J. Math. Anal. Appl. 474 (2019) 23–58

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Dominated and pointwise ergodic theorems with “weighted” averages for bounded Lamperti representations of amenable groups

Arkady Tempelman [a,b,*], Alexander Shulman [c,1]

[a] Department of Mathematics, The Pennsylvania State University, USA
[b] Department of Statistics, The Pennsylvania State University, USA
[c] Wells Fargo & Company, USA

A R T I C L E I N F O

Article history:
Received 9 June 2018
Available online 11 January 2019
Submitted by U. Stadtmueller

Keywords:
Ergodic theorems
Group actions
Group representations

A B S T R A C T

Group representations by bounded Lamperti operators in the spaces $L^\alpha(1 \leq \alpha < \infty)$ form a wide class of representations, including representations by bounded positive operators and (when $\alpha \neq 2$) representations by isometric operators. The Dominated and the Pointwise Ergodic Theorems (DET and PET) for Cesàro averages for the bounded Lamperti representations of amenable σ-compact locally compact groups in $L^\alpha(1 < \alpha < \infty)$ were proved by A. Tempelman in Proc. Amer. Math. Soc. 143 (2015) 4989–5004. By using a completely different, functional-analytical method, developed by A. Shulman in his PhD thesis in 1988, we generalize this result to “weighted” averages of such representations and discuss various conditions on the “weights” under which the DET and the PET hold. We conclude with applications of the general results to the bounded Lamperti representations of groups of polynomial growth and of the groups $\mathbb{R}^m$ and $\mathbb{Z}^m$.

1. Introduction

1.1. Introductory remarks

This paper is devoted to the study of the dominated and the pointwise ergodic theorems (DETs and PETs) for the bounded Lamperti L^α-representations of amenable σ-compact locally compact topological groups (in short, we call them “amenable groups”). In this Subsection, we provide a short review of the DETs and the PETs for the representations and the actions of such groups. For a locally compact group, μ and ν denote its left and right Haar measures. Let $(\Omega, \mathcal{F}, m)$ be a σ-finite measure space. For

* Corresponding author at: 219 Oakley Dr., State College, PA 16803, USA.
E-mail address: axt12@psu.edu (A. Tempelman).
[1] The views expressed in the article are those of the author and don’t necessarily reflect the position of the Wells Fargo & Company.

https://doi.org/10.1016/j.jmaa.2019.01.013

the wide class of bounded Lamperti operator representations $T : x \mapsto T_x$ of X in $L^\alpha(\Omega, \mathcal{F}, m)$, and for all $f \in L^\alpha(\Omega, \mathcal{F}, m)$ with arbitrary fixed $\alpha > 1$, we find conditions on a sequence $\{\kappa_n\}$ of probability measures ("weights") on X which guarantee the validity of the dominated inequalities involving the majorants $\hat{f}(\omega) = \sup_{n\geq 1} |\int_X T_x f(\omega)\kappa_n(dx)|$, as well as conditions for m-almost everywhere convergence, as $n \to \infty$, of averages $\tilde{f}(\omega) := \int_X T_x f(\omega)\kappa_n(dx)$ to a T-invariant function $\hat{f} \in L^\alpha(\Omega, \mathcal{F}, m)$.

When κ_n are uniform distributions on measurable sets A_n of finite μ-measure, our averages become the Cesàro type averages: $\tilde{f}_n(\omega) = \frac{1}{\mu(A_n)} \int_{A_n} T_x f(\omega)\mu(dx)$, and the conditions of convergence of these averages are stated in terms of the sequence $\{A_n\}$ (see also [47]).

An important class of the bounded Lamperti representations consists of the bounded representations by positive operators (including those induced by measure preserving actions τ_x of X on $(\Omega, \mathcal{F}, m)$: the operators T_x are defined as $T_x f(\omega) := f(\tau_x\omega))$ – see Proposition 3.1 in [28]. Also, when $\alpha \neq 2$, the representations by isometric operators in L^α are Lamperti (see [31]).

First we turn to the ergodic theorems for the representations by measure preserving transformations. Wiener [50] was the first to generalize the Birkhoff–Khinchin PET for the Cesàro averages over increasing balls in the groups $\mathbb{R}^m, m \geq 2$; he also proved the DET for these averages; this result was generalized to a more general class of Cesàro averages by Pitt [40]. Calderon [10] extended the DET and the PET for the measure preserving actions to a wide class of "ergodic" groups for averages over some subsequence of each family $\{A_t, t > 0\}$ of symmetric compact neighborhoods of the identity satisfying the conditions $A_t A_s \subset A_{t+s}$ for all $s, t > 0$ and $\sup_{t>0} \frac{\mu(A_{2t})}{\mu(A_t)} < \infty$; further generalizations were given by Cotlar [14]. Greenleaf and Emerson [23] proved the DET and the PET for the Cesàro averages over sequences of "semidirect rectangles" in connected solvable groups. Tempelman [44,45] and Emerson [20] proved the PET for functions $f \in L^\alpha(\Omega, \mathcal{F}, m), \alpha \geq 1$, with averaging over increasing sequences of sets satisfying the "regularity" condition

