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Elasto-plastic contact problems with heat exchange and fatigue ☆

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ABSTRACT

We deal with a one-dimensional temperature dependent model for fatigue accumulation in a moving visco-elasto-plastic material in contact with an elasto-plastic obstacle. The problem for the unknown displacement and temperature is formulated using hysteresis operators as solution operators of the underlying variational inequalities. The existence result for this problem, consisting of the momentum and energy balance equations and an evolution equation for the fatigue, is obtained using a priori estimates established for the space discretized problem. The uniqueness result follows from the Lipschitz continuity of the nonlinearities.

1. Introduction

The aim of this paper is to present a new model accounting for the fatigue accumulation in a moving inhomogeneous visco-elasto-plastic bar with a contact boundary condition and its mathematical analysis. The main novelty is the combination of fatigue accumulation in (visco-)elasto-plastic structures with a contact boundary condition, that (at least from the mathematical point of view) is generally a challenging problem (see for instance [16]).

Concerning the problem of fatigue accumulation in oscillating visco-elasto-plastic beams and plates, our basic modeling idea is the assumption that the fatigue accumulation is proportional to the dissipated energy. This is motivated by the so-called *rainflow method* for cycling fatigue accumulation in uniaxial processes, where damage is assumed to be proportional to the total area of closed hysteresis loops, which can in turn

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be interpreted as energy dissipation (see [11]). As we already mentioned in our previous contributions, this viewpoint is new in the literature, either because our focus is related to the dynamics of the processes (compared, for instance, with [26], [40], [48] that go into the direction of rate-independent damage processes in nonlinear elasticity) and also because the approach used in other papers, that interpret damage processes as a kind of phase transition in the material, (see for instance [4], [8], [9], [41]) is based on the idea that damage processes are driven by large deformations. We presented our main modeling ideas in [18], where also the thermodynamic consistency of the model was shown. Mathematical analysis of fatigue problems started in [19], where we dealt with the case of an oscillating elasto-plastic plate under the simplified situation of given temperature history. We obtained existence and uniqueness of a solution locally in time; according to the model indeed, material softening takes place under increasing fatigue and material failure is manifested in finite time. From [20], we started dealing with the non-isothermal case. We first treated the 1D case (beam), showing existence and uniqueness of a strong solution in the simplified setting where only the elastic component of the model depends on the fatigue. The extension to the 2D case (plate) has been considered in [21], where we proved existence of solutions for the whole time interval, assuming that the fatigue accumulation rate is proportional only to the plastic part of the dissipation rate. Finally in [10], [22], [17] we pursued the study of fatigue accumulation in oscillating visco-elasto-plastic structures by presenting a new phase field model under the additional hypothesis that the material can partially recover by the effect of local melting. We were able to treat both the 1D (beam) case and the 2D case (plate), showing global existence in time of a solution of the underlying system of momentum and energy balances coupled with the evolution equation for the fatigue rate and a differential inclusion for the phase dynamics. In the 1D case also uniqueness was obtained, in 2D it remains an open problem.

In the present paper unlike [19] and similarly to [24], we assume that out of all dissipative components in the energy balance, only the purely plastic dissipation produces damage. This different perspective is usually considered in engineering literature. From the mathematical viewpoint, the problem does not exhibit singularities and the expected solutions are global in time. On the other hand we consider here an additional difficulty that the weight function φ in the definition of the Prandtl–Ishlinskii operator depends also on the fatigue parameter m; this creates nontrivial mathematical complications when dealing with the fatigue terms in the estimates.

Concerning the contact problem, this is classically described by the so-called Signorini boundary conditions where the obstacle is assumed to be rigid. The original problem − modeling of an anisotropic non-homogeneous elastic body, resting on a rigid frictionless surface and subject only to its mass forces − was posed by A. Signorini during a course taught in 1959: he explicitly invited young analysts to study the problem and to determine if it is well-posed in a physical sense, i.e. if its solution exists and is unique or not. The Signorini problem was then solved by G. Fichera [23] and later interpreted as a free boundary problem [36]; a weak formulation of the problem can be given in terms of variational inequalities, after the fundamental work by J.L. Lions and G. Stampacchia [38].

The general problem is very complicated and attempts in many directions have been made. Problem was considered with or without friction, the obstacle was supposed to be non-deformable or moving, material to be elastic or viscoelastic, viscoelasticity has been often considered as a way to overcome the difficulty of the original problem.

We refer to [16] together with the references therein for a general and wide survey for the mathematical analysis of contact problems with friction and a major part of the analysis for contact problems without friction.

There are several different ways how to deal with the non-smoothness of the problem: the most classical approach is the penalization method (see for instance [25], [16], [34]): it consists in penalizing the obstacle constraint with elasticity modulus considered as penalty parameter, and then solving the penalized problem; among the other methods we mention also the characteristic method (see [46], [13] and references therein), valid only for 1D problems.

Models for contact, delamination and damage in elastic media are recently becoming very popular. In [5], [6], [7] and related references, contact problems between a viscoelastic body and a rigid support were considered, and the effects due to adhesion, friction and the evolution of temperature were taken into account. In [32] a quasi-static approach to delamination and adhesive contact was considered, see also delamination and contact problems in [42], [43], [44], [45].

It is physically more reasonable to consider dynamic contact problems, but the mathematical analysis is even more complicated. The first construction of a solution to a physically well posed dynamic problem was obtained in [1] for a vibrating string. No significant results have been obtained for elastic materials in dimension greater than 1, despite considerable efforts by mathematicians; we quote for instance the contributions of [2], [46], [47]. In higher space dimensions, in [25] the existence of a weak solution to the wave equation with contact at the boundary was proved. In [39] the existence result for unilateral contact problems where the space of admissible functions is a subset of the space of continuous functions was presented and finally in [35] the existence of a strong solution to the wave equation in a halfspace with contact at the boundary and conservation of energy was considered; this result essentially depends on this special geometric assumption.

From more recent years we can quote the paper [33], where a dynamic point of view was taken into account; the authors consider a visco-elastic rod and a deformable obstacle. This modeling situation is also the point of view of the new approach suggested in recent papers [30] and [31], where the authors dealt with the more complicated setting of (visco)-elasto-plasticity. In [30] the modeling of the contact boundary condition using hysteresis operators was presented and it was combined with an elasto-plastic dynamical problem. In [31] a full thermomechanical 1D model taking into account the exchange between different types of energy in an oscillating visco-elasto-plastic body in contact with an elasto-plastic obstacle is considered and analyzed. These two papers constitute also a novelty because they consider elasto-plastic dynamical contact problems which have not been considered in literature so far (if we exclude some results already presented in [15]). Indeed, while in the classical approach such kinds of problems are solved by means of the idea to penalize the constraints, derive energy estimates independent of the penalty and let the penalty parameter tend to 1, by the hysteresis approach variational inequalities are solved independently of the momentum balance equation, finer analytical properties of the solution operators are derived and the momentum balance equation is solved as an operator–differential equation. The advantage of the hysteresis method is that hysteresis operators in mechanics are typically (almost) monotone, Lipschitz continuous, and satisfy two-level energy inequalities. PDEs with hysteresis can thus be solved by standard techniques (Galerkin, discretization, etc.).

In the present paper, we combine the mathematical difficulties coming from modeling the dynamic contact problem and the material fatigue. The result is an involved model formulated using hysteresis operators as solution operators of the underlying variational inequalities to control the contact boundary conditions, where the unknown functions are displacement, temperature and material fatigue. We take into account irreversible deformations both of the body and of the obstacle, as well as the fact that the plastic deformation of elasto-plastic bodies in contact dissipates energy which is transformed into heat. This in turn increases the temperature of the body and by thermal expansion, the motion of the body is affected. The existence result for this problem, consisting of the momentum and energy balance equations and an evolution equation for the fatigue is obtained using a priori estimates established for the space discretized problem. The uniqueness result follows from the Lipschitz continuity of the nonlinearities. We refer in particular to Section 4.1 for a detailed description of the main mathematical difficulties (related to hysteresis, material fatigue and contact boundary conditions) together with the main novelty of our contribution.

Our model is described in detail in Section 2. The main result is stated in Section 3, together with the Hypotheses on the data. We introduce the space discretization of the problem in Section 4 and derive suitable a priori estimates needed to show the convergence of the approximated solutions to the original problem. We perform this limit procedure in Section 4, which therefore also contains the proof of the existence of solutions to our problem. Then, in Section 5, a continuous dependence of the solutions on the data is proved, which implies uniqueness. The Appendix is devoted to Sobolev interpolation inequalities (for more details see [31]), which are used in the proof of the main result.

2. Description of the model

We consider an inhomogeneous elasto-plastic bar of length L which vibrates longitudinally. The bar is free to move on one end as long as it does not hit a material obstacle, while on the other end a force is applied. Let $u(x,t)$ be the displacement at time t of the material point of spatial coordinate $x \in \Omega$ with $\Omega := (0,L)$, and let σ be the σ_{11} component of the stress tensor. The motion is governed by the equation

$$\rho u_{tt} - \sigma_x = 0, \tag{2.1}$$

where ρ denotes the mass density (see (H6) below). Here and in the sequel, we denote with $(\cdot)_x := \frac{\partial(\cdot)}{\partial x}$ and $(\cdot)_t := \frac{\partial(\cdot)}{\partial t}$ the partial derivatives; when dealing with ODEs (see for instance Definition 2.1), we will use instead the notation $\dot{(\cdot)}$ (except for operators, for which we will always use the notation $(\cdot)_t$), to indicate the time derivative. The stress σ is assumed to satisfy the constitutive equation

$$\sigma := B\varepsilon + \mathcal{P}[m,\varepsilon] + \nu\varepsilon_t - \beta(\theta - \theta^{\mathrm{ref}}) \quad \text{and} \quad \varepsilon := u_x, \tag{2.2}$$

where B is a constant hardening modulus, ε is the ε_{11} component of the strain tensor, $\theta(x,t) > 0$ is the absolute temperature which is one of the unknowns of the problem, ν is the viscosity modulus, β is the thermal expansion coefficient, θ^{ref} is a given referential temperature, $m(x,t) \geq 0$ is a scalar time and space dependent parameter describing the accumulation of fatigue, where $m = 0$ corresponds to zero fatigue and $P[m,\varepsilon]$ is a fatigue dependent Prandtl–Ishlinskii constitutive operator of elasto-plasticity defined below (see also (H6) below for the assumptions on the parameters).

The Prandtl–Ishlinskii model is constructed as a linear combination of basic stop operators $\mathfrak{s}_r[\varepsilon](t)$ with all possible yield points $r > 0$. Given a measurable function $\varphi : [0,\infty)\times(0,\infty) \to [0,\infty)$ satisfying Hypothesis (H1) below, we define the fatigue dependent Prandtl–Ishlinskii operator $P : (W^{1,1}(0,T))^2 \to W^{1,1}(0,T)$ by the integral

$$P[m,\varepsilon](t) = \int_0^\infty \varphi(m(t),r)\, \mathfrak{s}_r[\varepsilon](t)\, \mathrm{d}r\,. \tag{2.3}$$

Let us recall the definition of the stop operator $\mathfrak{s}_r[\varepsilon](t)$.

Definition 2.1. Let $\varepsilon \in W^{1,1}(0,T)$ and $r > 0$ be given. The variational inequality

$$\left.\begin{array}{lll} \varepsilon(t) = \sigma(t) + \xi(t), & & \forall t \in [0,T], \\ |\sigma(t)| \leq r, & & \forall t \in [0,T], \\ \dot{\xi}(t)(\sigma(t) - z) \geq 0, & \text{a.e.} & \forall |z| \leq r\,, \\ \sigma(0) = Q_r(\varepsilon(0)), & & \end{array}\right\} \tag{2.4}$$

where Q_r is the projection of $\mathbb{R}$ onto the interval $[-r,r]$, defines the stop and play operators $\mathfrak{s}_r$ and $\mathfrak{p}_r$ by the formula

$$\sigma(t) = \mathfrak{s}_r[\varepsilon](t)\,, \quad \xi(t) = \mathfrak{p}_r[\varepsilon](t)\,. \tag{2.5}$$

The stop and play operators were introduced in [27]. The parameter r is the *memory variable*, and for each given time t_0, the functions $r \mapsto \mathfrak{p}_r[\varepsilon](t_0)$, $r \mapsto \mathfrak{s}_r[\varepsilon](t_0)$ represent the memory state of the system.

Let us list here some basic properties of the play and stop operators. The proofs are elementary and can be found e.g. in [28].

Proposition 2.2. *Let $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$, $r > 0$ and data $\sigma_1^0, \sigma_2^0 \in [-r, r]$ be given, $\sigma_i = \mathfrak{s}_r[\varepsilon_i]$, $\xi_i = \varepsilon_i - \sigma_i = \mathfrak{p}_r[\varepsilon_i]$, $i = 1, 2$. Then*

(i) $(\sigma_1(t) - \sigma_2(t))(\dot\varepsilon_1(t) - \dot\varepsilon_2(t)) \ \geq\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}(\sigma_1(t) - \sigma_2(t))^2$ *a.e. in* $[0,T]$*;*

(ii) $|\dot\xi_1(t) - \dot\xi_2(t)| + \dfrac{\mathrm{d}}{\mathrm{d}t}|\sigma_1(t) - \sigma_2(t)| \ \leq\ |\dot\varepsilon_1(t) - \dot\varepsilon_2(t)|$ *a.e. in* $[0,T]$*;*

(iii) $|\sigma_1(t) - \sigma_2(t)| \ \leq\ |\sigma_1^0 - \sigma_2^0| + 2 \max\limits_{0 \leq \tau \leq t} |\varepsilon_1(\tau) - \varepsilon_2(\tau)|\ \forall t \in [0,T]$*;*

(iv) $\dot\xi_i(t)\dot\varepsilon_i(t) = \dot\xi_i(t)^2$ *a.e. in* $[0,T]$*.*

We can rewrite (2.4) equivalently in "energetic" form

$$\dot\varepsilon(t)\sigma_r(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\sigma_r^2(t)\right) + r|\dot\xi(t)|. \tag{2.6}$$

Indeed, $\dot\varepsilon(t)\sigma_r(t)$ is the power supplied to the system, part of it is used for the increase of the potential $\frac{1}{2}\sigma_r^2(t)$, and the rest $r|\dot\xi(t)|$ is dissipated.

