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Eulerian droplet model: Delta-shock waves and solution of the Riemann problem

Sana Keita, Yves Bourgault *

Department of Mathematics and Statistics, University of Ottawa, K1N 6N5, Ottawa, Ontario, Canada

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ABSTRACT

We study an Eulerian droplet model which can be seen as the pressureless gas system with a source term, a subsystem of this model and the inviscid Burgers equation with source term. The condition for loss of regularity of a solution to Burgers equation with source term is established. The same condition applies to the Eulerian droplet model and its subsystem. The Riemann problem for the Eulerian droplet model is constructively solved by going through the solution of the Riemann problems for the inviscid Burgers equation with a source term and the subsystem, respectively. Under suitable generalized Rankine–Hugoniot relations and entropy condition, the existence of delta-shock solution is established. The existence of a solution to the generalized Rankine–Hugoniot conditions is proven. Some numerical illustrations are presented.

1. Introduction

In this paper we consider the one-dimensional Eulerian droplet model [7] in conservative form

$$\begin{cases} \partial_t \alpha + \partial_x(\alpha u) = 0, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2) = \mu\alpha(u_a - u), \end{cases} \tag{1.1}$$

where α and u denote, the volume fraction and velocity of the particles (droplets), respectively, u_a is the velocity of the carrier fluid (air), and μ is the drag coefficient between the carrier fluid and the particles. Since the density of particles exceeds the air density by orders of magnitude, the virtual mass force is neglected. The lift force, gravity, and other interfacial effects are also negligible when compared to the viscous drag force. These forces could be important in other applications [7]. The Eulerian droplet model (1.1) corresponds to a dispersed phase subsystem in its simplest form, for instance a multi-phase system for particles suspended in a carrier fluid. For smooth solutions, the second equation of (1.1) is equivalent to

* Corresponding author.
E-mail addresses: skeit085@uottawa.ca (S. Keita), ybourg@uottawa.ca (Y. Bourgault).

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$$\alpha(\partial_t u + u\partial_x u) + u(\partial_t \alpha + \partial_x(\alpha u)) = \mu\alpha(u_a - u). \tag{1.2}$$

Using the first equation of (1.1) and simplifying by $\alpha \neq 0$, (1.2) reduces to

$$\partial_t u + u\partial_x u = \mu(u_a - u) \tag{1.3}$$

which can be rewritten in conservative form as

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = \mu(u_a - u). \tag{1.4}$$

Hence, for smooth solutions with $\alpha \neq 0$, the Eulerian droplet model (1.1) is equivalent to

$$\begin{cases} \partial_t \alpha + \partial_x(\alpha u) = 0, \\ \partial_t u + \partial_x(\frac{1}{2}u^2) = \mu(u_a - u). \end{cases} \tag{1.5}$$

One easily shows that any smooth solution of (1.5) is also a solution to (1.1). In the following, equation (1.4) will be also referred to as the *inviscid Burgers equation with source term*.

If $\mu = 0$ then (1.4) reduces to the classical *inviscid Burgers equation* which has been studied in most textbooks on conservation laws [23,30,40,44]. It is well known that the solution of the inviscid Burgers equation develops discontinuities in finite time provided that the slope of the initial condition is negative at some point.

For the homogeneous case $\mu = 0$, system (1.1) can be seen as the *zero-pressure gas dynamics system* [3] or as the *sticky particle system* [10,20] that arises in the modeling of particles hitting and sticking to each other to explain the formation of large scale structures in the universe. The system of zero-pressure gas dynamics has been studied by several authors [3–5,10,20,27,32,33,42]. In particular, the existence of measure solutions for the Riemann problem was first presented by Bouchut [3]. Under suitable generalized Rankine–Hugoniot relation and entropy condition, the Riemann problem is constructively solved in [42].

If $\mu > 0$ then system (1.1) is known as the *Eulerian droplet model* [2,6–9,25,38]. This model is successfully used for the prediction of droplets impingement on airfoils and airplane wings during in-flight icing events [7,9]. Extension to particle flows in airways was more recently attempted [6,8]. The Eulerian droplet model has been studied by several authors at the numerical level [7] and at the practical level [2,6,8,9,25,38]. To our knowledge, there is no theoretical study related to the system of zero-pressure gas dynamics including explicitly a right-hand side term as in (1.1). In this paper, we are interested in the theoretical study of the Eulerian droplet model (1.1). In reality, the drag coefficient μ is function of the droplet Reynolds number (see [7]). For performing analysis, we assume in the following that the drag coefficient μ and the carrier fluid velocity u_a are constant.

The Eulerian droplet model (1.1) is a first-order system of conservation laws for the volume fraction α and the momentum αu. For smooth solutions, it is equivalent to (1.5) which can be written in quasilinear form as

$$\begin{pmatrix} \alpha \\ u \end{pmatrix}_t + \begin{pmatrix} u & \alpha \\ 0 & u \end{pmatrix} \begin{pmatrix} \alpha \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ \mu(u_a - u) \end{pmatrix}. \tag{1.6}$$

The Jacobian matrix has one double eigenvalue u and is not diagonalizable. Hence, system (1.1) is *weakly hyperbolic*. Systems of conservation laws in which hyperbolicity fails (because of eigenvalue coincidence) can encounter many difficulties, particularly in terms of boundedness of their solutions. To illustrate this recurrent difficulty with boundedness, consider the following linear first-order system

$$\begin{cases} \begin{pmatrix} \alpha \\ u \end{pmatrix}_t + \begin{pmatrix} \lambda & \beta \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \alpha \\ u \end{pmatrix}_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ (\alpha, u)(x,0) = (\alpha_0, u_0)(x), \quad \forall x \in \mathbb{R}, \end{cases} \tag{1.7}$$

where $\lambda, \beta \neq 0$ are constant. System (1.7) is weakly hyperbolic with one double eigenvalue λ. One can first solve the second equation of (1.7) by the method of characteristics to find

$$u(x,t) = u_0(x - \lambda t) \tag{1.8}$$

and then, considering $-\beta \partial_x u$ as a source term, we calculate the solution of the first equation

$$\alpha(x,t) = \alpha_0(x - \lambda t) - \beta t u_0'(x - \lambda t). \tag{1.9}$$

We immediately see that α is not defined in the classical sense at points where the initial condition u_0 is not differentiable. For instance, if u_0 is a Heaviside function then the expression for α would contain a Dirac δ-function. The Cauchy problem, for bounded measurable data, is not of classical type. The concept of singular solutions incorporating Dirac δ-functions along shock trajectories was first introduced in [29]. Tan and Zhang [45] studied a new type of waves, delta-shock waves, as solution of nonlinear hyperbolic systems for which hyperbolicity fails. They proved that delta-shock waves are limit solutions to some reasonable viscous perturbations as the viscosity vanishes. A delta-shock wave is a generalization of an ordinary shock wave. Speaking informally, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a Dirac δ-function with the discontinuity in an other variable as its support. From the physical point of view, a delta-shock wave represents the process of concentration of mass. For related results on delta-shock waves, we refer to [12–14,28,32,42,49–51] and the references therein.

The general purpose of this work is to solve the Riemann problem for the Eulerian droplet model (1.1). The arrangement of this paper is as follows. In section 2, we derive the condition for loss of regularity for a smooth solution of (1.1). In sections 3, 4 and 5, we solve the Riemann problem for the inviscid Burgers equation with source term (1.4), system (1.5) and the Eulerian droplet model (1.1), respectively. In section 6, we investigate the existence of a solution to the generalized Rankine–Hugoniot conditions for (1.1). Test cases illustrating theoretical results are presented in section 7.

2. Loss of regularity for a smooth solution of the Eulerian droplet model

This section is devoted to the loss of regularity for smooth solutions of (1.1). For more details on loss of regularity for smooth solutions, we refer the readers to the work [1,48] on blowup of nonlinear hyperbolic systems, and to Whitham's classic text [47]. By a method similar to that in [48], we prove that α and $\partial_x u$ blow up simultaneously in finite time even starting from smooth initial data.

Let (α, u) be a smooth solution of (1.1) satisfying the initial condition

$$(\alpha, u)(x,0) = (\alpha_0, u_0)(x), \quad \alpha_0, u_0 \in \mathcal{C}^1(\mathbb{R}). \tag{2.1}$$

The characteristic curves $\chi = \chi(x,t;s)$ associated to (1.1) are solutions of

$$\begin{cases} \dfrac{d\chi}{ds}(x,t;s) = u\big(\chi(x,t;s), s\big), \quad s \in [0,T], \\ \chi(x,t;t) = x. \end{cases} \tag{2.2}$$

System (1.1) reduces along these characteristics to

$$\begin{cases} \dfrac{D\alpha}{dt} = \alpha \partial_x u, \\ \dfrac{Du}{dt} = \mu(u_a - u), \end{cases} \tag{2.3}$$

where $\frac{D}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ is the total derivative. An integration of the second equation of (2.3) gives

$$u\big(\chi(x,t;t),t\big) = u(x,t) = u_a + \big(u_0(x_0) - u_a\big)e^{-\mu t}, \tag{2.4}$$

where $x_0 = \chi(x,t;0)$. Substituting (2.4) in (2.2) and integrating, we get

$$x = \chi(x,t;t) = x_0 + u_a t + \frac{(u_0(x_0) - u_a)(1 - e^{-\mu t})}{\mu}. \tag{2.5}$$

Hence, $x = x(x_0,t)$ can be seen as a function of x_0 and t, and thus $\partial_x u$ can be written as

$$\partial_x u = \Big(\frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\Big). \tag{2.6}$$

As long as the characteristics do not intersect, the map

$$\begin{aligned} h : \mathbb{R}\times[0,\infty) &\to \mathbb{R}\times[0,\infty) \\ (x_0,t) \quad &\mapsto \quad (x,t) \end{aligned}$$

is bijective. The Jacobian matrix of h and its inverse h^{-1} are given by

$$J_h(x_0,t) = \begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial t} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J_{h^{-1}}(x,t) = \begin{pmatrix} \frac{\partial x_0}{\partial x} & \frac{\partial x_0}{\partial t} \\ \frac{\partial t}{\partial x} & 1 \end{pmatrix}, \tag{2.7}$$

respectively. Since $J_{h^{-1}}(x,t) = J_h^{-1}(x_0,t)$ then

$$\frac{\partial x_0}{\partial x} = \frac{1}{\frac{\partial x}{\partial x_0}} \quad \text{and} \quad \frac{\partial t}{\partial x} = 0. \tag{2.8}$$

Hence, (2.6) reduces to

$$\partial_x u = \frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial x} = \frac{\partial u}{\partial x_0}\frac{1}{\frac{\partial x}{\partial x_0}}. \tag{2.9}$$

Using (2.4) and (2.5) in (2.9), we obtain

$$\partial_x u = \frac{\mu e^{-\mu t}u_0'(x_0)}{\mu + (1 - e^{-\mu t})u_0'(x_0)}. \tag{2.10}$$

The first equation of (2.3) can be written now as

$$\frac{D\alpha}{dt} = -\frac{\mu\alpha e^{-\mu t}u_0'(x_0)}{\mu + (1 - e^{-\mu t})u_0'(x_0)}. \tag{2.11}$$

