



Radius of analyticity of analytic functions on Banach spaces

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ARTICLE INFO

Article history:

Received 15 December 2017
Available online 6 March 2018
Submitted by R.M. Aron

Keywords:

Real analytic function
Radius of convergence
Banach lattice
Regular polynomial

ABSTRACT

We examine the question of when the radius of analyticity of a real analytic function on a real Banach space is equal to its radius of uniform convergence. We will see that a positive solution to this problem on ℓ_1 implies a positive solution on all Banach spaces. In the final section we show that our question has an affirmative answer for power series of positive polynomials on Banach lattices.

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1. Introduction

Many of the classical functions which we consider on the real line or the complex plane can be approximated by polynomials. It is natural therefore that when we study functions on infinite dimensional spaces that we should look for a class of functions that can also be approximated by polynomials. A first step is to introduce what we mean by a polynomial or, more precisely, what we mean by a homogeneous polynomial.

Let E and F be Banach spaces over the same field \mathbb{K} , ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). A continuous function $P_n: E \rightarrow F$ is said to be a continuous n -homogeneous polynomial if there is a (unique symmetric) continuous n -linear mapping $A_n: \underbrace{E \times \cdots \times E}_{n\text{-times}} \rightarrow F$ such that $P_n(x) = A_n(x, \dots, x)$ for all x in E . A function $P: E \rightarrow F$ is

said to be a polynomial of degree n if $P(x) = \sum_{j=0}^n P_j(x)$ with $P_j: E \rightarrow F$ a j -homogeneous polynomial and

$P_n \neq 0$. We denote by $\mathcal{P}(^n E, F)$ the Banach space of all n -homogeneous polynomials from E into F . The norm of P in $\mathcal{P}(^n E, F)$ is defined by $\|P\| = \sup_{x \in B_E} \|P(x)\|_F$.

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Let a be a point in E . A series of the form

$$\sum_{j=0}^{\infty} P_j(x-a)$$

where P_j is a j -homogeneous polynomial is called a power series in x about a . The *radius of uniform convergence* of $\sum_{j=0}^{\infty} P_j(x-a)$ is defined as

$$\sup \left\{ r : \sum_{j=0}^{\infty} P_j(x-a) \text{ converges uniformly in } a + rB_E \right\}.$$

Let U be an open subset of a Banach space E . The Cauchy–Hadamard formula tells us that the radius of uniform convergence of $\sum_j P_j$ is equal to $(\limsup_{j \rightarrow \infty} \|P_j\|^{1/j})^{-1}$. Given an infinitely differentiable function f from U into F the Taylor series is the natural power series we associate to f . For $j = 0, 1, \dots$ and a in U we let $\hat{d}^j f(a)$ denote the j -th (Fréchet) derivative of f at a . We shall say that f is analytic on U if for each point a in U the series $\sum_{j=0}^{\infty} \frac{1}{j!} \hat{d}^j f(a)$ converges uniformly to f in a neighbourhood of a . As we will see shortly, there are considerable differences in what is known in the real and complex cases. To distinguish between the two settings the term *real analytic* is used for functions which are analytic on a real Banach space while *holomorphic* is reserved for functions which are analytic on a complex Banach space. For further information on real analytic functions we refer to [3,4] while for further information on holomorphic functions we refer to [7].

The Cauchy estimates tell us that if f is holomorphic in $B(a, R)$, the open ball with centre a and radius R , then its Taylor series at a has radius of uniform convergence R . In addition, for each point b in $B(a, R)$, the Taylor series of f about b , $\sum_{j=0}^{\infty} \frac{1}{j!} \hat{d}^j f(b)$, converges uniformly to f in $B(b, R - \|b - a\|)$.

The corresponding result for real analytic functions is unknown. According to [17] a real analytic function $f: B(a, R) \rightarrow F$ is *fully analytic* if for each point b in $B(a, R)$, the series $\sum_{j=0}^{\infty} \frac{1}{j!} \hat{d}^j f(b)$ converges uniformly to f in $B(b, R - \|b - a\|)$. The *radius of analyticity of f at a* is the largest $\rho > 0$ so that f is fully analytic in $B(a, \rho)$. One's first approach in calculating the radius of analyticity is to use complexification and the Cauchy–Hadamard formula. In [19] Taylor showed that if f is a real analytic function on the ball $B(a, R)$ whose Taylor series at a has radius of uniform convergence R , then the complexification of f has radius of uniform convergence $R/e\sqrt{2}$. From this it follows that when f is analytic in $B(a, R)$ then it is fully analytic in $B(a, R/e\sqrt{2})$. An improvement in the complexification constant by Muñoz, Sarantopoulos and Tonge [16] meant that the radius of analyticity could be increased to $R/2$. However, as this estimate uses the optimal value of the complexification constant, $R/2$ is the best we can hope to achieve for a general Banach space through complexification.

