



A question of Kühnau

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ABSTRACT

It is well known that the square is not a Strebel's point (i.e., its extremal Beltrami coefficient is not Teichmüller). Many years ago, Reiner Kühnau raised the question: does there exist in the case of a “long” rectangle the corresponding holomorphic quadratic differential?

We prove that the answer is negative for any bounded convex quadrilateral and establish for rectangles a stronger result.

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1. Extremal quasiconformality

The extremal quasiconformal maps $w(z)$ whose Beltrami coefficients $\mu_w(z) = \bar{\partial}w/\partial w$ minimize the dilatation $k(w) = \|\mu_w\|_\infty$ play a crucial role in geometric complex analysis and in the Teichmüller space theory.

Consider the disks $\mathbb{D} = \{z : |z| < 1\}$, $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : |z| > 1\}$, and let Σ^0 be the class of univalent functions $F^\mu(z) = z + b_0 + b_1 z^{-1} + \dots$ on \mathbb{D}^* with quasiconformal extension to $\widehat{\mathbb{C}}$, so their Beltrami coefficients range over the ball

$$\text{Bel}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu(z)|_{\mathbb{D}^*} = 0, \|\mu\|_\infty < 1\}.$$

Define in this ball the equivalence classes $[\mu]$ and $[F^\mu]$, letting μ_1, μ_2 be equivalent if the corresponding maps F^{μ_1} and F^{μ_2} coincide on the unit circle $S^1 = \partial\mathbb{D}$. These classes are in one-to-one correspondence with the Schwarzians S_{w^μ} on \mathbb{D}^* which fill a bounded domain in the space \mathbf{B} of hyperbolically bounded holomorphic functions on \mathbb{D}^* with norm $\|\varphi\| = \sup_{\mathbb{D}^*} (|z|^2 - 1)^2 |\varphi(z)|$; it models the *universal Teichmüller space* \mathbf{T} , and the quotient map $\phi_{\mathbf{T}} : \text{Bel}(\mathbb{D})_1 \rightarrow \mathbf{T}$, $\phi_{\mathbf{T}}(\mu) = S_{F^\mu}$ is a holomorphic split-submersion (has local sections).

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The intrinsic *Teichmüller metric* on \mathbf{T} is defined by

$$\tau_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi_{\mathbf{T}}(\nu)) = \frac{1}{2} \inf \{ \log K(w^{\mu*} \circ (w^{\nu*})^{-1}) : \mu_* \in \phi_{\mathbf{T}}(\mu), \nu_* \in \phi_{\mathbf{T}}(\nu) \},$$

where $K(w) = (1 + k(w))/(1 - k(w))$. It is the integral form of the infinitesimal Finsler metric

$$F_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi'_{\mathbf{T}}(\mu)\nu) = \inf \{ \|\nu_*/(1 - |\mu|^2)\|_{\infty} : \phi'_{\mathbf{T}}(\mu)\nu_* = \phi'_{\mathbf{T}}(\mu)\nu \}$$

on the tangent bundle $\mathcal{T}\mathbf{T}$ of \mathbf{T} .

The Beltrami coefficient $\mu \in \text{Bel}(\mathbb{D})_1$ is *extremal* in its class if $\|\mu\|_{\infty} = \inf \{ \|\nu\|_{\infty} : \phi_{\mathbf{T}}(\nu) = \phi_{\mathbf{T}}(\mu) \}$ and *infinitesimally extremal* if

$$\|\mu\|_{\infty} = \inf \{ \|\nu\|_{\infty} : \nu \in L_{\infty}(\mathbb{D}^*), \phi'_{\mathbf{T}}(\mathbf{0})\nu = \phi'_{\mathbf{T}}(\mathbf{0})\mu \}.$$

Any infinitesimally extremal Beltrami coefficient μ is globally extremal (and vice versa), and by the basic Hamilton–Krushkal–Reich–Strebel theorem either of these extremalities is equivalent to the equality

$$\|\mu\|_{\infty} = \inf \{ |\langle \mu, \psi \rangle_{\mathbb{D}}| : \psi \in A_1(\mathbb{D}) : \|\psi\| = 1 \}, \quad (1)$$

where $A_1(\mathbb{D})$ is the subspace of $L_1(\mathbb{D})$ formed by holomorphic quadratic differentials $\psi = \psi(z)dz^2$ on \mathbb{D} and the pairing

$$\langle \mu, \psi \rangle_{\mathbb{D}} = \iint_{\mathbb{D}} \mu(z)\psi(z)dxdy, \quad \mu \in L_{\infty}(\mathbb{D}), \psi \in L_1(\mathbb{D}), \quad z = x + iy.$$