We extend Prandtl–Ishlinskii operator (2.3) for space dependent inputs in the following way

$$P[m,\varepsilon](x,t) := P[m(x,\cdot),\varepsilon(x,\cdot)]$$

and similarly for the other operators we will deal with later.

Equation (2.6) enables us to establish the energy balance for the Prandtl–Ishlinskii operator (2.3). Indeed, if we define the Prandtl–Ishlinskii potential

$$V[m,\varepsilon](t) = \frac{1}{2}\int_0^\infty \varphi(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r\,, \tag{2.7}$$

and the dissipation operator

$$D[m,\varepsilon](t) = \int_0^\infty r\varphi(m,r)|\mathfrak{p}_r[\varepsilon]_t(t)|\,\mathrm{d}r\,, \tag{2.8}$$

we can write the Prandtl–Ishlinskii energy balance in the form

$$\varepsilon_t(t)P[m,\varepsilon](t) = V[m,\varepsilon]_t(t)\ + D[m,\varepsilon](t) - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r \quad \text{a.e. in } \Omega. \tag{2.9}$$

As a consequence of Proposition 2.2 (iv), we have

$$D[m,\varepsilon](t) \leq |\varepsilon_t(t)|\int_0^\infty r\varphi(m,r)\,\mathrm{d}r \quad \text{a.e. in } \Omega. \tag{2.10}$$

The analysis of the so-called *rainflow method of cyclic fatigue accumulation* in elasto-plastic materials carried out in [11] has shown a close relation between accumulated fatigue and dissipated energy, similarly as in [24]. Mathematically, this is expressed in terms of the evolution equation for the fatigue variable m

$$m_t(x,t) = \int_0^L \lambda(x-y) D[m,\varepsilon](y,t)\,\mathrm{d}y, \tag{2.11}$$

where λ is a nonnegative smooth function with (small) compact support and $D[m,\varepsilon]$ is the fatigue dependent dissipation operator, see (2.8).

The meaning of (2.11) is simple: the fatigue at a point x increases proportionally to the energy dissipated in a neighborhood of the point x; this is our main assumption.

We define the free energy $\mathscr{F}$ associated with the constitutive law (2.2) in the form

$$\mathscr{F} := \mathscr{F}[\theta,\varepsilon,m] := c\theta\left(1-\log\left(\frac{\theta}{\theta^{\mathrm{ref}}}\right)\right) + \frac{B}{2}\varepsilon^2 + V[m,\varepsilon] - \beta(\theta-\theta^{\mathrm{ref}})\varepsilon, \tag{2.12}$$

where the specific heat capacity c is assumed to be constant (see (H6) below). The corresponding entropy $\mathscr{S}$ and internal energy $\mathscr{U}$ are then given by the following formulas

$$\mathscr{S} = \mathscr{S}[\theta,\varepsilon] = -\frac{\partial}{\partial\theta}\mathscr{F}[\theta,\varepsilon] = c\log\left(\frac{\theta}{\theta^{\mathrm{ref}}}\right) + \beta\varepsilon, \tag{2.13}$$

$$\mathscr{U} = \mathscr{U}[\theta,\varepsilon.m] = \mathscr{F}[\theta,\varepsilon,m] + \theta\mathscr{S}[\theta,\varepsilon] = c\theta + \frac{B}{2}\varepsilon^2 + V[m,\varepsilon] + \beta\theta^{\mathrm{ref}}\varepsilon. \tag{2.14}$$

We require the first and the second principle of thermodynamics to hold in the form

$$\mathscr{U}[\theta,\varepsilon,m]_t + q_x = \sigma\varepsilon_t \quad \text{(energy conservation)}, \tag{2.15}$$

$$\mathscr{S}[\theta,\varepsilon]_t + \left(\frac{q}{\theta}\right)_x \geq 0 \quad \text{(Clausius–Duhem inequality)}, \tag{2.16}$$

where q is the heat flux that is assumed to be in the form of Fourier law

$$q = -\kappa\theta_x, \tag{2.17}$$

with a constant heat conductivity κ (see (H6) below).

In terms of the variables θ, ε and m the energy balance (2.15) then reads

$$c\theta_t - \kappa\theta_{xx} = \nu\varepsilon_t^2 + D[m,\varepsilon] - \beta\theta\varepsilon_t - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r, \tag{2.18}$$

where we also used (2.9).

On the other hand, we note that (2.16) formally follows from (2.13), (2.17), (2.18): indeed we have

$$\mathscr{S}[\theta,\varepsilon]_t + \left(\frac{q}{\theta}\right)_x = \frac{\kappa\theta_x^2}{\theta^2} + \frac{\nu\varepsilon_t^2}{\theta} + \frac{1}{\theta}\left(D[m,\varepsilon](t) - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r\right) \geq 0,$$

provided Hypothesis (H1) given in Section 3 for φ_m and (2.11) hold, and we check that the absolute temperature θ stays positive. Concerning this last point, we will find below a positive lower bound for the discrete approximations of the temperature, independent of the discretization parameter, which therefore is preserved in the limit and implies the positivity of the temperature.

We prescribe Cauchy initial data

$$u(x,0) = u^0(x), \quad u_t(x,0) = v^0(x), \quad \theta(x,0) = \theta^0(x), \quad m(x,0) = 0 \tag{2.19}$$

and boundary conditions at $x = 0$ and $x = L$, $t > 0$, given by

$$\sigma(0,t) = -p(t) \quad \text{and} \quad \sigma(L,t) = -f[u(L,\cdot)](t), \tag{2.20}$$

$$\kappa\theta_x(0,t) = 0 \quad \text{and} \quad \kappa\theta_x(L,t) = \alpha(\theta^{\mathrm{ext}} - \theta(L,t)) + c^{\mathrm{bdy}}\theta_t(L,t) - |d[u(L,\cdot)]_t(t)|, \tag{2.21}$$

with constants (see (H6) below) α (a boundary heat transfer coefficient), c^{bdy} (specific heat capacity of the boundary), and θ^{ext} (external temperature); moreover $p(t)$ is a given time dependent external force and f is a boundary contact hysteresis operator satisfying an energy balance equation analogous to (2.6), namely (3.2). For the contact boundary operator f we will assume moreover that Hypotheses (H2)–(H4) below are satisfied. As an example, we may consider, similarly to [30], an operator f in the form $f[u] = g(\mathcal{S}[u])$, where $\mathcal{S}$ is the solution operator $\mathcal{S} : u \mapsto w = \mathcal{S}[u]$ to the variational inequality

$$\begin{cases} w(t) - au(t) \le \widehat{c} & \text{for every } t \in [0,T], \\ w(0) = \min\{au(0) + \widehat{c}, bu(0)\}, & \\ (bu_t(t) - w_t(t))(w(t) - au(t) - z) \ge 0 & \text{a.e. for every } z \le \widehat{c}, \end{cases} \tag{2.22}$$

with constants $a > b > 0$, $\widehat{c} > 0$; here a is the elasticity modulus of the obstacle, b is its hardening modulus, $\widehat{c}$ is its yield point, and g is a twice continuously differentiable nondecreasing function with uniformly bounded derivative vanishing for negative arguments. It is shown in [30] that for this operator the Hypothesis (H2)–(H4) hold. In particular, see [30], the energy balance (3.2) holds provided we choose

$$e[u] := \frac{1}{b}\left(G(w) + \frac{b-a}{a}G\left(\frac{a}{b-a}(bu - w)\right)\right),$$

$$d[u] := \frac{b-a}{ab}\left(G\left(\frac{a}{b-a}(bu - w) + \frac{b\widehat{c}}{b-a}\right) - G\left(\frac{a}{b-a}(bu - w)\right)\right),$$

where $G(z) := \int_0^z g(s)\,\mathrm{d}s$. Identity (3.2) can be easily checked by a straightforward differentiation, taking into account the fact that $bu_t - v_t \ge 0$ almost everywhere, and if $bu_t - v_t > 0$, then $w = au + \widehat{c}$. The boundary condition (2.21) for $x = L$ has to be understood as follows: the terms $\alpha(\theta^{\mathrm{ext}} - \theta(L,t))$ and $|d[u(L,\cdot)]_t|$ represent heat sources. They partially contribute to the inflow $-q$ of heat, and partially to the boundary temperature increase $c^{\mathrm{bdy}}\theta_t(L,t)$.

We consider the problem in the following weak form

$$\int_0^1 (\rho u_{tt}\phi + \sigma\phi_x)\,\mathrm{d}x = -f[u(L,\cdot)](t)\phi(L) + p(t)\phi(0), \;\; \forall \phi \in W^{1,2}(\Omega), \tag{2.23}$$

$$\begin{aligned} \int_0^1 (c\theta_t\psi + \kappa\theta_x\psi_x)\,\mathrm{d}x &= \int_0^1 \left(\nu\varepsilon_t^2 + D[m,\varepsilon] - \beta\theta\varepsilon_t - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r\right)\psi\,\mathrm{d}x \\ &+ |d[u(L,\cdot)]_t(t)|\psi(L) + (\alpha(\theta^{\mathrm{ext}} - \theta(L,t)) - c^{\mathrm{bdy}}\theta_t(L,t))\psi(L), \;\; \forall \psi \in W^{1,2}(\Omega), \end{aligned} \tag{2.24}$$

together with (2.11). The value of L is not relevant for the subsequent mathematical analysis, therefore we assume from now on that $L = 1$.

3. Existence and uniqueness results

We begin this section by introducing some hypotheses on the data f, φ, e and d as well as obvious consequences following from these hypotheses, which we will use later on in this work.

(H1) The Prandtl–Ishlinskii density function φ satisfies the following assumptions: φ is a measurable distribution function: $\varphi : [0, \infty) \times (0, \infty) \to [0, \infty)$, locally Lipschitz continuous in the first variable, and there exist $\widetilde{\varphi}, \varphi^* \in L^1(0, \infty)$ such that $\varphi(m, r) \le \widetilde{\varphi}(r)$, $0 \le -\varphi_m(m, r) \le \varphi^*(r)$, $|\varphi_{mm}(m, r)| \le \varphi^*(r)$ a.e., and $\widetilde{M} := \int_0^\infty r\widetilde{\varphi}(r)\,\mathrm{d}r < \infty$, $M := \int_0^\infty r^2\varphi^*(r)\,\mathrm{d}r < \infty$.

(H2) The operator $f : C^0([0, T]) \to C^0([0, T])$ is Lipschitz continuous in the following sense

$$|f[u_1] - f[u_2]|(t) \le L_f\, |u_1 - u_2|_{[0,t]}\,, \tag{3.1}$$

for every $t \in [0, T]$ and every $u_1, u_2 \in C^0([0, T])$, where L_f is a positive constant and where we denote with $|w|_{[0,t]} := \max\{|w(t)| : t \in [0, t]\}$ the norm of $w \in C^0([0, T])$.

(H3) The operator f maps $W^{1,1}(0, T)$ into $W^{1,1}(0, T)$, and there exist a potential energy operator $e : W^{1,1}(0, T) \to W^{1,1}(0, T)$ and a dissipation operator $d : W^{1,1}(0, T) \to W^{1,1}(0, T)$, both locally Lipschitz continuous, such that for all $u \in W^{1,1}(0, T)$ we have

$$e[u](t) \ge c_0|f[u](t)|^2, \qquad e[u](0) \le c_1|u(0)|^2,$$
$$|d[u]_t(t)| \le c_1|u_t(t)|,$$

for all $t \in [0, T]$, with constants $c_0, c_1 > 0$, and the identity

$$\dot{\omega} f[\omega] - e[\omega]_t = |d[\omega]_t| \qquad \text{a.e. in } [0, T], \tag{3.2}$$

holds for almost every $t \in (0, T)$, for every absolutely continuous input ω, with potential energy operator $e[\omega]$ and dissipation operator $d[\omega]$.

(H4) For $u \in W^{2,1}(0, T)$ we have $f[u] \in W^{1,\infty}(0, T)$, and the "second order energy inequality"

$$\int_0^t f[u]_t u_{tt}\,\mathrm{d}t \ge -c_2|u_t(0)|^2 - c_3 \int_0^t |u_t|^3\,\mathrm{d}t \tag{3.3}$$

holds for all $t \in [0, T]$ with some constants $c_i > 0$, $i = 2, 3$.

