\right) a_{0,0}(\omega,x)\Psi(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega} \\ &\quad + \sum_{p,l=1}^{P,L} k^{2/3-p-2l/3} \left((1-\mathbf{n}\cdot\omega) - \frac{\mathcal{C}(x)}{ik}\right) a_{p,l}(\omega,x)\Psi^{(l)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega} \end{aligned}$$

$$= k^{2/3}\left((1-\mathbf{n}\cdot\omega)-\frac{\mathcal{C}(x)}{ik}\right)a_{0,0}(\omega,x)\Psi(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega}$$
$$+\sum_{\beta,\alpha=0}^{P-1,L-1}k^{-1-\beta-2\alpha/3}\left((1-\mathbf{n}\cdot\omega)-\frac{\mathcal{C}(x)}{ik}\right)a_{\beta+1,\alpha+1}(\omega,x)\Psi^{(\alpha+1)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega}. \quad (74)$$

From the preceding lemma, we have $\Psi(k^{1/3}Z(x,\omega)) \sim -ik^{1/3}Z(x,\omega)$ for $k \to +\infty$ and taking $a_{0,0}(x,\omega) = \frac{1}{Z(x,\omega)}$ [40], we obtain

$$a_Q(\omega,x,k) \sim (-ik(1-\mathbf{n}\cdot\omega)+\mathcal{C}(x))\,e^{ikx\cdot\omega}$$
$$+\sum_{\beta,\alpha=0}^{P-1,L-1}k^{-1-\beta-2\alpha/3}\left((1-\mathbf{n}\cdot\omega)-\frac{\mathcal{C}(x)}{ik}\right)a_{\beta+1,\alpha+1}(\omega,x)\Psi^{(\alpha+1)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega}. \quad (75)$$

Knowing that $|\partial_\tau^{\alpha+1}\Psi(\tau)| \le M_\alpha\tau^{-\alpha}$ for each $\tau\in\mathbb{R}$ and $|1-\mathbf{n}\cdot\omega|\le 2$, we get

$$\left|a_Q(\omega,x,k)-\partial_{\mathbf{n}}w^t(x,k)\right|$$
$$\le\sum_{\beta,\alpha=0}^{P-1,L-1}\left|k^{-1-\beta-2\alpha/3}\left((1-\mathbf{n}\cdot\omega)-\frac{\mathcal{C}(x)}{ik}\right)a_{\beta+1,\alpha+1}(\omega,x)\Psi^{(\alpha+1)}(k^{1/3}Z(\omega,x))\right|$$
$$\le\sum_{\beta,\alpha=0}^{P-1,L-1}Mk^{-1-\beta-\alpha}\left|2+\frac{\max_{x\in B}\mathcal{C}(x)}{k}\right|$$
$$\le Ck^{-1}, \quad (76)$$

and C is a real constant. □

The following result estimates (52) near the shadow boundary.

Proposition 10. *If a_Q is the amplitude given by (52), then*

$$|a_Q(\omega,x,k)|\le Mk^{2/3}\quad \textit{for } k\to+\infty \quad (77)$$

where M is a real constant.

Proof. From the definition of $a_Q(\omega,x,k)$, it follows that

$$|a_Q(\omega,x,k)|\le\sum_{p,l=0}^{P,L}k^{2/3-p-2l/3}\left|\left((1-\mathbf{n}\cdot\omega)-\frac{\mathcal{C}(x)}{ik}\right)\right|\left|a_{p,l}(\omega,x)\Psi^{(l)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega}\right|, \quad (78)$$

with $p\in\{0,1,...,P\}$ and $l\in\{0,1,...,L\}$. We know that in the shadow boundary $|1-\mathbf{n}\cdot\omega|\le 1$. We assume that the curvature is constant, so $\mathcal{C}(x)=C$, then

$$|a_Q(\omega,x,k)|\le(k^{2/3}+Ck^{-1/3})\left|a_{0,0}(\omega,x)\Psi(k^{1/3}Z(\omega,x))\right|$$
$$+\sum_{\alpha,\gamma=0}^{P-1,L-1}k^{-1-\alpha-2\gamma/3}(1+Ck^{-1})\left|a_{\alpha+1,\gamma+1}(\omega,x)\Psi^{(\gamma+1)}(k^{1/3}Z(\omega,x))\right|. \quad (79)$$