Let $F_0 := F^{\mu_0}$ be an extremal representative of its class $[F_0]$ with dilatation

$$k(F_0) = \|\mu_0\|_{\infty} = \inf \{ k(F^{\mu}) : F^{\mu}|S^1 = F_0|S^1 \},$$

and assume that there exists in this class a quasiconformal map w_1 whose Beltrami coefficient μ_1 satisfies the inequality $\text{ess sup}_{V_r} |\mu_1(z)| < k(F_0)$ in some annulus $V_r = \{r < |z| < 1\}$, $r > 0$. Any such w_1 is called the *frame map* for the class $[w_0]$, and the point of the space \mathbf{T} corresponding to the class $[F_0]$ is called the *Strebel point*.

Due to Strebel [15] (see also [3]), in any class $[w]$ having a frame map, the extremal map w_0 is unique and either conformal or a Teichmüller map with Beltrami coefficient $\mu_0 = k|\psi_0|/\psi_0$ on \mathbb{D} , defined by an integrable holomorphic quadratic differential ψ_0 on \mathbb{D} and a constant $k \in (0, 1)$. This is valid, for example, for domains $w(\mathbb{D})$ bounded by asymptotically conformal curves (such curves do not have angular points).

The theorem of Lakic [13], [4] yields that the set of Strebel's points is open and dense in \mathbf{T} . Similar results have been established also for arbitrary Riemann surfaces.

One can define the boundary dilatation $b_x(w)$ of the maps w taking the Beltrami coefficients supported in the neighborhoods of boundary points $x \in S^1$.

The points with $b_x(w) = k(w)$ are called *substantial* for w and for its equivalence class (see, e.g., [4, Ch. 17]).

2. Kühnau's question. Results

It was established in [12], [16] that in the case of the square P_4^0 there are different extremal quasiconformal reflections across its boundary (an orientation reversing quasiconformal automorphisms of the sphere $\widehat{\mathbb{C}}$

preserving pointwise the boundary ∂P_4^0), all not of Teichmüller type. Hence the corresponding class $[\mu]$ defining the maps with $w^\mu(\mathbb{D}) = P_4^0$ is not a Strebel point.

In this connection, Reiner Kühnau raised many years ago (personal communication) the following

Question. *Does there exist in the case of a “long” rectangle the corresponding holomorphic quadratic differential?*

We show that the answer to this question is negative, proving the following stronger results:

Theorem 1. *Every bounded convex quadrilateral P_4 has a non-Strebel representation in \mathbf{T} (hence, its outer conformal map $\mathbb{D}^* \rightarrow P_4^*$ does not admit an extremal extension of Teichmüller type).*

For the rectangles, we have more:

Theorem 2. *There exists a common substantial point $z_0 \in S^1$ at which the extremal quasiconformal dilatation $k(F^*)$ of any rectangle is attained.*

The arguments applied in the proof of these theorems extend straightforwardly to arbitrary bounded convex polygons having a common outwardly tangent ellipse, in particular to polygons obtained by affine transformations of regular polygons. So all such polygons have substantial boundary points and thus are not Strebel.

We present briefly the needed notions and results underlying the above theorems adapting those to our case; for details see, e.g. [1], [2], [4], [9], [12].

3. A glimpse at Grunski inequalities

The proof of the theorem essentially relies on a basic result for the Grunski inequalities. We present briefly needed facts from this theory.

The fundamental Grunsky theorem (extended to multiply connected domains by Milin) states that a holomorphic function $F(z) = z + \text{const} + O(1/z)$ in a neighborhood U_0 of the infinite point is extended to a univalent function on the disk \mathbb{D}^* if and only if it satisfies the inequality

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq 1,$$

where the Grunsky coefficients $\alpha_{mn}(F)$ are determined by

$$\log \frac{F(z) - F(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\mathbb{D}^*)^2,$$

taking the principal branch of the logarithmic function, and $\mathbf{x} = (x_n)$ ranges over the unit sphere $S(l^2)$ of the Hilbert space l^2 of sequences with $\|\mathbf{x}\|^2 = \sum_1^\infty |x_n|^2$ (cf. [5]). The quantity

$$\varkappa(F) = \sup \left\{ \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| : \mathbf{x} = (x_n) \in S(l^2) \right\}$$

is called the *Grunsky norm* of the map F .