(H5) The data have the regularity $p \in W^{2,2}(0, T)$, $u^0 \in W^{2,2}(\Omega)$, $v^0 \in W^{2,2}(\Omega)$, $\theta^0 \in W^{1,2}(\Omega)$, and there exists a constant $\theta_* > 0$ such that $\theta^0(x) \ge \theta_*$ almost everywhere. Furthermore, the compatibility conditions

$$p(0) = \beta(\theta^0(0) - \theta^{\mathrm{ref}}) - Bu^0_x(0) - \widehat{P}(u^0_x(0)) - \nu v^0_x(0), \tag{3.4}$$

$$\widehat{f}(u^0(1)) = \beta(\theta^0(1) - \theta^{\mathrm{ext}}) - Bu^0_x(1) - \widehat{P}(u^0_x(1)) - \nu v^0_x(1), \tag{3.5}$$

hold, where $\widehat{P}$ and $\widehat{f}$ are the initial value mappings $f[u](0) = \widehat{f}(u(0))$ and $\mathcal{P}[m, \varepsilon](0) = \widehat{P}(\varepsilon(0))$. (Note that m satisfies the zero initial condition, (2.19).)

(H6) $\rho, B, \beta, \nu, c, \kappa, \alpha, \theta^{\mathrm{ref}}, \theta^{\mathrm{ext}}$ and c^{bdy} are given positive constants.

(H7) $\lambda : \mathbb{R} \to [0, \infty)$ is a C^1 function with compact support, $\Lambda := \max\{\lambda(x) + |\lambda'(x)|, x \in \mathbb{R}\}$.

Remark 3.1. The assumption that $\varphi(m, r)$ decreases with increasing m corresponds to the observation that the stress of the material decreases with increasing fatigue m. Moreover, it follows from (3.1) and

Proposition 2.2 part (iii), that the initial value functions $\widehat{f}, \widehat{P} : \mathbb{R} \to \mathbb{R}$ are well defined and Lipschitz continuous.

For simplicity we set from now on

$$\mathcal{K}[m,\varepsilon](x,t) := -\frac{1}{2}\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon](x,t)\,\mathrm{d}r. \tag{3.6}$$

We denote here and in the sequel the set $Q_t := (0,1)\times(0,t)$, for $t \in [0,T]$.

The main result of this paper reads as follows.

Theorem 3.2. *Assume that* (H1)–(H7) *hold. Then the system* (2.2), (2.23)–(2.24) *and* (2.11) *with* $L = 1$ *has a unique solution* (u,θ,m) *such that* $u, u_x, u_{xt}, \theta, m_t \in C^0(\overline{Q}_T)$, $u_{xxt}, \theta_x \in L^\infty(0,T;L^2(0,1))$, $u_{xtt}, \theta_t \in L^2(Q_T)$, *and* $\theta_t(1,\cdot) \in L^2(0,T)$.

4. Proof of Theorem 3.2: existence

4.1. Strategy of the proof: novelties and main difficulties

The strategy of the existence proof is classical (see for instance [37]): discretization, a priori estimates and passage to the limit by compactness. The presence of the beam equation suggests to use a space discretization scheme which turns to be more convenient to deal with space derivatives instead of other kinds of discretization schemes (like e.g. [49, Chapter IX]).

The main goal when dealing with hysteresis problems is to get enough regularity to pass to the limit in the discrete equations, in particular with respect to the nonlinear terms. We recall indeed that non-differentiability and non-locality in time of hysteresis operators entail a loss of compactness, so standard techniques for the derivation of a priori estimates do not apply, and for limit processes with hysteresis nonlinearities the usual approach using weak convergence in L^p spaces does not work; instead, uniform convergence with respect to the time variable is mandatory. As a consequence, new techniques have to be designed to recover the compactness necessary for the existence proofs, which is a challenging mathematical task (see for instance [27], [12], [28], [49]).

In the particular case of the present paper, we need to retrieve uniform convergence both in the fatigue and in the strain terms. The strong convergence in the fatigue term (4.66) constitutes the main novelty (and also the principal difficulty) of the paper; it is performed through a complicated procedure based essentially on the properties of the fatigue equation (see Section 4.5). Concerning the strain terms, higher order estimates (4.54)–(4.55) are required; it turns out indeed that the energy estimate (4.25) (which is important because it allows us to deduce that $\dot{u}_k$ and θ_k remain globally bounded, so that the existence of solutions in the whole interval $[0,T]$ can be deduced) is not enough to perform the limit procedure.

Note the role of the temperature: a key estimate is (4.48), but to be able to test the equation for the temperature by $\dot{\theta}_k$, it is necessary to achieve more regularity than the one obtained by the energy estimate (which gives only L^1-estimate in the space variable). Here the Dafermos estimate (see Section 4.4.2) comes into play and gives more regularity in the space variable; more in details (4.28) provides space regularity in a L^q space with an exponent sufficiently large (i.e. $q > 2$). It is worth noticing that the Dafermos estimate is a useful trick that can be applied because the model is one-dimensional. In other situations where the model studied was multi-dimensional, more complicated procedures have been performed to achieve the necessary regularity for the temperature (see for instance [21]).

Let us note that compared to the computations performed in [31], we have to deal here with additional terms coming from the presence of fatigue in the model, and we use Hypothesis (H1) and (H6) to estimate these terms.

4.2. Discretization

We fix a discretization parameter $n \in \mathbb{N}$ and consider the following system

$$\rho\, \ddot{u}_k = n(\sigma_k - \sigma_{k-1}), \quad k = 1, \dots, n, \tag{4.1}$$

$$c\, \dot{\theta}_k = n^2 \kappa(\theta_{k+1} - 2\theta_k + \theta_{k-1}) + \nu\, \dot{\varepsilon}_k^2 + \, \dot{m}_k\, \mathcal{K}_k + D_k - \beta\theta_k\, \dot{\varepsilon}_k\,, \quad k = 1, \dots, n-1, \tag{4.2}$$

$$\sigma_k = B\varepsilon_k + \mathcal{P}[m_k, \varepsilon_k] + \nu\, \dot{\varepsilon}_k - \beta(\theta_k - \theta^{\mathrm{ref}})\,, \quad k = 1, \dots, n-1, \tag{4.3}$$

$$\varepsilon_k = n(u_{k+1} - u_k), \quad k = 1, \dots, n-1, \tag{4.4}$$

$$m_k = \int_0^t \mathcal{D}_k^*(\tau)\, \mathrm{d}\tau, \tag{4.5}$$

where

$$\mathcal{P}[m_k, \varepsilon_k](t) = \int_0^\infty \varphi(m_k(t), r)\mathfrak{s}_r[\varepsilon_k](t)\, \mathrm{d}r,$$

$$\mathcal{K}_k[m_k, \varepsilon_k](t) = -\frac{1}{2}\int_0^\infty \varphi_m(m_k(t), r)\mathfrak{s}_r^2[\varepsilon_k](t)\, \mathrm{d}r \in \left[0, \frac{M}{2}\right],$$

$$D_k[m_k, \varepsilon_k](t) = \int_0^\infty \varphi(m_k(t), r)\, \mathfrak{s}_r[\varepsilon_k](t)(\varepsilon_k - \mathfrak{s}_r[\varepsilon_k])_t(t)\, \mathrm{d}r \geq 0,$$

$$\mathcal{D}_k^*(t) = \frac{1}{n}\sum_{j=1}^{n-1} \lambda_{k-j} D_j(t) \geq 0 \qquad \lambda_i = \lambda(i/n),$$

with “boundary conditions”

$$\sigma_0(t) = -p(t) \quad \text{and} \quad \sigma_n(t) = -f[u_n](t), \tag{4.6}$$

$$\theta_0(t) = \theta_1(t) \quad \text{and} \quad n\kappa(\theta_n(t) - \theta_{n-1}(t)) = \alpha(\theta^{\mathrm{ext}} - \theta_n(t)) - c^{\mathrm{bdy}}\dot{\theta}_n(t) + |d[u_n]_t(t)|, \tag{4.7}$$

as a discrete counterpart of (2.20)–(2.21). The second equation in (4.7) is the definition of θ_n as a solution of the differential equation

$$\frac{c^{\mathrm{bdy}}}{\alpha + n\kappa}\dot{\theta}_n + \theta_n = \frac{1}{\alpha + n\kappa}(n\kappa\theta_{n-1} + \alpha\theta^{\mathrm{ext}} + |d[u_n]_t|)\,. \tag{4.8}$$

Furthermore, we define $\varepsilon_0(t)$ and $\varepsilon_n(t)$ as solutions to the differential equation (4.3) for $k = 0$ and $k = n$, with $\sigma_0, \sigma_n, \theta_0, \theta_n, m_0, m_n$ given by (4.5)–(4.7), and with initial conditions $\varepsilon_0(0) = u_x^0(0)$ and $\varepsilon_n(0) = u_x^0(1)$.

Observe that (4.1)–(4.5) is a system of ordinary differential equations with a locally Lipschitz continuous right hand side. Hence, for every choice of initial conditions

$$u_k(0) = u_k^0\,, \quad \dot{u}_k(0) = v_k^0\,, \quad \theta_k(0) = \theta_k^0, \quad m_k(0) = 0, \quad \text{for } \ k = 1, \dots, n, \tag{4.9}$$

it admits a unique absolutely continuous solution on a maximal interval $[0, T_n)$, $T_n \le T$. In view of (2.19), we choose the initial data ($k = 1, \dots, n-1$) as

$$u_k^0 = n \int_{(k-1)/n}^{k/n} u^0(x)\,\mathrm{d}x, \qquad v_k^0 = n \int_{(k-1)/n}^{k/n} v^0(x)\,\mathrm{d}x, \qquad \theta_k^0 = n \int_{(k-1)/n}^{k/n} \theta^0(x)\,\mathrm{d}x. \tag{4.10}$$

Using the summation by parts formulas for arbitrary test sequences $\phi_1, \dots, \phi_n$ and $\psi_1, \dots, \psi_n$,

$$\sum_{k=1}^{n} (\sigma_k - \sigma_{k-1})\phi_k = \sigma_n \phi_n - \sigma_0 \phi_1 - \sum_{k=1}^{n-1} (\phi_{k+1} - \phi_k)\sigma_k, \tag{4.11}$$

$$\sum_{k=1}^{n-1} (\theta_{k+1} - 2\theta_k + \theta_{k-1})\psi_k = (\theta_n - \theta_{n-1})\psi_n - (\theta_1 - \theta_0)\psi_1 - \sum_{k=1}^{n-1} (\theta_{k+1} - \theta_k)(\psi_{k+1} - \psi_k), \tag{4.12}$$

taking into account (4.6)–(4.7), we may rewrite (4.1)–(4.2) in a variational form as

$$\frac{\rho}{n} \sum_{k=1}^{n} \ddot{u}_k \phi_k + \sum_{k=1}^{n-1} (\phi_{k+1} - \phi_k)\sigma_k + f[u_n]\phi_n = p\phi_1\,, \tag{4.13}$$

$$\begin{aligned} &\frac{c}{n} \sum_{k=1}^{n-1} \dot{\theta}_k \psi_k + n\kappa \sum_{k=1}^{n-1} (\theta_{k+1} - \theta_k)(\psi_{k+1} - \psi_k) \\ &= \frac{1}{n} \sum_{k=1}^{n-1} \left(\nu \dot{\varepsilon}_k^2 + \dot{m}_k\, \mathcal{K}_k + D_k - \beta\theta_k \dot{\varepsilon}_k\right) \psi_k + \left(\alpha(\theta^{\mathrm{ext}} - \theta_n) - c^{\mathrm{bdy}} \dot{\theta}_n + |d[u_n]_t|\right) \psi_n, \end{aligned} \tag{4.14}$$

with σ_k ε_k and m_k defined by (4.3)–(4.5).

4.3. Positivity of the temperature

We first check that on $[0, T_n)$, all θ_k remain strictly positive. To prove this, we first choose in (4.14) all ψ_k nonnegative. Then it easily follows from the Hypotheses that the right hand side of (4.14) is bounded from below by

$$-\gamma \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^2 \psi_k + (\alpha(\theta^{\mathrm{ext}} - \theta_n) - c^{\mathrm{bdy}} \dot{\theta}_n)\psi_n \tag{4.15}$$

with $\gamma = \beta^2/(4\nu)$. This is the essential estimate and the rest of the proof of the positivity of the temperature follows the lines of [31] and we can show that for the solution $v : [0, \infty) \to (0, \infty)$ to the differential equation

$$\begin{cases} c\dot{v}(t) = -\gamma v^2(t), \\ v(0) = \min\{\theta_*, \theta^{\mathrm{ext}}\} \end{cases} \tag{4.16}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{c}{n} \sum_{k=1}^{n-1} (v - \theta_k)^+ + c^{\mathrm{bdy}} (v - \theta_n)^+ \right) \le -\gamma \frac{1}{n} \sum_{k=1}^{n-1} (v - \theta_k)^+ (v + \theta_k) - \alpha(v - \theta_n)^+, \tag{4.17}$$

where $r^+ := \max(r, 0)$. According to Hypothesis (H5) and (4.10), we may infer that there exists $t_n \in (0, T_n)$ such that $\theta_k(t) > 0$ for $t \in [0, t_n)$. Let us set

$$\bar{t}_n := \inf\{t \in [0, T_n) : \exists k : \theta_k(t) \le 0\} \ge t_n.$$

By (4.17), we have $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c}{n}\sum_{k=1}^{n}(v-\theta_k)^+\right) \le 0$ for almost every $t \in (0, \bar{t}_n)$ and $\theta_k(0) \ge v(0)$ for all k, hence $\theta_k(\bar{t}_n) \ge v(\bar{t}_n) > 0$ for all k, which is a contradiction. We conclude that

$$\theta_k(t) \ge v(t) > 0 \quad \text{for all } k = 1, \dots, n \quad \text{in } (0, T_n). \tag{4.18}$$

4.4. A priori estimates

Here we denote by $C > 0$ suitable constants depending on the data and independent of n.