Another line of attack was therefore needed. By considering a more focused analysis of the norm of an n -linear mapping, using Hoeffding's Inequality from probability theory, in 2009, Nguyen [17] improved this again to R/\sqrt{e} . The current best estimate for the radius of analyticity for general Banach spaces, due to Hájek and Johanis [10, p. 81], is $R/\sqrt{2}$, obtained using an inequality of Harris [11]. Subsequently, using methods similar to Nguyen, this value has also been obtained by Papadimitris and Sarantopoulos [18]. However, for certain spaces this constant can be greatly improved. Using the result of Banach [1] that the polarisation constant for Hilbert spaces is 1, it can be shown, see [17], that the radius of uniform convergence and the radius of analyticity coincide on Hilbert spaces.

The above results motivate the following definition.

Definition 1. Let E be a real Banach space. The *constant of analyticity of E* is the supremum of the set of positive real numbers ρ for which every power series at the origin in E , with unit radius of uniform convergence, has radius of analyticity at least ρ . We denote this number by $\mathcal{A}(E)$.

In general, in view of the above results, we have

$$\frac{1}{\sqrt{2}} \leq \mathcal{A}(E) \leq 1$$

and when E is a Hilbert space, $\mathcal{A}(E) = 1$ [17].

In this paper we undertake a detailed study of the radius of analyticity of a real analytic function. We show that in order to understand the radius of analyticity it suffices to understand the convergence of real analytic functions on ℓ_1 , in the sense that ℓ_1 possesses the smallest constant of analyticity. We show that, with additional structure on our space and our function, the radii of uniform convergence and analyticity coincide. For us, Banach lattices and positive analytic functions will provide a natural setting for this problem.

2. Real analytic functions on ℓ_1

As mentioned in the introduction, the question of whether the radius of uniform convergence and the radius of analyticity of a real analytic function coincide in general is open. In this section we show that a positive solution to this question for ℓ_1 would imply a positive solution for all Banach spaces. We begin with the following proposition.

Proposition 2. *Let E, F be real Banach spaces such that E is a quotient space of F . Then $\mathcal{A}(E) \geq \mathcal{A}(F)$.*

Proof. Let $Q: F \rightarrow E$ be a quotient map. If g is a mapping with domain E^n for $n \geq 1$, we shall denote by \bar{g} the mapping on F^n defined by composition with Q , namely, $\bar{g}(x_1, \dots, x_n) = g(Qx_1, \dots, Qx_n)$.

Let $f = \sum_n P_n = \sum_n \hat{A}_n$ be a power series at the origin in E with radius of uniform convergence R . Since Q is a quotient operator, it follows that $\|P_n\| = \|\bar{P}_n\|$ for every n and so the power series $\bar{f} = \sum_n \bar{P}_n$ on F has the same radius of uniform convergence.

Let $a \in E$ with $\|a\| < R$ and $\varepsilon > 0$ be small enough so that $\|a\| + \varepsilon < R$. Choose $\bar{a} \in F$ such that $Q(\bar{a}) = a$ and $\|\bar{a}\| < \|a\| + \varepsilon$. We claim that the Taylor series of f at a and of \bar{f} at \bar{a} have the same radius of uniform convergence. This follows from the fact that

$$\begin{aligned} \frac{1}{n!} \hat{d}^n \bar{f}(\bar{a})(u) &= \sum_{k \geq n} \binom{k}{n} \bar{A}_k(\bar{a})^{k-n}(u)^n \\ &= \sum_{k \geq n} \binom{k}{n} A_k(a)^{k-n}(Qu)^n = \frac{1}{n!} \hat{d}^n f(a)(Qu) \end{aligned}$$

for every $u \in F$. It follows from the Cauchy–Hadamard formula that the radii of convergence are the same. Therefore $\mathcal{A}(E) \geq \mathcal{A}(F)$. \square

Recall that $\ell_1(I)$ is the Banach space of families $u = (u_i)$ of real numbers indexed by the set I such that

$$\|u\|_1 = \sum_{i \in I} |u_i| < \infty.$$

For every $u \in \ell_1(I)$, the support of u ,

$$\text{supp}(u) = \{i \in I : u_i \neq 0\},$$

is a countable subset of I .