Assuming $\alpha \neq 0$, one can divide by α and integrate (2.11) on both sides to obtain

$$\log\big(\alpha(x,t)\big) = -\log\big(\mu + (1-e^{-\mu t})u_0'(x_0)\big) + \log(\mu) + \log\big(\alpha_0(x_0)\big). \tag{2.12}$$

This last equality leads to

$$\alpha(x,t) = \frac{\mu\alpha_0(x_0)}{\mu + (1-e^{-\mu t})u_0'(x_0)}. \tag{2.13}$$

The following result holds:

Proposition 2.1. *Let (α, u) be a smooth solution of (1.1) and (2.1). Then α and $\partial_x u$ blow up if and only if there exists x_0 in the domain such that*

$$u_0'(x_0) < -\mu. \tag{2.14}$$

Moreover, the blowup occurs at

$$t = \inf_{u_0'(x_0)<-\mu}\Big\{-\frac{\log(1+\frac{\mu}{u_0'(x_0)})}{\mu}\Big\}. \tag{2.15}$$

Proof. As $\alpha_0, u_0 \in \mathcal{C}^1(\mathbb{R})$ then α and $\partial_x u$ blow up if and only if $\mu + (1-e^{-\mu t})u_0'(x_0) = 0$. This happens if and only if

$$u_0'(x_0) < 0 \quad \text{and} \quad 1 - e^{-\mu t} = \frac{-\mu}{u_0'(x_0)} \Longleftrightarrow t = -\frac{\log(1+\frac{\mu}{u_0'(x_0)})}{\mu}. \tag{2.16}$$

Since $1 - e^{-\mu t} < 1, \forall t \geq 0$ then $u_0'(x_0) < -\mu$. The smallest time t satisfying (2.16) is given by (2.15). $\square$

Remark 2.2. Inequality (2.14) is also a necessary and sufficient condition for the characteristics to overlap. In fact, two characteristics $\chi_1(x,t;s)$ and $\chi_2(x,t;s)$ with distinct foots x_1 and x_2, respectively, cross each other if and only if there is $s^* > 0$ such that $\chi_1(x,t;s^*) = \chi_2(x,t;s^*)$. By using (2.5) and the inequalities $0 < 1 - e^{-\mu t} < 1, \forall t > 0$, the equality $\chi_1(x,t;s^*) = \chi_2(x,t;s^*)$ gives rise to

$$\frac{u_0(x_2) - u_0(x_1)}{x_2 - x_1} < -\mu, \tag{2.17}$$

and by the mean value theorem, there exists a point x_0 such that

$$u_0'(x_0) = \frac{u_0(x_2) - u_0(x_1)}{x_2 - x_1} < -\mu. \tag{2.18}$$

Thus, a smooth solution to (1.1) loses its regularity if and only if the slope of the initial condition for u is sufficiently negative with respect to the coefficient μ. This loss of regularity is reflected in a blowup of α and $\partial_x u$. This blowup leads to unboundedness and discontinuities in the solution. Therefore, no solution exists in the space of functions with bounded variation. We will investigate the form of the solution to the Riemann problem for (1.1) by going through the solution of the Riemann problem for equation (1.4) and then system (1.5), respectively.

3. Riemann problem for the inviscid Burgers equation with source term

In this section we study the inviscid Burgers equation with a zeroth order source term (1.4) satisfying the initial condition

$$u(x,0) = u_0(x), \tag{3.1}$$

where u_0 is a piecewise smooth function. The solution of the Riemann problem to the Burgers equation without a source term is either a rarefaction or a shock wave [30,44]. The solution to the Riemann problem for the Burgers equation with a discontinuous source term is constructed in [22]. It turns out that the discontinuity of the source term has clear influences on the shock or rarefaction waves generated by the initial Riemann data. In [52], the shock wave solution for the inviscid Burgers equation with a linear forcing term is obtained by combining the Rankine–Hugoniot jump condition together with the method of characteristics, which reflects the impact of the inhomogeneous forcing term on the shock front. In these references, the term source is function of x and t. In this section we solve the Riemann problem for the Burgers equation with a source term that depends on the solution u, using the method of characteristics. In addition, we refer to the work [11,37,43] on how to use the method of characteristics to solve the Riemann problem for scalar conservation law with source term.

Due to the breaking of waves and formation of shocks, the initial value problem for (1.4) does not generally possess globally defined smooth solutions, even when the initial data are very smooth. We showed in the previous section that discontinuity appears in the solution of (1.4) if condition (2.14) is satisfied. Here, we look for the solution of the Riemann problem for (1.4), i.e. the solution of (1.4) and (3.1), where

$$u_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \qquad u_-, u_+ \in \mathbb{R}. \tag{3.2}$$

We are particularly interested in the solution of the Riemann problem (1.4) and (3.2) because it will be useful in the resolution of the Riemann problem for system (1.5) in the next section.

The solution of (1.4) along the characteristics (2.2) is given by (2.4). Substituting (3.2) in (2.5), one obtains

$$\chi(x,t;s) = \begin{cases} x_0 + u_a s + \dfrac{u_a - u_-}{\mu}\left(e^{-\mu s} - 1\right), & x_0 < 0, \\[2ex] x_0 + u_a s + \dfrac{u_a - u_+}{\mu}\left(e^{-\mu s} - 1\right), & x_0 > 0. \end{cases} \tag{3.3}$$

3.1. Shock waves

We first assume that $u_- > u_+$. In this case, condition (2.14) is satisfied, and thus characteristics intersect within finite time. Some characteristic curves for different values of u_a are represented in Fig. 1. The solution is a shock wave, i.e. a smooth curve $\Gamma = \{(x,t): \ x = \xi(t), t \geq 0\}$ in the x–t plane moving at speed $\sigma(t) = \xi'(t)$ and separating a left and right states denoted by $u_l(x,t)$ and $u_r(x,t)$, respectively. Solutions that may be discontinuous are taken in the weak sense. We have the following definition:

Definition 3.1. We say that u is a *weak solution* of (1.4) and (3.1) if

$$\int_0^\infty \int_{-\infty}^\infty \left(u\psi_t + \frac{u^2}{2}\psi_x + \mu(u_a - u)\psi\right) dxdt = -\int_{-\infty}^\infty u_0(x)\psi(x,0)\, dx, \tag{3.4}$$

for all test functions $\psi \in \mathcal{C}_0^\infty(\overline{\mathbb{R} \times \mathbb{R}^+})$.

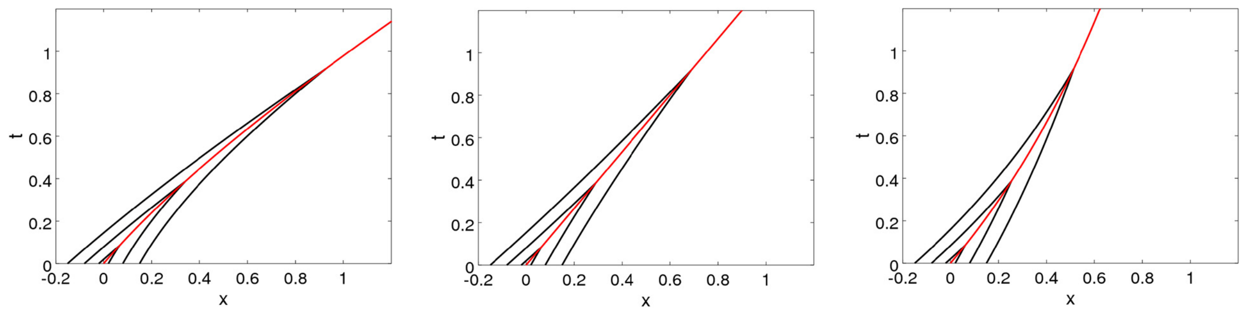

Fig. 1. Characteristic curves on the x–t plane for $u_- = 1.0$, $u_+ = 0.5$ and $\mu = 1.0$. Left: $u_a = 1.5$, middle: $u_a = 0.75$ and right: $u_a = 0.2$.

Let u be a regular function on both sides of the curve Γ, while being discontinuous across this curve. We have the following characterization for a weak solution of (1.4) and (3.1).

Theorem 3.2. *The function u is a weak solution of* (1.4) *and* (3.1) *if and only if the following properties hold:*

i) u satisfies (1.4) *in the classical sense on both sides of the curve Γ;*
ii) $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$;
iii) the following Rankine–Hugoniot conditions are satisfied:

$$\big(u_r(t) - u_l(t)\big)\sigma(t) = \frac{1}{2}\big(u_r(t)^2 - u_l(t)^2\big), \tag{3.5}$$

where $u_l(t)$ and $u_r(t)$ are the limit of the solution u when (x,t) approaches $(\xi(t),t)$ from the left and the right, respectively.

Proof. The proof is performed as for the classical Burgers equation (see [23,40]) but with an appropriate treatment of the source term which disappears in the Rankine–Hugoniot conditions. □

Definition 3.3. A discontinuity propagating with speed σ given by (3.5) satisfies the entropy condition if

$$u_r(t) < \sigma(t) < u_l(t). \tag{3.6}$$

Inequality (3.6) is known as *Lax's entropy condition* [30,40,44]. It means that all characteristics on both sides of the discontinuity are in-coming. This additional condition ensures the uniqueness of the Riemann solution to the Burgers equation without source term [21]. The Lax's entropy condition is also used in [22,52] for the Burgers equation with source term.

Returning to the Riemann problem for (1.4), Theorem 3.2 states that the solution u satisfies (1.4) in the classical sense on both sides of the curve Γ. The left and right states are determined from (2.4), that is

$$u_l(x,t) = u_a + (u_- - u_a)e^{-\mu t}, \qquad u_r(x,t) = u_a + (u_+ - u_a)e^{-\mu t}. \tag{3.7}$$

These states are independent of x away from the discontinuity, hence the limit states $u_l(t) = u_l(x,t)$ and $u_r(t) = u_r(x,t)$. The shock speed of the shock wave

$$\sigma(t) = \frac{1}{2}\big(u_l(t) + u_r(t)\big) = u_a + \left(\frac{u_- + u_+}{2} - u_a\right)e^{-\mu t}, \tag{3.8}$$

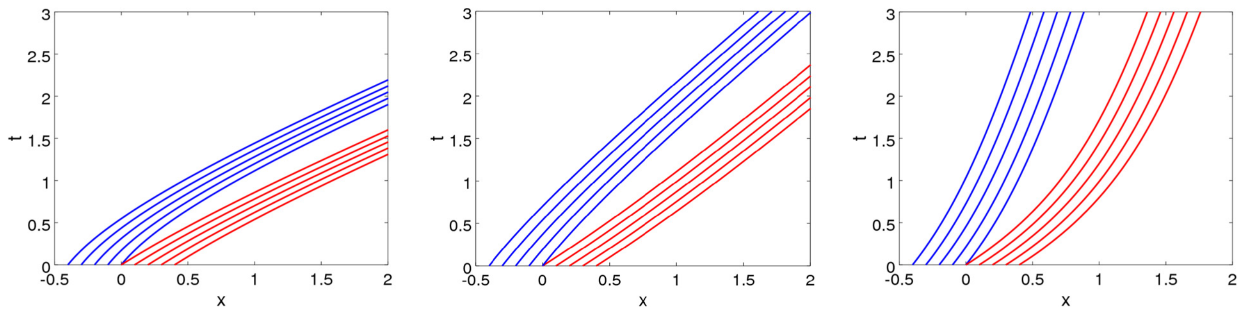

Fig. 2. Characteristic curves on the x–t plane for $u_- = 0.5$, $u_+ = 1.0$ and $\mu = 1.0$. Left: $u_a = 1.5$, middle: $u_a = 0.75$ and right: $u_a = 0.2$.

from (3.5) satisfies the entropy condition (3.6). The trajectory of the shock is given by

$$\xi(t) = \int_0^t \sigma(s)ds = u_a t + \left(\frac{u_- + u_+ - 2u_a}{2\mu}\right)\left(1 - e^{-\mu t}\right). \tag{3.9}$$

We reach the following result:

Corollary 3.4. *If $u_- > u_+$ then the solution of the Riemann problem (1.4) and (3.2) is given by*

$$u(x,t) = \begin{cases} u_l(x,t), & x < \xi(t), \\ \sigma(t), & x = \xi(t), \\ u_r(x,t), & x > \xi(t), \end{cases} \tag{3.10}$$

where u_l, u_r are given in (3.7), σ and ξ are given in (3.8) and (3.9), respectively.