This norm is dominated by the *Teichmüller norm* $k(F)$ of this map, i.e., with the minimal dilatation among quasiconformal extensions of F onto \mathbb{D} (see [11], [10]); so,

$$\varkappa(F) \leq k(F) = \tanh \tau_{\mathbf{T}}(\mathbf{0}, S_F), \quad (2)$$

where $\tau_{\mathbf{T}}$ denotes the Teichmüller distance on \mathbf{T} . The second norm is intrinsically connected with integrable holomorphic quadratic differentials on \mathbb{D} (the elements of the subspace $A_1 = A_1(\mathbb{D})$ of $L_1(\mathbb{D})$ formed by holomorphic functions), while the Grunsky norm naturally relates to the *abelian* structure determined by the set of quadratic differentials

$$A_1^2 = \{\psi \in A_1 : \psi = \omega^2\}$$

having only zeros of even order on \mathbb{D} . It is shown in [6] that such differentials have the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn} x_m x_n z^{m+n-2} \quad (3)$$

with $\mathbf{x} = (x_n) \in l^2$, and $\|\mathbf{x}\|_{l^2} = \|\psi\|_{A_1}$.

A crucial point here is that for a generic function $F \in \Sigma^0$ in (2) the strict inequality $\varkappa(F) < k(F)$ is valid; moreover, it holds on the (open) dense subset of Σ^0 in both strong and weak topologies (i.e., in the Teichmüller distance and in locally uniform convergence on \mathbb{D}^*) (see, e.g., [6], [9], [12]).

So it is important to know whether for a concrete function F , we have $\varkappa(F) = k(F)$. In terms of the pairing

$$\langle \mu, \psi \rangle_{\mathbb{D}} = \iint_{\mathbb{D}} \mu(z) \psi(z) dx dy, \quad \mu \in L_{\infty}(\mathbb{D}), \quad \psi \in L_1(\mathbb{D}) \quad (z = x + iy),$$

such functions are completely characterized by the following

Lemma 1. [6], [10] For all $F \in \Sigma^0$,

$$\varkappa(F) \leq k \frac{k + \alpha(F)}{1 + \alpha(F)k}, \quad k = k(F),$$

and $\varkappa(F) < k$ unless

$$\alpha(F) := \sup \{ |\langle \mu, \psi \rangle_{\mathbb{D}}| : \psi \in A_1^2, \|\psi\|_{A_1(\mathbb{D})} = 1 \} = \|\mu\|_{\infty}; \quad (4)$$

the last equality is equivalent to $\varkappa(F) = k(F)$. Moreover, for small $\|\mu\|_{\infty}$,

$$\varkappa(F) = \sup |\langle \mu, \psi \rangle_{\mathbb{D}}| + O(\|\mu\|_{\infty}^2), \quad \|\mu\|_{\infty} \rightarrow 0,$$

with the same supremum as in (4).

If $\varkappa(F) = k(F)$ and the equivalence class of F (the collection of maps equal to F on $S^1 = \partial D^*$) is a Strebel point, then the extremal μ_0 in this class is necessarily of the form

$$\mu_0 = \|\mu_0\|_{\infty} |\psi_0| / \psi_0 \quad \text{with} \quad \psi_0 \in A_1^2. \quad (5)$$

Geometrically, (4) means the equality of the Carathéodory and Teichmüller distances on the image of the geodesic disk $\mathbb{D}(\mu_0) = \{t\mu_0/\|\mu_0\|_\infty : t \in \mathbb{D}\}$ in the space \mathbf{T} . For functions $F \in \Sigma^0$ holomorphic in the closed disk $\overline{\mathbb{D}}^*$, the relation (5) was also obtained by a different method in [12].

Note also that the Grunsky coefficients $\alpha_{mn}(F)$ are holomorphic functions of the Schwarzians $\varphi = S_F$ on the universal Teichmüller space \mathbf{T} , and for each $\mathbf{x} = (x_n) \in S(l^2)$ the series

$$h_{\mathbf{x}}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(\varphi) x_m x_n \quad (6)$$

defines a holomorphic map of the space \mathbf{T} into the unit disk \mathbb{D} so that $\varkappa(F) = \sup_{\mathbf{x}} |h_{\mathbf{x}}(S_F)|$.