Theorem 3. Let E be any real Banach space. Then $\mathcal{A}(E) \geq \mathcal{A}(\ell_1)$.

Proof. We first recall that there is a set I such that E is a quotient of $\ell_1(I)$ (for example, we can take I to be a dense subset of the unit sphere of E). Thus we have $\mathcal{A}(E) \geq \mathcal{A}(\ell_1(I))$. To conclude the proof, we show that the constant of analyticity of $\ell_1(I)$ is independent of the set I . Let I be any indexing set and let $f = \sum_n P_n$ be a power series at the origin in $\ell_1(I)$ with radius of uniform convergence R . Let $u \in \ell_1(I)$ with $\|u\| < R$ and, let $\varepsilon > 0$.

For each n , we may choose $v^{(n)}$ in the closed unit ball of $\ell_1(I)$ such that

$$\|P_n\| \leq |P_n(v^{(n)})|(1 - \varepsilon)^{-1}.$$

Similarly, we may choose points $w^{(n)}$ in the unit ball of $\ell_1(I)$ such that

$$\left\| \frac{1}{n!} \hat{d}^n f(u) \right\| \leq \left| \frac{1}{n!} \hat{d}^n f(u)(w^{(n)}) \right| (1 - \varepsilon)^{-1}$$

Now

$$J = \text{supp}(u) \cup \bigcup_n \text{supp}(v^{(n)}) \cup \bigcup_n \text{supp}(w^{(n)})$$

is a countable subset of I . There is a canonical embedding of $\ell_1(J)$ into $\ell_1(I)$ given by $u \mapsto u'$, where $u'_i = u_i$ for $i \in J$ and $u'_i = 0$ otherwise. We shall identify $\ell_1(J)$ with its image in $\ell_1(I)$ in this way.

We use a superscripted J to denote restriction of a function on $\ell_1(I)$ to the subspace $\ell_1(J)$; thus f^J and P_n^J denote the restrictions of f and P_n . Since $\ell_1(J)$ contains the points $v^{(n)}$, we have

$$\|P_n^J\| \leq \|P_n\| \leq \|P_n^J\|(1 - \varepsilon)^{-1}$$

for every n . Then the Taylor series of f^J at the origin is $\sum_n P_n^J$ and its radius of uniform convergence is R . It is easy to see that the Taylor series of f^J at the point u is

$$\sum_n \frac{1}{n!} \hat{d}^n f^J(u) = \sum_n \frac{1}{n!} (\hat{d}^n f(u))^J = \left(\sum_n \frac{1}{n!} \hat{d}^n f(u) \right)^J.$$

Since the points $w^{(n)}$ lie in $\ell_1(J)$, it follows that the radius of uniform convergence of the Taylor series of f at u is equal to the radius of uniform convergence of the Taylor series of f^J at u in $\ell_1(J)$.

Therefore $\mathcal{A}(\ell_1(I)) \geq \mathcal{A}(\ell_1(J)) = \mathcal{A}(\ell_1)$. \square

3. Power series on Banach lattices

Let E and F be Banach lattices with F Dedekind complete (every order bounded subset of F has a supremum), see [15]. An n -homogeneous polynomial $P_n = \hat{A}_n$ from E into F is *positive* if $A_n(x_1, \dots, x_n) \geq 0$ whenever $x_1, \dots, x_n \geq 0$. We write $P_n \geq 0$. A partial order is defined for n -homogeneous polynomials by $P_n \geq Q_n$ if $P_n - Q_n \geq 0$. An n -homogeneous polynomial P_n is said to be *regular* if it can be written as the difference of two positive n -homogeneous polynomials. This is equivalent to the existence of a positive n -homogeneous polynomial, $|P_n|$, called the *absolute value of P_n* , with the property that $|P_n|$ is the smallest positive n -homogeneous polynomial satisfying $P_n \leq |P_n|$ and $-P_n \leq |P_n|$. When P_n is a regular n -homogeneous polynomial we shall say that A_n is *regular*. For further information on polynomials on Banach lattices we refer to [5,6,14].