4.4.1. Estimate 1: energy estimate

We derive now the first energy estimate similarly as in [31], we additionally have to deal with the terms entering the equation in connection with fatigue. On the one hand, we test (4.13) with $\phi_k = \dot{u}_k$ and we use (4.3) and (4.4) to get

$$\frac{\rho}{n}\sum_{k=1}^{n}\ddot{u}_k\dot{u}_k + \frac{1}{n}\sum_{k=1}^{n-1}(B\varepsilon_k + \mathcal{P}[m_k, \varepsilon_k] + \nu\dot{\varepsilon}_k - \beta(\theta_k - \theta^{\mathrm{ref}}))\dot{\varepsilon}_k + f[u_n]\dot{u}_n = p\,\dot{u}_1,$$

which, by employing (2.9) and (3.2), gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{2n}\sum_{k=1}^{n}|\dot{u}_k|^2 + \frac{1}{n}\sum_{k=1}^{n-1}\left(B\varepsilon_k^2 + V[m_k, \varepsilon_k] + \beta\theta^{\mathrm{ref}}\varepsilon_k\right) + e[u_n]\right) \\
&+ \frac{1}{n}\sum_{k=1}^{n-1}(\nu|\dot{\varepsilon}_t|^2 + D_k + \dot{m}_k\mathcal{K}_k + |d[u_n]_t| = p\,\dot{u}_1 + \frac{1}{n}\sum_{k=1}^{n-1}\beta\theta_k\dot{\varepsilon}_k.
\end{aligned} \tag{4.19}$$

On the other hand, we test (4.14) with $\psi_k = 1$ and we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c}{n}\sum_{k=1}^{n-1}\theta_k + c^{\mathrm{bdy}}\theta_n\right) = \frac{1}{n}\sum_{k=1}^{n-1}(\nu|\dot{\varepsilon}_k|^2 + D_k + \dot{m}_k\mathcal{K}_k) + \alpha(\theta^{\mathrm{ext}} - \theta_n) + |d[u_n]_t| - \frac{1}{n}\sum_{k=1}^{n-1}\beta\theta_k\dot{\varepsilon}_k. \tag{4.20}$$

Therefore, adding (4.19) and (4.20) some terms cancel out and we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{2n}\sum_{k=1}^{n}|\dot{u}_k|^2 + \frac{1}{n}\sum_{k=1}^{n-1}\left(B\varepsilon_k^2 + V[m_k, \varepsilon_k] + \beta\theta^{\mathrm{ref}}\varepsilon_k + c\theta_k\right) + e[u_n] + c^{\mathrm{bdy}}\theta_n\right) \\
&+ \alpha(\theta_n - \theta^{\mathrm{ext}}) = p\dot{u}_1.
\end{aligned} \tag{4.21}$$

Notice that the term

$$\mathcal{E}_n(t) := \frac{\rho}{2n}\sum_{k=1}^{n}|\dot{u}_k(t)|^2 + \frac{1}{n}\sum_{k=1}^{n-1}\left(B\varepsilon_k^2(t) + V[m_k, \varepsilon_k](t) + \beta\theta^{\mathrm{ref}}\varepsilon_k(t) + c\theta_k(t)\right) + e[u_n](t) + c^{\mathrm{bdy}}\theta_n(t) \tag{4.22}$$

under the time derivative in (4.21) represents the total energy of the system; we also observe that we already know that all θ_k are positive. Hence, by using Hypothesis (H1) (and in particular its consequence that $V[m_k, \varepsilon_k](t) \geq 0$ for all $t \in [0, T_n]$), (H3) and (2.7) it follows that $\mathcal{E}_n(t)$ is bounded from below by a constant.

At this point, we integrate (4.21) over $(0,t)$ and after integration by parts, we get (recall that $T_n \leq T$)

$$\forall t \in [0, T_n): \mathcal{E}_n(t) \leq \mathcal{E}_n(0) + T\alpha\theta^{\text{ext}} + |p(t)u_1(t) - p(0)u_1(0)| + \int_0^t |p_t(\tau)||u_1(\tau)| \, \mathrm{d}\tau. \tag{4.23}$$

By virtue of Hypotheses (H1), (H3), (H5) and Proposition 2.2 (iii), using the same idea as in [31] for the bound of the term $u_1(0)$, we estimate the initial energy as

$$\begin{aligned} \mathcal{E}_n(0) &\leq C \left(1 + \frac{1}{n} \sum_{k=1}^{n} |\dot{u}_k(0)|^2 + \frac{1}{n} \sum_{k=1}^{n-1} |\varepsilon_k(0)|^2 + \|\theta^0\|_{L^\infty(0,1)} \right) \\ &\leq C \left(1 + \|v^0\|^2_{L^2(0,1)} + \|u^0_x\|^2_{L^2(0,1)} + \|\theta^0\|_{L^\infty(0,1)} \right), \end{aligned} \tag{4.24}$$

and we obtain from (4.23), (4.24), the discrete Hölder inequality and the discrete Gronwall's lemma that

$$\forall t \in [0, T_n): \frac{1}{n} \sum_{k=1}^{n} |\dot{u}_k(t)|^2 + \frac{1}{n} \sum_{k=1}^{n-1} \left(|\varepsilon_k(t)|^2 + \theta_k(t) \right) + e[u_n](t) + \theta_n(t) \leq C. \tag{4.25}$$

As a consequence of (4.25), $\dot{u}_k$ and θ_k for each k remain globally bounded in $(0, T_n)$, and we can extend it to the whole interval $[0, T]$ by classical results of ODEs theory. The final bound is independent of n and the system (4.1)–(4.7) with initial conditions (4.9) admits for an arbitrary $n \in \mathbb{N}$ a unique absolutely continuous solution in the whole interval $[0, T]$.

4.4.2. Estimate 2: the Dafermos estimate

Following the idea developed in [14], we take in (4.14) $\psi_k = -\theta_k^{-1/3}$ and similarly as in [31] we obtain for all $t \in [0, T]$, after integrating over $(0, t)$, and estimating the non-positive terms by 0 from above, the following inequality

$$\begin{aligned} &\int_0^t \left(\frac{\nu}{n} \sum_{k=1}^{n-1} \theta_k^{-1/3}(\tau) \dot{\varepsilon}_k^2(\tau) + 3n\kappa \sum_{k=1}^{n-1} |\theta_{k+1}^{1/3}(\tau) - \theta_k^{1/3}(\tau)|^2 \right) \mathrm{d}\tau \\ &\leq \int_0^t \frac{\beta}{n} \sum_{k=1}^{n-1} \theta_k^{2/3}(\tau) |\dot{\varepsilon}_k(\tau)| \, \mathrm{d}\tau + \frac{3c}{2n} \sum_{k=1}^{n-1} \theta_k^{2/3}(t) + \alpha \int_0^\tau \theta_n^{2/3}(\tau) \, \mathrm{d}\tau + c^{\text{bdy}} \theta_n^{2/3}(t). \end{aligned} \tag{4.26}$$

On the other hand, we may deduce from (4.25) that the last three terms on the right hand side of (4.26) are bounded by a constant. Using the Hölder's inequality, (A.4) applied to the particular case where $v_k = \theta_k^{1/3}$ for $k = 1, \ldots, n-1$, with the choice $s = 3$, $p = 2$ and $q = 5$ and consequently by (A.1) with $\gamma = 4/25$, we deduce (for more details see [31])

$$\int_0^t \left(\frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{-1/3}(\tau) |\dot{\varepsilon}_k(\tau)|^2 + n \sum_{k=1}^{n-1} |\theta_{k+1}^{1/3}(\tau) - \theta_k^{1/3}(\tau)|^2 \right) \mathrm{d}\tau \leq C. \tag{4.27}$$

Using once again (A.4) for $v_k = \theta_k^{1/3}$ for $k = 1, \ldots, n-1$, with the choice $s = 3$, $p = 2$ and $q = 8$, and consequently by (A.1) $\gamma = 1/4$, we may deduce that

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}(\tau)\,\mathrm{d}\tau \le C. \tag{4.28}$$

On the other hand, we integrate (4.19) over $(0,t)$ and thanks to assumptions (H1), (H3) and (2.7), we find that

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\nu|\dot\varepsilon_k(\tau)|^2\,\mathrm{d}\tau \le C + |\beta|\left(\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{7/3}(\tau)\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{-1/3}(\tau)|\dot\varepsilon_k(\tau)|^2\,\mathrm{d}\tau\right)^{1/2},$$

which according to (4.27) and (4.28) leads to

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k(\tau)|^2\,\mathrm{d}\tau \le C. \tag{4.29}$$

4.4.3. Estimate 3: higher order estimates

• **First higher order estimate:** First of all we differentiate (4.1) with respect to time and consider the corresponding variational formulation,

$$\frac{\rho}{n}\sum_{k=1}^{n}\dddot{u}_k\phi_k + \sum_{k=1}^{n-1}(\phi_{k+1}-\phi_k)\dot\sigma_k + f[u_n]_t\phi_n = \dot p\phi_1, \tag{4.30}$$

where we used (4.4) and the formula of summation by parts (4.11). We take $\phi_k = \ddot u_k$, use (4.3) and we obtain

$$\frac{\rho}{n}\sum_{k=1}^{n}\dddot{u}_k\ddot u_k + \frac{1}{n}\sum_{k=1}^{n-1}\ddot\varepsilon_k(B\dot\varepsilon_k + \mathcal{P}[m_k,\varepsilon_k]_t + \nu\ddot\varepsilon_k - \beta\dot\theta_k) + f[u_n]_t\ddot u_n = (\dot p\,\dot u_1)_t - \ddot p\,\dot u_1. \tag{4.31}$$

We integrate now (4.31) over $(0,t)$ and using Hypotheses (H1), (H3), (H4) and (H5) we obtain

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(t)|^2 + \frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k(t)|^2 + \frac{1}{n}\sum_{k=1}^{n-1}\int_0^t|\ddot\varepsilon_k(\tau)|^2\,\mathrm{d}\tau \le C\Bigg(1 + \frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(0)|^2 \\ + |\dot u_1(t)| + \int_0^t(|\dot u_1(\tau)|^2 + |\dot u_n(\tau)|^3)\,\mathrm{d}\tau + \frac{1}{n}\sum_{k=1}^{n-1}\int_0^t|\dot\theta_k(\tau)|^2\,\mathrm{d}\tau\Bigg).\end{aligned} \tag{4.32}$$

Notice that we used here the direct estimate for $P[m_k,\varepsilon_k]_t$.

The initial acceleration term $\frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(0)|^2$ is estimated by using the compatibility conditions (3.4)–(3.5). We distinguish two situations: the case $k = 2,\dots,n-1$ and the cases $k = 1$, $k = n$.

▷ *Estimate of the initial acceleration term for* $k = 2,\dots,n-1$: To this aim, we first observe that (4.1) and (4.3) imply

$$|\ddot u_k(0)| \le C\,n\,(|\varepsilon_k(0) - \varepsilon_{k-1}(0)| + |\dot\varepsilon_k(0) - \dot\varepsilon_{k-1}(0)| + |\theta_k(0) - \theta_{k-1}(0)|) \tag{4.33}$$

for all $k = 2,\dots,n-1$. Indeed, to estimate the term $\mathcal{P}[m_k,\varepsilon_k](0) - \mathcal{P}[m_{k-1},\varepsilon_{k-1}](0)$, we used the following estimate, which we evaluate at $t = 0$

$$
\begin{aligned}
&|\mathcal{P}[m_k, \varepsilon_k](t) - \mathcal{P}[m_{k-1}, \varepsilon_{k-1}](t)| \qquad (4.34)\\
&= \left| \int_0^\infty \varphi(m_k, r)\mathfrak{s}_r[\varepsilon_k] - \varphi(m_{k-1}, r)\mathfrak{s}_r[\varepsilon_{k-1}]\,\mathrm{d}r \right| \\
&= \left| \int_0^\infty (\varphi(m_k, r) - \varphi(m_{k-1}, r))\mathfrak{s}_r[\varepsilon_k]\,\mathrm{d}r + \int_0^\infty \varphi(m_{k-1}, r)(\mathfrak{s}_r[\varepsilon_k] - \mathfrak{s}_r[\varepsilon_{k-1}])\,\mathrm{d}r \right| \\
&\le C \int_0^\infty |m_k - m_{k-1}|(t)\, r\,\mathrm{d}r + \left(|\varepsilon_k(0) - \varepsilon_{k-1}(0)| + C \max_k |\varepsilon_k - \varepsilon_{k-1}| \right) \int_0^\infty \widetilde{\varphi}(r)\,\mathrm{d}r
\end{aligned}
$$

and we use the fact that, from (4.9), $m_k(0) = 0$ for $k = 1, \dots, n$. At this point, we deduce from (4.4), (4.9), (4.10) and from the Cauchy–Schwarz inequality that

$$
\begin{aligned}
|\varepsilon_k(0) - \varepsilon_{k-1}(0)| &= n^2 \left| \int_{(k-1)/n}^{k/n} u^0\left(x + \frac{1}{n}\right) - 2u^0(x) + u^0\left(x - \frac{1}{n}\right)\mathrm{d}x \right| \\
&\le \frac{C}{\sqrt{n}} \left(\int_{(k-2)/n}^{(k+1)/n} |u^0_{xx}|^2\,\mathrm{d}x \right)^{1/2}
\end{aligned}
$$

for all $k = 2, \dots, n-1$. Hence, it follows that

$$
n \sum_{k=2}^{n-1} |\varepsilon_k(0) - \varepsilon_{k-1}(0)|^2 \le C \int_0^1 |u^0_{xx}|^2\,\mathrm{d}x. \qquad (4.35)
$$