4. Proof of Theorem 2

We first prove the second theorem answering directly Kühnau's question and illustrating the main features.

Let P_4^0 be the unit square centered at the origin, and let $\mu_0 \in \text{Bel}(\mathbb{D})_1$ be an extremal Beltrami coefficient for P_4^0 , i.e., for the extremal extension of the outer conformal map $F_{P_0^*} : \mathbb{D}^* \rightarrow P_0^*$.

As was mentioned above, this μ_0 is not Teichmüller; hence, there exist a point z_0 on the unit circle S^1 and a degenerating sequence $\{\psi_n\} \subset A_1(\mathbb{D})$ such that $\psi_n(z) \rightarrow 0$ locally uniformly on \mathbb{D} , $\|\psi_n\|_{A_1(\mathbb{D})} = 1$, and the boundary dilatation at z_0 satisfies

$$b_{z_0}(F^{\mu_0}) = k(w^{\mu_0}) = \lim_{p \rightarrow \infty} |< \mu_0, \psi_p >_{\mathbb{D}}|. \quad (7)$$

On the other hand, since for the square (as well as for any rectangle) its outer conformal mapping function $F^* = F_{P_0^*}$ has equal Grunsky and Teichmüller norms, Lemma 1 and (3) yield that all ψ_p in (7) belong to $A_1^2(\mathbb{D})$ and hence,

$$\psi_p(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn} x_m^p x_n^p z^{m+n-2}, \quad (8)$$

where $\mathbf{x}^p = (x_1^p, \dots, x_n^p, \dots) \in S(l^2)$. So from (7) and (8), we have

$$b_{z_0}(F^*) = \varkappa(F^*) = k(F^*) = \lim_{p \rightarrow \infty} h_{\mathbf{x}^p}(S_{F^*}). \quad (9)$$

Note also that, by definition of the Grunsky norm, each $h_{\mathbf{x}^p}$ satisfies

$$|h_{\mathbf{x}^p}(S_{F^*})| \leq \varkappa(F^*) = k(F^*), \quad p = 1, 2, \dots \quad (10)$$

We shall need also the differential version of the relations (9). Using the variation of $F^\mu(z) = z + b_0 + b_1 z^{-1} + \dots \in \Sigma^0$ with small $\|\mu\|_\infty$ given by

$$F^\mu(z) = z - \frac{1}{\pi} \iint_{\mathbb{D}} \frac{\mu(w) du dv}{w - z} + O(\|\mu^2\|_\infty), \quad w = u + iv,$$

one gets

$$b_n = \frac{1}{\pi} \iint_{\mathbb{D}} \mu(w) w^{n-1} du dv + O(\|\mu^2\|_\infty), \quad n = 1, 2, \dots,$$

and

$$\alpha_{mn}(\mu) = -\pi^{-1} \iint_{\mathbb{D}} \mu(z) z^{m+n-2} dx dy + O(\|\mu\|_{\infty}^2).$$

Combining this with (3) and (6), one derives that the differential at zero of the corresponding map $h_{\mathbf{x}}(t\mu)$ in the direction determined by μ equals

$$dh_{\mathbf{x}}(0)\mu = -\frac{1}{\pi} \iint_{\mathbb{D}} \mu(z) \sum_{m+n=2}^{\infty} \sqrt{mn} x_m x_n z^{m+n-2} dx dy = -\langle \mu, \psi \rangle_{\mathbb{D}}. \quad (11)$$

One can assume that the vertices of a rectangle P_4 are the points $(\pm 1, 0), (\pm 1, \pm 1 + ia)$, $a > 0$, so each P_4 is obtained from the square P_4^0 by an affine transform

$$g^t(z) = t_1 z + t_2 \bar{z}$$

with real t_1, t_2 , and use the parameter $t = t_2/t_1 \in (-1, 1)$ measuring the affinity. The map $\mathbf{b}(t) = S_{F_t^*} : (-1, 1) \rightarrow \mathbf{T}$, where F_t^* is the conformal map $\mathbb{D}^* \rightarrow P_4 = g^t(P_4^0)$, defines a curve Γ in \mathbf{T} whose points represent all rectangles P_4 .