We list some facts about regular polynomials which we will use later:

- (a) If P_n is regular, then $|P_n(x)| \leq |P_n|(|x|)$ for every $x \in E$.
- (b) Every regular n -homogeneous polynomial is bounded.
- (c) The space of regular n -homogeneous polynomials on E is a Banach lattice with the *regular norm*, defined by

$$\|P_n\|_r = \| |P_n| \| = \sup\{|P_n|(x) : \|x\| \leq 1, x \geq 0\}.$$

We have $\|P_n\| \leq \|P_n\|_r$ and in most cases, these norms are not equivalent.

- (d) $E = C(K)$: an n -homogeneous polynomial on E is regular if and only if it is integral (Fremlin, [8]).
- (e) $E = L_1(\mu)$: Every bounded n -homogeneous polynomial is regular. The regular norm is equivalent to the uniform norm (Wittstock, [20]).
- (f) If E has an unconditional normalised Schauder basis, an n -homogeneous polynomial on E is regular if and only if it has a monomial expansion that converges unconditionally at every point of E . If $\sum_m a_m x^m$ is the monomial expansion of a regular n -homogeneous polynomial P_n , then the absolute value of P_n is given by $|P_n|(x) = \sum_m |a_m| x^m$ for every $x \in E$ [9].

The following result gives an estimate for the absolute values of the derivatives of a regular homogeneous polynomial.

Lemma 4. *Let $P_n = \hat{A}_n$ be a regular n -homogeneous polynomial on a Banach lattice E . Then, for every $x \in E$ and $0 \leq k \leq n$, $\frac{1}{k!} \hat{d}^k P_n(x)$ is a regular k -homogeneous polynomial and its absolute value satisfies*

$$\left| \frac{1}{k!} \hat{d}^k P_n(x) \right| \leq \frac{1}{k!} \hat{d}^k |P_n|(|x|)$$

Proof. The k -homogeneous polynomial $\frac{1}{k!} \hat{d}^k P_n(x)$ is generated by the symmetric k -linear form

$$(y_1, \dots, y_k) \mapsto \binom{n}{k} A_n(x^{n-k}, y_1, \dots, y_k).$$

Writing x as $x^+ - x^-$ and expanding we see that $\frac{1}{k!} \hat{d}^k P_n(x)$ is regular. Therefore,

$$\begin{aligned} \left| \frac{1}{k!} \hat{d}^k P_n(x)(y_1, \dots, y_k) \right| &= \left| \binom{n}{k} A_n(x^{n-k}, y_1, \dots, y_k) \right| \\ &\leq \binom{n}{k} |A_n|(|x|^{n-k}, |y_1|, \dots, |y_k|) = \frac{1}{k!} \hat{d}^k |P_n|(|x|)(|y_1|, \dots, |y_k|) \end{aligned}$$

for all y_1, \dots, y_k and the result follows. \square

Let E be a real Banach lattice. Let us recall how the complexification of E , $(E)_{\mathbb{C}} := E \oplus iE = \{x + iy : x, y \in E\}$, can be given the structure of a complex Banach lattice. A modulus is defined on the algebraic complexification $(E)_{\mathbb{C}}$ by the formula

$$|z| = |x + iy| = \sup\{x \cos \theta + y \sin \theta : 0 \leq \theta \leq 2\pi\}.$$

It can be shown that this supremum always exists in E . See [15, Section 2.2] for details. The modulus is also given by

$$|z| = \sqrt{x^2 + y^2},$$

where the expression on the right hand side is defined using the Krivine Functional Calculus [13, pp. 42–43].

A norm is defined on $(E)_{\mathbb{C}}$ by $\|z\| = \|\|z\|\|$, making $(E)_{\mathbb{C}}$ into a complex Banach space. A *complex Banach lattice* is a Banach space $E_{\mathbb{C}}$, where E is a Banach lattice, together with the modulus and norm defined above. A complex Banach lattice $E_{\mathbb{C}}$ is said to be Dedekind complete if the Banach lattice E is Dedekind complete.