Remark 3.5. We have:

$$\lim_{t\to\infty} \left(u_l(t) - \sigma(t)\right) = \lim_{t\to\infty} \left(\sigma(t) - u_r(t)\right) = 0. \tag{3.11}$$

This means that the Lax's entropy condition for (1.4) degenerates as time goes to infinity. This degeneracy is not observed with the classical inviscid Burgers equation ($\mu = 0$) because the two limits states are constant.

3.2. Rarefaction waves

Secondly, we assume that $u_- < u_+$. Condition (2.14) is not satisfied, and thus characteristics do not intersect, but do not cover the whole x–t plane. Some characteristic curves for different values of u_a are represented in Fig. 2. The uncovered region $\mathcal{S}$ is delimited by the curves

$$X_1(t) = \int_0^t u_l(s)ds = u_a t + \frac{(u_a - u_-)(e^{-\mu t} - 1)}{\mu} \tag{3.12}$$

and

$$X_2(t) = \int_0^t u_r(s)ds = u_a t + \frac{(u_a - u_+)(e^{-\mu t} - 1)}{\mu}. \tag{3.13}$$

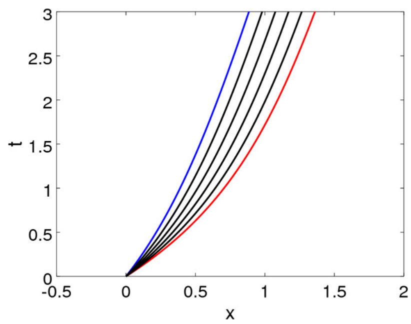

Fig. 3. Region $\mathcal{S}$ filled with characteristics starting at the origin. $u_- = 0.5$, $u_+ = 1.0$, $\mu = 1.0$ and $u_a = 0.2$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

By the method of characteristics, the solution of the Riemann problem is a rarefaction wave, i.e. a continuous function satisfying (1.4). This solution is given by $u_l(x,t)$ for $x < X_1(t)$ and $u_r(x,t)$ for $x > X_2(t)$. To find the solution inside $\mathcal{S}$, we have to fill this region by a family of characteristics starting at the origin, i.e. to solve (2.2) with the initial condition $\chi(x,t;0) = 0$. Fig. 3 shows the region $\mathcal{S}$ (delimited by $x = X_1(t)$ (in blue) and $x = X_2(t)$ (in red)) filled with a family of characteristics (in black) starting at the origin. From (2.4), we calculate the solution u at the foot of the characteristics. We obtain

$$u(\chi(x,t;0),0) = u(x_0,0) = u_a + \big(u(x,t) - u_a\big)e^{\mu t}. \tag{3.14}$$

Substituting (3.14) in (2.5), one gets

$$\chi(x,t;s) = u_a s - \frac{\big(u(x,t) - u_a\big)\big(e^{\mu(t-s)} - e^{\mu t}\big)}{\mu}. \tag{3.15}$$

From this last equation, we get for $s = t$ that

$$u(x,t) = \overline{u}(x,t) = u_a + \frac{\mu\big(x - u_a t\big)}{e^{\mu t} - 1}. \tag{3.16}$$

The following result holds:

Corollary 3.6. *If $u_- < u_+$ then the solution of the Riemann problem (1.4) and (3.2) is given by*

$$u(x,t) = \begin{cases} u_l(x,t), & x < X_1(t), \\ \overline{u}(x,t), & X_1(t) \le x \le X_2(t), \\ u_r(x,t), & x > X_2(t), \end{cases} \tag{3.17}$$

where u_l, u_r are given in (3.7), X_1, X_2 are given in (3.12) and (3.13), respectively, and $\overline{u}$ is given in (3.16).

Proof. The function u satisfies (1.4) inside and outside $\mathcal{S}$, and is continuous at points $X_1(t)$ and $X_2(t)$ for all $t > 0$. In fact, by replacing $X_1(t)$ (resp. $X_2(t)$) in (3.16), one gets u_l (resp. u_r). $\square$

The Burgers equation with source term develops discontinuities if and only if the slope of the initial condition is sufficiently negative with respect to the coefficient μ. This is an extension of the condition for loss of regularity for the classical Burgers equation (for $\mu = 0$). The characteristic curves associated to the inviscid Burger equation with source term are no longer straight lines but are curves that tend asymptotically to straight lines as time grows, as opposed to the classical Burgers equation. The solution of the Riemann problem is either a shock or a rarefaction wave as in the homogeneous case. However, the left and right states are no longer constant, while asymptotically approaching u_a which behaves as an equilibrium point as time goes to infinity. The zeroth order linear source term acts as a relaxation force preventing shocks from occurring and the Lax's entropy condition degenerates as time goes to infinity.

4. Riemann problem for system (1.5)

In this section we study system (1.5) satisfying the initial conditions

$$\big(\alpha, u\big)(x, 0) = \big(\alpha_0, u_0\big)(x), \tag{4.1}$$

where α_0 and u_0 are piecewise smooth functions. For $\mu = 0$, system (1.5) is used to model the evolution of density inhomogeneities in matter in the universe (see [41], section II.B.3). Although viscosity is mentioned in this reference, it is set to zero before solving. For a complete solution of system (1.5) in the homogeneous case $\mu = 0$, we refer the reader to [20]. Here, we solve the Riemann problem for (1.5) with the initial condition

$$\big(\alpha_0, u_0\big)(x) = \begin{cases} (\alpha_-, u_-), & x < 0, \\ (\alpha_+, u_+), & x > 0, \end{cases} \qquad \alpha_-, \alpha_+ \in \mathbb{R}^+ \text{ and } u_-, u_+ \in \mathbb{R}. \tag{4.2}$$

The solution of (1.5) and (4.2) will be useful in the resolution of the Riemann problem for (1.1) in the next section.

4.1. Delta-shock waves

Assume $u_- > u_+$. The characteristics overlap. As pointed out in section 2, the solution is not bounded, more precisely, α blows up and u is discontinuous. This leads to the fundamental question of defining products of non-smooth solutions. A suitable notion of weak solutions for nonconservative systems involving product of non-smooth functions was proposed by Dal Maso, LeFloch, and Murat [18] and the nonlinear stability of such solutions was investigated therein. We are interested here in the conservative form and solutions are sought in the sense of distributions. Motivated by [14,16,42,45,48], we seek solutions with δ-distribution at the jump, i.e. we look for a solution in the form

$$\alpha(x,t) = \alpha^0(x,t) + \omega(t)\delta(x - \xi(t)), \quad u(x,t) = u^0(x,t), \tag{4.3}$$

where α^0, u^0 are smooth functions on both sides of the curve

$$\Gamma = \{(x,t) : x = \xi(t), t \geq 0\}, \tag{4.4}$$

while being discontinuous across this curve, $\delta = \delta(x)$ is the Dirac mass centered at the origin and ω is a smooth function defined on $\mathbb{R}_0^+$ and satisfying the initial condition

$$\omega(0) = \omega_0 \in \mathbb{R}_0^+. \tag{4.5}$$

We define a weighted δ-function $\omega(t)\delta_\xi$ supported on the curve Γ as

$$\langle \omega(t)\delta_\xi, \psi\rangle = \int_0^\infty \omega(t)\psi(\xi(t), t)dt, \tag{4.6}$$

for all test function $\psi \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^+)$. We also define the duality products between the functions α and u (seen as distributions) and test functions in $\mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ as

$$\langle \alpha, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \alpha^0 \psi \, dxdt + \langle \omega(t)\delta_\xi, \psi \rangle, \tag{4.7}$$

$$\langle \alpha u, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \alpha^0 u^0 \psi \, dxdt + \langle \omega(t)\xi'(t)\delta_\xi, \psi \rangle, \tag{4.8}$$

and we introduce the following definition:

Definition 4.1. We say that a pair of distributions (α, u) as given in (4.3)–(4.6) is a **weak solution** of (1.5) and (4.1) if

$$\langle \alpha, \psi_t \rangle + \langle \alpha u, \psi_x \rangle = -\int_{-\infty}^\infty \alpha_0(x)\psi(x,0)\, dx - \omega_0 \psi(\xi(0), 0), \tag{4.9}$$

$$\int_0^\infty \int_{-\infty}^\infty \Big(u\psi_t + \frac{u^2}{2}\psi_x + \mu(u_a - u)\psi \Big) dxdt = -\int_{-\infty}^\infty u_0(x)\psi(x,0)\, dx, \tag{4.10}$$

hold for all test functions $\psi \in \mathcal{C}_0^\infty(\overline{\mathbb{R} \times \mathbb{R}^+})$.

