



Evolution by mean curvature flow of Lagrangian spherical surfaces in complex Euclidean plane



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ABSTRACT

We describe the evolution under the mean curvature flow of embedded Lagrangian spherical surfaces in the complex Euclidean plane \mathbb{C}^2 . In particular, we answer the Question 4.7 addressed in [10] by A. Neves about finding out a condition on a starting Lagrangian torus in \mathbb{C}^2 such that the corresponding mean curvature flow becomes extinct at finite time and converges after rescaling to the Clifford torus.

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1. Introduction

Let $F_0 : M^n \rightarrow \mathbb{R}^m$ be an immersion of a compact manifold of dimension $n \geq 2$ into Euclidean space. The mean curvature flow with initial condition F_0 is a smooth family of immersions $F : M \times [0, T) \rightarrow \mathbb{R}^m$ satisfying

$$\frac{\partial}{\partial t} F(p, t) = H(p, t), \quad p \in M, \quad t \geq 0; \quad F(\cdot, 0) = F_0, \quad (\text{MCF})$$

where $H(p, t)$ is the mean curvature vector of the submanifold $M_t = F(M, t)$ at p . It is well-known that (MCF) is a quasilinear parabolic system that is invariant under reparametrizations of M and isometries of the ambient space and short-time existence and uniqueness is guaranteed, being $T < \infty$ the maximal time of existence.

The first classical works in this topic studied the evolution of hypersurfaces by their mean curvature. We emphasize Huisken's paper [6] on the flow of convex surfaces into spheres, proving that if the initial

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hypersurface is uniformly convex, then the mean curvature flow converges to a round point in finite time. That is, the shape of M_t approaches the shape of a sphere very rapidly and no singularities will occur before the hypersurfaces M_t shrink down to a single point after a finite time. Recently, mean curvature flow of higher codimension submanifolds has also received interest by many authors who have paid attention mainly to graphical submanifolds and symplectic or Lagrangian submanifolds. We recall that Huisken's monotonicity formula [7], relating the formation of singularities to self-shrinking solutions of the mean curvature flow, also applies in any codimension. Concretely, the so-called Type I singularities forming in Euclidean space look like self-similar contracting solutions after an appropriate rescaling procedure. According to [13], this type of singularities usually occur when there exists some kind of pinching of the second fundamental form. Andrews and Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying suitable pinching condition and showed that such submanifolds contract to round points.

In this paper we are interested in the class of Lagrangian immersions in complex Euclidean space $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, which is a preserved class under the mean curvature flow. We notice that there do not exist Lagrangian self-shrinking spheres (see [2] or [13] and references therein) and, in addition, Smoczyk showed that the class of smooth closed Lagrangian immersions in \mathbb{C}^n is not δ -pinchable for any δ (see [13, Section 4.1]). The authors do not know any available result regarding convergence of compact Lagrangians in \mathbb{C}^n . In fact, the following problem was posed by André Neves [10, Question 7.4] as a Lagrangian analogue of Huisken's classical result [6] for the mean curvature flow of convex spheres:

Find a condition on a Lagrangian torus in \mathbb{C}^2 , which implies that the Lagrangian mean curvature flow $(M_t)_{0 < t < T}$ will become extinct at time T and, after rescale, M_t converges to the Clifford torus.

Our contribution to this problem is the following main result.

Theorem A. *Let M_0 be an embedded Lagrangian compact surface of \mathbb{C}^2 which is contained in some hypersphere $\mathbb{S}^3(R_0)$ of radius $R_0 > 0$. Then the mean curvature flow (MCF) with initial condition M_0 has a unique solution defined on a maximal interval $[0, T)$, $T \leq R_0^2/4$. In addition:*

- (a) *If M_0 divides $\mathbb{S}^3(R_0)$ in two connected components of equal volume, then $T = R_0^2/4$, the limit M_T of the evolving surfaces M_t when $t \rightarrow T$ is a point and, after rescaling the flow by multiplication by $1/\sqrt{R_0^2 - 4t}$, the limit is a Clifford torus in $\mathbb{S}^3 := \mathbb{S}^3(1) \subset \mathbb{C}^2$.*
- (b) *If M_0 divides $\mathbb{S}^3(R_0)$ in two connected components of different volumes (being $2\pi A_0 R_0^3$ the lowest volume), then $T = A_0 R_0^2/2\pi$, the limit M_T of the evolving surfaces M_t when $t \rightarrow T$ is a circle of radius $R_0 \sqrt{1 - 2A_0/\pi}$ and, after rescaling the flow by multiplication by $\sqrt{A_0(R_0^2 - 4t)/(A_0 R_0^2 - 2\pi t)}$, the limit is a cylinder in $\mathbb{R}^3 \subset \mathbb{C}^2$.*

In Proposition 2.1 (see also [14, Corollary 1]) we show that any compact Lagrangian surface of complex Euclidean plane contained in some hypersphere must be the preimage of a spherical closed curve by the corresponding Hopf fibration, providing in general an immersed torus that was called a Hopf torus by Pinkall in [11]. As we shall see in the proof of Theorem A, A_0 coincides with the area enclosed by the spherical curve $\pi(M_0/R_0)$, projection of $(1/R_0)M_0 \subset \mathbb{S}^3$ on $\mathbb{S}^2(1/2)$ by the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$. The isometry type of the torus M_0 depends not only on the length of the spherical curve $\pi(M_0/R_0)$ but also on the enclosed area A_0 . It was proved in [11] that a Hopf torus M_0 is a critical point of the Willmore functional if and only if its corresponding spherical curve is an elastic curve.