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The function Ψ and all its derivatives are bounded,

$$|a_{0,0}(\omega, x)\Psi(k^{1/3}Z(\omega, x))| \leq M_1, \tag{80}$$

for all $\gamma \in N$, where M_1 is a real constant. Thus

$$|a_Q(\omega, x, k)| \leq Mk^{2/3} + \sum_{\alpha,\gamma=0}^{P-1,L-1} k^{-1-\alpha-2\gamma/3}(1 + Ck^{-1})|a_{\alpha+1,\gamma+1}(\omega, x)|M_\gamma, \tag{81}$$

such that $M = (1 + C)M_1$ and $|\Psi^{(\gamma+1)}(\tau)(k^{1/3}Z(\omega, x))| \leq M_\gamma$. Taking $k \to +\infty$, we obtain (77). □

4. Expansion of the Kirchhoff amplitude using the second order approximation

We are now interested in computing the asymptotic expansion in the case where we have

$$\begin{aligned}\partial_{\mathbf{n}} w^s(x, k) &= -ikw^i(x, k) + \frac{c(x)}{2} w^i(x, k) - \frac{c(x)^2}{8(c(x) - ik)} w^i(x, k) - \frac{1}{2(c(x) - ik)} \partial_x^2 w^i(x, k) \\ &= -ikw^i(x, k) + \frac{c(x)}{2} w^i(x, k) - \frac{1}{2(c(x)^2 + k^2)}(c(x) + ik)(\frac{c(x)^2}{4} + \partial_x^2)w^i(x, k).\end{aligned} \tag{82}$$

As for the first order case, we compute first the kernel associated to the operator Q (7) in the case where the DtN is approximated by (82). For the sake of simplicity, we denote the operator (17) by Q_1.

Theorem 11. *Let $O \subset \mathbb{R}^{n+1}$ be a strictly convex bounded obstacle such that $\partial O = B$, where B is a C^∞ hypersurface with positive curvature in $\mathbb{R}^{n+1}$. Suppose that Ω is an open set of $\mathbb{R}^{n+1}$ such that $\Omega = \mathbb{R}^{n+1} \backslash O$. Let w^s be a solution of (5). Using the approximation (82), the operator Q can be written as*

$$Q = Q_1 - \frac{\pi}{2c(x)} T_{e^{-c(x)|t|}}(c(x) - \partial_t)\tilde{F}, \tag{83}$$

where $\kappa_Q(\omega, x, t)$ is its kernel given by

$$\kappa_Q(\omega, x, t) = ((1 - \mathbf{n} \cdot \omega) + \frac{c(x)}{2})\kappa_F(\omega, x, t) - \frac{\pi}{2c(x)} e^{-c(x)|t|} * (c(x) - \partial_t)\, \kappa_{\tilde{F}}(\omega, x, t), \tag{84}$$

and $T_{e^{-c(x)|t|}}$ denotes the convolution operator of $e^{-c(x)|t|}$, $\tilde{F} = (\frac{c^2(x)}{4} + \partial_x^2)F$, and $\kappa_{\tilde{F}}(\omega, x, t) = (\frac{c^2(x)}{4} + \partial_x^2)\kappa_F(\omega, x, t)$ where $\kappa_F(\omega, x, t) = \delta(t - \omega \cdot x)$.

Proof. Using the approximation (82) and the definition of the total field, we can write

$$\begin{aligned}\partial_{\mathbf{n}} w^t(x, k) &= \left(-ik(1 - \mathbf{n} \cdot \omega) + \frac{c(x)}{2}\right) e^{ikx\cdot\omega} - \frac{1}{2(c(x)^2 + k^2)}(c(x) + ik)(\frac{c(x)^2}{4} + \partial_x^2)e^{ikx\cdot\omega} \\ &= \mathcal{Q}_1(x, k) + \mathcal{Q}_2(x, k),\end{aligned} \tag{85}$$

with

$$\mathcal{Q}_1(x, k) = \left(-ik(1 - \mathbf{n} \cdot \omega) + \frac{c(x)}{2}\right) e^{ikx\cdot\omega}, \tag{86}$$

$$\mathcal{Q}_2(x, k) = \left(-\frac{1}{2(c(x)^2 + k^2)}(c(x) + ik)(\frac{c(x)^2}{4} + \partial_x^2)\right) e^{ikx\cdot\omega}. \tag{87}$$