The other terms in (4.33) are treated similarly, so we may conclude that

$$
\frac{1}{n} \sum_{k=2}^{n-1} |\ddot{u}_k(0)|^2 \le C \left(\int_0^1 \left(|u^0_{xx}|^2 + |v^0_{xx}|^2 + |\theta^0_x|^2\right)\,\mathrm{d}x \right). \qquad (4.36)
$$

▷ *Estimate of the initial acceleration term for $k = 1$ and $k = n$*: On the other hand, (4.1), (4.3), (4.6) together with (3.4)–(3.5) give

$$
\rho\ddot{u}_1(0) = n\left(B(\varepsilon_1(0) - u^0_x(0)) + \widehat{P}(\varepsilon_1(0)) - \widehat{P}(u^0_x(0)) + \nu(\dot{\varepsilon}_1(0) - v^0_x(0)) - \beta(\theta_1(0) - \theta^0(0))\right), \quad (4.37)
$$

$$
\begin{aligned}
\rho\ddot{u}_n(0) = &-n\Big(B(\varepsilon_{n-1}(0) - u^0_x(1)) + \widehat{P}(\varepsilon_{n-1}(0)) - \widehat{P}(u^0_x(1)) + \nu(\dot{\varepsilon}_{n-1}(0) - v^0_x(1)) \\
&- \beta(\theta_{n-1}(0) - \theta^0(1)) + \widehat{f}(u_{n-1}(0)) - \widehat{f}(u^0(1))\Big). \qquad (4.38)
\end{aligned}
$$

We may observe that, by virtue of Proposition 2.2 (iii) and (2.19),

$$
n|\widehat{P}(\varepsilon_{n-1}(0)) - \widehat{P}(u^0_x(1))| \le Cn|\varepsilon_{n-1}(0) - u^0_x(1)|.
$$

Once again using (4.4), (4.9), (4.10) and the Cauchy–Schwarz inequality, it comes that (for more details, see [31])

$$n|\varepsilon_{n-1}(0) - u_x^0(1)| \le 2\sqrt{2n}\left(\int\limits_{1-(2/n)}^{1} |u_{xx}^0(z)|^2 \, \mathrm{d}z\right)^{1/2}.$$

All the other differences appearing in (4.37)–(4.38) are treated similarly, and in combination with (4.36), we find

$$\frac{1}{n}\sum_{k=1}^{n} |\ddot{u}_k(0)|^2 \le C\left(\int\limits_0^1 |u_{xx}^0|^2 + |v_{xx}^0|^2 + |\theta_x^0|^2 \, \mathrm{d}x\right) \le C. \tag{4.39}$$

▷ *Estimate of the boundary terms*: We estimate the boundary terms in (4.32) involving $\dot{u}_1$ and $\dot{u}_n$ using (A.4) with $q = \infty$, $p = s = 2$. Then $\gamma = 1/2$ and we see by virtue of (4.25) that they are absorbed by the left hand side. We may conclude that

$$\frac{1}{n}\sum_{k=1}^{n}|\ddot{u}_k(t)|^2 + \frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k(t)|^2 + \frac{1}{n}\sum_{k=1}^{n-1}\int\limits_0^t|\ddot{\varepsilon}_k(\tau)|^2 \, \mathrm{d}\tau \le C\left(1 + \frac{1}{n}\sum_{k=1}^{n-1}\int\limits_0^t|\dot{\theta}_k(\tau)|^2 \, \mathrm{d}\tau\right). \tag{4.40}$$

• **Second higher order estimate:** First of all we have by (4.3) that

$$|\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1}| \le \frac{1}{\nu}\left(B(\varepsilon_k - \varepsilon_{k-1}) + |\mathcal{P}[m_k, \varepsilon_k] - \mathcal{P}[m_{k-1}, \varepsilon_{k-1}]| + |\beta||\theta_k - \theta_{k-1}| + |\sigma_k - \sigma_{k-1}|\right).$$

We square this inequality, sum over k and substitute from (4.1) to obtain for all $t \in [0, T]$ that

$$\begin{aligned} &n\sum_{k=1}^{n} |\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1}|^2(t) \\ &\le \frac{C}{n}\sum_{k=1}^{n}\left(|\ddot{u}_k|^2 + n^2(\varepsilon_k - \varepsilon_{k-1})^2 + n^2|\mathcal{P}[m_k, \varepsilon_k] - \mathcal{P}[m_{k-1}, \varepsilon_{k-1}]|^2 + n^2|\theta_k - \theta_{k-1}|^2\right)(t). \end{aligned} \tag{4.41}$$

We estimate now the right hand side. The following estimate holds because of (4.5), (2.10) and (4.29)

$$\begin{aligned} |m_k - m_{k-1}|(t) &\le C\int\limits_0^t\left(\frac{1}{n}\sum_{j=1}^{n}|\lambda_{k-j} - \lambda_{k-j-1}|D_j(\tau)\right)\mathrm{d}\tau \\ &\le C\int\limits_0^t\left(\frac{1}{n}\sum_{j=1}^{n}|\dot{\varepsilon}_j(\tau)||\lambda_{k-j} - \lambda_{k-j-1}|\right)\mathrm{d}\tau \\ &\le C\left(\frac{1}{n^2}\int\limits_0^t\sum_{j=1}^{n}|\dot{\varepsilon}_j(\tau)| \, \mathrm{d}\tau\right) \le \frac{C}{n}. \end{aligned} \tag{4.42}$$

Note that by (4.34) and (4.42) we have

$$|\mathcal{P}[m_k, \varepsilon_k](t) - \mathcal{P}[m_{k-1}, \varepsilon_{k-1}](t)|^2 \le C\left(\frac{1}{n^2} + |\varepsilon_k(0) - \varepsilon_{k-1}(0)|^2 + \int\limits_0^t|\dot{\varepsilon}_k(\tau) - \dot{\varepsilon}_{k-1}(\tau)|^2 \, \mathrm{d}\tau\right). \tag{4.43}$$

Therefore using (4.43) in (4.41), we find

$$
\begin{aligned}
n\sum_{k=1}^{n}|\dot\varepsilon_k(t)-\dot\varepsilon_{k-1}(t)|^2 &\le C\Big(1+n\sum_{k=1}^{n}|\varepsilon_k(0)-\varepsilon_{k-1}(0)|^2+\frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(t)|^2 \\
&+n\sum_{k=1}^{n}|\theta_k(t)-\theta_{k-1}(t)|^2+\int_0^t n\sum_{k=1}^{n}|\dot\varepsilon_k(\tau)-\dot\varepsilon_{k-1}(\tau)|^2\,\mathrm{d}\tau\Big).
\end{aligned}
\tag{4.44}
$$

We deal now with the last term on the right hand side of (4.44). To this aim, let us introduce

$$
w(t):=\int_0^t n\sum_{k=1}^{n}|\dot\varepsilon_k(\tau)-\dot\varepsilon_{k-1}(\tau)|^2\,\mathrm{d}\tau
$$

and

$$
g(t):=1+n\sum_{k=1}^{n}|\varepsilon_k(0)-\varepsilon_{k-1}(0)|^2+\frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(t)|^2+n\sum_{k=1}^{n}|\theta_k(t)-\theta_{k-1}(t)|^2.
$$

Clearly with these notations, (4.44) can be rewritten as a differential inequality

$$
\dot w(t)-Cw(t)\le Cg(t). \tag{4.45}
$$

Multiplying (4.45) by $\exp(-C\,t)$, we get

$$
w(t)\le\int_0^t Cg(\tau)\exp(-C(t-\tau))\,\mathrm{d}\tau,
$$

which gives, using (4.35),

$$
\int_0^t n\sum_{k=1}^{n}|\dot\varepsilon_k(\tau)-\dot\varepsilon_{k-1}(\tau)|^2\,\mathrm{d}\tau\le C\left(1+\int_0^t\left(\frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(\tau)|^2+n\sum_{k=1}^{n}|\theta_k(\tau)-\theta_{k-1}(\tau)|^2\right)\mathrm{d}\tau\right). \tag{4.46}
$$

We may conclude by combining (4.46) with (4.44) and (4.35) to get

$$
\begin{aligned}
n\sum_{k=1}^{n}|\dot\varepsilon_k(t)-\dot\varepsilon_{k-1}(t)|^2 &\le C\Bigg(1+\frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(t)|^2+n\sum_{k=1}^{n}|\theta_k(t)-\theta_{k-1}(t)|^2 \\
&+\int_0^t\left(\frac{1}{n}\sum_{k=1}^{n}|\ddot u_k(\tau)|^2+n\sum_{k=1}^{n}|\theta_k(\tau)-\theta_{k-1}(\tau)|^2\right)\mathrm{d}\tau\Bigg) \\
&\le C\Bigg(1+n\sum_{k=1}^{n}|\theta_k(t)-\theta_{k-1}(t)|^2+\int_0^t\left(\frac{1}{n}\sum_{k=1}^{n-1}|\dot\theta_k(\tau)|^2+n\sum_{k=1}^{n}|\theta_k(\tau)-\theta_{k-1}(\tau)|^2\right)\mathrm{d}\tau\Bigg),
\end{aligned}
\tag{4.47}
$$

where in the last line we used (4.40).

• **Estimate for the temperature:** We take $\psi_k=\dot\theta_k$ in (4.14). The right hand side is estimated via Hölder's inequality, we integrate the equation over $(0,t)$, note that the last term on the right hand side is bounded after integration by virtue of (H3) and (4.29) and we get

$$
\begin{aligned}
&\int_0^t \left(\frac{1}{n} \sum_{k=1}^{n-1} |\dot{\theta}_k(\tau)|^2 + |\dot{\theta}_n(\tau)|^2 \right) \mathrm{d}\tau + n \sum_{k=1}^{n-1} |\theta_{k+1}(t) - \theta_k(t)|^2 \\
&\le C \left(1 + \frac{1}{n} \int_0^t \sum_{k=1}^{n-1} (|\dot{\varepsilon}_k(\tau)|^4 + |\theta_k(\tau)|^4) \right) \mathrm{d}\tau.
\end{aligned}
\tag{4.48}
$$

• **Final estimates and conclusions:** For later purposes we define for a generic sequence $\{\varphi_k : k = 0, 1, \dots, n\}$ with the notations $\Delta_k \varphi = n(\varphi_k - \varphi_{k-1})$, and $\Delta_k^2 \varphi = n^2(\varphi_{k+1} - 2\varphi_k + \varphi_{k-1})$, piecewise constant and piecewise linear interpolations

$$
\overline{\varphi}^{(n)}(x) = \begin{cases} \varphi_k & \text{for } \ x \in \left[\dfrac{k-1}{n}, \dfrac{k}{n} \right), \ k = 1, \dots, n-1\,, \\[2ex] \varphi_{n-1} & \text{for } \ x \in \left[\dfrac{n-1}{n}, 1 \right], \end{cases}
\tag{4.49}
$$

$$
\underline{\varphi}^{(n)}(x) = \begin{cases} \varphi_{k-1} & \text{for } \ x \in \left[\dfrac{k-1}{n}, \dfrac{k}{n} \right), \ k = 1, \dots, n-1\,, \\[2ex] \varphi_{n-1} & \text{for } \ x \in \left[\dfrac{n-1}{n}, 1 \right], \end{cases}
\tag{4.50}
$$

$$
\widehat{\varphi}^{(n)}(x) = \varphi_{k-1} + \left(x - \frac{k-1}{n} \right) \Delta_k \varphi \ \text{ for } \ x \in \left[\frac{k-1}{n}, \frac{k}{n} \right), \qquad k = 1, \dots, n\,.
\tag{4.51}
$$

We also define

$$
\lambda^{(n)}(x, y) = \lambda_{k-j} \qquad \text{for } \ (x, y) \in \left[\frac{k-1}{n}, \frac{k}{n} \right) \times \left[\frac{j-1}{n}, \frac{j}{n} \right).
\tag{4.52}
$$

At this point, we consider piecewise linear interpolations $\widehat{u}^{(n)}(x,t)$, $\widehat{\varepsilon}^{(n)}(x,t)$ and $\widehat{\theta}^{(n)}(x,t)$ constructed from the sequences u_k, ε_k and θ_k by the formula (4.51). We conclude in the same way as in [31] that

$$
\|\widehat{\varepsilon}_t^{(n)}\|^2_{W^{1,\mathbf{P}}(Q_T)} + \|\widehat{\theta}^{(n)}\|^2_{W^{1,\mathbf{P}}(Q_T)} + \int_0^T |\widehat{\theta}_t^{(n)}(1,t)|^2 \, \mathrm{d}t \le C,
\tag{4.53}
$$

or, in terms of series, we have for all $t \in [0, T]$ that

$$
\frac{1}{n} \sum_{k=1}^{n} |\ddot{u}_k(t)|^2 + \frac{1}{n} \sum_{k=0}^{n} |\dot{\varepsilon}_k(t)|^2 + n \sum_{k=1}^{n} |\dot{\varepsilon}_k(t) - \dot{\varepsilon}_{k-1}(t)|^2 + n \sum_{k=1}^{n} |\theta_k(t) - \theta_{k-1}(t)|^2 \le C,
\tag{4.54}
$$

$$
\int_0^t \left(\frac{1}{n} \sum_{k=1}^{n-1} |\dot{\theta}_k(\tau)|^2 + |\dot{\theta}_n(\tau)|^2 + \frac{1}{n} \sum_{k=0}^{n} |\ddot{\varepsilon}_k(\tau)|^2 \right) \mathrm{d}\tau \le C.
\tag{4.55}
$$