We have the following result:

Theorem 4.2. *A pair of distributions (α, u) as given in (4.3)–(4.6) is a weak solution of (1.5) and (4.1) if and only if the following properties are satisfied:*

i) (α^0, u^0) satisfies (1.5) in the classical sense on both sides of the curve Γ;
ii) $\alpha^0(x, 0) = \alpha_0(x)$ and $u^0(x, 0) = u_0(x)$ for all $x \in \mathbb{R}$;
iii) the following system of differential-algebraic equations (DAE) is satisfied on Γ:

$$\begin{cases} \dfrac{d\omega}{dt}(t) = \big(\alpha_r(t) - \alpha_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big), \\ \big(u_r(t) - u_l(t)\big)\sigma(t) = \dfrac{1}{2}\big(u_r(t)^2 - u_l(t)^2\big), \end{cases} \tag{4.11}$$

where $(\alpha_l(t), u_l(t))$ and $(\alpha_r(t), u_r(t))$ are the limit of the solution (α, u) when (x, t) approaches $(\xi(t), t)$ from the left and the right, respectively;
iv) the following initial condition is satisfied:

$$\omega(0) = \omega_0. \tag{4.12}$$

Proof. By Theorem 3.2, $u = u^0$ is a weak solution of the second equation of (1.5) satisfying the second equality in the initial conditions (4.1) if and only if the properties (i)–(iii) are satisfied by u. It remains to prove the properties (i)–(iv) for α. Let $\psi \in \mathcal{C}_0^\infty(\overline{\mathbb{R} \times \mathbb{R}^+})$. We have:

$$\langle \alpha, \psi_t\rangle + \langle \alpha u, \psi_x\rangle = \int_0^\infty \int_{-\infty}^\infty \left(\alpha^0\psi_t + \alpha^0 u^0 \psi_x\right) dxdt + \int_0^\infty \left(\omega(t)\psi_t(\xi(t),t) + \omega(t)\xi'(t)\psi_x(\xi(t),t)\right) dt$$
$$= \int_0^\infty \int_{-\infty}^\infty (\alpha^0\psi_t + \alpha^0 u^0 \psi_x)\, dxdt + \int_0^\infty \omega(t)\frac{d\psi}{dt}(\xi(t),t)\, dt.$$

Integrating by part the r.h.s. integrals, we obtain

$$\begin{aligned}\langle \alpha, \psi_t\rangle + \langle \alpha u, \psi_x\rangle = &- \int_0^\infty \int_{-\infty}^{\xi(t)} \left(\alpha_t^0 + (\alpha^0 u^0)_x\right)\psi\, dxdt - \int_0^\infty \int_{\xi(t)}^\infty \left(\alpha_t^0 + (\alpha^0 u^0)_x\right)\psi\, dxdt \\ &+ \int_0^\infty \Big(\big(\alpha_r(t) - \alpha_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big) - \frac{d\omega}{dt}(t)\Big)\psi(\xi(t),t)\, dt \\ &- \int_{-\infty}^\infty \alpha^0(x,0)\psi(x,0)\, dx - \omega(0)\psi(\xi(0),0). \end{aligned} \tag{4.13}$$

1) Suppose that (α, u) is a weak solution of (1.5) and (4.1). Then (4.13) reduces to

$$\begin{aligned} - \int_{-\infty}^\infty \alpha_0(x)\psi(x,0)\, dx - \omega_0\psi(\xi(0),0) = &- \int_0^\infty \int_{-\infty}^{\xi(t)} \left(\alpha_t^0 + (\alpha^0 u^0)_x\right)\psi\, dxdt \\ &- \int_0^\infty \int_{\xi(t)}^\infty \left(\alpha_t^0 + (\alpha^0 u^0)_x\right)\psi\, dxdt - \int_{-\infty}^\infty \alpha^0(x,0)\psi(x,0)\, dx - \omega(0)\psi(\xi(0),0) \\ &+ \int_{x=\xi(t)} \Big(\big(\alpha_r(t) - \alpha_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big) - \frac{d\omega}{dt}(t)\Big)\psi(\xi(t),t)\, dt. \end{aligned} \tag{4.14}$$

<u>*Case 1*</u>: Taking $\psi \in \mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{R}^+)$ satisfying $\psi(\xi(t),t) = 0$ for all $t \geq 0$ in (4.14), we get (i).
<u>*Case 2*</u>: Taking $\psi \in \mathcal{C}_0^\infty(\overline{\mathbb{R}\times\mathbb{R}^+})$ satisfying $\psi(\xi(t),t) = 0$ for all $t \geq 0$ and $\psi(x,0) \neq 0$ in (4.14) and using (i), we obtain (ii).
<u>*Case 3*</u>: Taking $\psi \in \mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{R}^+)$ in (4.14) and using (i), one gets (iii).
<u>*Case 4*</u>: Taking $\psi \in \mathcal{C}_0^\infty(\overline{\mathbb{R}\times\mathbb{R}^+})$ in (4.14) and using (i), (ii) and (iii) then we obtain (iv).
2) Conversely, if (i), (ii), (iii) and (iv) are satisfied then (4.13) reduces to (4.9). □

Remark 4.3. The DAE (4.11)–(4.12) reflect the exact relationship between the limit states on the two sides, the weight and propagation speed of the discontinuity, as the classical Rankine–Hugoniot conditions do for ordinary shocks. It is called the **generalized Rankine–Hugoniot (GRH) conditions**. The equations (4.11) state that the defect of mass conservation induced by the discontinuous velocity at the shock leads to a mass accumulation along the trajectory of the shock.

Definition 4.4. We call a **δ-shock solution** or **delta-shock wave** of system (1.5) and (4.1), a weak solution of (1.5) and (4.1) satisfying the entropy condition (3.6).

The above definition of delta-shock waves was used by Sheng and Zhang [42] to construct the solution to the Riemann problem for the zero-pressure gas dynamics system. The classical Rankine–Hugoniot conditions for ordinary shock waves have been generalized to those of delta-shock waves to describe the relationship among the limit states, propagation speed, location and weight of the discontinuity [42]. Delta-shock waves can be interpreted as particular measure solutions as defined in [15], and belong to the space of signed Borel measures on $\mathbb{R}$, denoted $\mathcal{M}(\mathbb{R})$. A measure-theoretic justification of delta-shock waves can also be found in [3,20,31,45]. Since the volume fraction α may develop a Dirac δ-function in finite time, it is natural to seek solutions for (1.5) in the sense of measures, i.e. in the sense of distributions which are signed measures. The solution of the Cauchy problem for the system of zero-pressure gas dynamics is constructed in [15] with measure solutions in the space $\mathcal{M}(\mathbb{R})$. For initial conditions taken in the space of bounded measurable functions, the uniqueness of solutions to the zero-pressure gas dynamics equations is established in [46] with the Oleinik entropy condition. Huang and Wang [27] established the uniqueness of the weak solution to zero-pressure gas dynamics equations when the initial condition is a Random measure. They showed that, besides the Oleinik entropy condition, it is also important to require the energy to be weakly continuous initially. This condition is called the energy condition. For initial data in the space of measures, it was proven in [31] that the Oleinik condition is not sufficient to ensure uniqueness for measure solutions, but it has to be complemented with a cohesion condition. The Lax's entropy condition (3.6) is used in [12, 13,42] as a first step towards the uniqueness of delta-shock solutions to the system of zero-pressure gas dynamics.

In the following, we construct a delta-shock solution to (1.5) that satisfies the Lax's entropy condition (3.6). Theorem 4.2 states that the solution (α, u) is smooth on both side of the curve Γ. From (2.13), we get $\alpha(x,t) = \alpha_0(x_0)$ on both sides of Γ since $u_0' = 0$ on both sides of this curve. Hence, the left and right states for the solution α are determined by the initial data, that is

$$\alpha_l(x,t) = \alpha_l(t) = \alpha_- \quad \text{and} \quad \alpha_r(x,t) = \alpha_r(t) = \alpha_+. \tag{4.15}$$

By Corollary 3.4, u is given by (3.10), and therefore the limit states are given by (3.7). Substituting (3.7), (3.8) and (4.15) in the first equation in (4.11) and integrating the latter, we obtain the weighted function

$$\omega(t) = \omega_0 + \frac{(\alpha_+ + \alpha_-)(u_- - u_+)}{2\mu}\big(1 - e^{-\mu t}\big). \tag{4.16}$$

For the Riemann problem, we take $\omega_0 = 0$. In general, one can start with a δ-shock solution as an initial condition and ω_0 is not necessarily zero. We can now state the following result:

Corollary 4.5. *If $u_- > u_+$ then the Riemann problem (1.5) and (4.2) has a δ-shock solution given by*

$$(\alpha, u)(x,t) = \begin{cases} \big(\alpha_-, u_l(x,t)\big), & x < \xi(t), \\ \big(\omega(t)\delta(x - \xi(t)), \sigma(t)\big), & x = \xi(t), \\ \big(\alpha_+, u_r(x,t)\big), & x > \xi(t), \end{cases} \tag{4.17}$$

where u_l, u_r are given in (3.7), ω is given in (4.16) with $\omega_0 = 0$, σ and ξ are given by (3.8) and (3.9), respectively.

4.2. Two contact discontinuities with a vacuum state

Assume $u_- < u_+$. By Corollary 3.6, u is given by (3.17) and is independent of x outside the region $\mathcal{S}$, hence

$$\frac{D\alpha}{dt} = -\alpha \partial_x u = 0, \quad \forall (x,t) \notin \mathcal{S}. \tag{4.18}$$

Outside $\mathcal{S}$, $\alpha(x,t) = \alpha_0(x_0)$ and is determined by the Riemann initial data, that is

$$\alpha(x,t) = \begin{cases} \alpha_-, & x < X_1(t), \\ \alpha_+, & x > X_2(t). \end{cases} \tag{4.19}$$

Inside $\mathcal{S}$, u is given by (3.16). We have:

$$\frac{D\alpha}{dt} = -\alpha \partial_x \overline{u}, \quad \forall (x,t) \in \overline{\mathcal{S}}. \tag{4.20}$$

The function $\alpha = 0$ satisfies (4.20). Let $\epsilon > 0$ and assume $\alpha \neq 0$. One divides (4.20) by α and integrates on both sides from $s = \epsilon$ to $s = t$ to obtain

$$\alpha(x,t) = \frac{\left(1 - e^{-\mu\epsilon}\right) e^{\mu t} \alpha(x,\epsilon)}{e^{\mu t} - 1}, \quad \forall (x,t) \in \overline{\mathcal{S}}. \tag{4.21}$$

Taking $\epsilon \to 0$ in (4.21), one obtains

$$\alpha(x,t) = 0, \quad \forall (x,t) \in \overline{\mathcal{S}}. \tag{4.22}$$

We reach the following result:

Corollary 4.6. *If $u_- < u_+$ then the solution of the Riemann problem (1.5) and (4.2) is given by*

$$\big(\alpha, u\big)(x,t) = \begin{cases} \big(\alpha_-, u_l(x,t)\big), & x < X_1(t), \\ \big(0, \overline{u}(x,t)\big), & X_1(t) \leq x \leq X_2(t), \\ \big(\alpha_+, u_r(x,t)\big), & x > X_2(t), \end{cases} \tag{4.23}$$

where u_l and u_r are given in (3.7), $\overline{u}$ is given in (3.16), X_1 and X_2 are given in (3.12) and (3.13), respectively.

Note that u is continuous while α might be discontinuous across the curves $x = X_1(t)$ and $x = X_2(t)$. This type of solution is called a *two-contact-discontinuity*. The two-contact-discontinuity solution (4.23) contains a vacuum state (region where $\alpha = 0$). Vacuum states are important physical states in fluid mechanics and often yield singularities in the physical systems, which cause essential analytical and numerical difficulties (see [19,22,26,34–36]).

5. Riemann problem for the Eulerian droplet model

In this section we construct a solution to the Riemann problem for (1.1) using the results from the previous sections.