Part (a) of Theorem A is our answer to the Neves question quoted before for a Lagrangian embedded torus M_0 . We point out that the hypotheses on M_0 established in Theorem A are preserved by the mean curvature flow (see Lemma 3.3). Thinking of the shape of the closed spherical elastic curves, M_0 could be

a Willmore torus. We remark that the first two authors provided in [2] four rigidity results for the Clifford torus in the class of compact self-shrinkers for the Lagrangian mean curvature flow.

All the singularities appearing in Theorem A are of Type I (see Remark 3.7).

1.1. The ideas behind the main results

We now expose some ideas showing that the evolutions considered in Theorem A are natural in some geometric sense since they (and some other studied in [5], [8] and [9]) can be regarded as evolutions related with geometric flows of plane and spherical curves.

Let $\alpha_0 : I_1 \rightarrow \mathbb{C}^*$ be a regular plane curve and $\gamma_0 : I_2 \rightarrow \mathbb{S}^2(1/2)$ be a regular spherical curve in \mathbb{C}^2 , where I_1 and I_2 are intervals in \mathbb{R} . Let

$$F_0 : I_1 \times I_2 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{C}^2, \quad F_0(x, y) = \alpha_0(x)\tilde{\gamma}_0(y),$$

with $\tilde{\gamma}_0 : I_2 \rightarrow \mathbb{S}^3 \subset \mathbb{C}^2$ a horizontal lift of γ_0 via the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$. We denote by $\langle \cdot, \cdot \rangle$ and J the Euclidean metric and the complex structure in \mathbb{C}^2 and consider simultaneously a one-parameter family of plane curves

$$\alpha = \alpha(x, t) \in \mathbb{C}^*, \quad t \geq 0, \quad \text{with } \alpha(x, 0) = \alpha_0(x), \quad x \in I_1,$$

and a one-parameter family of spherical curves

$$\gamma = \gamma(y, t) \in \mathbb{S}^2(1/2), \quad t \geq 0, \quad \text{with } \gamma(y, 0) = \gamma_0(y), \quad y \in I_2,$$

and define (see [12]) the Lagrangians

$$F = F(x, y, t) = \alpha(x, t)\tilde{\gamma}(y, t), \quad t \geq 0, \quad (x, y) \in I_1 \times I_2 \subseteq \mathbb{R}^2, \quad (1)$$

where $\tilde{\gamma} = \tilde{\gamma}(y, t) \in \mathbb{S}^3 \subset \mathbb{C}^2$ is a horizontal lift of $\gamma = \gamma(y, t)$ via the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$.

It is clear that $F(x, y, 0) = F_0(x, y)$. Our goal is to analyze the possible evolutions of α and γ in order to F be a solution of (MCF). Using [12] and the Lagrangian character of each $F_t := F(\cdot, \cdot, t)$, $t \geq 0$, it is not difficult to get that F is a solution of (MCF) if and only if the following two equations (corresponding to the normal directions $J(F_t)_x$ and $J(F_t)_y$) are satisfied:

$$\left\langle \frac{\partial \alpha}{\partial t}, i\alpha_x \right\rangle + \langle \alpha, \alpha_x \rangle \left\langle \frac{\partial \tilde{\gamma}}{\partial t}, J\tilde{\gamma} \right\rangle = |\alpha_x| \kappa_\alpha + \frac{\langle \alpha_x, i\alpha \rangle}{|\alpha|^2} \quad (2)$$

and

$$|\alpha|^2 \left\langle \frac{\partial \tilde{\gamma}}{\partial t}, J\tilde{\gamma}_y \right\rangle = |\tilde{\gamma}_y| \kappa_{\tilde{\gamma}}. \quad (3)$$

In formulae (2) and (3) and in the rest of this section, subscript x (resp. y) means derivative respect to x (resp. y) and κ will always denote curvature of the corresponding curve along the paper. Looking at (3) we distinguish two complementary cases:

Case (i): there is no (normal) evolution for $\tilde{\gamma} = \tilde{\gamma}(y, t)$ (and hence for $\gamma = \gamma(y, t)$) and so $\tilde{\gamma}$ (and γ) must be a static geodesic, say

$$\tilde{\gamma}(y, t) = (\cos y, \sin y) \in \mathbb{C}^2, \quad \forall t \geq 0.$$

Then equation (2) can be easily rewritten as

$$\left(\frac{\partial\alpha}{\partial t}\right)^\perp = \vec{\kappa}_\alpha - \frac{\alpha^\perp}{|\alpha|^2},$$

where $\vec{\kappa}_\alpha$ is the curvature vector of α and α^\perp denotes the normal component of α . Putting this information in (1) we arrive at the evolution studied in [5], [8] and [9].