$$\sup_n \frac{\mu(A_n^{-1} \cdot A_n)}{\mu(A_n)} < \infty \tag{1.1}$$

and the Følner condition

$$\lim_{n\to\infty} \frac{\mu(xA_n \triangle A_n)}{A_n} = 0, \quad x \in X; \tag{1.2}$$

see also [46] and other references therein. In [41] Shulman replaced the monotonicity condition and condition (1.1) by the condition

$$\sup_n \frac{\mu(\tilde{A}_n^{-1} \cdot A_n)}{\mu(A_n)} < \infty \tag{1.3}$$

where $\tilde{A}_n = \cup_{k=1}^n A_k$. In [48] Tempelman and Shulman extended the results of [44], [45] to some classes of non-amenable groups and symmetric homogeneous spaces (see Example 9.3, Ch. 6 in [46] for special cases).

It is well known that Følner sequences exist in every amenable σ-compact locally compact group X [22]. Tessera [49] and Breuillard [9] have constructed regular Følner sequences in the locally compact groups of polynomial growth. But there are solvable groups that do not possess regular sequences (see [33,34]). On the other hand, even in non-amenable groups there are always sequences of probability measures $\{\kappa_n\}$ such that the averages $\int_X T_x f(\omega)\kappa_n(dx)$ converge m-a.e. to an invariant mean for any T_x induced by a measure preserving action on a σ-finite measure space $(\Omega, \mathcal{F}, m)$ and $f \in L^\alpha(\Omega, \mathcal{F}, m),\ 1 \leq \alpha < \infty$ (see [46], Theorem 6.6.1).

In 1988 Shulman [41,42] extended the mentioned results due to Emerson and Tempelman to general σ-compact amenable locally compact groups and to functions $f \in L^\alpha(\Omega, \mathcal{F}, m)(1 < \alpha < \infty)$ by replacing the regularity property (1.3) by the weaker condition

$$\sup_n \frac{\mu(\tilde{A}_{n-1}^{-1} \cdot A_n)}{\mu(A_n)} < \infty. \tag{1.4}$$

assuming that $f \in L^\alpha(\Omega, \mathcal{F}, m), 1 < \alpha < \infty$ (see also [46]). Sequences $\{A_n\}$ satisfying (1.4) were later called "Shulman sequences" and "tempered sequences" by Lindenstrauss in [33] and [34], resp.; we also use the latter term. In fact, Shulman proved the Maximal Inequality and the PET for a more general class of sequences: "weakly tempered" sequences of sets defined in §2.

In 1999, Lindenstrauss [33] has proved that tempered Følner sequences of sets exist in all countable amenable groups and extended Shulman's PET for Cesàro averages to all functions $f \in L^1(\Omega, \mathcal{F}, m)$ by using a totally different approach; in [34] he extended these results to arbitrary σ-compact locally compact amenable groups.

The mentioned results are discussed in an excellent review by Nevo [39].

In this paper we generalize the results due to Shulman [41] in two directions: we prove the DET and the PET with averaging by ergodic weakly tempered sequences of probability measures as "weights" for the bounded Lamperti representations in $L^\alpha(\Omega, \mathcal{F}, m)$ when α is fixed and $1 < \alpha < \infty$. It is known (see Chacon [11] and Ionescu Tulcea [26]) that the PET may fail for some positive invertible isometries in $L^1(\Omega)$; therefore our restriction on α is quite natural. Also, as demonstrated by counterexamples due to Feder [21], Assani [4,5], and Cohen [12] it is quite natural to focus our study on the *Lamperti* representations.

Let us mention the main results related to DETs and PETs for operators and representations by operators in the spaces L^α.

In 1958, Dunford and Schwartz proved the PET for $L^1 - L^\infty$ contractions (see [18], Ch. VIII). The first PET for operators acting in one space $L^\alpha(1 < \alpha < \infty)$ was obtained by Ionescu Tulcea [25] who proved it for the positive invertible isometries. This was extended by Akcoglu [1] to the positive contractions in $L^\alpha(1 < \alpha < \infty)$; Akcoglu's result stimulated the study of the pointwise ergodic theorems for the power-bounded positive operators in L^α and for the bounded representations by positive operators in L^α. The multi-parameter version of Akcoglu's theorem was obtained by McGrath [38]. Martin-Reyes and de la Torre [36,37] proved the DET and the PET for the powers of invertible positive operators with uniformly bounded averages. Duncan [17] discovered a link between the maximal inequality for a sequence of uniformly bounded operators in $L^\alpha, \alpha > 1$, and a simpler inequality for the sequence of the conjugate operators; using this result he proved the pointwise convergence for sequences of a fairly large class of operators. Kan [29] proved the DET and the PET for the powers of bounded operators (not necessarily invertible) with Lamperti conjugates (all invertible bounded Lamperti operators possess this property). Unfortunately, it is still unknown whether the DET and the PET are valid for arbitrary power-bounded positive operators (see [35]). Asmar, Berkson and Gillespie [2,3] have proved the DET and the PET for Lamperti representations with convolution powers of a probability measure as "weights" on the commutative groups. The first DET and PET for the Lamperti representations of general amenable groups in $L^\alpha(\Omega, \mathcal{F}, m)$, $\alpha > 1$, are due to Lin and Wittmann [32] who considered the convolution powers of a symmetric probability measure as "weights". In [13] Cohen, Cuny and Lin proved the DET and the PET for Lamperti representations with convolution powers for a wide class of the non-symmetric probability measures; see also [27] for the case of representations by measure preserving transformations.