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To obtain the kernel of the operator Q, we compute the Fourier transform of the amplitude $\partial_n w^t(x,k)$ with respect to k. Let $\varphi_x(k) = \varphi(x,k) \in S(\mathbb{R})$, thus

$$\left\langle \partial_n \widehat{w^t(x,k)}, \varphi \right\rangle_{S',S} = \left\langle \widehat{\mathcal{Q}_1(x,k)}, \varphi \right\rangle_{S',S} + \left\langle \widehat{\mathcal{Q}_2(x,k)}, \varphi \right\rangle_{S',S}. \tag{88}$$

The quantity $\left\langle \widehat{\mathcal{Q}_1(x,k)}, \varphi \right\rangle_{S',S}$ is given by (20). For the one regarding $\mathcal{Q}_2(x,k)$, we have

$$\begin{aligned}
\left\langle \widehat{\mathcal{Q}_2(x,k)}, \varphi \right\rangle_{S',S} &= \langle \mathcal{Q}_2(x,k), \widehat{\varphi} \rangle_{S',S} \\
&= -\int\limits_{\mathbb{R}\times\mathbb{R}} \left[\frac{1}{2(c(x)^2+k^2)} (c(x)+ik)(\frac{c(x)^2}{4} + \partial_x^2) e^{ikx\cdot\omega} \right] \varphi_x(t) e^{-ikt} dt dk \\
&= -\int\limits_{\mathbb{R}} \left[\int\limits_{\mathbb{R}} \frac{1}{2(c(x)^2+k^2)} e^{-ikt} dk * \int\limits_{\mathbb{R}} (c(x)+ik)(\frac{c(x)^2}{4} + \partial_x^2) e^{ikx\cdot\omega} e^{-ikt} dk \right] \varphi_x(t) dt \\
&= -\frac{\pi}{2c(x)} \langle e^{-c(x)|t|} * (c(x)-\partial_t)\, (\frac{c(x)^2}{4} + \partial_x^2) \delta(t-\omega\cdot x), \varphi \rangle_{S',S} \\
&= \langle \kappa_{Q_2}(\omega,x,t), \varphi \rangle_{S',S}\,.
\end{aligned} \tag{89}$$

Taking

$$\kappa_{\tilde{F}}(\omega,x,t) = (\frac{c(x)^2}{4} + \partial_x^2)\kappa_F(\omega,x,t), \quad \kappa_F(\omega,x,t) = \delta(t-\omega\cdot x), \tag{90}$$

we get

$$\begin{aligned}
\kappa_Q(\omega,x,t) &= ((1-\mathbf{n}\cdot\omega)\partial_t + \frac{c(x)}{2})\kappa_F(\omega,x,t) - \frac{\pi}{2c(x)} e^{-c(x)|t|} * (c(x)-\partial_t)\, \kappa_{\tilde{F}}(\omega,x,t) \\
&= \kappa_{Q_1}(\omega,x,t) + \kappa_{Q_2}(\omega,x,t),
\end{aligned} \tag{91}$$

where $\kappa_{Q_1}(\omega,x,t)$ indicates the kernel of Q_1. This allows us to write

$$Q = Q_1 - \frac{\pi}{2c(x)} T_{e^{-c(x)|t|}} (c(x)-\partial_t)\tilde{F}, \tag{92}$$

where $T_{e^{-c(x)|t|}}$ is the convolution operator of $e^{-c(x)|t|}$ and $\tilde{F} = (\frac{c^2(x)}{4} + \partial_x^2)F$. $\square$

The next theorem is concerned with the computation of the amplitude of the operator (83).