Case (ii): necessarily $|\alpha|$ only depends on time variable t , say $R(t) := |\alpha|$. This means that the evolution of $\alpha = \alpha(x, t)$ consists of concentric circles centered at the origin and, up to reparametrizations, it can be given by $\alpha(x, t) = R(t)e^{ix}$. Now (2) translates into a simple o.d.e. for $R(t)$, concretely $-R \, dR/dt = 2$, whose general solution is $R(t) = \sqrt{R(0)^2 - 4t}$. Putting this in (1), we get that in this case F can be written as

$$F(x, y, t) = \sqrt{R_0^2 - 4t} e^{ix} \tilde{\gamma}(y, t), \quad 0 \leq t < \frac{R_0^2}{4}, \quad (4)$$

with $R_0 = R(0)$ and where $\tilde{\gamma}(y, t)$ satisfy now the equation, coming from (3), given by

$$\left\langle \frac{\partial \tilde{\gamma}}{\partial t}, \frac{J \tilde{\gamma}_y}{|\tilde{\gamma}_y|} \right\rangle = \frac{\kappa_{\tilde{\gamma}}}{R_0^2 - 4t}. \quad (5)$$

Using that the Hopf fibration π is a Riemannian submersion, we rewrite (5) as

$$\left\langle \frac{\partial \gamma}{\partial t}, \frac{\gamma \times \gamma_y}{|\gamma_y|} \right\rangle = \frac{2\kappa_\gamma}{R_0^2 - 4t}, \quad (6)$$

where \times denotes the cross product in \mathbb{R}^3 . We will check in Section 3 that (6) is essentially the curve shortening flow in $\mathbb{S}^2(1/2)$. The relation between this flow and the corresponding flow (4) of the initial Lagrangian surface will lead to different situations and their study in depth allows us to prove Theorem A in Section 3.

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2. Preliminaries

2.1. The geometry of Hopf tori in the Hopf fibration

Let $\mathbb{S}^3(R)$ be the 3-sphere of radius R in $\mathbb{C}^2 \equiv \mathbb{R}^4$, let $\mathbb{S}^2(R/2)$ be the 2-sphere of radius $R/2$ in \mathbb{R}^3 , and let $\pi_R : \mathbb{S}^3(R) \rightarrow \mathbb{S}^2(R/2)$ be the Hopf fibration

$$\pi_R(z, w) = \frac{1}{2R} (2z\bar{w}, |z|^2 - |w|^2), \quad (z, w) \in \mathbb{S}^3(R) \subset \mathbb{C}^2.$$

When $R = 1$, we will omit the subindex R . We shall denote by N the unit vector orthogonal to $\mathbb{S}^3(R)$ pointing inward. If J is the natural complex structure of \mathbb{C}^2 , then the fibers of the Hopf fibration are the integral curves of JN , which are geodesics of $\mathbb{S}^3(R)$. For every $p \in \mathbb{S}^3(R)$, the subspace $\mathcal{H}_p = \{JN_p\}^\perp$ of $T_p\mathbb{S}^3(R)$ orthogonal to JN_p is called horizontal, and it is invariant under J . Moreover π_{R*} restricted to \mathcal{H} is an isometry and, through this isometry, J induces on $\mathbb{S}^2(R/2)$ a complex structure that we shall denote again by J .

Let P_R be a closed curve in $\mathbb{S}^2(R/2)$ which we will parametrize by $\gamma_R(v)$, $v \in [0, 2\pi]$, where $\gamma_R : \mathbb{S}^1 \equiv [0, 2\pi]/\sim \rightarrow \mathbb{S}^2(R/2)$, and define $M_R \subset \mathbb{S}^3(R)$ the Riemannian surface $\pi_R^{-1}(P_R)$ given by its position vector F_R in $\mathbb{S}^3(R)$; we remark that $F_R : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3(R)$. We have the following diagram:

$$\begin{array}{ccccc} M_R \equiv \mathbb{S}^1 \times \mathbb{S}^1 & \xrightarrow{F_R} & \mathbb{S}^3(R) & \hookrightarrow & \mathbb{C}^2 \\ \pi_R \downarrow & & \downarrow \pi_R & & \\ P_R \equiv \mathbb{S}^1 & \xrightarrow{\gamma_R} & \mathbb{S}^2(R/2) & & \end{array}$$

If X is a vector field tangent to P_R or $\mathbb{S}^2(R/2)$, then X^* will denote its horizontal lift tangent to M_R or $\mathbb{S}^3(R)$, respectively.

Given $x \in \mathbb{S}^2(R/2)$, if we can write γ_R as the image of a curve $c(u)$ in $T_x\mathbb{S}^2(R/2)$ by the exponential map \exp_x , then we can parametrize F_R as $F_R(\beta, u) = \exp_{e^{i\beta}q} c(u)^*$, with $\pi(q) = x$ and $c(u)^*$ the horizontal lift of $c(u)$ at $e^{i\beta}q$.

Given X, Y vector fields tangent to $\mathbb{S}^2(R/2)$ we get that

$$\pi_{R*}(\bar{\nabla}_{X^*} Y^*) = \hat{\nabla}_X Y, \quad (7)$$

where $\bar{\nabla}$ and $\hat{\nabla}$ denote the Riemannian connections of $\mathbb{S}^3(R)$ and $\mathbb{S}^2(R/2)$, respectively. Moreover, we shall denote by $\tilde{\nabla}$ the covariant derivative (that is, the standard directional derivative) in \mathbb{C}^2 .

Let us denote by $e_1 = \gamma'_R(u)/|\gamma'_R(u)|$, by $\vec{\kappa}_R$ the curvature vector of γ_R in $\mathbb{S}^2(R/2)$, by σ_R and $\bar{\sigma}_R$ the second fundamental forms of M_R in \mathbb{C}^2 and $\mathbb{S}^3(R)$, respectively. H_R and \bar{H}_R will denote the respective mean curvatures. Moreover, $\tilde{\sigma}$ will denote the second fundamental form of $\mathbb{S}^3(R)$ in \mathbb{C}^2 .