5.1. Delta-shock waves

Assume $u_- > u_+$. The characteristics overlap. As pointed out in section 2, a bounded solution of (1.1) does not exist. Motivated by the discussion and results in the previous section, we seek for solution in the form given in (4.3)–(4.6). We define the following duality products between α and u (seen as distributions) and test functions in $\mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^+)$:

$$\langle \alpha, \psi \rangle = \int_{0}^{\infty} \int_{-\infty}^{\infty} \alpha^0 \psi dx dt + \langle \omega(t)\delta_\xi, \psi \rangle, \tag{5.1}$$

$$\langle \alpha u, \psi \rangle = \int_{0}^{\infty} \int_{-\infty}^{\infty} \alpha^0 u^0 \psi dx dt + \langle \omega(t)\xi'(t)\delta_\xi, \psi \rangle, \tag{5.2}$$

$$\langle \alpha u^2, \psi \rangle = \int_{0}^{\infty} \int_{-\infty}^{\infty} \alpha^0 (u^0)^2 \psi dx dt + \langle \sigma(t)\omega(t)\xi'(t)\delta_\xi, \psi \rangle, \tag{5.3}$$

$$\langle \alpha(u_a - u), \psi \rangle = \int_{0}^{\infty} \int_{-\infty}^{\infty} \alpha^0 (u_a - u^0) \psi dx dt + \langle (u_a - \sigma(t))\omega(t)\delta_\xi, \psi \rangle, \tag{5.4}$$

where $\sigma(t) = \xi'(t)$ satisfies

$$\sigma(0) = \sigma_0 \in \mathbb{R}. \tag{5.5}$$

Definition 5.1. We say that a pair of distributions (α, u) as given in (4.3)–(4.6) is a **weak solution** of (1.1) and (4.1) if

$$\langle \alpha, \psi_t \rangle + \langle \alpha u, \psi_x \rangle = -\int_{-\infty}^{\infty} \alpha_0(x)\psi(x,0)\, dx - \omega_0 \psi(\xi(0), 0), \tag{5.6}$$

$$\langle \alpha u, \psi_t \rangle + \langle \alpha u^2, \psi_x \rangle + \mu \langle \alpha(u_a - u), \psi \rangle = -\int_{-\infty}^{\infty} \alpha_0(x) u_0(x) \psi(x,0)\, dx - \sigma_0 \omega_0 \psi(\xi(0), 0), \tag{5.7}$$

hold for all test functions $\psi \in \mathcal{C}_0^\infty(\overline{\mathbb{R} \times \mathbb{R}^+})$.

We have the following characterization for a weak solution of (1.1) and (4.1):

Theorem 5.2. *A pair of distributions (α, u) as given in (4.3)–(4.6) is a weak solution of (1.1) and (4.1) if and only if the following properties are satisfied:*

i) *(α^0, u^0) satisfies (1.1) in the classical sense on both sides of the curve Γ;*
ii) *$\alpha^0(x,0) = \alpha_0(x)$ and $\alpha^0(x,0)u^0(x,0) = \alpha_0(x)u_0(x)$ for all $x \in \mathbb{R}$;*
iii) *the following ODEs are satisfied on the curve Γ:*

$$\begin{cases} \dfrac{d\omega}{dt} = \big(\alpha_r(t) - \alpha_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big), \\ \dfrac{d(\omega\sigma)}{dt} = \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t)^2 - \alpha_l(t)u_l(t)^2\big) + \mu\big(u_a - \sigma(t)\big)\omega(t), \end{cases} \tag{5.8}$$

where $(\alpha_l(t), u_l(t))$ and $(\alpha_r(t), u_r(t))$ are the limit of the solution (α, u) when (x,t) approaches $(\xi(t), t)$ from the left and the right, respectively;
iv) *the following initial conditions are satisfied:*

$$\omega(0) = \omega_0, \quad \sigma(0)\omega(0) = \sigma_0\omega_0. \tag{5.9}$$

Proof. Proceed as in the proof of Theorem 4.2. □

Definition 5.3. We call a δ**-shock solution** or **delta-shock wave** of system (1.1) and (4.1), a weak solution of (1.1) and (4.1) satisfying the Lax's entropy condition (3.6).

We now return to the solution of the Riemann problem for (1.1). By Theorem 5.2, the solution of (1.1) is smooth on both sides of the curve $x = \xi(t)$. Hence, it is given by the solution of (1.5) on both sides of this curve. We have the following result:

Corollary 5.4. *If $u_- > u_+$ then the Riemann problem (1.1) and (4.2) has a δ-shock solution given by*

$$(\alpha, u)(x,t) = \begin{cases} \big(\alpha_-, u_l(x,t)\big), & x < \xi(t), \\ \big(\omega(t)\delta(x-\xi(t)), \sigma(t)\big), & x = \xi(t), \\ \big(\alpha_+, u_r(x,t)\big), & x > \xi(t), \end{cases} \tag{5.10}$$

where u_l, u_r are given by (3.7), (ω, σ) is the solution of the GRH conditions (5.8)–(5.9) with $\omega_0 = 0$, and ξ is given by

$$\xi(t) = \int_0^t \sigma(s)\, ds. \tag{5.11}$$

We will prove in the next section the existence of a solution to the GRH conditions (5.8)–(5.9) satisfying the Lax's entropy condition (3.6).

5.2. Two contact discontinuities with a vacuum state

Assume $u_- < u_+$. The Riemann solution is constructed in the same manner as in subsection 4.2. It is given by the two-contact-discontinuity with vacuum state (4.23).

The solution of the Riemann problem for (1.1) is either a delta-shock wave or a two-contact-discontinuity. The systems (1.1) and (1.5) are equivalent for smooth and two-contact-discontinuity solutions but they differ for δ-shock solutions. In fact, one can check with a tedious calculation that the solution of the DAE (4.11)–(4.12) does not satisfy the GRH conditions (5.8)–(5.9).

6. Existence of a solution to the GRH conditions satisfying the Lax's entropy

In this section we prove the existence of a solution to (5.8)–(5.9) satisfying the Lax's entropy condition (3.6). The following results hold:

Lemma 6.1 *(Growth of the point mass ω). Assume that $\alpha_l(t), \alpha_r(t) > 0$ and $u_l(t) > u_r(t)$ for all $t \geq 0$. Suppose that there exists a solution $(\omega, \sigma) \in \mathcal{C}^1(\mathbb{R}^+) \times \mathcal{C}^1(\mathbb{R}^+)$ to the initial value problem (5.8)–(5.9). If σ satisfies (3.6) then*

$$\omega(t_2) > \omega(t_1), \quad \forall t_2 > t_1. \tag{6.1}$$

In particular, for the solution of the Riemann problem (1.1) and (4.2), we have

$$\omega(t) \geq \omega_0 + \frac{K(1-e^{-\mu t})}{\mu}, \quad \forall t \geq 0, \tag{6.2}$$

where $K = \min\{\alpha_-, \alpha_+\}(u_- - u_+)$.

Proof. We proceed by cases. Let $t \geq 0$.
Case 1: Assume that $\alpha_l(t) = \alpha_r(t)$. The first equation of (5.8) reduces to

$$\frac{d\omega}{dt}(t) = \alpha_l(t)\big(u_l(t) - u_r(t)\big) > 0.$$

Case 2: Assume that $\alpha_l(t) < \alpha_r(t)$. From the first inequality in (3.6), the first equation of (5.8) leads to

$$\frac{d\omega}{dt}(t) = \big(\alpha_r(t) - \alpha_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big) > \alpha_l(t)\big(u_l(t) - u_r(t)\big) > 0.$$

Case 3: Assume that $\alpha_l(t) > \alpha_r(t)$. From the second inequality in (3.6), the first equation of (5.8) leads to

$$\frac{d\omega}{dt}(t) = \big(\alpha_r(t) - \alpha_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big) > \alpha_r(t)\big(u_l(t) - u_r(t)\big) > 0.$$

Combining these three cases, we obtain

$$\frac{d\omega}{dt}(t) > 0, \quad \forall t \geq 0. \tag{6.3}$$

Hence, (6.1) holds. In particular, for the solution of the Riemann problem (1.1) and (4.2), the limit states are given by $\alpha_l(t) = \alpha_-$, $\alpha_r(t) = \alpha_+$, $u_l(t) = u_a + (u_- - u_a)e^{-\mu t}$ and $u_l(t) = u_a + (u_+ - u_a)e^{-\mu t}$. By setting $K = \min\{\alpha_-, \alpha_+\}(u_- - u_+)$, one obtains

$$\frac{d\omega}{dt}(t) \geq Ke^{-\mu t}, \quad \forall t \geq 0. \tag{6.4}$$

Integrating this last inequality on both sides from 0 to t, we get

$$\omega(t) \geq \omega(0) + K\int_0^t e^{-\mu s}\, ds = \omega_0 + \frac{K(1 - e^{-\mu t})}{\mu}, \quad \forall t \geq 0. \quad \square \tag{6.5}$$

Proposition 6.2 *(The shock speed satisfies the Lax's entropy condition). Assume that $\alpha_l(t), \alpha_r(t) > 0$, $u_l(t) > u_r(t)$ for all $t \geq 0$ and $\sigma_0 \in (u_r(0), u_l(0))$. If $(\omega, \sigma) \in \mathcal{C}^1(\mathbb{R}^+) \times \mathcal{C}^1(\mathbb{R}^+)$ is a solution to the initial value problem (5.8)–(5.9) then σ satisfies the Lax's entropy condition (3.6) for all $t \geq 0$.*

Proof. We proceed by contradiction. Suppose that there exists $t \geq 0$ such that (3.6) is not satisfied. From $\sigma(0) = \sigma_0 \in (u_r(0), u_l(0))$ and the continuity of σ, there exists a smallest $t > 0$, denoted t^*, such that

$$\sigma(t^*) = u_l(t^*) \text{ or } \sigma(t^*) = u_r(t^*) \text{ and } \sigma \text{ satisfies (3.6)}, \ \forall t \in [0, t^*). \tag{6.6}$$

By Lemma 6.1, ω satisfies (6.1) on $[0, t^*)$. Combining this with the continuity of ω, we get

$$\omega(t^*) > \omega(0) = \omega_0 \geqslant 0. \tag{6.7}$$

Assume that $\sigma(t^*) = u_l(t^*)$. Substituting $\sigma(t^*)$ by $u_l(t^*)$ in (5.8), one can calculate

$$\omega(t^*)\frac{d\sigma}{dt}(t^*) = \frac{d(\omega\sigma)}{dt}(t^*) - \sigma(t^*)\frac{d\omega}{dt}(t^*) = -\alpha_r(t^*)\big(u_l(t^*) - u_r(t^*)\big)^2 + \mu\big(u_a - u_l(t^*)\big)\omega(t^*). \tag{6.8}$$

Using this last equation, one gets

$$\omega(t^*)\frac{d(\sigma-u_l)}{dt}(t^*)=\omega(t^*)\frac{d\sigma}{dt}(t^*)-\omega(t^*)\frac{du_l}{dt}(t^*)=-\alpha_r(t^*)\big(u_l(t^*)-u_r(t^*)\big)^2<0. \tag{6.9}$$

Using (6.7), we deduce from (6.9) that

$$\frac{d(\sigma-u_l)}{dt}(t^*)<0.$$

By the continuity of the function $t\mapsto\dfrac{d(\sigma-u_l)}{dt}(t)$, there exists $\epsilon>0$ such that

$$\frac{d(\sigma-u_l)}{dt}(t)<0,\quad \forall t\in]t^*-\epsilon,t^*[.$$

Integrating this last inequality on both sides from $t^*-\epsilon$ to t^*, we obtain

$$0\geq\int\limits_{t^*-\epsilon}^{t^*}\frac{d(\sigma-u_l)}{ds}(s)ds=(\sigma-u_l)(t^*)-(\sigma-u_l)(t^*-\epsilon)=-\sigma(t^*-\epsilon)+u_l(t^*-\epsilon).$$

This last inequality implies that $\sigma(t^*-\epsilon)\geq u_l(t^*-\epsilon)$ which contradicts (6.6).