We also get as a consequence of (4.5) and the above estimates together with (A.12) that

$$
\max_{t \in [0,T]} \max_{i=1,\dots,n} |\dot{m}_i(t)| \le C.
\tag{4.56}
$$

4.5. Passage to the limit

With the notation introduced in (4.49)–(4.51), passing to a subsequence, if necessary, we find functions ε, θ, u such that $\varepsilon_t, \theta \in W^{1,\mathbf{P}}(Q_T)$, $\theta_t(1,\cdot) \in L^2(0,T)$, $u_{tt} \in L^2(Q_T)$ and such that

$$\widehat{u}_{tt}^{(n)} \rightharpoonup u_{tt}, \quad \widehat{\varepsilon}_{xt}^{(n)} \rightharpoonup \varepsilon_{xt}, \quad \widehat{\theta}_x^{(n)} \rightharpoonup \theta_x \quad \text{weakly* in } L^\infty((0,T), L^2(0,1)), \tag{4.57a}$$

$$\widehat{\varepsilon}_{tt}^{(n)} \rightharpoonup \varepsilon_{tt}, \quad \widehat{\theta}_t^{(n)} \rightharpoonup \theta_t \quad \text{weakly in } L^2(Q_T), \tag{4.57b}$$

$$\widehat{\varepsilon}_t^{(n)} \to \varepsilon_t, \quad \widehat{\theta}^{(n)} \to \theta \quad \text{uniformly in } C^0(\overline{Q}_T), \tag{4.57c}$$

$$\widehat{\theta}_t^{(n)}(1,\cdot) \rightharpoonup \theta_t(1,\cdot) \quad \text{weakly in } L^2(0,T). \tag{4.57d}$$

We have for $x \in [(k-1)/n, k/n)$

$$|\underline{\varepsilon}_t^{(n)}(x,t) - \widehat{\varepsilon}_t^{(n)}(x,t)|^2 \le |\varepsilon_{k,t}(t) - \varepsilon_{k-1,t}(t)|^2 \le \frac{1}{n}\left(n\sum_{k=1}^n |\varepsilon_{k,t}(t) - \varepsilon_{k-1,t}(t)|^2\right) \le \frac{C}{n}, \tag{4.58}$$

with some suitable $C > 0$. Hence $\underline{\varepsilon}_t^{(n)} \to \varepsilon_t$ uniformly in $L^\infty(Q_T)$, and similarly $\underline{\varepsilon}^{(n)} \to \varepsilon$, $\overline{\varepsilon}_t^{(n)} \to \varepsilon_t$, $\overline{\varepsilon}^{(n)} \to \varepsilon$, $\underline{\theta}^{(n)} \to \theta$, $\overline{\theta}^{(n)} \to \theta$ uniformly in $L^\infty(Q_T)$. We have indeed $\widehat{u}_x^{(n)} = \underline{\varepsilon}^{(n)}$ and $\widehat{u}_{xt}^{(n)} = \underline{\varepsilon}_t^{(n)}$, hence $\widehat{u}_x^{(n)} \to \varepsilon = u_x$, $\widehat{u}_{xt}^{(n)} \to \varepsilon_t = u_{xt}$ uniformly in $L^\infty(Q_T)$.

To check that the limit functions satisfy the initial conditions we proceed in the same way as in [31].

To prove the existence of solutions, we check that the limit functions satisfy (2.23)–(2.24). Let $\phi \in W^{1,2}(0,1)$ be an arbitrary test function, and let us define

$$\delta_n(t) := \int_0^1 (\overline{u}_{tt}^{(n)}(t)\phi(x) + \underline{\sigma}^{(n)}(t)\phi_x(x))\,\mathrm{d}x + f[\overline{u}^{(n)}(1,\cdot)](t)\phi(1) - p(t)\phi(0).$$

We now use (4.1) and (4.6) to rewrite δ_n in the form

$$\begin{aligned}\delta_n(t) &= \sum_{k=1}^n \ddot{u}_k(t) \int_{(k-1)/n}^{k/n} \phi(x)\,\mathrm{d}x + \sum_{k=1}^n \sigma_{k-1}(t)(\phi(k/n) - \phi((k-1)/n)) \\ &\quad + f[u_n](t)\phi(1) - p(t)\phi(0) \\ &= \sum_{k=1}^n \ddot{u}_k(t) \int_{(k-1)/n}^{k/n} \phi(x)\,\mathrm{d}x - \sum_{k=1}^n (\sigma_k - \sigma_{k-1})\phi(k/n) \\ &= \sum_{k=1}^n \ddot{u}_k(t) \int_{(k-1)/n}^{k/n} (\phi(x) - \phi(k/n))\,\mathrm{d}x.\end{aligned} \tag{4.59}$$

Clearly, there exists $C > 0$ such that

$$|\delta_n(t)| \le \left(\frac{1}{n}\sum_{k=1}^n |\ddot{u}_k(t)|^2\right)^{1/2} \left(\frac{1}{n^2}\sum_{k=1}^n \int_{(k-1)/n}^{k/n} |\phi_x(x)|^2\,\mathrm{d}x\right)^{1/2} \le \frac{C}{n}|\phi_x|_2.$$

In the identity

$$0 = \int_0^1 (\overline{u}_{tt}^{(n)}\phi + \underline{\sigma}^{(n)}\phi_x)(x,t)\,\mathrm{d}x + f[\overline{u}^{(n)}(1,\cdot)](t)\phi(1) - p(t)\phi(0) - \delta_n(t) \tag{4.60}$$

we now pass to the limit as $n \to \infty$. First of all we have for all $x \in (0,1)$ by Proposition 2.2 (ii) that

$$\begin{aligned} &\int_0^t \left|\overline{m}_t^{(n)} - \overline{m}_t^{(l)}\right|(x,\tau)\,\mathrm{d}\tau \qquad (4.61)\\ &\leq C\int_0^t\int_0^1\int_0^\infty \left|\lambda^{(n)}(x,y)\varphi(\overline{m}^{(n)},r)\delta^{(n)}(y,\tau,r) - \lambda^{(l)}(x,y)\varphi(\overline{m}^{(l)},r)\delta^{(l)}(y,\tau,r)\right|\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}\tau, \end{aligned}$$

where we denote

$$\delta^{(n)} = \delta^{(n)}(y,t,r) = \mathfrak{s}_r[\overline{\varepsilon}^{(n)}](\overline{\varepsilon}^{(n)} - \mathfrak{s}_r[\overline{\varepsilon}^{(n)}])_t(y,t) = r|\mathfrak{p}_r[\overline{\varepsilon}^{(n)}]_t(y,t)|.$$

By Proposition 2.2 (ii) we have (note that $||a|-|b|| \leq |a-b|$ for $a,b \in \mathbb{R}$)

$$\int_0^t |\delta^{(n)} - \delta^{(l)}|(y,\tau)\,\mathrm{d}\tau \leq r\int_0^t |\overline{\varepsilon}_t^{(n)} - \overline{\varepsilon}_t^{(l)}|(y,\tau)\,\mathrm{d}\tau,$$

hence, by Hypothesis (H1) and (H7)

$$\int_0^t\int_0^1\int_0^\infty \lambda^{(n)}(x,y)\varphi(\overline{m}^{(n)},r)|\delta^{(n)} - \delta^{(l)}|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}\tau \leq C\int_0^t\int_0^1 |\overline{\varepsilon}_t^{(n)} - \overline{\varepsilon}_t^{(l)}|(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \tag{4.62}$$

Similarly, by Hypothesis (H1),

$$\begin{aligned} &\int_0^t\int_0^1\int_0^\infty \delta^{(l)}\lambda^{(n)}(x,y)|\varphi(\overline{m}^{(n)},r) - \varphi(\overline{m}^{(l)},r)|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}\tau\\ &\leq C\int_0^t \left(\int_0^1 |\overline{\varepsilon}_t^{(l)}(y,\tau)|\,\mathrm{d}y\right)\max_{x\in(0,1)}|m^{(n)}(x,\tau) - m^{(l)}(x,\tau)|\,\mathrm{d}\tau. \qquad (4.63) \end{aligned}$$

Finally, we have the pointwise bound

$$|\lambda^{(n)}(x,y) - \lambda^{(l)}(x,y)| \leq \frac{4\Lambda}{\min\{n,l\}}, \tag{4.64}$$

where Λ has been introduced in (H7). Combining (4.61)–(4.64) gives the estimate

$$\max_{x\in(0,1)} |m^{(n)} - m^{(l)}|(x,t) \le \max_{x\in(0,1)} \int_0^t \left|\overline{m}_t^{(n)} - \overline{m}_t^{(l)}\right|(x,\tau)\,\mathrm{d}\tau$$
$$\le q_{nl} + C\int_0^t \left(\int_0^1 |\overline{\varepsilon}_t^{(l)}(y,\tau)|\,\mathrm{d}y\right) \max_{x\in(0,1)} |m^{(n)} - m^{(l)}|(x,\tau)\,\mathrm{d}\tau, \tag{4.65}$$

with

$$q_{nl} = C\left(\frac{1}{\min\{n,l\}} + \|\overline{\varepsilon}_t^{(n)} - \overline{\varepsilon}_t^{(l)}\|_1\right).$$

Inequality (4.65) can be interpreted as an inequality of the form

$$q(t) \le q_{nl} + \int_0^t s^{(l)}(\tau)q(\tau)\,\mathrm{d}\tau,$$

with $q(t) = \max_{x\in(0,1)} |\overline{m}^{(n)} - \overline{m}^{(l)}|(x,t)$, $s^{(l)}(t) = C\int_0^1 |\overline{\varepsilon}_t^{(l)}(y,t)|\,\mathrm{d}y$, with $s^{(l)}$ uniformly bounded in $L^1(0,T)$. We obtain using Gronwall's lemma that

$$q(t) \le q_{nl} e^{\int_0^t s^{(l)}(\tau)\,\mathrm{d}\tau} \le Cq_{nl}.$$

The convergences established at the beginning of this section imply that q_{nl} is small if n,l are large. Hence, $\overline{m}^{(n)}$ is a Cauchy sequence, so that

$$\overline{m}^{(n)} \to m \ \text{ strongly in } L^\infty(Q_T), \tag{4.66}$$

and, by (4.65),

$$\overline{m}_t^{(n)} \to m_t \ \text{ strongly in } L^\infty(0,1;L^1(0,T)).$$

Furthermore, by virtue of (4.56) $\overline{m}_t^{(n)}$ are uniformly bounded in $L^\infty(Q_T)$, hence $\overline{m}_t^{(n)} \to m_t$ in $L^\infty(Q_T)$ weakly star. Using the convergences established at the beginning of this section and Proposition 2.2, we conclude that $\overline{D}^{(n)}(x,\cdot)$, $\overline{K}^{(n)}(x,\cdot)$ converge for all $x \in (0,1)$ to $D[m,\varepsilon](x,\cdot)$, $\mathcal{K}[m,\varepsilon](x,\cdot)$, respectively, strongly in $L^\infty(0,T)$.

By continuity of the operator $\mathcal{P}[m,\varepsilon]$, we have that $\underline{\sigma}^{(n)}$ converge to $\sigma = B\varepsilon + \mathcal{P}[m,\varepsilon] + \nu\varepsilon_t - \beta(\theta - \theta^{\mathrm{ref}})$ uniformly in $L^\infty(Q_T)$. Similarly, the boundary term $f[\overline{u}^{(n)}(1,\cdot)]$ converges uniformly in $C^0([0,T])$ to $f[u(1,\cdot)]$. The sequence $\overline{u}_{tt}^{(n)}$ converges weakly in $L^2(Q_T)$ and δ_n converge uniformly to 0, hence the limit functions satisfy (2.2) and (2.23).