One has that $\langle \bar{\nabla}_{e_1^*} e_1^*, JN \rangle = -\langle e_1^*, \bar{\nabla}_{e_1^*} JN \rangle = -\langle e_1^*, \tilde{\nabla}_{e_1^*} JN \rangle = \frac{1}{R} \langle e_1^*, J e_1^* \rangle = 0$. That is, $\bar{\nabla}_{e_1^*} e_1^*$ is horizontal and, since it must be orthogonal to e_1^* , it is in the direction of $J e_1^*$.

As a consequence of this fact and (7) one has

$$\vec{\kappa}_R = (\hat{\nabla}_{e_1} e_1)^\perp = \pi_{R*}((\bar{\nabla}_{e_1^*} e_1^*)^\perp) = \pi_{R*}(\bar{\sigma}(e_1^*, e_1^*)). \quad (8)$$

That is, $\bar{\sigma}(e_1^*, e_1^*) = \vec{\kappa}_R^*$. Then, for the mean curvatures one has

$$\begin{aligned} H_R &= \sigma_R(e_1^*, e_1^*) + \sigma_R(JN, JN) = \bar{\sigma}_R(e_1^*, e_1^*) + \tilde{\sigma}_R(e_1^*, e_1^*) + \bar{\sigma}_R(JN, JN) + \tilde{\sigma}(JN, JN) \\ &= \vec{\kappa}_R^* + \frac{2}{R} N = \frac{1}{R} \vec{\kappa}^* - \frac{2}{R} F, \end{aligned} \quad (9)$$

where we have used, for the last equality, that under an homothety the curvature of a curve becomes divided by the magnitude of the homothety. Recall also that when we consider $R = 1$ we do not write the subindex R . Moreover, we have chosen N pointing inward, which gives $N = -F$.

2.2. Spherical Lagrangian submanifolds

In the complex Euclidean plane \mathbb{C}^2 we consider the bilinear Hermitian product defined by

$$(z, w) = z_1 \bar{w}_1 + z_2 \bar{w}_2, \quad z, w \in \mathbb{C}^2.$$

Then $\langle \cdot, \cdot \rangle = \text{Re}(\cdot, \cdot)$ is the Euclidean metric on \mathbb{C}^2 and $\omega = -\text{Im}(\cdot, \cdot)$ is the Kaehler two-form given by $\omega(\cdot, \cdot) = \langle J\cdot, \cdot \rangle$, where J is the complex structure on \mathbb{C}^2 .

Let $F : M \rightarrow \mathbb{C}^2$ be an isometric immersion of a surface M into \mathbb{C}^2 . F is said to be Lagrangian if $F^*\omega = 0$. This is equivalent to the orthogonal decomposition $T\mathbb{C}^2 = F_*TM \oplus JF_*TM$, where TM is the tangent bundle of M .

Proposition 2.1. *Let M be any compact Lagrangian surface of \mathbb{C}^2 contained in some hypersphere $\mathbb{S}^3(R)$, $R > 0$. Then M must be the preimage of a closed curve in $\mathbb{S}^2(R/2)$ by the Hopf fibration $\pi_R : \mathbb{S}^3(R) \rightarrow \mathbb{S}^2(R/2)$.*

Proof. Let N be the unit vector normal to $\mathbb{S}^3(R)$ in \mathbb{C}^2 . Then JN is a vector field on $\mathbb{S}^3(R)$ whose integral curves are the fibers of the Hopf fibration π_R . Since M is Lagrangian, the restriction of JN to M is a tangent vector field on M and its integral curves are contained in M . In this way, the restriction of π_R to M is a Riemannian submersion on its image $\pi_R(M) =: C$ with the same fibers that the Hopf fibration. That is, $M = \pi_R^{-1}(C)$ for some closed curve $C \subset \mathbb{S}^2(R/2)$. \square

Remark 2.2. If F_R denotes a Lagrangian immersion of M into $\mathbb{S}^3(R) \subset \mathbb{C}^2$, then Proposition 2.1 tells us that F_R can be regarded as $F_R : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3(R)$ and there exists a curve $\gamma_R : \mathbb{S}^1 \rightarrow \mathbb{S}^2(R/2)$ such that $(\pi_R \circ F_R)(u, v) = \gamma_R(v)$ and $F_R(\mathbb{S}^1 \times \{v_0\})$ is a fiber of the Hopf fibration for every $v_0 \in \mathbb{S}^1$. That is, a Lagrangian spherical immersion of a compact surface is a Hopf torus and viceversa. There are many Lagrangian tori in \mathbb{C}^2 which are not Hopf tori; e.g., see [2, Section 3].