The DET and the PET for Cesàro averages over tempered sequences of sets for bounded Lamperti representations of amenable groups in $L^\alpha(1 < \alpha < \infty)$ were proved by Tempelman [47].

In this paper, by using a completely different method, we generalize the results of [47] to "weighted" averages of the bounded Lamperti representations. We discuss various conditions on the "weights" under which these theorems hold.

In the rest of this section we remind the reader the definitions and properties of some notions considered in this paper. In §2, we introduce the notions of regular, tempered, and ergodic sequences of "weights". In §3 and §4, we prove the DET and the PET with such "weights" for all bounded Lamperti representations of amenable groups in $L^\alpha, \alpha > 1$. Some applications to bounded Lamperti representations of groups of polynomial growth and of groups $\mathbb{R}^m$ and $\mathbb{Z}^m$ are considered in §5. In §6, the Appendix, we discuss some properties of Lamperti representations that are used in the main part of the paper; also, we include counterexamples emphasizing the difference between the notions introduced in §2.

1.2. Notation and abbreviations

We use the following notation: X is an amenable non-compact σ-compact locally compact group; $\mathcal{B}$ is the Borel σ-algebra on X; μ is the left Haar measure on $\mathcal{B}$; $A^{-1}B := \{z : z = a^{-1}b, a \in A, b \in B\}, A, B \subset X$; e denotes the unit element in X.

$(\Omega, \mathcal{F}, m)$ is a σ-finite measure space; for a fixed $\alpha, 1 \leq \alpha \leq \infty$, $L^\alpha(\Omega, \mathcal{F}, m)$ (or simply $L^\alpha(\Omega)$) is the Lebesgue space with the norm $||f||_\alpha = (\int_\Omega |f|^\alpha dm)^{\frac{1}{\alpha}}$ and $L^\alpha_+(\Omega, \mathcal{F}, m)$ (or $L^\alpha_+(\Omega)$) denotes the set of all non-negative functions in this space; 1_Λ is the indicator of the set $\Lambda \subseteq \Omega$; we write 1 instead of 1_Ω.

Operator in a Banach space B = bounded linear operator in B. If T is an operator in B, T^* denotes the operator conjugate to T in B^*; if B and B' are dual spaces, T' denotes the conjugate to T in B'; if $B' = B^*$ we write T^* instead of T'. L-operator=Lamperti type operator (see Subsect. 6.2).

Representation (of a locally compact group) = bounded measurable representation (see Subsects. 1.4 and 1.5).

$\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$ are the sets of all positive integers, all integers, and all real numbers, respectively; $\mathbb{R}_+ = [0, \infty)$.

Two real numbers α and β are said to be *mutually conjugate* if $1 < \alpha < \infty, 1 < \beta < \infty$ and $\alpha^{-1}+\beta^{-1} = 1$, or if one of them equals 1 and the other one equals ∞.

For a complex number z and for a $b \in R_+$, we always consider the principal value of z^b.

$|F|$ denotes the cardinality of the set F; if a set F is contained in a specified space Y then $F^c = Y \setminus F$.

esup = essential supremum.

a.e. = almost everywhere, a.a. = almost all.

1.3. Densities

A (probability) *density* φ is a non-negative measurable function on X such that $\int_X \varphi d\mu = 1$. The Radon–Nikodym derivative of an absolutely continuous probability measure with respect to the left Haar measure is a density. We denote by $s(\varphi)$ the support of φ.

1.4. Group representations and actions

Let $(\Omega, \mathcal{F}, m)$ be a σ-finite measure space.

A (bounded)operator T in a Lebesgue space $L^\alpha(\Omega, \mathcal{F}, m), 1 \leq \alpha < \infty$, is called a *Lamperti operator* if for $f, g \in L^\alpha$, $f \cdot g = 0$ m-a.e. implies $Tf \cdot Tg = 0$ m-a.e. To any Lamperti operator T in L^α there corresponds its *modulus* $|T|$: a positive linear operator in L^α such that $|T||f| = |Tf|$ if $f \in L^\alpha; |T|$ is a Lamperti operator [28]. Some useful properties of Lamperti operators are summarized in Subsection 6.1 in the Appendix.