Theorem 12. *Let K and J be elliptic Fourier integral operators of order 0. Then the operator Q (83) and its kernel κ_Q can respectively be written as*

$$Q = Q_1 - \frac{\pi}{2c(x)} T_{e^{-c(x)|t|}} (c(x) J \mathcal{A}^{-1} P_1^{\#} K - J\mathcal{A}^{-1} P_2^{\#} K), \tag{93}$$

$$\kappa_Q(\omega,x,t) = \kappa_{Q_1}(\omega,x,t) + \kappa_{Q_2}(\omega,x,t), \tag{94}$$

where $\kappa_{Q_1}(\omega,x,t)$ is the kernel of Q_1, and κ_{Q_2} is defined by

$$\kappa_{Q_2}(\omega,x,t) = -\frac{\pi}{2} \int e^{-c(x)|t-r| + i\psi_1(x,\xi,k) - i\omega\cdot\xi - ikr} [b^{\#}(\omega,x,\xi,k) - \frac{1}{c(x)} a^{\#}(\omega,x,\xi,k)] \frac{1}{A_+(k^{-1/3}\xi_1)} d\xi dk dr.$$

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Here $P_1^\# \in OPS^{-n/2-1/6}$, $P_2^\# \in OPS^{-n/2+5/6}$, $a^\#(\omega, x, \xi, k) = (\frac{c^2(x)}{4}+\partial_x^2)a(\omega, x, \xi, k)$, *and* $b^\#(\omega, x, \xi, k) = (\frac{c^2(x)}{4} + \partial_x^2)b(\omega, x, \xi, k)$ *where* $a(\omega, x, \xi, k)$ *and* $b(\omega, x, \xi, k)$ *are defined in Theorem 4. Furthermore, the amplitude* a_Q *is given by*

$$a_Q(\omega, x, k) = a_{Q_1}(\omega, x, k) + a_{Q_2}(\omega, x, k), \tag{95}$$

where $a_{Q_1}(\omega, x, k)$ *is the amplitude (31) and*

$$a_{Q_2}(\omega, x, k) = -\frac{1}{2}\frac{c(x)}{c^2(x)+k^2}\int e^{ik\psi_2(\omega,x,\zeta)}[b_1^\#(\omega, x, \zeta, k) - \frac{1}{c(x)}a_1^\#(\omega, x, \zeta, k)]\frac{1}{A_+(k^{2/3}\zeta_1)}d\zeta,$$

such that $a_1^\#(\omega, x, \zeta, k) = k^n a(\omega, x, k\zeta, k) \in S^{n/2+7/6}$, $b_1^\#(\omega, x, \zeta, k) = k^n b(\omega, x, k\zeta, k) \in S^{n/2+1/6}$, *and* $\psi_2(\omega, x, \zeta)$ *is given in Theorem 4.*

Proof. Using (22) and (83) we obtain

$$Q = Q_1 - \frac{\pi}{2c(x)}T_{e^{-c(x)|t|}}(c(x)J(\tilde{E}_1\mathcal{A} + \tilde{E}_2\mathcal{A}')K - J(\tilde{E}_3\mathcal{A} + \tilde{E}_4\mathcal{A}')K), \tag{96}$$

where $\tilde{E}_1 \in OPS^{-n/2+1/6}$, $\tilde{E}_2 \in OPS^{-n/2-1/6}$, $\tilde{E}_3 \in OPS^{-n/2+1/6+1}$, and $\tilde{E}_4 \in OPS^{-n/2-1/6+1}$. Using Theorem 6.5 in [40], we get

$$Q = Q_1 - \frac{\pi}{2c(x)}T_{e^{-c(x)|t|}}(c(x)J\mathcal{A}^{-1}P_1^\# K - J\mathcal{A}^{-1}P_2^\# K), \tag{97}$$

such that $P_1^\# \in OPS^{-n/2-1/6}$ with symbol

$$p_1^\#(\omega, x, \xi, k) = (\frac{c^2(x)}{4} + \partial_x^2)p_2(\omega, x, \xi, k), \tag{98}$$

and $P_2^\# \in OPS^{-n/2+5/6}$ with symbol

$$p_2^\#(\omega, x, \xi, k) = (\frac{c^2(x)}{4} + \partial_x^2)p_1(\omega, x, \xi, k), \tag{99}$$

where $p_1(\omega, x, \xi, k)$ and $p_2(\omega, x, \xi, k)$ are described in Theorem 4. As in the first order case, we use now the Dirac delta function $\delta_{(\omega,0)} \in \mathcal{E}'(\mathcal{S}^n \times \mathbb{R})$ to find the kernel of Q at the base point. Let $\varphi(x,t) \in S(\mathbb{R}^n \times \mathbb{R})$, thus we have