Now, using the chain rule for Beltrami coefficients μ, ν from the unit ball in $L_{\infty}(\mathbb{C})$,

$$w^{\mu} \circ w^{\nu} = w^{\sigma_{\nu}(\mu)} \quad \text{with} \quad \sigma_{\nu}(\mu) = (\nu + \mu_1)/(1 + \bar{\nu}\mu_1)$$

and $\mu_1(z) = \mu(w^{\nu}(z))\overline{w_z^{\nu}}/w_z^{\nu}$ (so for ν fixed, $\sigma_{\nu}(\mu)$ depends holomorphically on μ in L_{∞} norm), we consider the composite maps

$$W_t = g^t \circ w^{\mu_0} \quad (12)$$

with complex $t \in \mathbb{D}$. Their Schwarzians S_{W_t} fill in the space \mathbf{T} a holomorphically embedded non-geodesic disk $\Omega_0 = \mathbf{b}(G_0)$ which is the image of a simply connected domain $G_0 \subset \mathbb{C}$ containing the interval $(-1, 1)$ corresponding to rectangles.

We now take the restrictions $\hat{h}_p := h_{\mathbf{x}^p}(S_{F^{\sigma_{\mu_0}(\mu)}})$ of functions $h_{\mathbf{x}^p}$ to the disk Ω_0 and apply these restrictions to pull back the hyperbolic metric $ds = |d\zeta|/(1 - |\zeta|^2)$ of \mathbb{D} . This provides on the disk Ω_0 the conformal metrics $ds = \lambda_{\hat{h}_p}(t)|dt|$ with

$$\lambda_{\hat{h}_p}(t) = \hat{h}_p^* \lambda_{\mathbb{D}} = \frac{|\hat{h}_p'(t)||dt|}{1 - |\hat{h}_p(t)|^2} \quad (13)$$

of Gaussian curvature -4 at noncritical points. Consider their upper envelope

$$\lambda_{\infty}(t) = \sup \lambda_{\hat{h}_p}(t),$$

taking the supremum over all $p = 1, 2, \dots$ and all $\mu = \mu(t) \in \text{Bel}(\mathbb{D})_1$ with $\phi_{\mathbf{T}}(\mu) = S_{F^{\sigma_{\mu_0}(\mu)}} \in \Omega_0$, and pick its upper semicontinuous regularization

$$\hat{\lambda}_{\infty}(t) = \limsup_{t' \rightarrow t} \hat{\lambda}_{\infty}(t').$$

The standard arguments (cf. e.g., [1], [8]) imply that $\hat{\lambda}_{\infty}(t)$ is a logarithmically subharmonic metric on Ω_0 of the generalized Gaussian curvature at most -4 , which means that

$$\Delta \log \widehat{\lambda}_\infty \geq 4\widehat{\lambda}_\infty^2,$$

where Δ denotes the *generalized* Laplacian

$$\Delta u(\zeta) = 4 \liminf_{r \rightarrow 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(\zeta + re^{i\theta}) d\theta - \lambda(\zeta) \right\}$$

(for $u \in C^2$, it coincides with the usual Laplacian).

Arguing similarly with all functions (6) and taking supremum over all $\mathbf{x} \in S(l^2)$ and $\mu \in \text{Bel}(\mathbb{D})_1$, one obtains on Ω_0 the canonical plurisubharmonic Finsler metric λ_\varkappa generated by the Grunsky structure on the space \mathbf{T} , also of curvature $\kappa(\lambda_\varkappa) \leq -4$. Both metrics λ_∞ and λ_\varkappa are dominated by the infinitesimal Kobayashi metric λ_K of the space \mathbf{T} (equal to its Teichmüller metric by the Royden–Gardiner theorem) via

$$\lambda_\infty \leq \lambda_\varkappa \leq \lambda_K.$$

It is shown in [8] that for any rectangle P_4 its outer conformal map F^* has equal Grunsky and Teichmüller norms, i.e., $\varkappa(F^*) = k(F^*)$, and this implies the equality of metrics λ_\varkappa and λ_K , on the whole disk Ω_0 .

The above construction yields (see the relations (9)–(11)) that the minimal metric λ_∞ coincides with its dominants λ_\varkappa and λ_K in the base point $t = 0$ representing the square P_4^0 :

$$\lambda_\infty(0) = \lambda_\varkappa(0) = \lambda_K(0). \quad (14)$$

To compare λ_∞ with either of two other metrics on the whole disk Ω_0 , we apply the following basic facts.