A mapping $T : x \mapsto T_x$ on X is said to be a *left (resp. right) L^α-representation of* X if, for each $x \in X$, T_x is a bounded operator in $L^\alpha(\Omega, \mathcal{F}, m)$, T_e is the identity operator, $T_{xy} = T_xT_y, x, y \in X$ (resp. $T_{xy} = T_yT_x, x, y \in X$), and $||T||_\alpha := \sup_{x\in X} ||T_x||_\alpha < \infty$. For the T_x-image of $f \in L^\alpha$ we use the notation T_xf if T is a left representation and fT_x if T is a right one. It is clear that if $T : x \mapsto T_x$ is a left (right) representation then $T^* : x \mapsto T^*_x$ is a right (resp. left) one. We say that T is a *Lamperti* (resp. *positive*, resp. *isometric*) if it is a L^α-representation and all T_x have the corresponding property. If $T : x \mapsto T_x$ is a Lamperti (resp. positive) representation in $L^\alpha(\Omega)$, then the conjugate L^β-representation $T^* : x \mapsto T^*_x$ has the corresponding property. When $1 \leq \alpha \leq \infty$, each positive L^α-representation is a Lamperti one (see [28]), and when $\alpha \neq 2$, each isometric L^α-representation is also Lamperti (see [31], [28]).

If $T : x \mapsto T_x$ is a Lamperti L^α-representation then its modulus $|T| : x \mapsto |T_x|$ is also a Lamperti L^α-representation.

We say that representations $T^{(\alpha')}$ in $L^{\alpha'}(\Omega, \mathcal{F}, m)$ and $T^{(\alpha'')}$ in $L^{\alpha''}(\Omega, \mathcal{F}, m)$ are *compatible* if $T_x^{(\alpha')} f = T_x^{(\alpha'')} f$ for all $f \in L^{\alpha'}(\Omega, \mathcal{F}, m) \cap L^{\alpha''}(\Omega, \mathcal{F}, m)$.

An $L^1 - L^\infty$-*representation* of X is a family of pairwise compatible L^α-representations $\{T^{(\alpha)} : X \mapsto L^\alpha, \quad 1 \le \alpha \le \infty\}$; by the Riesz convexity theorem, this implies $||T|| := \sup_{1 \le \alpha \le \infty} ||T^{(\alpha)}||_\alpha < \infty$. An $L^1 - L^\infty$-representation is said to be positive, isometric or Lamperti if each component $T^{(\alpha)}$ possesses the corresponding property.

We consider also *Lamperti type representations* (in short L-representations) by operators acting in some Banach spaces (see Subsect. 6.2).

We say that a point transformation σ of $(\Omega, \mathcal{F}, m)$ is *measurable* if for any $\Lambda \in \mathcal{F}$ we have $\sigma^{-1}\Lambda \in \mathcal{F}$; in this case we denote: $m\sigma^{-1}(\Lambda) := m(\sigma^{-1}\Lambda), \Lambda \in \mathcal{F}$; we call σ *bounded* (by a constant $K < \infty$) if $m(\sigma^{-1}\Lambda) \le Km(\Lambda), \Lambda \in \mathcal{B}$ or, equivalently, if $D(\omega) := \frac{d(m\sigma^{-1})}{dm}(\omega) \le K$; we then denote: $||\sigma|| := ||D||_\infty$.

A mapping $\tau : x \mapsto \tau_x, x \in X$ is said to be a left (resp. right) *action* of X in $(\Omega, \mathcal{F}, m)$ if all $\tau_x, x \in X$ are measurable point transformations of Ω and $\tau_x\tau_y = \tau_{xy}$ (resp. $\tau_x\tau_y = \tau_{yx}$); we denote the τ_x-image of $\omega \in \Omega$ by $\tau_x\omega$ when τ is a left action and by $\omega\tau_x$ when τ is a right one. An action τ in $(\Omega, \mathcal{F}, m)$ is *bounded* if $||\tau|| := \sup ||\tau_x|| < \infty$, and τ is said to be *measure preserving* if $m(\tau_x^{-1}\Lambda) = m(\Lambda), \Lambda \in \mathcal{F}, x \in X$ (in this case $||\tau_x|| = 1$). With any bounded left action $\tau = \{\tau_x, x \in X\}$ of X for each $\alpha, 1 \le \alpha \le \infty$, we associate a positive right Lamperti L^α representation T defined by $x \mapsto T_x : fT_x(\omega) = f(\tau_x\omega)$ $(||T||_\alpha = ||\tau||^{\frac{1}{\alpha}})$ (similarly, a bounded right action induces a left positive Lamperti L^α-representation T); if τ is measure preserving then T is isometric.