$$\begin{aligned}\langle Q\delta_{(\omega,0)}, \varphi\rangle_{S',S} &= \langle Q_1\delta_{(\omega,0)}, \varphi\rangle_{S',S} - \langle\frac{\pi}{2c(x)}e^{-c(x)|t|} * [c(x)JP_3^\#\mathcal{A}^{-1} - JP_4^\#\mathcal{A}^{-1}]\delta_0, \varphi\rangle_{S',S} \\ &= \langle Q_1\delta_{(\omega,0)}, \varphi\rangle_{S',S} + \langle Q_2\delta_{(\omega,0)}, \varphi\rangle_{S',S},\end{aligned} \tag{100}$$

with $P_3^\# = P_1^\# K$ and $P_4^\# = P_2^\# K$. The term $\langle Q_1\delta_{(\omega,0)}, \varphi\rangle_{S',S}$ is already computed (46). Therefore, we only need

$$\begin{aligned}\langle Q_2\delta_{(\omega,0)}, \varphi\rangle_{S',S} &= -\langle\frac{\pi}{2c(x)}e^{-c(x)|t|} * A\delta_0, \varphi\rangle_{S',S} \\ &= -\frac{\pi}{2c(x)}\int e^{-c(x)|t-r|}\langle A\delta_0, \varphi\rangle_{S',S}dr,\end{aligned} \tag{101}$$

with

$$A = c(x)JP_3^{\#}\mathcal{A}^{-1} - JP_4^{\#}\mathcal{A}^{-1}. \tag{102}$$

On the other hand, we can write

$$\langle A\delta_0, \varphi\rangle_{S',S} = \int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikt}[c(x)b^{\#}(\omega,x,\xi,k) - a^{\#}(\omega,x,\xi,k)]\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk, \tag{103}$$

where $b^{\#}(\omega,x,\xi,k) = p_3^{\#}(\omega,x,\xi,k)\#a_J(x,\xi,k)$ and $a^{\#}(\omega,x,\xi,k) = p_4^{\#}(\omega,x,\xi,k)\#a_J(x,\xi,k)$ (see (41) for the definition of $\#$). This leads to

$$\begin{aligned}&\langle Q_2\delta_{(\omega,0)}, \varphi\rangle_{S',S}\\ &= -\frac{\pi}{2c(x)}\int e^{-c(x)|t-r|+i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikr}[c(x)b^{\#}(\omega,x,\xi,k) - a^{\#}(\omega,x,\xi,k)]\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk dr.\end{aligned}$$

Finally, we find

$$\begin{aligned}&Q\delta_{(\omega,0)}\\ &= Q_1\delta_{(\omega,0)} - \frac{\pi}{2}\int e^{-c(x)|t-r|+i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikr}[b^{\#}(\omega,x,\xi,k) - \frac{1}{c(x)}a^{\#}(\omega,x,\xi,k)]\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk dr.\end{aligned}$$

This shows that the kernel κ_Q is as follows

$$\kappa_Q(\omega,x,t) = \kappa_{Q_1}(\omega,x,t) + \kappa_{Q_2}(\omega,x,t), \tag{104}$$

and

$$\kappa_{Q_2}(\omega,x,t) = -\frac{\pi}{2}\int e^{-c(x)|t-r|+i\psi_1(x,\xi,k)-i\omega\cdot\xi-ikr}[b^{\#}(\omega,x,\xi,k) - \frac{1}{c(x)}a^{\#}(\omega,x,\xi,k)]\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi dk dr,$$

with $a^{\#}(\omega,x,\xi,k) = (\frac{c^2(x)}{4} + \partial_x^2)a(\omega,x,\xi,k)$ and $b^{\#}(\omega,x,\xi,k) = (\frac{c^2(x)}{4} + \partial_x^2)b(\omega,x,\xi,k)$. To obtain the amplitude a_Q, we take the inverse Fourier transform of κ_Q. The one related to κ_{Q_1} is given by (31). First, we can write that

$$\kappa_{Q_2}(\omega,x,t) = -\frac{\pi}{2}e^{-c(x)|t|} * \kappa_A(\omega,x,t), \tag{105}$$

where $\kappa_A(\omega,x,t)$ is the kernel of the operator (102). Using the inverse Fourier transform, we find

$$\mathcal{F}^{-1}(\kappa_{Q_2})(\omega,x,k) = -\frac{\pi}{2}\mathcal{F}^{-1}(e^{-c(x)|t|})(k)\mathcal{F}^{-1}(\kappa_A)(\omega,x,k). \tag{106}$$