Lemma 2. [7] *The infinitesimal form λ_K of the Kobayashi–Teichmüller metric on the tangent bundle $\mathcal{T}(\mathbf{T})$ of \mathbf{T} is continuous, logarithmically plurisubharmonic in $\varphi \in \mathbf{T}$ and has constant holomorphic sectional curvature $\kappa_K(\varphi, v) = -4$ (hence $\Delta \log \lambda_K = 4\lambda_K^2$).*

The global Kobayashi and Teichmüller distances are logarithmically plurisubharmonic in each of their variables on $\mathbf{T} \times \mathbf{T}$ (cf. [7]).

Lemma 3 (Minda’s maximum principle [14]). *If a function $u : D \rightarrow [-\infty, +\infty)$ is upper semicontinuous in a domain $D \subset \mathbb{C}$ and its generalized Laplacian satisfies the inequality $\Delta u(z) \geq Ku(z)$ with some positive constant K at any point $z \in D$, where $u(z) > -\infty$, and if*

$$\limsup_{z \rightarrow \zeta} u(z) \leq 0 \quad \text{for all } \zeta \in \partial D,$$

then either $u(z) < 0$ for all $z \in D$ or else $u(z) = 0$ for all $z \in \Omega$.

Take a sufficiently small neighborhood U_0 of the point $t = 0$, and let

$$M = \{\sup \lambda_K(t) : t \in U_0\};$$

then in this neighborhood, $\lambda_K(t) + \lambda_\infty(t) \leq 2M$ and the function

$$u = \log \frac{\lambda_\infty}{\lambda_K} = \log \lambda_\infty - \log \lambda_K$$

satisfies

$$\Delta u = 4\lambda_\infty^2 - 4\lambda_\kappa^2 \geq 8M(\lambda_\infty - \lambda_\kappa).$$

The elementary estimate

$$M \log(t/s) \geq t - s \quad \text{for } 0 < s \leq t < M$$

(with equality only for $t = s$) implies that

$$M \log \frac{\lambda_\infty(t)}{\lambda_\kappa(t)} \geq \lambda_\infty(t) - \lambda_\kappa(t),$$

and hence,

$$\Delta u(t) \geq 4M^2 u(t).$$

The equality (14) for the square P_0 implies that all metrics λ_∞ , λ_κ , λ_κ must be equal in the entire disk Ω_0 (hence, are continuous on this disk).

Now observe that, due to a well-known basic result of potential theory, a subharmonic function in a domain $\Omega \subset \mathbb{R}^n$ can be different (smaller) than its upper semicontinuous envelope only on a subset $E \subset \Omega$ of the capacity zero, hence also of the Lebesgue n -measure zero.

Applying this to our metric (13), one derives from the above construction and relations (1), (9), (11) that omitting a nowhere dense subset $E \subset \Omega_0$, the extremal Beltrami coefficients $\mu_0(\cdot, t)$ corresponding to $t \in \Omega_0 \setminus E$ satisfy the equality

$$\|\mu_0(\cdot, t)\|_\infty = \sup_p |\langle \mu_0(\cdot, t), \psi_p \rangle_{\mathbb{D}}|. \quad (15)$$

Such an equality can hold only when $\mu_0(\cdot, t)$ are not of Teichmüller type (since the Teichmüller extremal coefficients do not have the substantial boundary points and cannot attain their dilatation on the degenerated sequences).

The remaining rectangles $P(t)$ with $t \in E$ also cannot be Strebel points, since by the Lakic theorem the set of such points is open in the space \mathbf{T} .

It remains to establish the last conclusion of Theorem 2 on existence of a common substantial point $z_0 \in \partial\mathbb{D}$ for all rectangles $P(t)$, $t \in \Omega_0$.