1.5. Measurability and integration of representations and actions

We say that a representation T in a space $L^\alpha = L^\alpha(\Omega, \mathcal{F}, m)(1 \le p < \infty)$ is *measurable* if for each $f \in L^\alpha$ the function $(\omega, x) \mapsto T_x f(\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable; let us note that, for each $x \in X$, $\|T_x f\|_\alpha \le ||T||_\alpha ||f||_\alpha$, and hence the function $x \mapsto T_x f(\omega)$ is integrable m-a.e. with respect to each finite Borel measure κ on X.

An action τ is *measurable* if the associated representation $x \mapsto f(\tau_x \cdot)$ in each $L^\alpha, 1 \le \alpha \le \infty$, is measurable; this is equivalent to the following property: $\{(\omega, x) : \tau_x\omega \in \Lambda\} \in \mathcal{F} \times \mathcal{B}$ for each set $\Lambda \in \mathcal{F}$.

In this paper, along with a given Lamperti representation, we consider its conjugate representation and its modulus. *If $x \mapsto T_x$ is a measurable Lamperti representation in $L^\alpha(1 \le \alpha \le \infty)$, then $x \mapsto |T_x|$ is a measurable positive Lamperti representation.* This follows from the construction of the operators $|T_x|$ (see [28]).

In Subsection 6.1 we prove (see Proposition 6.6) that *measurability of a Lamperti representation T implies measurability of the conjugate representation T^*.*

From now on we assume that all representations under consideration are measurable and hence have measurable conjugates.

2. Tempered, regular and ergodic sequences of densities and sets

In this section we consider various classes of sequences of probability measures, densities and sets. We will use them as "weights" in the DET and in the PET we will prove in §3 and in §4. We start with the class based on the strongest (and most transparent) property and then turn to two wider classes of sequences possessing weaker properties.

2.1. Tempered and regular sequences of densities

Let $\Phi = \{\varphi_n\}$ be a sequence of densities on the group X. For each $n \ge 2$ we denote: $S_n := s(\varphi_n)$, $\tilde{S}_n := \bigcup_{1 \le i \le n} S_i$;

$$\psi_n(x) := \sup\{\varphi_n(ux) | u \in \tilde{S}_{n-1}\} = \sup\{\varphi_n(z) | z \in \tilde{S}_{n-1}x\}. \tag{2.1}$$

It is clear that $s(\psi_n) = \tilde{S}_{n-1}^{-1} S_n$. *In the sequel we assume that these sets are measurable. Moreover, we assume that the functions ψ_n are measurable, too* (of course, this is the case when all S_n are compact the densities φ_n are continuous on their supports).

Definition 2.1. The quantity $t(\Phi) := \sup_{n\geq 2} \int \psi_n(x)\mu(dx)$ is called *the tempering index* of the sequence Φ. If $t(\Phi) < \infty$ we say that the sequence Φ is *tempered*.

If we replace $\tilde{S}_{n-1}$ with $\tilde{S}_n$ in the above definition, we come to the notion of *regular* sequences of functions and we consider the *regularity index* $r(\Phi)$ instead of $t(\Phi)$ (certainly, $t(\Phi) \leq r(\Phi)$ so *each regular sequence is tempered*). The notions of a tempered sequence and of a regular sequence of densities were introduced in [41].

It is easy to see that if Φ is a sequence of densities then $t(\Phi) \geq 1$ and $r(\Phi) \geq 1$.

The definitions of regular and tempered sequences imply the following statement.

Proposition 2.1. *Let $\Phi = \{\varphi_n\}$ and $\Theta = \{\theta_n\}$. If there is a constant C such that $\varphi_n(x) \leq C\theta_n(x)$, $x \in X, n \in \mathbb{N}$, then $t(\Phi) \leq Ct(\Theta)$ and $r(\Phi) \leq Cr(\Theta)$.*

For any non-negative μ-measurable function ψ we denote: $G(\psi) := \cup_{x\in s(\psi)}\{(x, y) : 0 < y \leq \psi(x)\} \subset s(\psi) \times \mathbb{R}$, the subgraph of ψ; it is clear that $G(\psi) = \cup_{x\in s(\psi)} G_x(\psi)$ where $G_x(\psi) := (\{x\} \times [0, \psi(x)]$. If $\Phi = \{\varphi_n\}$ is a sequence of densities, we denote: $G_n := G(\varphi_n)$. Let $\tilde{\mu}$ be the left Haar measure in $X \times \mathbb{R}$, i.e. $\tilde{\mu} = \mu \times l$ where l is the Lebesgue measure in $\mathbb{R}$. The set $G(\psi)$ is $\tilde{\mu}$-measurable and $\tilde{\mu}(G(\psi)) = \int \psi d\mu$. For each $A \subset X \times \mathbb{R}, x \in X$ we put $xA = (x, 0)A$.

In Remark 2.1 we consider tempered sequences; changes needed for regular sequences are obvious.