Assume that $\sigma(t^*)=u_r(t^*)$. A similar reasoning as above leads also to a contradiction of (6.6). Thus, the shock speed σ satisfies (3.6) for all $t\geq 0$. $\square$

We next state the result for the existence of a solution to (5.8)–(5.9) satisfying the (3.6). To simplify our notation, we set

$$\begin{aligned}&a(t)=\alpha_r(t)-\alpha_l(t),\quad b(t)=\alpha_r(t)u_r(t)-\alpha_l(t)u_l(t),\\&c(t)=\alpha_r(t)u_r(t)^2-\alpha_l(t)u_l(t)^2,\quad \theta(t)=\omega(t)\sigma(t).\end{aligned} \tag{6.10}$$

The functions a, b and c are continuous and bounded for all $t\geqslant 0$ since the limit states (α_l,u_l) and (α_r,u_r) are continuous and bounded. Substituting a, b, c and θ in (5.8), this system can be rewritten now as

$$\begin{cases}\dfrac{d\omega}{dt}(t)=a(t)\dfrac{\theta(t)}{\omega(t)}-b(t),\\[2mm]\dfrac{d\theta}{dt}(t)=b(t)\dfrac{\theta(t)}{\omega(t)}+\mu\big(u_a\omega(t)-\theta(t)\big)-c(t).\end{cases} \tag{6.11}$$

The initial value problem (5.8)–(5.9) can then be rewritten in the following condensed form

$$\begin{cases}\dfrac{d\mathbf{z}}{dt}=\mathbf{f}(\mathbf{z},t),\\[2mm]\mathbf{z}(0)=(\omega_0,\theta_0)^T,\ \text{with }\theta_0=\omega_0\sigma_0,\end{cases} \tag{6.12}$$

where

$$\mathbf{z}(t)=\begin{pmatrix}\omega(t)\\\theta(t)\end{pmatrix}\quad\text{and}\quad \mathbf{f}(\mathbf{z},t)=\begin{pmatrix}f_1(\mathbf{z},t)\\f_2(\mathbf{z},t)\end{pmatrix}=\begin{pmatrix}a(t)\dfrac{\theta(t)}{\omega(t)}-b(t)\\[2mm] b(t)\dfrac{\theta(t)}{\omega(t)}+\mu\big(u_a\omega(t)-\theta(t)\big)-c(t)\end{pmatrix}. \tag{6.13}$$

Theorem 6.3 *(Existence of a solution to the GRH conditions). If $\alpha_-, \alpha_+ > 0$, $u_- > u_+$ and $\sigma_0 \in (u_+, u_-)$ then the GRH conditions (5.8)–(5.9) for the solution of the Riemann problem (1.1) and (4.2) have a $\mathcal{C}^1$-solution (ω, σ) satisfying (3.6) for all $t \geqslant 0$.*

Proof. i) Suppose that $\omega_0 > 0$. Cauchy–Péano theorem (see [17], p. 59) ensures the existence of a $\mathcal{C}^1$-solution (ω, θ) to (6.12) on some interval $[0, t_1]$. From Proposition 6.2 and Lemma 6.1, $\omega(t) > 0$ for all $t \in [0, t_1]$. This implies that the partial derivatives $\frac{\partial f_i}{\partial z_j}$ exist and are continuous on $\mathbb{R}^+ \times \mathbb{R} \times [0, t_1]$. Cauchy–Lipschitz existence theorem (see [17], p. 65) ensures the uniqueness of this solution on the interval $[0, t_1]$. Define

$$\sigma(t) = \frac{\theta(t)}{\omega(t)}, \quad \forall t \in [0, t_1]. \tag{6.14}$$

Clearly, (ω, σ) is a $\mathcal{C}^1$-solution of (5.8)–(5.9) on $[0, t_1]$. Using the Lemmas 3.2 and 3.3 in [17], Proposition 6.2 and Lemma 6.1, this solution can be extended to all $t \geq 0$.

ii) Suppose that $\omega_0 = 0$. Take any finite $T > 0$. Let $F_1 = \{(\omega_n, \theta_n)\}_{n \geq 1}$ be the family of functions such that (ω_n, θ_n) is a $\mathcal{C}^1$-solution of (6.12) on the interval $[0, T]$ satisfying the initial conditions $\omega_n(0) = \frac{1}{n}$ and $\theta_n(0) = \frac{\sigma_0}{n}$. Moreover, the functions ω_n satisfy (6.1) and (6.2). The existence of the family F_1 follows from part (i). Let us prove that F_1 is a family of bounded and equicontinuous functions on the interval $[0, T]$. For all $n \geq 1$, we have:

$$\begin{cases} \dfrac{d\omega_n}{dt}(t) = a(t)\sigma_n(t) - b(t), \\ \dfrac{d\theta_n}{dt}(t) = b(t)\sigma_n(t) + \mu\big(u_a - \sigma_n(t)\big)\omega_n(t) - c(t). \end{cases} \tag{6.15}$$

Taking the absolute value in the first equation of (6.15), we get

$$\left|\frac{d\omega_n}{dt}(t)\right| \leqslant |a(t)|\,|u_l(t)| + |b(t)| \leqslant M, \tag{6.16}$$

where $M = \max_{s\in[0,T]} \big\{|a(s)||u_l(s)| + |b(s)|\big\}$. Integrating ω_n' from 0 to t, taking the absolute value and using (6.16), we get

$$|\omega_n(t)| = \left|\omega_n(0) + \int_0^t \omega_n'(s)\,ds\right| \leqslant \frac{1}{n} + \int_0^t |\omega_n'(s)|\,ds \leqslant 1 + MT, \quad \forall t \in [0, T].$$

Hence, the functions ω_n and their first derivative are uniformly bounded on $[0, T]$. For all $n \geq 1$, $\theta_n = \omega_n\sigma_n$ is bounded as a product of two bounded functions since σ_n satisfies (3.6) and $u_r(t)$, $u_l(t)$ are bounded. Furthermore, $\theta_n'(t)$ is also bounded since

$$\begin{aligned} \left|\frac{d\theta_n}{dt}(t)\right| &= \big|b(t)\sigma_n(t) + \mu\big(u_a - \sigma_n(t)\big)\omega_n(t) - c(t)\big| \\ &\leqslant |b(t)|\,|u_l(t)| + \mu\big(1 + MT\big)\big(\,|u_a| + |u_l(t)|\,\big) + |c(t)| \\ &\leqslant Q = \max_{t\in[0,T]} \Big\{\,|b(t)|\,|u_l(t)| + \mu\big(1 + MT\big)\big(\,|u_a| + |u_l(t)|\,\big) + |c(t)|\Big\} < \infty. \end{aligned}$$

Hence, F_1 is a family of bounded and equicontinuous functions at every point of the interval $[0, T]$. Arzelà–Ascoli theorem ensures the existence of a subsequence $\{(\omega_{n_k}, \theta_{n_k})\} \subset F_1$ that converges uniformly to the continuous functions (ω, θ) on the interval $[0, T]$. Since ω_{n_k} satisfies (6.2) then ω is positive on any interval of the form $[\eta, T]$, with $0 < \eta \leqslant T$. Define the sequence of functions

$$\sigma_{n_k}(t) = \frac{\theta_{n_k(t)}}{\omega_{n_k}(t)}, \quad \forall t \in [0, T], \tag{6.17}$$

and the function

$$\sigma(t) = \frac{\theta(t)}{\omega(t)}, \quad \forall t \in [\eta, T]. \tag{6.18}$$

Clearly, σ is continuous on the interval $[\eta, T]$ as a quotient of two continuous functions. Let us prove that σ_{n_k} converges uniformly to σ on the interval $[\eta, T]$. Let $t \in [\eta, T]$. We have:

$$\begin{aligned} |\sigma_{n_k}(t) - \sigma(t)| &\leqslant \left| \frac{\theta_{n_k}(t)\omega(t) + \theta(t)\omega(t) - \theta(t)\omega(t) - \omega_{n_k}(t)\theta(t)}{\omega_{n_k}(\eta)\omega(\eta)} \right| \\ &\leqslant \frac{\mu^2}{K^2(1 - e^{-\mu\eta})^2} \Big(|\theta(t)|\, |\omega_{n_k}(t) - \omega(t)| + |\omega(t)|\, |\theta_{n_k}(t) - \theta(t)| \Big). \end{aligned} \tag{6.19}$$

Set

$$P = \frac{\max_{t \in [\eta, T]} \big\{ |\theta(t)|, |\omega(t)| \big\} \mu^2}{K^2(1 - e^{-\mu\eta})^2} < \infty.$$

Taking the supremum in (6.19), we get

$$\sup_{t \in [\eta, T]} |\sigma_{n_k}(t) - \sigma(t)| \leqslant P\big(\sup_{t \in [\eta, T]} |\omega_{n_k}(t) - \omega(t)| + \sup_{t \in [\eta, T]} |\theta_{n_k}(t) - \theta(t)| \big). \tag{6.20}$$

Hence, the sequence σ_{n_k} converges uniformly to σ on the interval $[\eta, T]$ since ω_{n_k} and θ_{n_k} converge uniformly to ω and θ, respectively, on the interval $[0, T]$. Let us prove that (ω, σ) is a $\mathcal{C}^1$-solution of (5.8) on the interval $[\eta, T]$. For all $n_k \geq 1$, we have:

$$\begin{cases} \dfrac{d\omega_{n_k}}{dt}(t) = a(t)\sigma_{n_k}(t) - b(t), \\ \dfrac{d(\omega_{n_k}\sigma_{n_k})}{dt} = b(t)\sigma_{n_k}(t) + \mu\big(u_a - \sigma_{n_k}(t)\big)\omega_{n_k}(t) - c(t). \end{cases} \tag{6.21}$$

As ω_{n_k} and σ_{n_k} converge uniformly to ω and σ, respectively, on the interval $[\eta, T]$ then the terms on the r.h.s. of each equation of (6.21) also converge uniformly on the interval $[\eta, T]$, i.e. the sequence $\{(\frac{d\omega_{n_k}}{dt}, \frac{d(\omega_{n_k}\sigma_{n_k})}{dt})\}_{k \geq 1}$ converges uniformly on the interval $[\eta, T]$. Take $n_k \to \infty$ in (6.21). Then, by using theorem 7.17 in [39], we obtain

$$\begin{cases} \dfrac{d\omega}{dt}(t) = \lim\limits_{n_k \to \infty} \dfrac{d\omega_{n_k}}{dt}(t) = \lim\limits_{n_k \to \infty} \Big(a(t)\sigma_{n_k}(t) - b(t) \Big) = a(t)\sigma(t) - b(t), \\ \dfrac{d(\omega\sigma)}{dt}(t) = \lim\limits_{n_k \to \infty} \dfrac{d(\omega_{n_k}\sigma_{n_k})}{dt}(t) = \lim\limits_{n_k \to \infty} \Big(b(t)\sigma_{n_k}(t) + \mu\big(u_a - \sigma_{n_k}(t)\big)\omega_{n_k}(t) - c(t) \Big) \\ \qquad\qquad\qquad = b(t)\sigma(t) + \mu\big(u_a - \sigma(t)\big)\omega(t) - c(t). \end{cases} \tag{6.22}$$