Every bounded linear mapping, T , between real Banach spaces E and F extends to (complex) linear mapping $T_{\mathbb{C}}: E_{\mathbb{C}} \rightarrow F_{\mathbb{C}}$ via the formula

$$T_{\mathbb{C}}(x + iy) = T(x) + iT(y).$$

Moreover, a mapping $S: E_{\mathbb{C}} \rightarrow F_{\mathbb{C}}$ is complex linear if and only if there exist real linear mappings $T_0, T_1: E \rightarrow F$ such that

$$S = (T_0)_{\mathbb{C}} + i(T_1)_{\mathbb{C}}.$$

The relationship between the complex operator S and the real operators T_0, T_1 is given by

$$T_0(x) = (\operatorname{Re} S(x))|_E \quad T_1(x) = (\operatorname{Im} S(x))|_E.$$

It follows that the vector space $L((E)_{\mathbb{C}}, (F)_{\mathbb{C}})$ is canonically isomorphic to $(L(E, F))_{\mathbb{C}}$. If E, F are Banach lattices, then S is bounded if and only if both T_0 and T_1 are bounded. The complex operator S is defined to be regular if both T_0 and T_1 are regular. If F is Dedekind complete (for example, F is a dual Banach lattice) and S is regular, then it has a modulus that satisfies

$$|S|(a) = \sup\{|S(z)| : z \in E_{\mathbb{C}}, |z| \leq a\}$$

for every $a \in E$, $a \geq 0$; furthermore, $|S(z)| \leq |S|(|z|)$ for every $z \in E_{\mathbb{C}}$, see [15, Proposition 2.2.6]. In this case, the space of regular complex operators is a Dedekind complete complex Banach lattice:

$$\mathcal{L}_r(E_{\mathbb{C}}, F_{\mathbb{C}}) = (\mathcal{L}_r(E, F))_{\mathbb{C}}.$$

Here $\mathcal{L}_r(X, Y)$ denotes the space of regular linear operators from X to Y .

In particular, there is a canonical isometric isomorphism between $(E')_{\mathbb{C}}$ and $(E_{\mathbb{C}})'$ and the modulus of $\varphi \in (E')_{\mathbb{C}}$ satisfies

$$|\varphi|(a) = \sup\{|\varphi(z)| : z \in E_{\mathbb{C}}, |z| \leq a\}$$

for $a \in E$, $a \geq 0$. Every dual Banach lattice is Dedekind complete. We refer to [15] for further information on regular operators.

Let $P_n = \hat{A}_n: E \rightarrow F$ be an n -homogeneous polynomial. We can apply the complexification process to each of the n variables in A_n to define the complexified polynomial $(P_n)_{\mathbb{C}} = \overline{(A_n)_{\mathbb{C}}}$ on $E_{\mathbb{C}}$.

We now prove the following result.

Proposition 5. *Let E, F be Banach lattices with F Dedekind complete and let $P_n: E \rightarrow F$ be a regular n -homogeneous polynomial. Then its complexification satisfies*

$$|(P_n)_{\mathbb{C}}(z)| \leq |P_n|(|z|)$$

for every $z \in (E_{\mathbb{C}})$.

Proof. The proof is by induction. We show that, if A_n is a regular n -linear mapping on E^n , then

$$|(A_n)_{\mathbb{C}}(z_1, \dots, z_n)| \leq |A_n|(|z_1|, \dots, |z_n|)$$

for all $z_1, \dots, z_n \in E_{\mathbb{C}}$.

If $n = 1$, then $A_1 = T$ is a regular linear mapping. We have

$$\begin{aligned} |T_{\mathbb{C}}(z)| &= \sup\{T(x) \cos \theta + T(y) \sin \theta : 0 \leq \theta \leq 2\pi\} \\ &= \sup\{T(x \cos \theta + y \sin \theta) : 0 \leq \theta \leq 2\pi\} \\ &\leq \sup\{|T|(x \cos \theta + y \sin \theta) : 0 \leq \theta \leq 2\pi\} \end{aligned}$$

As $|T|$ is positive, it is an increasing function and so

$$|T_{\mathbb{C}}(z)| \leq |T|(\sup\{x \cos \theta + y \sin \theta : 0 \leq \theta \leq 2\pi\}) = |T|(|z|).$$