3. Proof of Theorem A

Let M_t be a one-parameter family of Lagrangian surfaces of \mathbb{C}^2 contained in the spheres $\mathbb{S}^3(R(t))$ of radius $R(t) > 0$. Using Proposition 2.1 and Remark 2.2, this family can be parametrized in the following way:

$$F_{R(t)}(u, v, t) = R(t)F(u, v, t), \quad (10)$$

where $F(\cdot, \cdot, t) : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3$ is a family of Lagrangian immersions of a torus in \mathbb{C}^2 contained in the unit hypersphere, and there exists a family of curves $\gamma(\cdot, t) : \mathbb{S}^1 \rightarrow \mathbb{S}^2(1/2)$ such that $(\pi_{R(t)} \circ F_{R(t)})(u, v, t) = R(t)\gamma(v, t)$, which is equivalent to

$$(\pi \circ F)(u, v, t) = \gamma(v, t). \quad (11)$$

Lemma 3.1. *The family of Lagrangian immersions given in (10) satisfies the mean curvature flow equation (MCF) in \mathbb{C}^2 if and only if $R(t) = \sqrt{R_0^2 - 4t}$ and $F(\cdot, t)$ is the preimage by the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$ of a curve $\gamma(\cdot, \bar{t}(t))$ satisfying the mean curvature flow equation in the 2-sphere, with the change of parameter given by $\bar{t} = \frac{1}{4} \ln \frac{R_0^2}{R_0^2 - 4t}$, where $R_0 = R(0)$.*

Proof. The left side of (MCF) is obviously

$$\frac{\partial F_R}{\partial t}(u, v, t) = R'(t) F(u, v, t) + R(t) \frac{\partial F}{\partial t}(u, v, t). \quad (12)$$

To compute the right side of (MCF), we will use (9) at each time t . Using (9) and (12), the evolution equation $\partial F_R / \partial t = H_R$ becomes

$$R'F + R \frac{\partial F}{\partial t} = \frac{1}{R} \bar{\kappa}_\gamma^* - \frac{2}{R} F.$$

Since $|F| = 1$, necessarily $\frac{\partial F}{\partial t}$ is orthogonal to F , and so the above equation separates in two coupled ones:

$$\begin{cases} R' = -\frac{2}{R}, \\ R \frac{\partial F}{\partial t} = \frac{1}{R} \vec{\kappa}_\gamma^*. \end{cases}$$

Putting $R(0) = R_0$, the solution of the first equation is $R^2(t) = R_0^2 - 4t$. Plugging this solution in the second one, we obtain that

$$\frac{\partial F}{\partial t} = \frac{1}{R_0^2 - 4t} \vec{\kappa}_\gamma^*. \quad (13)$$

Using (11), the composition with π_* of the above equation implies that

$$\frac{\partial \gamma}{\partial t} = \frac{1}{R_0^2 - 4t} \vec{\kappa}_\gamma. \quad (14)$$

This is not exactly the mean curvature flow for $\gamma(v, t)$; but we consider the change of parameter $t = t(\bar{t})$ given by

$$\bar{t} = \bar{t}(t) = \int_0^t \frac{1}{R_0^2 - 4s} ds = -\frac{1}{4} \ln \frac{R_0^2 - 4t}{R_0^2} = \ln \left(\frac{R_0^2}{R_0^2 - 4t} \right)^{1/4}. \quad (15)$$

In this way, we arrive at

$$\frac{\partial \gamma}{\partial \bar{t}} = \frac{\partial t}{\partial \bar{t}} \frac{1}{R_0^2 - 4t} \vec{\kappa}_\gamma = \vec{\kappa}_\gamma, \quad (16)$$

which is the mean curvature flow for $\gamma(u, t(\bar{t}))$. \square

Next we employ Lemma 3.1 to prove the following result. In particular, we deduce that the spherical condition is preserved by the Lagrangian mean curvature flow.

Theorem 3.2. *Let F_{R_0} be a Lagrangian immersion of a surface in \mathbb{C}^2 , contained in the hypersphere $\mathbb{S}^3(R_0)$ of radius $R_0 > 0$. Then F_{R_0} evolves under the mean curvature flow following the formula:*

$$F_{R_0}(\cdot, t) = \sqrt{R_0^2 - 4t} F(\cdot, t), \quad (17)$$

where $F(\cdot, t)$ is the preimage by the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$ of a curve $\gamma(\cdot, \bar{t}(t))$ satisfying the evolution equation (16), where $\bar{t}(t)$ is the function given in (15).

Proof. Define $F_0 = (1/R_0)F_{R_0}$, and $\gamma_0 : \mathbb{S}^1 \rightarrow \mathbb{S}^2(1/2)$ satisfying $\pi \circ F_0 = \gamma_0$ as in Remark 2.2. Let $\gamma(\cdot, \bar{t}) : \mathbb{S}^1 \times [0, \bar{T}] \rightarrow \mathbb{S}^2(1/2)$ be a solution of the curve shortening problem (16) satisfying $\gamma(\cdot, 0) = \gamma_0(\cdot)$. After the reparametrization of time given by (15), the family $\gamma(\cdot, t)$ is a solution of (14). Then $F(\cdot, \cdot, t) : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3$ defined as the family of Lagrangian surfaces of \mathbb{C}^2 contained in \mathbb{S}^3 which are liftings of the Hopf fibration is a solution of (13) satisfying $F(\cdot, 0) = F_0$, and $\sqrt{R_0^2 - 4t} F(\cdot, t)$ is (by Lemma 3.1) a solution of the mean curvature flow equal to F_{R_0} at $t = 0$. By the uniqueness of the solution of the mean curvature flow with given initial condition, the statement of the theorem follows. \square

In order to continue with the proof of Theorem A, we need the following lemma.