Let $\mu_t = \mu_0(\cdot, t)$ and consider the maps $F^{s\mu_t}$ with sufficiently small $|s|$ so that $\|S_{F^{s\mu_t}}\|_{\mathbf{B}} < 2$. Then

$$s \langle \mu_t, \psi_p \rangle_{\mathbb{D}} = \langle \nu_{F^{s\mu_t}}, \psi_p \rangle_{\mathbb{D}}, \quad (16)$$

where

$$\nu_\varphi(z) = \frac{1}{2}(1 - |z|^2)^2 \varphi(1/\bar{z}) 1/\bar{z}^4$$

means the harmonic Beltrami coefficient of the Ahlfors–Weill extension of the map $F^\mu \in \Sigma^0$ whose Schwarzian $S_{F^\mu} = \varphi$, provided that $\|\varphi\|_{\mathbf{B}} < 2$. These coefficients are connected with the initial extremal coefficients μ_t via

$$\nu_{F^{s\mu_t}} = s\mu_t + \sigma(s, t),$$

where $\sigma(s, t)$ belong to the annihilating set for $A_1(\mathbb{D})$, i.e., $\langle \sigma(s, t), \psi \rangle_{\mathbb{D}} = 0$ for all $\psi \in A_1(\mathbb{D})$ (this set coincides with $\ker \phi_{\mathbf{T}}'(\mathbf{0})$).

Noting that the harmonic coefficients ν_φ depend holomorphically on $\varphi \in \mathbf{T}$ and the metric λ_K is continuous on the space \mathbf{T} , one extends by applying (16) the initial equality (15) to $t \in \Omega_0 \setminus E$. This implies that for each $t \in \Omega_0$, the L_∞ -length $\|\mu_0(\cdot, t)\|_\infty$ is attained on a subsequence from the initial sequence $\{\psi_p\}$. This completes the proof of the theorem.

5. Proof of Theorem 1

The idea of the proof of this theorem (without assertion on a common substantial point) is similar to the case of rectangles taking into account that the basic property $\kappa(F^*) = k(F^*)$ holds for every bounded rectilinear convex quadrilateral P_4 .

We first establish the assertion of Theorem 1 for trapezoids. Given a trapezoid P_4 , one can inscribe an ellipse \mathcal{E} tangent to all sides of P_4 . Now take an affine map $g(w)$ moving \mathcal{E} into a circle. Then in view of affinity, the image $P_4^0 = g(P_4)$ is a rectilinear quadrilateral with a common tangent from the outward circle, and Werner's construction in [16] implies an extremal quasiconformal reflection across the boundary of P_4^0 with a variable dilatation function, thus not Teichmüller; equivalently, P_4^0 is not a Strebel point in \mathbf{T} .

The collection of all affine deformations of P_4^0 , by applying the composed maps (12), generates a holomorphic disk $\Omega \subset \mathbf{T}$ representing quadrilaterals. One can repeat for this disk all the above constructions from the proof of Theorem 2 giving the corresponding subharmonic infinitesimal metrics on Ω . After a similar comparison, one obtains that the original quadrilateral P_4 by itself cannot be obtained from \mathbb{D} by a Teichmüller extremal map.

We now pass to the generic quadrilaterals. It suffices to prove the theorem for bounded quadrilaterals $P_4 = A_1A_2A_3A_4$ (with vertices A_j ordered accordingly to positive direction of ∂P_4) which have a line L inside P_4 drawn from the vertex A_1 parallel to the opposite edge A_2A_3 such that L separates this edge from the remaining vertex A_4 .

Fix such a quadrilateral $P_4^0 = A_1^0A_2^0A_3^0A_4^0$ and consider the collection \mathcal{P}^0 of quadrilaterals $P_4 = A_1^0A_2^0A_3^0A_4$ with the same first three vertices and variable A_4 ; the corresponding A_4 runs over a subset E of the thrice punctured sphere $\widehat{\mathbb{C}} \setminus \{A_1^0, A_2^0, A_3^0\}$. One such quadrilateral is a trapezoid, which we denote by $P_4^* = A_1^0A_2^0A_3^0A_4^*$. We know that its extremal Beltrami coefficient μ_0 is not Teichmüller and that its outer conformal map has equal Grunsky and Teichmüller norms.

For any $P_4 \in \mathcal{P}^0$, the conformal map F of disk \mathbb{D}^* onto the complementary complement P_4^* is represented by the Schwarz–Christoffel integral

$$F(z) = d_1 \int_0^z \prod_{j=1}^4 (\zeta - e_j)^{\alpha_j - 1} \frac{d\zeta}{\zeta^2} + d_0, \quad (17)$$

where $e_j = F^{-1}(A_j) \in S^1$, $\pi\alpha_j$ is the interior angle at A_j for P_4^* , and d_0, d_1 are two complex constants. Let F^0 denote the conformal map for the complement of P_4^0 .