Remark 2.1. Let ψ_n be the functions defined by (2.1). This formula implies: $G_x(\psi_n) = \cup_{u\in\tilde{S}_{n-1}} G_x(\varphi_n(ux)) = \cup_{u\in\tilde{S}_{n-1}} u^{-1}G_x(\varphi_n)$. Therefore $G(\psi_n) = \cup_{u\in\tilde{S}_{n-1}} u^{-1}G_n = \tilde{S}_{n-1}^{-1}G_n$. Hence

$$t(\Phi) = \sup_n \tilde{\mu}(\tilde{S}_{n-1}^{-1}G_n); \tag{2.2}$$

similarly

$$r(\Phi) = \sup_n \tilde{\mu}(\tilde{S}_n^{-1}G_n). \tag{2.3}$$

The simple relation between sequences of the subgraphs of non-negative measurable functions and the indexes t and r of the sequences of the corresponding densities is covered by the following Remark.

Remark 2.2. If $\varphi_n(x) := \frac{1}{\int g_n d\mu} g_n(x)$ where g_n are non-negative measurable functions then

$$t(\Phi) = \sup_n \frac{1}{\int g_n d\mu} \tilde{\mu}(\tilde{S}_{n-1}^{-1}G(g_n)); \quad r(\Phi) = \sup_n \frac{1}{\int g_n d\mu} \tilde{\mu}(\tilde{S}_n^{-1}G(g_n)). \tag{2.4}$$

Choose $0 < a_n \leq \sup_x \varphi_n(x)$ and denote $M_{n,a_n} := \{x : \varphi_n(x) \geq a_n\}$. It is clear that $M_{n,a_n} \times [0, a_n] \subset G_n$; therefore

$$a_n\mu(\tilde{S}_{n-1}^{-1}M_{n,a_n}) \leq t(\Phi), n \in \mathbb{N}; \tag{2.5}$$

hence $\varphi_n(x_n)\mu(\tilde{S}_{n-1}^{-1}x_n) \leq t(\Phi), n \in \mathbb{N}, x_n \in M_{n,a_n}$. Thus *if Φ is tempered then the right Haar measure $\nu(S_n) < \infty, n \in \mathbb{N}$.* The obvious relation $G_n \subset S_n \times [0, \sup_x \varphi_n(x)]$ implies

$$t(\Phi) \le \sup_{n\ge 2}[\sup_x \varphi_n(x)\mu(\tilde{S}_{n-1}^{-1}S_n)], n \in \mathbb{N}. \tag{2.6}$$

Let $\{A_n\}$ be a sequence of integrable sets; $\tilde{A}_n := \bigcup_{i=1}^{n} A_i$ and $g_n(x) := 1_{A_n}(x)$; then $\tilde{\mu}(\tilde{S}_{n-1}^{-1}G(g_n)) = \mu(\tilde{A}_{n-1}^{-1}A_n)$ and $\tilde{\mu}(\tilde{S}_n^{-1}G(g_n)) = \mu(\tilde{A}_{n-1}^{-1}A_n)$; therefore Remark 2.2 implies the following statement.

Proposition 2.2. *Let $\mathcal{A} = \{A_n\}$ be a sequence of measurable sets with $0 < \mu(A_n) < \infty$ and let the sets $\tilde{A}_{n-1}^{-1}A_n$ and $\tilde{A}_n^{-1}A_n$ be measurable. Put $\varphi_n^{\mathcal{A}}(x) := \dfrac{1}{\mu(A_n)}1_{A_n}(x), n = 1, 2, \ldots$. Then the functions ψ_n, defined by (2.1), are measurable and*

$$t(\Phi^{\mathcal{A}}) = \sup_{n\ge 2}\frac{\mu(\tilde{A}_{n-1}^{-1}A_n)}{\mu(A_n)}, \quad r(\Phi^{\mathcal{A}}) = \sup_{n\ge 2}\frac{\mu(\tilde{A}_n^{-1}A_n)}{\mu(A_n)}.$$

Definition 2.2. We say that a sequence $\mathcal{A}$ is *tempered* (resp. *regular*) if all sets $A_n, \tilde{A}_{n-1}^{-1}A_n$ (resp. $A_n^{-1}A_n$) are measurable, $\mu(A_n) > 0$ for all $n \ge 2$, and $t(\mathcal{A}) := t(\Phi^{\mathcal{A}}) < \infty$ (resp. $r(\mathcal{A}) := r(\Phi^{\mathcal{A}}) < \infty$).

Proposition 2.3. *Let H be a normal subgroup of a connected locally compact group F, let $K = F/H$ be compact, $\mu_K(K) = 1$, let $\imath$ be the natural homomorphism $F \mapsto H$. Consider a sequence of densities $\Lambda = \{\lambda_n\}$ on H and let $\Phi = \{\varphi_n = \lambda_n \circ \imath\}$. Then $t(\Phi) = t(\mathcal{K})$ and $r(\Phi) = r(\mathcal{K})$.*

Proof. It is easy to check that φ_n are densities and it is clear that $\tilde{S}_{n-1}^{\Lambda} = \imath(S_{n-1}^{\Phi})$ and $G_{\imath(h)}^{\Phi} = G_h^{\Lambda}$, $h \in H$; hence, by (2.2), $t(\Phi) = \tilde{\mu}_F((S_{n-1}^{\Phi})^{-1}G_n^{\Phi}) = \tilde{\mu}_H(\tilde{S}_{n-1}^{\Lambda})^{-1}G_n^{\Lambda}) = t(\mathcal{K})$; the proof of the second relation is quite similar. □

Relation (2.6) implies the following statement.