Lemma 3.3. Let γ_0 be a closed simple curve in $\mathbb{S}^2(1/2)$ enclosing a domain with area $A_0 \leq \pi/2$. If $A(\bar{t})$ denotes the area enclosed by a solution $\gamma(\cdot, \bar{t})$ of (16) with initial condition $\gamma(\cdot, 0) = \gamma_0(\cdot)$, then $A(\bar{t}) = \pi/2 - (\pi/2 - A_0)e^{4\bar{t}}$, and the extinction time of $\gamma(\cdot, \bar{t})$ is given by $\tau = \ln\left(\frac{\pi}{\pi - 2A_0}\right)^{1/4} \leq \infty$.

Proof. It is well known that the rate at which the area $A(\bar{t})$ decrease with time \bar{t} is given by $\partial A/\partial \bar{t} = -\int_{\gamma} \kappa_{\gamma} ds$, which implies using the Gauss–Bonnet formula that $A'(\bar{t}) = 4A(\bar{t}) - 2\pi$, taking into account that γ lies in a sphere of radius $1/2$. Solving the former equation, we obtain that $\ln(2\pi - 4A(\bar{t}))^{1/4} = \ln(2\pi - 4A_0)^{1/4} + \bar{t}$, and this proves the statement. \square

Corollary 3.4. Under the hypotheses of Theorem 3.2 and Lemma 3.3, there are only two possibilities for the evolution under the mean curvature flow of a Lagrangian embedding F_{R_0} of a compact surface in \mathbb{C}^2 :

- (a) If $F_{R_0}(\mathbb{S}^1 \times \mathbb{S}^1)$ divides $\mathbb{S}^3(R_0)$ in two connected components of equal volume, then $F_{R_0}(\cdot, t)$ is defined for $t \in [0, R_0^2/4)$, the limit of $F_{R_0}(\cdot, t)$ when $t \rightarrow R_0^2/4$ is the center of $\mathbb{S}^3(R_0)$, and rescaling t by \bar{t} according to (15) and $F_{R_0}(\cdot, t)$ by $\tilde{F}_{R_0}(\cdot, t) = \frac{1}{\sqrt{R_0^2 - 4t}} F_{R_0}(\cdot, t)$, then $\lim_{\bar{t} \rightarrow \infty} \tilde{F}_{R_0}(\cdot, \bar{t})$ is the Clifford torus in \mathbb{S}^3 .
- (b) If $F_{R_0}(\mathbb{S}^1 \times \mathbb{S}^1)$ divides $\mathbb{S}^3(R_0)$ in two connected components of different volumes, then $F_{R_0}(\cdot, t)$ is defined for $t \in [0, T)$, $T = A_0 R_0^2/2\pi < R_0^2/4$, and the limit of $F_{R_0}(\cdot, t)$ when $t \rightarrow T$ is a circle of radius $\sqrt{R_0^2 - 4T} = R_0 \sqrt{1 - 2\pi/A_0} > 0$, where A_0 is the area enclosed by the curve $\gamma_0(\cdot) = \gamma(\cdot, 0) \subset \mathbb{S}^2(1/2)$.

Proof. From Theorem 3.2 it follows that the flow $F_{R_0}(\cdot, t)$ given in (17) is defined in $[0, T)$, the intersection of the intervals where $\sqrt{R_0^2 - 4t}$ and $\gamma(\cdot, \bar{t}(t))$ are defined. On the one hand, this implies immediately that $T \leq R_0^2/4$. On the other hand, $\gamma(\cdot, \bar{t})$ is well defined on $[0, \tau)$ (see Lemma 3.3). Using (15), we get that

$$t(\tau) = \frac{R_0^2}{4} (1 - e^{-4\tau}). \quad (18)$$

It is well known for the curve shortening flow in the 2-sphere (see for instance [4] and also [3]) that there are only two possibilities:

- (a) $\tau = \infty$ and $\lim_{\bar{t} \rightarrow \infty} \gamma(\cdot, \bar{t})$ is a geodesic of $\mathbb{S}^2(1/2)$.
This case corresponds to $A_0 = \pi/2$. Then it follows from (18) that $t(\infty) = R_0^2/4$ and so $\lim_{t \rightarrow R_0^2/4} \gamma(\cdot, \bar{t}(t))$ is a geodesic in $\mathbb{S}^2(1/2)$. Thus, the limit of the preimage $F(\cdot, t)$ when $t \rightarrow R_0^2/4$ is the preimage of a geodesic in $\mathbb{S}^2(1/2)$, which is the Clifford torus in \mathbb{S}^3 . Therefore, rescaling t to get \bar{t} and $F_{R_0}(\cdot, t)$ to $\tilde{F}_{R_0}(\cdot, t) = \frac{1}{\sqrt{R_0^2 - 4t}} F_{R_0}(\cdot, t)$, we obtain that

$$\tilde{F}_{R_0}(\cdot, \bar{t}) := \tilde{F}_{R_0}(\cdot, t(\bar{t})) = F(\cdot, t(\bar{t}))$$

and, as we have just deduced, $\lim_{\bar{t} \rightarrow \infty} F(\cdot, t(\bar{t}))$ is the Clifford torus in \mathbb{S}^3 .