One obtains from the general properties of quasiconformal maps and (17) that the logarithmic derivatives $b_F = (\log F')' = F''/F'$ of maps F defining the quadrilaterals $P_4 \in \mathcal{P}^0$ are (for a fixed z) real analytic functions of $t = A_4$. Passing to their Schwarzians

$$S_F = b'_F - \frac{1}{2}b_F^2 \in \mathbf{T},$$

one can find a smooth real arc $\Gamma = \mathbf{b}(t) \subset \mathbf{T}$ containing the point S_{F^0} and the points corresponding to trapezoids; here \mathbf{b} denotes the map $t = A_4 \rightarrow S_F$.

Since \mathbf{T} is a domain, there is a tubular neighborhood containing Γ ; therefore, Γ is located on some nonsingular holomorphic disk of the form $\Omega = \mathbf{F}(G) \subset \mathbf{T}$, where G again is a simply connected planar

domain containing G . This disk is not geodesic in the Teichmüller–Kobayashi metric on \mathbf{T} and does not pass through the base point of this space, but one can apply to it the same arguments as in the proof of Theorem 2 constructing by (6) the holomorphic maps $h_{\mathbf{x}}$ for the Schwarzians of compositions $W_t = g^t \circ F^*$, where the initial map $W_0 = F^*$ is the outer conformal map of a trapezoid $P_4^* \in \mathcal{P}^0$ and g^t runs over this collection.

The restrictions of these maps to the disk Ω determine by pulling back the hyperbolic metric of the unit disk the corresponding conformal metrics of type (13), and one can apply to their upper semicontinuous envelope λ_∞ the same arguments as in the proof of Theorem 2, getting that for any $P_4 \in \mathcal{P}^0$ its extremal Beltrami coefficients is determined by a degenerating sequences and hence not Teichmüller. This completes the proof of the theorem.

References

- [1] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.
- [2] C.J. Earle, I. Kra, S.L. Krushkal, Holomorphic motions and Teichmüller spaces, *Trans. Amer. Math. Soc.* 944 (1994) 927–948.
- [3] C.J. Earle, Zong Li, Isometrically embedded polydisks in infinite dimensional Teichmüller spaces, *J. Geom. Anal.* 9 (1999) 51–71.
- [4] F.P. Gardiner, N. Lakic, *Quasiconformal Teichmüller Theory*, Amer. Math. Soc., Providence, RI, 2000.
- [5] H. Grunsky, Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen, *Math. Z.* 45 (1939) 29–61.
- [6] S.L. Krushkal, Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings, *Comment. Math. Helv.* 64 (1989) 650–660.
- [7] S.L. Krushkal, Plurisubharmonic features of the Teichmüller metric, *Publ. Inst. Math. (Beograd) (N.S.)* 75 (89) (2004) 119–138.
- [8] S.L. Krushkal, Quasireflections, Fredholm eigenvalues and Finsler metrics, *Dokl. Math.* 69 (2004) 221–224.
- [9] S.L. Krushkal, Strengthened Moser’s conjecture, geometry of Grunsky coefficients and Fredholm eigenvalues, *Cent. Eur. J. Math.* 5 (3) (2007) 551–580.
- [10] S.L. Krushkal, Strengthened Grunsky and Milin inequalities, *Contemp. Math.* 667 (2016) 159–179.
- [11] R. Kühnau, Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q -quasikonforme Fortsetzbarkeit?, *Comment. Math. Helv.* 61 (1986) 290–307.
- [12] R. Kühnau, Möglichst konforme Spiegelung an einer Jordankurve, *Jahresber. Dtsch. Math.-Ver.* 90 (1988) 90–109.
- [13] N. Lakic, Strebel Points, Lipa’s Legacy, *Contemporary Mathematics*, vol. 211, Amer. Math. Soc., Providence, RI, 2001, pp. 417–431.
- [14] D. Minda, The strong form of Ahlfors’ lemma, *Rocky Mountain J. Math.* 17 (1987) 457–461.
- [15] K. Strebel, On the existence of extremal Teichmueller mappings, *J. Anal. Math.* 30 (1976) 464–480.
- [16] S. Werner, Spiegelungskoeffizient und Fredholmscher Eigenwert für gewisse Polygone, *Ann. Acad. Sci. Fenn. Ser. AI. Math.* 22 (1997) 165–186.