Similarly, with the intention to prove that (2.24) holds, we consider now an arbitrary test function $\psi \in W^{1,2}(0,1)$ and define the quantity

$$\Delta_n(t) := \int_0^1 (c\overline{\theta}_t^{(n)}(x,t)\psi(x) + \kappa\widehat{\theta}_x^{(n)}(x,t)\psi_x(x) - |\nu(\overline{\varepsilon}_t^{(n)}(x,t)|^2 + \overline{m}_t^{(n)}(x,t)\overline{K}^{(n)}(x,t) + \overline{D}^{(n)}(x,t)$$
$$- \beta\overline{\theta}^{(n)}(x,t)\overline{\varepsilon}_t^{(n)}(x,t))\psi(x))\,\mathrm{d}x - (|d[\widehat{u}^{(n)}(1,\cdot)]_t(t)| + \alpha(\theta^{\mathrm{ref}} - \widehat{\theta}^{(n)}(1,t)) - c^{\mathrm{bdy}}\widehat{\theta}_t^{(n)}(1,t))\psi(1), \tag{4.67}$$

where

$$\overline{m}^{(n)}(x,t) = \int_0^t \left(\int_0^1 \lambda^{(n)}(x,y)\overline{D}^{(n)}(y,\tau)\,\mathrm{d}y \right) \mathrm{d}\tau, \tag{4.68}$$

$$\overline{D}^{(n)}(x,t) = \int_0^\infty \varphi(\overline{m}^{(n)},r)\ \mathfrak{s}_r[\overline{\varepsilon}^{(n)}](\overline{\varepsilon}^{(n)} - \mathfrak{s}_r[\overline{\varepsilon}^{(n)}])_t(x,t)\,\mathrm{d}r, \tag{4.69}$$

$$\overline{K}^{(n)}(x,t) = -\frac{1}{2}\int_0^\infty \varphi_m(\overline{m}^{(n)},r)\mathfrak{s}_r^2[\overline{\varepsilon}^{(n)}](x,t)\,\mathrm{d}r. \tag{4.70}$$

Notice that we have

$$\begin{aligned}\int_0^1 \widehat{\theta}_x^{(n)}\psi_x\,\mathrm{d}x &= n\sum_{k=1}^{n}(\theta_k(t)-\theta_{k-1}(t))(\psi(k/n)-\psi((k-1)/n))\\ &= n\sum_{k=1}^{n-1}(\theta_{k+1}(t)-\theta_k(t))(\psi((k+1)/n)-\psi(k/n)).\end{aligned}$$

We use (4.14) with $\psi_k = \psi(k/n)$ to obtain

$$\begin{aligned}\Delta_n(t) = &\sum_{k=1}^{n}\left(c\dot\theta_k(t) - \nu|\dot\varepsilon_k(t)|^2 - \ \dot m_k\ \mathcal{K}_k - D_k + \beta\theta_k(t)\dot\varepsilon_k(t)\right)\int_{(k-1)/n}^{k/n}\psi(x)\,\mathrm{d}x\\ &+\kappa n\sum_{k=1}^{n-1}(\theta_{k+1}(t)-\theta_k(t))(\psi((k+1)/n)-\psi(k/n))\\ &-(|d[u_n]_t| + \alpha(\theta^{\mathrm{ext}}-\theta_n) - c^{\mathrm{bdy}}\dot\theta_n)(t)\psi(1)\\ =&\sum_{k=1}^{n}\left(c\dot\theta_k - \nu\dot\varepsilon_k^2 - \ \dot m_k\ \mathcal{K}_k - D_k + \beta\theta_k\dot\varepsilon_k\right)(t)\int_{(k-1)/n}^{k/n}(\psi(x)-\psi(k/n))\,\mathrm{d}x,\end{aligned}$$

hence, arguing as in the estimate of δ_n, we may infer that there exists $C > 0$ such that

$$|\Delta_n(t)| \le \frac{C}{n}\left(\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n}\psi_x^2(x)\,\mathrm{d}x\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n}(\dot\theta_k^2 + \dot\varepsilon_k^4 + \theta_k^4)(t)\right)^{1/2},$$

hence Δ_n converge to 0 strongly in $L^2(0,T)$. Passing to the weak limit in $L^2(0,T)$ in (4.67) we check that (2.24) holds, so that (u,θ,m) is a desired solution from Theorem 3.2.

5. Proof of Theorem 3.2: uniqueness

It remains to prove uniqueness. Instead, we prove here a stronger continuous data dependence result which implies uniqueness if the data coincide. Here we follow the ideas from [31], but we have to face additionally many technical difficulties caused with the presence of fatigue.

Theorem 5.1. *Let $p_i, u_i^0, v_i^0, \theta_i^0$ and θ_i^{ext}, $i = 1, 2$, be sets of data satisfying Hypotheses* (H5) and (H6) *(recall that, by* (2.19) *$m(x, 0) = 0$ a.e. in $(0, 1)$), and let (u_i, θ_i, m_i), $i = 1, 2$, be the corresponding solutions as in Theorem 3.2. Set $p^* := p_1 - p_2$, $u^{0*} := u_1^0 - u_2^0$, $v^{0*} := v_1^0 - v_2^0$, $\theta^{0*} := \theta_1^0 - \theta_2^0$, $u^* := u_1 - u_2$, $\theta^* := \theta_1 - \theta_2$, $m^* := m_1 - m_2$ and $\theta^{\mathrm{ext}*} := \theta_1^{\mathrm{ext}} - \theta_2^{\mathrm{ext}}$. Then there exists $C > 0$ depending only on the norms of the data in their respective spaces such that for all $t \in [0, T]$, the following inequality holds*

$$\int_0^1 |u_t^*(x,t)|^2 \,\mathrm{d}x + \int_{Q_t} (|u_{xt}^*(x,\tau)|^2 + |\theta^*(x,\tau)|^2) \,\mathrm{d}x \,\mathrm{d}\tau + \int_0^t |\theta^*(1,\tau)|^2 \,\mathrm{d}\tau$$
$$\leq C \left(|\theta^{0*}(1)|^2 + |\theta^{\mathrm{ext}*}|^2 + \int_0^t |p^*(\tau)|^2 \,\mathrm{d}\tau + \int_0^1 (|u^{0*}|^2 + |u_x^{0*}|^2 + |v^{0*}|^2 + |\theta^{0*}|^2)(x) \,\mathrm{d}x \right).$$

Proof. First of all, integrating the difference of (2.24) for the two solutions in time from 0 to t, for all $t \in [0, T]$, for all $\psi \in W^{1,2}(0, 1)$, we obtain

$$\int_0^1 c(\theta^*(x,t) - \theta^{0*}(x))\psi(x) \,\mathrm{d}x + \kappa \int_{Q_t} \theta_x^*(x,\tau)\psi_x(x) \,\mathrm{d}x \,\mathrm{d}\tau + c^{\mathrm{bdy}}(\theta^*(1,t) - \theta^{0*}(1))\psi(1)$$
$$= \nu \int_{Q_t} (\varepsilon_{1,t}^2 - \varepsilon_{2,t}^2)(x,\tau)\psi(x) \,\mathrm{d}x \,\mathrm{d}\tau + \int_{Q_t} (D[m_1,\varepsilon_1](x,\tau) - D[m_2,\varepsilon_2](x,\tau))\,\psi(x) \,\mathrm{d}x \,\mathrm{d}\tau$$
$$- \beta \int_{Q_t} (\theta_1\varepsilon_{1,t} - \theta_2\varepsilon_{2,t})(x,\tau)\psi(x) \,\mathrm{d}x \,\mathrm{d}\tau - \int_{Q_t} (m_{1,t}\mathcal{K}[m_1,\varepsilon_1](x,\tau) - m_{2,t}\mathcal{K}[m_2,\varepsilon_2](x,\tau))\,\psi(x) \,\mathrm{d}x \,\mathrm{d}\tau$$
$$+ \int_0^t (|d[u_1(1,\cdot)]_t(\tau)| - |d[u_2(1,\cdot)]_t(\tau)|)\,\psi(1) \,\mathrm{d}\tau + \alpha \int_0^t (\theta^{\mathrm{ext}*} - \theta^*(1,\tau))\psi(1) \,\mathrm{d}\tau.$$

We test now by $\psi(x) = \theta^*(x,t)$. We observe that $\varepsilon_{i,t}$, $u_{i,t}$, θ_i and m_i are bounded in $L^\infty(Q_T)$, $i = 1, 2$, and the dissipation operators are Lipschitz continuous in $W^{1,1}(0,T)$ by Proposition 2.2 (ii) and Hypothesis (H2) that is,

$$\int_0^t |(D[m_1,\varepsilon_1] - D[m_2,\varepsilon_2)(x,\tau)]| \,\mathrm{d}\tau \leq C \left(|u_x^{0*}(x)| + \int_0^t (|(m_1 - m_2)(x,\tau)||\varepsilon_{1,t}(x,\tau)| + |\varepsilon_t^*(x,\tau)|) \,\mathrm{d}\tau \right), \tag{5.1}$$

$$\int_0^t (|d[u_1(1,\cdot)]_t(\tau)| - |d[u_2(1,\cdot)]_t(\tau)|) \,\mathrm{d}\tau \leq C \left(|u^{0*}(1)| + \int_0^t |u_t^*(1,\tau)| \,\mathrm{d}\tau \right), \tag{5.2}$$

for all $x \in (0, 1)$ and $t \in [0, T]$, with some $C > 0$. The fatigue term is estimated using (H1) as

$$\int_0^t |(m_{1,t}\mathcal{K}[m_1,\varepsilon_1] - m_{2,t}\mathcal{K}[m_2,\varepsilon_2])(x,\tau)| \,\mathrm{d}\tau \tag{5.3}$$
$$\leq C \int_0^t \left(|(m_{1,t} - m_{2,t})(x,\tau)| + |m_{1,t}(x,\tau)| \left(|(m_1 - m_2)(x,\tau)| + |u_x^{0*}(x)| + \int_0^\tau |\varepsilon_t^*(y,\tau)| \,\mathrm{d}y \right) \right) \mathrm{d}\tau.$$

Now, $|m_{1,t}(x,t)| \leq C$ by the existence part of Theorem 3.2, and

$$\int_0^t |m_{1,t} - m_{2,t}|(x,\tau)\,\mathrm{d}\tau \leq C \int_0^t \int_0^1 |u_x^{0*}(y)|^2 \,\mathrm{d}y\,\mathrm{d}\tau \\ + C \int_0^t \left(\int_0^1 \left(|m_1 - m_2||\varepsilon_{1,t}| + \int_0^\tau |\varepsilon_t^*|\,\mathrm{d}s \right)(y,\tau)\,\mathrm{d}y \right) \mathrm{d}\tau \tag{5.4}$$

by Proposition 2.2 (ii), together with (2.11) and (5.2); moreover

$$\int_0^t \int_0^1 (|m_1 - m_2|\,|\varepsilon_{1,t}|)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \leq C \int_0^t \left(\int_0^1 |m_1 - m_2|^2(y,\tau)\,\mathrm{d}y \right)^{1/2} \mathrm{d}\tau \tag{5.5}$$

by (4.54). On the other hand,

$$|m_1 - m_2|(x,t) \leq \int_0^t |m_{1,t} - m_{2,t}|(x,\tau)\,\mathrm{d}\tau$$

for almost every x, hence, using (5.4) and (5.5) we have

$$|m_1 - m_2|^2(x,t) \leq C\left(\int_0^1 |u_x^{0*}(x)|^2\,\mathrm{d}x + \int_0^t \int_0^1 (|m_1 - m_2|^2 + |\varepsilon_t^*|^2)(y,t)\,\mathrm{d}y\,\mathrm{d}\tau \right).$$

Integrating in space and using Gronwall's argument, we obtain from (5.4) that

$$\int_0^t |m_{1,t} - m_{2,t}|(x,\tau)\,\mathrm{d}\tau \leq C\left(\left(\int_0^1 |u_x^{0*}(x)|^2\,\mathrm{d}x \right)^{1/2} + \left(\int_0^t \int_0^1 |\varepsilon_t^*(y,\tau)|^2\,\mathrm{d}y\,\mathrm{d}\tau \right)^{1/2} \right). \tag{5.6}$$

Using the L^∞ bounds for θ_i and $\varepsilon_{i,t}$ and the inequalities (5.2)–(5.2), we may infer that there exists $C > 0$ such that

$$\int_0^1 c\,\theta^*(x,t)(\theta^*(x,t) - \theta^{0*}(x))\,\mathrm{d}x + \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\int_0^t \theta_x^*(x,s)\,\mathrm{d}s \right)^2 \mathrm{d}x + c^{\mathrm{bdy}}\theta^*(1,t)(\theta^*(1,t) - \theta^{0*}(1)) \\ \leq C\left(\int_0^1 |\theta^*(x,t)| \left(|u_x^{0*}(x)| + \int_0^t (|\varepsilon_t^*(x,s)| + |\theta^*(x,s)|)\,\mathrm{d}s + \left(\int_0^1 |u_x^{0*}(x)|^2\,\mathrm{d}x \right)^{1/2} \right. \right. \\ \left. \left. + \left(\int_0^t \int_0^1 |\varepsilon_t^*|^2(y,s)\,\mathrm{d}y\,\mathrm{d}s \right)^{1/2} \right) \mathrm{d}x + |\theta^*(1,t)| \left(|u^{0*}(1)| + |\theta^{\mathrm{ext}*}| + \int_0^t (|\theta^*(1,s)| + |u_t^*(1,s)|)\,\mathrm{d}s \right) \right).$$

Hence, also by virtue of (5.15), it follows that there exist constants $C, \kappa^* > 0$ such that

$$
\begin{aligned}
&\int_0^1 |\theta^*(x,\tau)|^2 \,\mathrm{d}x + \kappa^* \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(\int_0^t \theta_x^*(x,s)\,\mathrm{d}s \right)^2 \mathrm{d}x + |\theta^*(1,t)|^2 \\
&\le C \Bigg(|\theta^{\mathrm{ext}*}|^2 + |\theta^{0*}(1)|^2 + \int_0^1 (|u^{0*}|^2 + |u_x^{0*}|^2 + |\theta^{0*}|^2)(x)\,\mathrm{d}x \\
&+ \int_{Q_t} (|u_t^*|^2 + |\varepsilon_t^*|^2 + |\theta^*|^2)(x,s)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t |\theta^*(1,s)|^2\,\mathrm{d}s \Bigg).
\end{aligned}
\tag{5.7}
$$

Integrating (5.7) in time over $(0,t)$ we obtain that

$$
\begin{aligned}
&\int_{Q_t} |\theta^*(x,\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t |\theta^*(1,\tau)|^2\,\mathrm{d}\tau \\
&\le C\,T \left(|\theta^{\mathrm{ext}*}|^2 + |\theta^{0*}(1)|^2 + \int_0^1 (|u^{0*}|^2 + |u_x^{0*}|^2 + |\theta^{0*}|^2)(x)\,\mathrm{d}x \right) \\
&+ C \left(\int_0^t \int_{Q_t} (|u_t^*|^2 + |\varepsilon_t^*|^2 + |\theta^*|^2)(x,t)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t \int_0^\tau |\theta^*(1,s)|^2\,\mathrm{d}s\,\mathrm{d}\tau \right).
\end{aligned}
\tag{5.8}
$$