Proposition 2.4. *Let $\Phi = \{\varphi_n\}$ be a sequence of bounded densities with integrable supports S_n. If $\sup_{x\in X}\varphi_n(x) \le \frac{c}{\mu(S_n)} < \infty$ for all $n \in \mathbb{N}$ and the sequence $\mathcal{S} = \{S_n\}$ is tempered (resp. regular) then Φ is tempered (resp. regular); $t(\Phi) \le c\, t(\mathcal{S})$ (resp. $r(\Phi) \le c\, r(\mathcal{S})$).*

Remark 2.3. Let $\{A_n\}$ and $\{B_n\}$ be sequences of integrable sets, $A_n \subset B_n$ and let B_n be tempered (resp. regular); if $s := \sup_n \frac{\mu(B_n)}{\mu(A_n)} < \infty$ then $\{A_n\}$ is tempered (resp. regular), too and $t(\{A_n\}) \le s\ t(\{B_n\})$ (resp. $r(\{A_n\}) \le s\ r(\{B_n\})$.

Remark 2.4. Let $\Phi = \{\varphi_n\}$ be a sequence of densities with integrable supports S_n on X. Remark 2.1 implies that if there is a sequence of positive numbers $\{c_n\}$ such that $\{G(c_n\varphi_n)\} = \{(e, c_n)G(\varphi_n)\}$ is a tempered (resp. regular) sequence of sets in the group $X \times \mathbb{R}$ then the sequence $\{\varphi_n\}$ is tempered (resp. regular) and $t(\Phi) \le t(\{G(c_n\varphi_n)\})$ (resp. $r(\Phi) \le r(\{G(c_n\varphi_n)\})$.

2.2. Weakly tempered sequences of probability measures and densities

Let $\mathcal{K} = \{\kappa_n\}$ be a sequence of probability measures, $k \in \mathbb{N}$. We often write $\int_{X_i}$ instead of $\int_X$ in order to emphasize that the integration is with respect to the variable x_i.

Definition 2.3. We define the *weakly tempering index of order k* of the sequence $\mathcal{K}$ as

$$\tau_k(\mathcal{K}) := \sup_{n_k:n_k<\infty}\int\limits_{X_k}\max_{n_{k-1}:n_{k-1}<n_k}\int\limits_{X_{k-1}}\ldots\max_{n_2:n_2<n_3}\int\limits_{X_2}\max_{n_1:n_1<n_2}\int\limits_{X_1}\times$$

$$\kappa_{n_k}(x_1x_2\ldots x_{k-1}dx_k)\kappa_{k-1}(x_1x_2\ldots x_{k-2}dx_{k-1})\ldots\ \kappa_2(x_1dx_2)\kappa_1(dx_1), \tag{2.7}$$

We say that the sequence $\mathcal{K}$ is *weakly tempered* if $\tau_k(\mathcal{K}) < \infty$.

This definition is rather formal. We provide an explicit step by step definition. Let $A_2, ..., A_k \in \mathcal{B}$, $x_1, ..., x_{k-1} \in X$. Of course, for each fixed $A_2, ..., A_k$ the function

$$(x_1, ..., x_{k-1}) \mapsto \kappa_{n_k}(x_1 x_2 \dots x_{k-1} A_k)\kappa_{n_{k-1}}(x_1 x_2 \dots x_{k-2} A_{k-1})...\kappa_{n_2}(x_1 A_2)$$

is bounded and measurable, hence integrable with respect to every probability measure.

$$M^{(2)}_{n_2}(A_2, x_2 A_3, x_2 x_3 A_4, ..., x_2 ... x_{k-1} A_k) :=$$
$$\max_{n_{n_1}: n_1 < n_2} \int_X \kappa_{n_k}(x_1 x_2 \dots x_{k-1} A_k)\kappa_{n_{k-1}}(x_1 x_2 \dots x_{k-2} A_{k-1})...\kappa_{n_2}(x_1 A_2)\kappa_{n_1}(dx_1).$$

It is easy to verify that, for each fixed $x_1, ..., x_{k-1}$, $M^{(2)}_{n_2}$ is a finite measure with respect to each set $A_2, ..., A_k$. Define:

$$M^{(3)}_{n_3}(A_3, x_3 A_4, ..., x_3 ... x_{k-1} A_k) :=$$
$$\max_{n_2: n_2 < n_3} \int_X M^{(2)}_{n_2}(dx_2, x_2 A_3, x_2 x_3 A_4, ..., x_2 ... x_{k-1} A_k).$$