Now suppose the result is true for all regular n -linear mappings. Let $A_{n+1}: E^{n+1} \rightarrow F$ be a regular $(n+1)$ -linear mapping. We define an n -linear mapping B_n from E^n into $L(E, F)$ by

$$B_n(x_1, \dots, x_n)(x) = A_{n+1}(x_1, \dots, x_n, x).$$

Writing $x_i = x_i^+ - x_i^-$ and expanding, we see that $B_n(x_1, \dots, x_n)$ is the difference of two positive operators. Thus, B_n takes its values in the Banach lattice $\mathcal{L}_r(E, F)$ of regular operators. It follows from [15, Theorem 1.3.5] that $\mathcal{L}_r(E, F)$ is Dedekind complete. Furthermore, B_n is regular. Thus the induction hypothesis can be applied to B_n . It is easy to see that

$$(B_n)_{\mathbb{C}}(z_1, \dots, z_n)(z) = (A_{n+1})_{\mathbb{C}}(z_1, \dots, z_n, z)$$

for all $z_1, \dots, z_{n+1} \in (E)_{\mathbb{C}}$. Therefore by [15, Proposition 2.2.6]

$$\begin{aligned} |(A_{n+1})_{\mathbb{C}}(z_1, \dots, z_{n+1})| &= |(B_n)_{\mathbb{C}}(z_1, \dots, z_n)(z_{n+1})| \\ &\leq |(B_n)_{\mathbb{C}}(z_1, \dots, z_n)| |z_{n+1}| \\ &\leq |B_n|(|z_1|, \dots, |z_n|) |z_{n+1}| \\ &= |A_{n+1}|(|z_1|, \dots, |z_n|, |z_{n+1}|) \end{aligned}$$

So the result holds for every $n \in \mathbb{N}$. \square

Corollary 6. Let E, F be Banach lattices with F Dedekind complete and let $P_n: E \rightarrow F$ be a regular n -homogeneous polynomial.

(a) If P_n is regular then

$$\|(P_n)_{\mathbb{C}}\| \leq \|P_n\| = \|P_n\|_r.$$

(b) If P_n is positive then

$$\|(P_n)_{\mathbb{C}}\| = \|P_n\|.$$

Theorem 7. *Let E be a real Banach lattice and for each $n \geq 0$ let P_n be a positive n -homogeneous polynomial on E . Then the radius of analyticity and the radius of uniform convergence of the power series $\sum_n P_n$ are equal.*

Proof. Let us suppose that $f = \sum_n P_n$ has radius of uniform convergence R . We extend each P_n to an n -homogeneous polynomial $(P_n)_\mathbb{C}$ from $E_\mathbb{C}$ to $F_\mathbb{C}$. By Corollary 6 we have $\|(P_n)_\mathbb{C}\| = \|P_n\|$ and therefore the Cauchy–Hadamard Theorem implies that $g := \sum_n (P_n)_\mathbb{C}$ converges uniformly on $B_E(0, R)$. Consider a in $B_E(0, R)$. The Cauchy estimates imply that the series $\sum_k \frac{1}{k!} \hat{d}^k g(a)$ has radius of uniform convergence at least $R - \|a\|$ in $E_\mathbb{C}$. Then, for x in E , $\|x - a\| < R - \|a\|$, we have

$$\begin{aligned} \frac{1}{k!} \hat{d}^k (f_\mathbb{C})(a)(x) &= \sum_{n \geq k} \frac{1}{k!} \hat{d}^k ((P_n)_\mathbb{C})(a)(x) \\ &= \sum_{n \geq k} \binom{n}{k} (A_n)_\mathbb{C}(a)^{n-k} x^k \\ &= \sum_{n \geq k} \binom{n}{k} (A_n)(a)^{n-k} x^k \\ &= \frac{1}{k!} \hat{d}^k f(a)(x). \end{aligned}$$

Hence, the Taylor series of f at a has radius of uniform convergence at least $R - \|a\|$. \square

Corollary 8. *Let E be a real Banach space with a normalised 1-unconditional basis. Suppose f is a real analytic function on $B(0, R)$ with unconditionally convergent monomial expansion $\sum_{m \in \mathbb{N}^{\mathbb{N}}} a_m z^m$ about the origin. If $a_m \geq 0$ for all m in $\mathbb{N}^{(\mathbb{N})}$ then the radius of analyticity of f is equal to its radius of uniform convergence.*

The radius of analyticity of a power series of regular polynomials is related to the radius of convergence of the series of absolute values.