Knowing that $\mathcal{F}^{-1}(e^{-c(x)|t|}) = \frac{1}{\pi}\frac{c(x)}{c^2(x)+k^2}$, we obtain

$$a_{Q_2}(\omega,x,k) = -\frac{1}{2}\frac{c(x)}{c^2(x)+k^2}\int e^{i\psi_1(x,\xi,k)-i\omega\cdot\xi}[b^{\#}(\omega,x,\xi,k) - \frac{1}{c(x)}a^{\#}(\omega,x,\xi,k)]\frac{1}{A_+(k^{-1/3}\xi_1)}d\xi.$$

Applying the change of variable $\xi = k\zeta$ with $\xi \in \mathbb{R}^n$, we get

$$a_{Q_2}(\omega,x,k) = -\frac{1}{2}\frac{c(x)}{c^2(x)+k^2}\int e^{ik\psi_2(\omega,x,\zeta)}[b_1^{\#}(\omega,x,\zeta,k) - \frac{1}{c(x)}a_1^{\#}(\omega,x,\zeta,k)]\frac{1}{A_+(k^{2/3}\zeta_1)}d\zeta, \tag{107}$$

such that $a_1^{\#}(\omega,x,\zeta,k) = k^n a^{\#}(\omega,x,k\zeta,k)$, $b_1^{\#}(\omega,x,\zeta,k) = k^n b^{\#}(\omega,x,k\zeta,k)$, and $\psi_2(\omega,x,\zeta)$ is described in Theorem 4. □

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The next theorem gives the asymptotic expansion of (95).

Theorem 13. *The asymptotic expression of the Kirchhoff amplitude (95) is given by*

$$
\begin{aligned}
a_Q(\omega, x, k) = {} & a_{Q_1}(\omega, x, k) \\
& - \frac{1}{2}\frac{c(x)}{c^2(x)+k^2}\sum_{p,l=0}^{P,L} k^{2/3-p-2l/3}\left[b^{\#}_{p,l}(\omega, x) - \frac{1}{c(x)}a^{\#}_{p,l}(\omega, x)\right]\psi^{(l)}(k^{1/3}Z(\omega, x))e^{ikx\cdot\omega} \\
& + R_{P,L}(k),
\end{aligned}
\tag{108}
$$

where

$$
a_{Q_1}(\omega, x, k) = \sum_{p,l=0}^{P,L} k^{2/3-p-2l/3}\left((1-\mathbf{n}\cdot\omega)a_{p,l}(\omega, x) + \mathcal{C}(x)b_{p,l}(\omega, x)\right)\psi^{(l)}(k^{1/3}Z(\omega, x))e^{ikx\cdot\omega}. \tag{109}
$$

Here $a^{\#}_{p,l}(\omega, x) = (\frac{c^2(x)}{4} + \partial_x^2)a_{p,l}(\omega, x)$, $b^{\#}_{p,l}(\omega, x) = (\frac{c^2(x)}{4} + \partial_x^2)b_{p,l}(\omega, x)$, $p \in \{0, 1.., P\}$, $l \in \{0, 1.., L\}$, ω *is the incidence direction, and* $c(x) > 0$ *is the curvature. In addition,* $Z(\omega, x)$ *is a continuous real function that is positive on the illuminated region, negative on the shadow region, and vanishing on the shadow boundary. The functions* $a_{p,l}$, $b_{p,l}$, $a^{\#}_{p,l}$, *and* $b^{\#}_{p,l}$ *result from the expansion of the symbol and the application of the stationary phase method.*