The derivative of the functions ω and θ are continuous on the interval $[\eta, T]$ since the terms on the r.h.s. of (6.22) are continuous. Hence, the couple (ω, σ) is a $\mathcal{C}^1$-solution of (5.8) on the interval $[\eta, T]$. Since $\eta > 0$

is arbitrary then the $\mathcal{C}^1$-solution (ω, σ) can be extended to all t in $(0, T]$. From the uniform convergence of ω_{n_k} to ω on the interval $[0, T]$, we get

$$\omega(0) = \lim_{n_k \to \infty} \omega_{n_k}(0) = \lim_{n_k \to \infty} \frac{1}{n_k} = 0 = \omega_0. \tag{6.23}$$

We set

$$\sigma(0) = \sigma_0 \in (u_+, u_-).$$

Thus, the $\mathcal{C}^1$-functions (ω, σ) satisfy the initial value problem (5.8)–(5.9) for all $t \in [0, T]$. This solution can be extended to all $t \geqslant 0$ since $T > 0$ is arbitrary. By Proposition 6.2, the shock speed σ satisfies (3.6) for all $t \geqslant 0$. $\square$

Remark 6.4. The generalized Rankine–Hugoniot conditions (5.8)–(5.9) reduce to the classical Rankine–Hugoniot conditions for a contact discontinuity or a two-contact-discontinuity solution. In fact, if $u_- = u_+$ then $\omega = 0$. Hence, (5.8) reduces to the classical Rankine–Hugoniot conditions given by

$$\begin{cases} \big(\alpha_r(t) - \alpha_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big) = 0, \\ \big(\alpha_r(t)u_r(t) - \alpha_l(t)u_l(t)\big)\sigma(t) - \big(\alpha_r(t)u_r(t)^2 - \alpha_l(t)u_l(t)^2\big) = 0. \end{cases} \tag{6.24}$$

Proposition 6.2 stipulates that the initial shock speed σ_0 should belong to the interval (u_+, u_-). The exact value for σ_0 is given by the following result:

Proposition 6.5. *Assume that $\alpha_l(0), \alpha_r(0) > 0$. Let $(\omega, \sigma) \in \mathcal{C}^1(\mathbb{R}_0^+) \times \mathcal{C}^1(\mathbb{R}_0^+)$ be a solution to the GRH conditions (5.8)–(5.9). If $\omega_0 = 0$ then the initial shock speed σ_0 satisfying the GRH conditions and the Lax's entropy condition (3.6) at the origin is given by*

$$\sigma_0 = \frac{\sqrt{\alpha_r(0)}u_r(0) + \sqrt{\alpha_l(0)}u_l(0)}{\sqrt{\alpha_r(0)} + \sqrt{\alpha_l(0)}}. \tag{6.25}$$

Proof. By using the first equation of (5.8), the second equation can be written as

$$\omega(t)\frac{d\sigma}{dt}(t) = -\,a(t)\sigma(t)^2 + 2b(t)\sigma(t) - c(t) + \mu\big(u_a - \sigma(t)\big)\omega(t). \tag{6.26}$$

At $t = 0$, $\omega(0) = \omega_0 = 0$ and (6.26) reduces to

$$-a(0)\sigma(0)^2 + 2b(0)\sigma(0) - c(0) = 0. \tag{6.27}$$

If $a(0) = 0$ then (6.27) has one solution given by

$$\sigma(0) = \frac{u_r(0) + u_l(0)}{2} \in (u_r(0), u_l(0)) \tag{6.28}$$

that satisfies (3.6) at the origin. If $a(0) \neq 0$ then (6.27) has two roots

$$\sigma(0)_1 = \frac{\sqrt{\alpha_r(0)}u_r(0) - \sqrt{\alpha_l(0)}u_l(0)}{\sqrt{\alpha_r(0)} - \sqrt{\alpha_l(0)}} \quad \text{and} \quad \sigma(0)_2 = \frac{\sqrt{\alpha_r(0)}u_r(0) + \sqrt{\alpha_l(0)}u_l(0)}{\sqrt{\alpha_r(0)} + \sqrt{\alpha_l(0)}}. \tag{6.29}$$

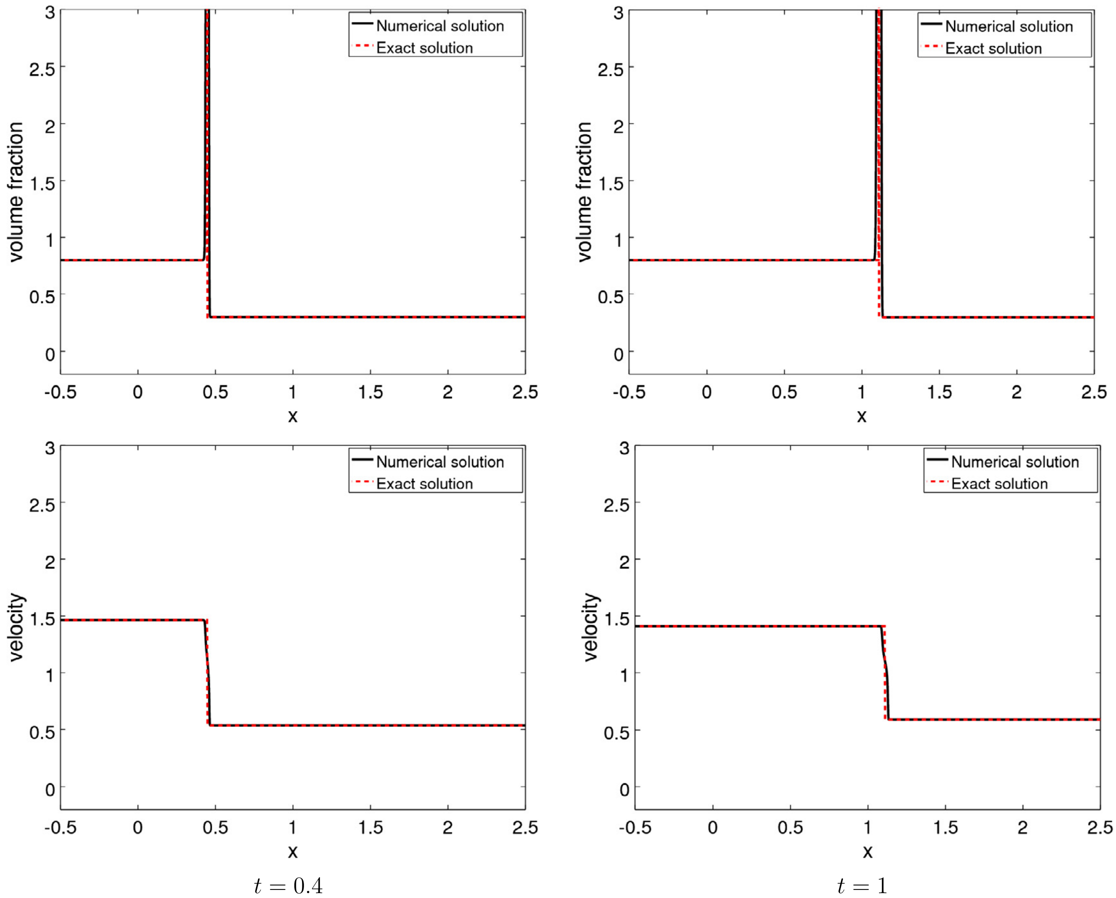

Fig. 4. A delta-shock wave of system (1.1): exact and numerical solutions for two different times. $\mu = 0.2$, $u_a = 1$, $\Delta x = 10^{-3}$ and $\Delta t = 10^{-4}$.

The first root $\sigma(0)_1$ does not always satisfy (3.6) at the origin. In fact,

$$\sigma_{01} - u_l(0) = \frac{\sqrt{\alpha_r(0)}(u_r(0) - u_l(0))}{\sqrt{\alpha_r(0)} - \sqrt{\alpha_l(0)}} > 0, \quad \text{if } \alpha_l(0) > \alpha_r(0). \tag{6.30}$$

The second root $\sigma(0)_2 \in (u_r(0), u_l(0))$ for all $\alpha_l(0), \alpha_r(0) > 0$. Moreover, if $\alpha_l(0) = \alpha_r(0)$ then $\sigma(0)_2$ reduces to (6.28). □

For the solution of Riemann problem (1.1) and (4.2), we proved the existence of a solution to the GRH conditions (5.8)–(5.9) satisfying the Lax's entropy condition (3.6). In general, it might be hard to find the analytical solution of (5.8)–(5.9). At least for the Riemann problem, we are lucky to find an analytical solution of (5.8)–(5.9) given by

$$\begin{aligned} \omega(t) &= \omega_0 + \frac{\sqrt{\alpha_- \alpha_+}(u_- - u_+)}{\mu}(1 - e^{-\mu t}), \\ \sigma(t) &= u_a + \left(\frac{\sqrt{\alpha_-}u_- + \sqrt{\alpha_+}u_+}{\sqrt{\alpha_-} + \sqrt{\alpha_+}} - u_a \right) e^{-\mu t}. \end{aligned} \tag{6.31}$$