Proposition 9. *Let E be a real Banach lattice and for each $n \geq 0$ let P_n be a regular n -homogeneous polynomial on E . Then the radius of analyticity of the power series $\sum_n P_n$ is greater than or equal to the radius of uniform convergence of the power series $\sum_n |P_n|$.*

Proof. This follows from Lemma 4. \square

Let $\sum_n P_n$ be a power series with regular terms in a Banach lattice. A natural question is the relationship between the radius of uniform convergence of $\sum_n P_n$ and the radius of uniform convergence of $\sum_n |P_n|$. Let us consider the case of a power series that converges uniformly on the ball of real ℓ_∞^k . We let $I^k := \{(x_j)_{j=1}^k : |x_j| < 1\}$ be the open unit polydisc in \mathbb{R}^k and $\Delta^k := \{(z_j)_{j=1}^k : |z_j| < 1\}$ be the open unit polydisc in \mathbb{C}^k . We note that these sets are the open unit balls of real and complex ℓ_∞^k respectively. Let $A(I^k)$ denote the space of real analytic functions on I^k and $\mathcal{H}(\Delta^k)$ denote the space of holomorphic functions on Δ^k . We recall some of the norms on $\mathcal{P}({}^m \ell_\infty^k)$ introduced by Beauzamy, Bombieri, Enflo and Montgomery in [2]. Given P in $\mathcal{P}({}^m \ell_\infty^k)$ let $\sum_{|\alpha|=m} a_\alpha z^\alpha$ be the monomial expansion of P . We let

$$|P|_2 = \left(\sum_{|\alpha|=m} |a_\alpha|^2 \right)^{1/2},$$

$$\|P\|_2 = \left(\int_0^{2\pi} \dots \int_0^{2\pi} |P(e^{i\theta_1}, \dots, e^{i\theta_k})|^2 \frac{d\theta_1}{2\pi} \dots \frac{d\theta_k}{2\pi} \right)^{1/2}$$

and

$$\|P\|_\infty = \sup_{\theta_1, \dots, \theta_k} |P(e^{i\theta_1}, \dots, e^{i\theta_k})|.$$

From [2, p. 220] we have that

$$\|P\|_2 = |P|_2 \leq \|P\|_\infty$$

for every P in $\mathcal{P}({}^m\ell_\infty^k)$.

Suppose we are given a real analytic function $f(x) = \sum_{m=1}^\infty P_m(x)$ in $A(I^k)$. Let the monomial expansion of P_m be

$$P_m(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha.$$

We note that the complexification of P_m is $(P_m)_\mathbb{C}(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$. Then, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \sum_{|\alpha|=m} |a_\alpha x^\alpha| &\leq \left(\sum_{|\alpha|=m} |a_\alpha|^2 \right)^{1/2} \left(\sum_{|\alpha|=m} |x|^{2\alpha} \right)^{1/2} \\ &\leq |(P_m)_\mathbb{C}|_2 \left(\sum_{|\alpha|=m} \binom{m}{\alpha} x^{2\alpha} \right)^{1/2} \\ &= |(P_m)_\mathbb{C}|_2 \|x\|_2^m \\ &\leq \|(P_m)_\mathbb{C}\|_\infty \|x\|_2^m \\ &\leq 2^{m-1} \|P_m\|_\infty \|x\|_2^m, \end{aligned}$$

where the last inequality follows from [16, Equation 4]. Hence we have shown that

$$\sum_{|\alpha|=m} |a_\alpha| |x|^\alpha \leq 2^{m-1} \|P_m\|_\infty \|x\|_2^m.$$

This means that the power series $\sum_{m=1}^\infty |P_m|$ converges uniformly on $\frac{1}{2}B_{\ell_2^k}$.

We conclude with a discussion of some related results of Hayman, [12], on power series expansions of harmonic functions. Let f be a harmonic function on the open Euclidean ball of radius R in \mathbb{R}^k . Then the Taylor series of f at the origin, $\sum_n P_n$, has radius of uniform convergence R . Hayman studied the convergence of the monomial expansion of f . He observed that if u is harmonic on the hyperball $|x| < r_o$ in \mathbb{R}^k , then u possesses a polynomial expansion

$$u(x) = \sum_0^\infty P_n(x)$$

where the P_n are harmonic homogeneous polynomials of degree n in the coordinates x_1, x_2, \dots, x_k . This series converges uniformly and absolutely in $|x| \leq r$ where $r < r_o$. Hayman's result is as follows.