We now consider the difference of (2.23) taken for the two solutions (u_1, θ_1, m_1), (u_2, θ_2, m_2), tested by $\phi = u_t^*$, then we use (2.2) and finally we integrate this expression over $(0,t)$ to get

$$
\begin{aligned}
&\frac{\rho}{2} \int_0^1 |u_t^*(x,t)|^2 \,\mathrm{d}x + \nu \int_{Q_t} |\varepsilon_t^*(x,\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\tau = \frac{\rho}{2} \int_0^1 |v^{0*}(x)|^2\,\mathrm{d}x \\
&+ \beta \int_{Q_t} \theta^*(x,\tau)\varepsilon_t^*(x,t)\,\mathrm{d}x\,\mathrm{d}\tau - \int_{Q_t} (B\varepsilon^*(x,\tau) + \mathcal{P}[m_1,\varepsilon_1](x,\tau) - \mathcal{P}[m_2,\varepsilon_2](x,\tau))\varepsilon_t^*(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&- \int_0^t (f[u_1](1,\tau) - f[u_2](1,\tau))u_t^*(1,\tau)\,\mathrm{d}\tau + \int_0^t p^*(\tau)u_t^*(0,\tau)\,\mathrm{d}\tau.
\end{aligned}
\tag{5.9}
$$

The terms on the right hand side of (5.9) will be estimated using a suitable constant $\mu > 0$ that will be specified later. We have

$$
\beta \int_{Q_t} \theta^*(x,\tau)\varepsilon_t^*(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \le \frac{\beta^2}{2\mu} \int_{Q_t} |\theta^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\mu}{2} \int_{Q_t} |\varepsilon_t^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau, \tag{5.10}
$$

$$
-\int_{Q_t} B\varepsilon^*(x,\tau)\ \varepsilon_t^*(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \le \frac{B^2}{2\mu} \int_{Q_t} |\varepsilon^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\mu}{2} \int_{Q_t} |\varepsilon_t^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau, \tag{5.11}
$$

$$
\begin{aligned}
&- \int_{Q_t} (\mathcal{P}[m_1,\varepsilon_1](x,\tau) - \mathcal{P}[m_2,\varepsilon_2](x,\tau))\varepsilon_t^*(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \frac{1}{2\mu} \int_{Q_t} |\mathcal{P}[m_1,\varepsilon_1](x,\tau) - \mathcal{P}[m_2,\varepsilon_2](x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\mu}{2} \int_{Q_t} |\varepsilon_t^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
&-\int_0^t (f[u_1](1,\tau) - f[u_2](1,\tau))u_t^*(1,\tau)\,\mathrm{d}\tau \\
&\leq \frac{1}{2\mu}\int_0^t |f[u_1](1,\tau) - f[u_2](1,\tau)|^2\,\mathrm{d}\tau + \frac{\mu}{2}\int_0^t |u_t^*(1,\tau)|^2\,\mathrm{d}\tau,
\end{aligned} \tag{5.13}
$$

$$
\int_0^t p^*(\tau)u_t^*(0,\tau)\,\mathrm{d}\tau \leq \frac{1}{2\mu}\int_0^t |p^*|^2(\tau)\,\mathrm{d}\tau + \frac{\mu}{2}\int_0^t |u_t^*(0,\tau)|^2\,\mathrm{d}\tau. \tag{5.14}
$$

By (A.2) for $r = s = p = 2$ we have for all $(y,\tau) \in Q_T$ that

$$
|u_t^*(y,\tau)|^2 \leq \int_0^1 |u_t^*(x,\tau)|^2\,\mathrm{d}x + \sqrt{2}\left(\int_0^1 |u_t^*(x,\tau)|^2\,\mathrm{d}x\right)^{1/2}\left(\int_0^1 |\varepsilon_t^*(x,\tau)|^2\,\mathrm{d}x\right)^{1/2},
$$

hence, by Hölder's inequality,

$$
\int_0^t |u_t^*(y,\tau)|^2\,\mathrm{d}\tau \leq \frac{3}{2}\int_{Q_t} |u_t^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_{Q_t} |\varepsilon_t^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau. \tag{5.15}
$$

Furthermore,

$$
\int_{Q_t} |\mathcal{P}[m_1,\varepsilon_1](x,\tau) - \mathcal{P}[m_2,\varepsilon_2](x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C\left(\int_0^1 |u_x^{*0}(x)|^2\,\mathrm{d}x + \int_0^t\int_{Q_\tau} |\varepsilon_t^*(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau\right), \tag{5.16}
$$

$$
\begin{aligned}
\int_0^t |f[u_1](1,\tau) - f[u_2](1,\tau)|^2\,\mathrm{d}\tau &\leq 2L_f^2\left(|u^{*0}(1)|^2 + \int_0^t\int_0^\tau |u_t^*(1,s)|^2\,\mathrm{d}s\,\mathrm{d}\tau\right) \\
&\leq 2L_f^2\left(|u^{*0}(1)|^2 + \frac{3T}{2}\int_{Q_t} |u_t^*(x,t)|^2\,\mathrm{d}t + \int_0^t\int_{Q_\tau} |\varepsilon_t^*(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau\right).
\end{aligned} \tag{5.17}
$$

Similarly we have

$$
\int_{Q_t} |\varepsilon^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C\left(\int_0^1 |u_x^{*0}(x)|^2\,\mathrm{d}x + \int_0^t\int_{Q_\tau} |\varepsilon_t^*(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau\right). \tag{5.18}
$$

Choosing now $\mu = \nu/4$ and inserting the estimates (5.10)–(5.17) into (5.9), we conclude that there exists a constant $C^* > 0$ depending only on the physical constants of the problem such that

$$
\begin{aligned}
&\int_0^1 |u_t^*(x,t)|^2\,\mathrm{d}x + \int_{Q_t} |\varepsilon_t^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C^*\left(\int_0^1 \left(|v^{0*}|^2 + |u^{0*}|^2 + |u_x^{0*}|^2\right)(x)\,\mathrm{d}x\right. \\
&\left. + \int_0^t |p^*(\tau)|^2\,\mathrm{d}\tau + \int_{Q_t} |\theta^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_{Q_t} |u_t^*(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_{Q_\tau} |\varepsilon_t^*(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau\right).
\end{aligned} \tag{5.19}
$$

We now multiply (5.8) by $2C^*$ and add the result to (5.19), apply the Gronwall's argument and complete the proof in the same way as in [31]. $\square$

Appendix A. Sobolev interpolation inequalities

Let $p, q, s \in [1, \infty]$ be such that $q > s$, and let $|\cdot|_p$ denote the norm in $L^p(0,1)$. The *Gagliardo–Nirenberg inequality* states that there exists a constant $C^{\mathrm{GN}} > 0$ such that for every $v \in W^{1,p}(0,1)$ we have

$$|v|_q \le C^{\mathrm{GN}} \left(|v|_s + |v|_s^{1-\gamma} |v'|_p^{\gamma}\right) \quad \text{with} \quad \gamma := \frac{\frac{1}{s} - \frac{1}{q}}{1 + \frac{1}{s} - \frac{1}{p}}. \tag{A.1}$$

Note that (A.1) is straightforward. Indeed if we introduce an auxiliary parameter $r := 1 + s(1 - \frac{1}{p})$ and use the chain rule $\frac{\mathrm{d}}{\mathrm{d}x}|v(x)|^r \le r|v(x)|^{r-1}|v'(x)|$ almost everywhere, we obtain from Hölder's inequality that

$$|v|_\infty \le |v|_r + C|v|_s^{1-(1/r)}|v'|_p^{1/r} \quad \text{with} \quad C := r^{1/r}. \tag{A.2}$$

Combined with the obvious interpolation inequality $|v|_h \le |v|_\infty^{1-(s/h)}|v|_s^{s/h}$ for $h = q$ if $r \ge s$, and for both $h = q$ and $h = r$ if $r > s$, this yields (A.1).

Let now $\mathbf{v} := (v_0, v_1, \ldots, v_n)^{\mathsf{T}}$ be a vector, and let us denote

$$|\mathbf{v}|_p := \left(\frac{1}{n}\sum_{k=0}^{n} |v_k|^p\right)^{1/p} \quad \text{and} \quad |\mathbf{D}\mathbf{v}|_p := \left(n^{p-1}\sum_{k=1}^{n} |v_k - v_{k-1}|^p\right)^{1/p}. \tag{A.3}$$

The discrete counterpart of (A.1) reads

$$|\mathbf{v}|_q \le C^{\mathrm{GND}} \left(|\mathbf{v}|_s + |\mathbf{v}|_s^{1-\gamma}|\mathbf{D}\mathbf{v}|_p^{\gamma}\right), \tag{A.4}$$

where $C^{\mathrm{GND}} > 0$ is a constant depending on the data and independent of n.

Let us recall here the following embedding formula for anisotropic Sobolev spaces from [29, Theorem A.1]. For a vector $\mathbf{p} := (p_1, \ldots, p_N)^{\mathsf{T}}$, $1 \le p_i < \infty$, we define the space $L^{\mathbf{p}}(\mathbb{R}^N)$ as the subspace of $L^1(\mathbb{R}^N)$ of functions v such that the norm

$$\|v\|_{\mathbf{p}} := \left(\int_{\mathbb{R}} \left(\ldots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |v(x)|^{p_1}\, \mathrm{d}x_1\right)^{p_2/p_1} \mathrm{d}x_2 \,\ldots \right)^{p_N/p_{N-1}} \mathrm{d}x_N \right)^{1/p_N} \tag{A.5}$$

is finite, with obvious modifications if $p_i = \infty$. If $p_1 = p_2 = \cdots = p_N$, we write simply $\|v\|_p$. For a matrix $\mathbf{P} := (P_{ij})_{i,j=1}^N$ with $P_{ij} := 1/p_{ij}$, $1 \le p_{ij} \le \infty$, we define the anisotropic Sobolev space

$$W^{1,\mathbf{P}}(\mathbb{R}^N) := \left\{ v \in L^1(\mathbb{R}^N) : \frac{\partial v}{\partial x_i} \in L^{\mathbf{p}_i}(\mathbb{R}^N), i = 1, \ldots, N \right\}, \tag{A.6}$$

where $\mathbf{p}_i := (p_{i1}, \ldots, p_{iN})$. The proof in [29] is carried out explicitly only for $p_{ij} < \infty$ using the methods of [3], but the case $p_{ij} = \infty$ works exactly in the same way.

We denote by $\mathbf{I}$ the identity $N \times N$ matrix, and by $\mathbf{1}$ the vector $\mathbf{1} := (1, 1, \ldots, 1)^{\mathsf{T}}$. The spectral radius $\varrho(\mathbf{P})$ of $\mathbf{P}$ is defined as

$$\varrho(\mathbf{P}) := \max\{|\lambda| : \lambda \in \mathbb{C},\ \det(\mathbf{P} - \lambda\mathbf{I}) = 0\} = \limsup_{n\to\infty} |\mathbf{P}^n|^{1/n}. \tag{A.7}$$

Theorem A.1. *Let $\varrho(\mathbf{P}) < 1$, and let*

$$(\mathbf{I} - \mathbf{P})^{-1}\mathbf{1} := \mathbf{b} := (b_1, \ldots, b_N). \tag{A.8}$$

Then $W^{1,\mathbf{P}}(\mathbb{R}^N)$ is embedded in $L^\infty(\mathbb{R}^N)$, and there exists a constant $C^{\mathbf{P}} > 0$ such that each $v \in W^{1,\mathbf{P}}(\mathbb{R}^N)$ has for all $x, z \in \mathbb{R}^N$ the Hölder property

$$|v(z) - v(x)| \leq C^{\mathbf{P}} \|v\|_{W^{1,\mathbf{P}}(\mathbb{R}^N)} \sum_{i=1}^{N} |z_i - x_i|^{1/b_i}, \tag{A.9}$$

and putting $|\mathbf{b}| := \sum_{i=1}^N b_i$, we have for every $\delta \in (0, 1]$ and every $q \in [1, \infty)$ that

$$\forall x \in \mathbb{R}^N : \ |v(x)| \leq C^{\mathbf{P}} \left(\delta^{-|\mathbf{b}|/q} \|v\|_q + \delta \|v\|_{W^{1,\mathbf{P}}(\mathbb{R}^N)} \right). \tag{A.10}$$

Corollary A.2. *In the situation of the previous Theorem for $r > q$ the following interpolation inequality holds:*

$$\|v\|_r \leq C^{\mathbf{P}} \left(\|v\|_q + \|v\|_q^{1-\gamma^*} \|v\|_{W^{1,\mathbf{P}}(\mathbb{R}^N)}^{\gamma^*} \right), \tag{A.11}$$

with $\gamma^* := |\mathbf{b}|(1 - (q/r))/(q + |\mathbf{b}|)$.

For a detailed proof see [31]. This result will be applied to our situation in the following particular case.

Corollary A.3. *Let $\mathbf{P}$ be the matrix*

$$\mathbf{P} := \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{pmatrix}. \tag{A.12}$$

Then the space $W^{1,\mathbf{P}}(Q_T)$ defined as in (A.6) with $x_1 = x$ and $x_2 = t$, is compactly embedded in $C^0(\overline{Q_T})$.

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