Proof. The derivation of (108) is based on the application of the stationary phase method to the amplitude $a_Q(\omega, x, k) = a_{Q_1}(\omega, x, k) + a_{Q_1}(\omega, x, k)$ where

$$
a_{Q_1}(\omega, x, k) = k^{1/3}\int e^{ik\psi_2(\omega,x,\zeta)+iktζ_1+ik\frac{t^3}{3}}\left[(1-\mathbf{n}\cdot\omega)a_1(\omega, x, \zeta, k) + \mathcal{C}(x)b_1(\omega, x, \zeta, k)\right]\Psi(k^{1/3}t)dtd\zeta, \tag{110}
$$

see (55), and

$$
a_{Q_2}(\omega, x, k) = -\frac{1}{2}\frac{k^{1/3}c(x)}{c^2(x)+k^2}\int e^{ik\psi_2(\omega,x,\zeta)+ikt\zeta_1+ikt^3/3}[b^{\#}_1(\omega, x, \zeta, k) - \frac{1}{c(x)}a^{\#}_1(\omega, x, \zeta, k)]\Psi(k^{1/3}t)d\zeta dt, \tag{111}
$$

obtained using (54) in (107). The critical point are the same as the ones given in the paper [40] and used when applying the stationary phase method to (110) to derive (52). Therefore, the latter method for the amplitude a_Q leads to the asymptotic expression

$$
\begin{aligned}
a_Q(\omega, x, k) = {} & a_{Q_1}(\omega, x, k) \\
& - \frac{1}{2}\frac{c(x)}{c^2(x)+k^2}\sum_{p,l=0}^{P,L} k^{2/3-p-2l/3}\left[b^{\#}_{p,l}(\omega, x) - \frac{1}{c(x)}a^{\#}_{p,l}(\omega, x)\right]\psi^{(l)}(k^{1/3}Z(\omega, x))e^{ikx\cdot\omega} + R_{P,L}(k),
\end{aligned}
$$

such that $a_{Q_1}(\omega, x, k)$ is (109) (first order expansion (52)), $a^{\#}_{p,l}(\omega, x) = (\frac{c^2(x)}{4} + \partial_x^2)a_{p,l}(\omega, x)$, and $b^{\#}_{p,l}(\omega, x) = (\frac{c^2(x)}{4} + \partial_x^2)b_{p,l}(\omega, x)$. The remainder $R_{P,L}$ satisfies

$$
|R_{P,L}(k)| \leq C_{PL}k^{-min(2L/3,P+1/3)}, \tag{112}
$$

and C_{PL} is a constant depending on P and L. $\square$

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Remark 14. As is mentioned in Remark 2, the second order condition (82) is derived in two dimensions. If the three dimensional absorbing boundary condition

$$\partial_{\mathbf{n}} w^s(x,k) = -(ik - c(x))w^i(x,k) - \frac{c^2(x)(c(x)+ik)}{2(c(x)^2+k^2)}\partial_x^2 w^i(x,k), \tag{113}$$

is used (see condition (29) in [39]), then we obtain

$$\begin{aligned} a_Q(\omega,x,k) &\sim a_{Q_1}(\omega,x,k) \\ &- \frac{1}{2}\frac{c^2(x)}{c^2(x)+k^2}\sum_{p,l=0}^{P,L} k^{2/3-p-2l/3}\left[c(x)\widetilde{b}_{p,l}(\omega,x) - \widetilde{a}_{p,l}(\omega,x)\right]\psi^{(l)}(k^{1/3}Z(\omega,x))e^{ikx\cdot\omega}, \end{aligned} \tag{114}$$

where $a_{Q_1}(\omega,x,k)$ is given by (109) with $\mathcal{C}(x) = c(x)$, $\widetilde{a}_{p,l}(\omega,x) = \partial_x^2 a_{p,l}(\omega,x)$, and $\widetilde{b}_{p,l}(\omega,x) = \partial_x^2 b_{p,l}(\omega,x)$.

5. Conclusion

In this paper, we derived some new expansions of the normal derivative of the total field solution of the Helmholtz equation. The original expansions are based on a pseudo-differential decomposition of the Dirichlet to Neumann operator. In this work, we used approximations of this operator to derive new expansions. One of the goals is to facilitate construction of a new ansatz class that can be used in the development of numerical solvers that are able to produce more accurate solutions.

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