Theorem B. (Hayman, [12]) *With the hypotheses above, the multiple Taylor series expansion*

$$u = \sum_{n_v=0}^{\infty} a_{n_1, n_2, \dots, n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

of u converges uniformly and absolutely for $|x| \leq r$ where $r < r_0/\sqrt{2}$, but the series may diverge at some points of the hypersphere $|x| = r_0/\sqrt{2}$.

We note that in the language of Banach lattices, Hayman's result can be stated in the following way:

If $\sum_n P_n$ is a power series in \mathbb{R}^k with harmonic terms having radius of convergence R then the power series of $\sum_n |P_n|$ has radius of convergence $R/\sqrt{2}$ and this is sharp.

Acknowledgments

The authors would like to thank Stephen Gardiner for the reference to the work of Hayman and Michal Johanis for drawing our attention to [10].

References

- [1] Stefan Banach, Über homogene Polynome in (L^2) , *Studia Math.* 7 (1938) 36–44.
- [2] Bernard Beauzamy, Enrico Bombieri, Per Enflo, Hugh L. Montgomery, Products of polynomials in many variables, *J. Number Theory* 36 (2) (1990) 219–245.
- [3] Jacek Bochnak, Józef Siciak, Analytic functions in topological vector spaces, *Studia Math.* 39 (1971) 77–112.
- [4] Jacek Bochnak, Józef Siciak, Polynomials and multilinear mappings in topological vector spaces, *Studia Math.* 39 (1971) 59–76.
- [5] G. Buskes, C. Schwanke, Complex vector lattices via functional completions, *J. Math. Anal. Appl.* 434 (2) (2016) 1762–1778.
- [6] James Cruickshank, John Loane, Raymond A. Ryan, Positive polynomials on Riesz spaces, *Positivity* 21 (3) (2017) 885–895.
- [7] Seán Dineen, *Complex Analysis on Infinite-Dimensional Spaces*, Springer Monogr. Math., Springer-Verlag London, Ltd., London, 1999.
- [8] D.H. Fremlin, Tensor products of Banach lattices, *Math. Ann.* 211 (1974) 87–106.
- [9] Bogdan C. Grecu, Raymond A. Ryan, Polynomials on Banach spaces with unconditional bases, *Proc. Amer. Math. Soc.* 133 (4) (2005) 1083–1091.
- [10] Petr Hájek, Michal Johanis, *Smooth Analysis in Banach Spaces*, De Gruyter Ser. Nonlinear Anal. Appl., vol. 19, De Gruyter, Berlin, 2014.
- [11] Lawrence A. Harris, A Bernstein–Markov theorem for normed spaces, *J. Math. Anal. Appl.* 208 (2) (1997) 476–486.
- [12] W.K. Hayman, Power series expansions for harmonic functions, *Bull. Lond. Math. Soc.* 2 (1970) 152–158.
- [13] Joram Lindenstrauss, Lior Tzafriri, *Classical Banach Spaces. II*, *Ergeb. Math. Grenzgeb.*, vol. 97, Springer-Verlag, Berlin–New York, 1979.
- [14] John Loane, Polynomials on Riesz spaces, *J. Math. Anal. Appl.* 364 (1) (2010) 71–78.
- [15] Peter Meyer-Nieberg, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1991.
- [16] Gustavo A. Muñoz, Yannis Sarantopoulos, Andrew Tonge, Complexifications of real Banach spaces, polynomials and multilinear maps, *Studia Math.* 134 (1) (1999) 1–33.
- [17] T. Nguyen, A lower bound on the radius of analyticity of a power series in a real Banach space, *Studia Math.* 191 (2009) 171–179.
- [18] Marios K. Papadimitris, Yannis Sarantopoulos, Radius of analyticity of a power series on real Banach spaces, *J. Math. Anal. Appl.* 434 (2) (2016) 1281–1289.
- [19] A.E. Taylor, Additions to the theory of polynomials on normed linear spaces, *Tohoku Math. J.* 44 (1938) 302–308.
- [20] Gerd Wittstock, Ordered normed tensor products, in: *Lecture Notes in Phys.*, vol. 29, 1974, pp. 67–84.