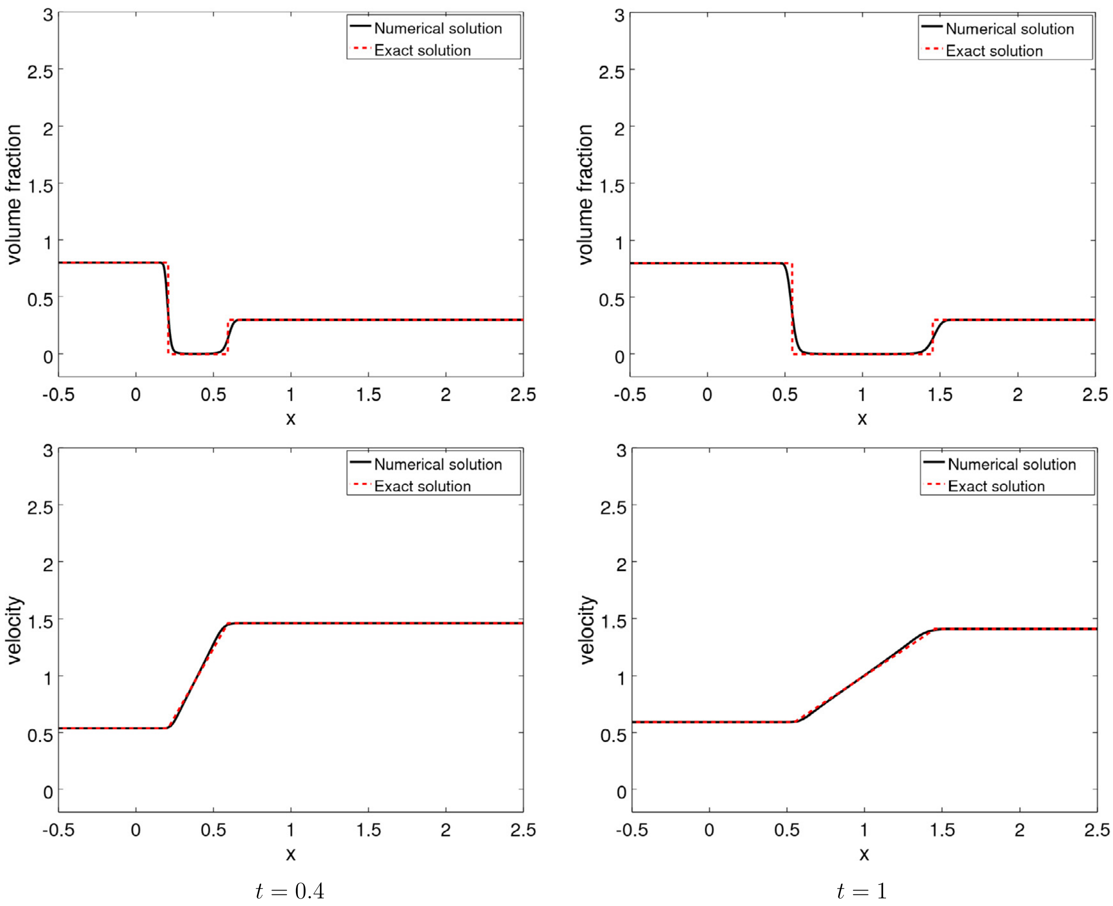

Fig. 5. A two-contact-discontinuity solution with a vacuum state of system (1.1): exact and numerical solutions for two different times. $\mu = 0.2$, $u_a = 1$, $\Delta x = 10^{-3}$ and $\Delta t = 10^{-4}$.

7. Numerical results

We perform some representative test cases to illustrate the theoretical analysis of delta-shock waves and vacuum states for the Eulerian droplet model. We employ the Transport-Collapse method [3] to discretize the equations (1.1) and we take $u_a = 1$.

7.1. Riemann problem solution: numerical solutions vs theoretical analysis

We compute Riemann solutions for a delta-shock wave and a vacuum state. Recall that in real applications, the drag coefficient μ is not constant (depends on the Reynolds number of the particles). For physical applications with non-constant drag coefficient, we refer to [6,8,9,24]. Here we assume $\mu = 0.2$ is constant. For a delta-shock wave, we take the initial conditions

$$(\alpha, u)(x, 0) = \begin{cases} (0.008, 1.5), & x \leqslant 0, \\ (0.003, 0.5), & x > 0, \end{cases} \tag{7.1}$$

which correspond to a physical case where initially the particles behind move faster. Theoretical and numerical solutions are shown in Fig. 4. The particles behind catch those in front resulting in a huge concentration of particles leading the formation of point mass on the shock trajectory.

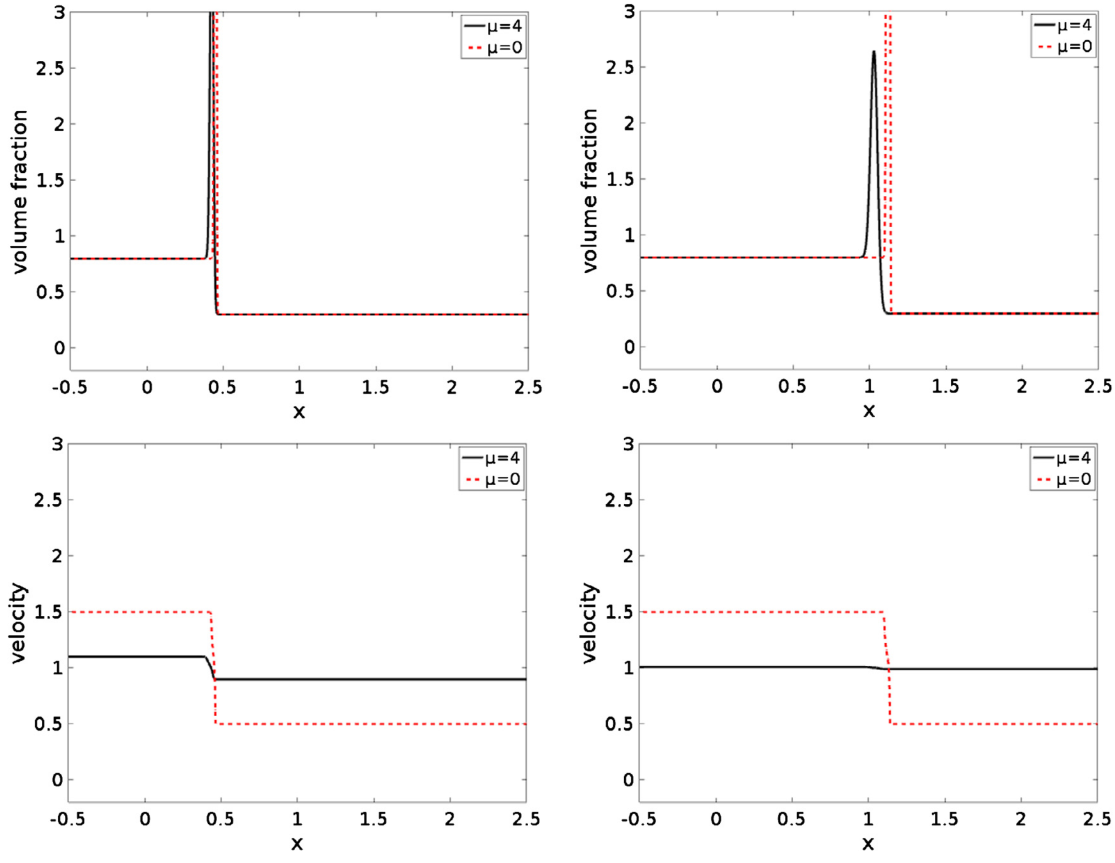

Fig. 6. Evolution of a delta-shock wave of system (1.1). Solutions with ($\mu = 4$)/without ($\mu = 0$) the source term. $u_a = 1$, $\Delta x = 10^{-3}$ and $\Delta t = 10^{-4}$.

For a vacuum state, we take the initial conditions

$$(\alpha, u)(x, 0) = \begin{cases} (0.008, 0.5), & x \leqslant 0, \\ (0.003, 1.5), & x > 0, \end{cases} \tag{7.2}$$

which correspond to a physical case where initially the particles in front move faster. Theoretical analysis and numerical solutions are represented in Fig. 5. We observe left and right nonvacuum states of particles delimiting a vacuum state, and moving at a continuous velocity. This theoretically corresponds to a two-contact-discontinuity with a vacuum state. Note that in the above test cases and in the following, the displayed volume fraction is rescaled ($\times 100$).

In both cases the numerical results are in complete agreement with the theoretical analysis.

7.2. Impact of the source term

Recall that if $\mu = 0$, i.e. there is no source term then system (1.1) can be seen as the zero-pressure gas dynamics system whose Riemann problem was solved in [42]. We wish to highlight the impact of the zeroth order source term on the Riemann solution. We take $\mu = 4$. Numerical results without ($\mu = 0$) and with ($\mu = 4$) the source term, computed with the initial conditions (7.1) and (7.2), are displayed in Fig. 6 and Fig. 7, respectively. The solutions shown are obtained numerically, hence the delta-shocks can only have limited amplitude. The amplitude of the delta-shocks goes to infinity as the mesh is refined. We notice that the zeroth order source term has significant impact on the solution. In fact, it acts as a relaxation term by

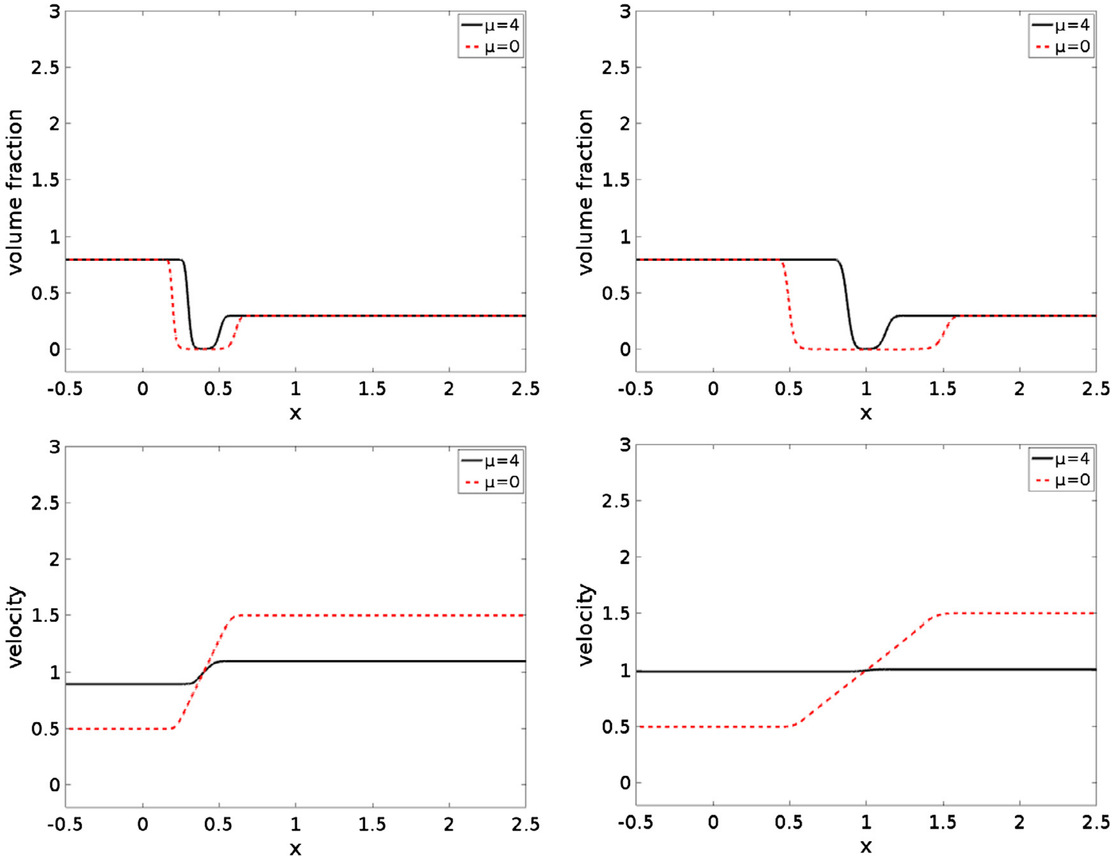

Fig. 7. Evolution of a vacuum state of system (1.1). Solutions with ($\mu = 4$)/without ($\mu = 0$) the source term. $u_a = 1$, $\Delta x = 10^{-3}$ and $\Delta t = 10^{-4}$.

weakening the delta-shock (seen as difference of amplitude in the numerical solution for the volume fraction) in Fig. 6, and reducing the extent of the vacuum region in Fig. 7. The left and right states of the velocity are no longer constant over time and tend to the air velocity which behaves as an equilibrium point.

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