- (b) $\tau < \infty$ and $\lim_{\bar{t} \rightarrow \tau} \gamma(\cdot, \bar{t})$ is a point of $\mathbb{S}^2(1/2)$.
This case corresponds to $A_0 < \pi/2$. Using Lemma 3.3 and (18), we have that $T = t(\tau) = A_0 R_0^2/2\pi < R_0^2/4$. Moreover, the limit when $t \rightarrow T$ of $\gamma(\cdot, \bar{t}(t))$ is a point of $\mathbb{S}^2(1/2)$, whose preimage is a circle of radius 1 in \mathbb{S}^3 . Thus $\lim_{t \rightarrow T} F_{R_0}(\cdot, t)$ is a circle of radius $\sqrt{R_0^2 - 4T} > 0$ in $\mathbb{S}^3(\sqrt{R_0^2 - 4T})$. \square

In the case (a) of Corollary 3.4 we have used the total space to rescale. However, in the case (b) we will use the base space to rescale. A natural rescaling for the curve γ in $\mathbb{S}^2(1/2)$ shrinking to a point $x \in \mathbb{S}^2(1/2)$ is to consider the 2-sphere in \mathbb{R}^3 and to multiply $\gamma - x$ by a function of \bar{t} such that the area enclosed by the rescaled curves be constant. According to Lemma 3.3, this rescaling is given by

$$\tilde{\gamma}(\cdot, \bar{t}) - x = \sqrt{\frac{A_0}{\pi/2 - (\pi/2 - A_0)e^{4\bar{t}}}} (\gamma(\cdot, t(\bar{t})) - x). \quad (19)$$

Now a well known result on the curve shortening flow in a surface (see [15]) implies that the limit of the rescaling (19) when $\bar{t} \rightarrow \tau$ (that is, $t \rightarrow T$) is a planar circle centered at x of radius $\sqrt{A_0/\pi}$.

Hence, taking into account the formula given in (15), for the Lagrangian surface $F_{R_0}(\cdot, t)$ we will use the rescaling

$$\tilde{F}_{R_0}(\cdot, t) - R(t)q = \sqrt{\frac{A_0}{\pi/2 - (\pi/2 - A_0)\left(\frac{R_0^2}{R_0^2 - 4t}\right)}} (F_{R_0}(\cdot, t) - R(t)q) \quad (20)$$

where $R(t) = \sqrt{R_0^2 - 4t}$ and q is a point in the limit circle of F when $t \rightarrow T$. Notice that $\pi(q) = x$ and that the rescaling factor in (20) coincides with that in (19) when we consider the relation (15).

Proposition 3.5. *When $T < R_0^2/4$, the limit of the rescaling (20) when $t \rightarrow T$ is a cylinder passing through $\sqrt{R_0^2 - 4T}q$, which is the product of a circle of radius $\sqrt{(R_0^2 - 4T)A_0/\pi}$ and a line.*

Proof. Let us denote

$$\lambda \equiv \lambda(t) := \sqrt{\frac{A_0}{\pi/2 - (\pi/2 - A_0)\left(\frac{R_0^2}{R_0^2 - 4t}\right)}}. \quad (21)$$

We remark that $\lambda \rightarrow \infty$ when $t \rightarrow T = A_0 R_0^2/2\pi$ and recall that $R(t) = \sqrt{R_0^2 - 4t}$.

The rescalings \tilde{F}_{R_0} and $\tilde{\gamma}_{R_0} := R(t)\tilde{\gamma}$, of F_{R_0} and $\gamma_{R_0} := R(t)\gamma$ respectively (see equations (20) and (19)), are just the restrictions to F_{R_0} and γ_{R_0} of the maps

$$\begin{aligned} \mu_t : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2; & \mu_t(z) &= R(t)q + \lambda(z - R(t)q), & \text{and} \\ \nu_t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3; & \nu_t(w) &= R(t)x + \lambda(w - R(t)x), \end{aligned}$$

which transform spheres in the following way:

$$\mu_t(\mathbb{S}_{R(t)}^3) = \mathbb{S}^3((1 - \lambda)R(t)q, \lambda R(t)), \text{ and } \nu_t(\mathbb{S}_{R(t)/2}^2) = \mathbb{S}^2((1 - \lambda)R(t)x, \lambda R(t)/2),$$

where $\mathbb{S}^m(y, r)$ indicates a sphere in \mathbb{R}^{m+1} of radius r and center y . Then the map

$$\tilde{\pi}_t = \nu_t \circ \pi_{R(t)} \circ \mu_t^{-1} : \mathbb{S}^3((1 - \lambda)R(t)q, \lambda R(t)) \longrightarrow \mathbb{S}^2((1 - \lambda)R(t)x, \lambda R(t)/2)$$

is a Hopf fibration which, for every $z \in \mathbb{S}^3 \subset \mathbb{C}^2$, takes the geodesic circle $(1 - \lambda)R(t)q + \lambda R(t)e^{i\beta}z \in \mathbb{S}^3((1 - \lambda)R(t)q, \lambda R(t))$ into the point $(1 - \lambda)R(t)x + \lambda R(t)\pi(e^{i\beta}z) \in \mathbb{S}^2((1 - \lambda)R(t)x, \lambda R(t)/2)$. Moreover, $\tilde{\pi}_t \circ \tilde{F}_{R_0}(\cdot, t) = \nu_t \circ \pi_{R(t)} \circ \mu_t^{-1} \circ \tilde{F}_{R_0}(\cdot, t) = \nu_t \circ \pi_{R(t)} \circ F_{R_0}(\cdot, t) = \nu_t \circ \gamma_{R_0}(\cdot, t) = \tilde{\gamma}_{R_0}(\cdot, t)$. Moreover, since $\tilde{F}_{R_0}(\cdot, t)$ is a Lagrangian submanifold of \mathbb{C}^2 , Proposition 2.1 applies to the Hopf map $\tilde{\pi}_t$ and so $\tilde{F}_{R_0}(\cdot, t)$ is the preimage of $\tilde{\gamma}_{R_0}(\cdot, t)$ by $\tilde{\pi}_t$.

Let $\{e_1, e_2\}$ be an orthonormal basis of $T_{R(t)x}\mathbb{S}^2((1 - \lambda)R(t)x, \lambda R(t)/2)$, and let $\{e_1^*, e_2^*\}$ be its corresponding lifting to the fiber on $R(t)x$ in $\mathbb{S}^3((1 - \lambda)R(t)q, \lambda R(t))$. Since $\tilde{\gamma}_{R_0}(\cdot, t)$ converges to a circle with center at $R(T)x$ and radius $R(T)\sqrt{A_0/\pi}$, when t is near T , $\tilde{\gamma}_{R_0}(\cdot, t)$ becomes convex near its limit, and can be parametrized in the form

$$\begin{aligned}\tilde{\gamma}_{R_0}(\varphi, t) &= \exp_{R(t)x} r(\varphi, t)(\cos \varphi e_1 + \sin \varphi e_2) \\ &= (1 - \lambda)R(t)x + \frac{\lambda}{2}R(t) \left(2x \cos \frac{r(\varphi, t)}{\lambda R(t)} + (\cos \varphi e_1 + \sin \varphi e_2) \sin \frac{r(\varphi, t)}{\lambda R(t)} \right)\end{aligned}$$

with $\lim_{t \rightarrow T} r(\varphi, t) = R(T)\sqrt{A_0/\pi}$, where \exp denotes the exponential map in $\mathbb{S}^2((1 - \lambda)R(t)x, \lambda R(t)/2)$. As a consequence, since $\tilde{F}_{R_0}(\cdot, t)$ is the preimage of $\tilde{\gamma}_{R_0}(\cdot, t)$ by $\tilde{\pi}_t$, it can be parametrized (as was recalled in section 2.1) by

$$\begin{aligned}\tilde{F}_{R_0}(\varphi, s, t) &= \exp_{(1-\lambda)R(t)q + e^{i(s/(\lambda R(t)))}\lambda R(t)q} r(\varphi, t)(\cos \varphi e_1^* + \sin \varphi e_2^*) \\ &= (1 - \lambda)R(t)q + \lambda R(t) \left(\left(q \cos \frac{s}{\lambda R(t)} + Jq \sin \frac{s}{\lambda R(t)} \right) \cos \frac{r(\varphi, t)}{\lambda R(t)} \right. \\ &\quad \left. + (\cos \varphi e_1^* + \sin \varphi e_2^*) \sin \frac{r(\varphi, t)}{\lambda R(t)} \right),\end{aligned}\tag{22}$$

where s is the arclength of the curve $(1 - \lambda)R(t)q + e^{i(s/(\lambda R(t)))}\lambda R(t)q$.

Now, taking the limit in (22) when $t \rightarrow T$ (which implies $\lambda \rightarrow \infty$), we obtain the cylinder

$$\tilde{F}_{R_0}(\varphi, s, T) = R(T)q + sJq + (\cos \varphi e_1^* + \sin \varphi e_2^*)R(T)\sqrt{A_0/\pi},$$

which is the cylinder indicated in the statement of the Proposition. \square

Remark 3.6. We observe that the rescaling (20) is not exactly the standard one given in [6]. Nevertheless, they only differ in the product by a bounded function and consequently they are equivalent. Thus the blow up will be again a cylinder in $\mathbb{R}^3 \subset \mathbb{C}^2$.

Remark 3.7. All the singularities appearing in Theorem A are Type I singularities. In fact, following Section 2 and using Theorem 3.2, it is not difficult to check that the second fundamental form σ of the evolution (17) is given by

$$|\sigma|^2 = \frac{4 + \kappa_\gamma^2}{R_0^2 - 4t}, \quad t \in [0, T).\tag{23}$$

In case (a), we have that $T = R_0^2/4$ and we know that κ_γ is bounded by some constant L ; then we get that $(T - t)|\sigma|^2 = 1 + \kappa_\gamma^2/4 \leq 1 + L/4$, which implies the condition of being a Type I singularity.

In case (b), we have that $T = t(\tau) = A_0 R_0^2/2\pi < R_0^2/4$ and we know that γ develops a Type I singularity. So there exists a constant C such that $(\tau - \bar{t})\kappa_\gamma^2 \leq C$. Using that $A_0 < \pi/2$ and (15), we get that

$$(T - t)|\sigma|^2 < 1 + \frac{(T - t)\kappa_\gamma^2}{R_0^2 - 4t} = 1 + \frac{1 - e^{4(\bar{t} - \tau)}}{4}\kappa_\gamma^2.$$

If we define $G(\bar{t}) = (1 - e^{4(\bar{t} - \tau)})/4 - (\tau - \bar{t})$, it is easy to check that $G'(\bar{t}) > 0$ and so $G(\bar{t}) < G(\tau) = 0$. Hence we conclude that $(T - t)|\sigma|^2 < 1 + (\tau - \bar{t})\kappa_\gamma^2 \leq 1 + C$, that shows that the behavior of $|\sigma|$ in case (b) is determined by the one of $|\kappa_\gamma|$, which corresponds to a Type I singularity.

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