This is also a finite measure with respect to each set $A_3, ..., A_k$. For each $i, 4 \le i \le k-1$ we consider the measure

$$M^{(i+1)}_{n_{i+1}}(A_{i+1}, x_{i+1} A_{i+2}, ..., x_{i+1} ... x_{k-1} A_k) :=$$
$$\max_{n_i: n_i < n_{i+1}} \int_X M^{(i)}_{n_i}(dx_i, x_i A_{i+1}, x_i x_{i+1} A_{i+2}, ..., x_i ... x_{k-1} A_k),$$

and at last

$$M^{(k)}_{n_k}(A_k) := \max_{n_{k-1}: n_{k-1} < n_k} \int_X M^{(k-1)}_{n_{k-1}}(dx_{k-1}, x_{k-1} A_k).$$

So the exact definition of the weakly tempering index of order k is

$$\tau_k(\mathcal{K}) := \sup_{n_k: n_k < \infty} \int_X M^{(k)}_{n_k}(dx_k).$$

The definition (2.7) can be written shorter by introducing the operations $F^i_{n_{i+1}} = \sup_{n_i: n_i < n_{i+1}} \int_{X_i}$:

$$\tau_k(\mathcal{K}) := \prod_{i=1}^{k} F^i_{n_{i+1}} \prod_{j=1}^{k} \kappa_j(x_1 x_2 \dots x_{j-1} dx_j) \tag{2.8}$$

where $n_{k+1} = \infty, x_0 = e$.

If all probability measures κ_n are absolutely continuous (with respect to the Haar measure μ), we consider the densities $\varphi_n := \frac{d\kappa_n}{d\mu}$ and denote $\Phi := \{\varphi_n\}$. It is natural to define: $\tau_k(\Phi) := \tau_k(\mathcal{K})$. For the sake of references we specify the definition of $\tau_k(\Phi)$.

Definition 2.4. Let Φ be sequence of densities. The *weakly tempering index of Φ of order $k \geq 2$*

$$\tau_k(\Phi) := \sup_{n_k<\infty} \int\limits_X \mu(dx_k) \max_{n_{k-1}:n_{k-1}<n_k} \int\limits_X \mu(dx_{k-1}) \ldots \max_{n_2:n_2<n_3} \int\limits_X \mu(dx_2) \times$$

$$\max_{n_1:n_1<n_2} \int\limits_X \mu(dx_1)\varphi_{n_1}(x_1)\varphi_{n_2}(x_1x_2)\ldots\varphi_{n_k}(x_1x_2\ldots x_k),$$

or, in short, considering (for a fixed n) operations $f \mapsto \max_{s<n} \int_X \mu(dx)f(x)$ defined in the obvious way on measurable positive functions on X

$$\tau_k(\Phi) := \prod_{i=1}^{k} [\sup_{n_i:n_i<n_{i+1}} \int\limits_X \mu(dx_i)] \prod_{j=1}^{k} \varphi_{n_j}(x_1x_2\ldots x_j) \tag{2.9}$$

where $n_{k+1} = \infty$. If $\tau_k(\Phi) < \infty$ we say that *the sequence Φ is weakly tempered of order k.*

If $\varphi_n(x) = \frac{1}{\mu(A_n)} 1_{A_n}(x), n \in \mathbb{N}$, is a sequence of densities on X, weakly tempered of order k, we say also that $\{A_n\}$ has this property.

Remark 2.5. $\tau_2(\Phi) = \sup_{n_2<\infty} \int_X \max_{n_1:n_1<n_2} \tilde{\varphi}_{n_2} * \varphi_{n_1}(x)\nu(dx)$ where $\tilde{\varphi}_{n_2}(x) = \varphi_{n_2}(x^{-1})$, $\tilde{\varphi}_{n_2} * \varphi_{n_1}(x) = \int_X \tilde{\varphi}_{n_2}(xy^{-1})\varphi_{n_1}(y)\mu(dy)$, the convolution of $\tilde{\varphi}_{n_2}$ and φ_{n_1}, and $\nu(A) = \mu(A^{-1}), A \in \mathcal{B}$ (the right Haar measure on X). This remark simplifies the estimation of $\tau_2(\Phi)$ when the above convolutions are known (see, e.g., Remark 5.1).

Proposition 2.5. *Any sequence $\mathcal{K}$ which is weakly tempered of order k is also weakly tempered of every lower order. More precisely,*

$$\tau_k(\mathcal{K}) \geq \tau_{k-1}(\mathcal{K}), k \geq 2 \quad \text{and} \quad \rho_k(\mathcal{K}) \geq \rho_{k-1}(\mathcal{K}), k \geq 2,$$

hence $\rho_k(\mathcal{K}) \geq \tau_k(\mathcal{K}) \geq \tau_1(\mathcal{K}) = 1$.

Proof. We consider here the weakly tempering indexes; the proof for weakly regular sequences is quite similar. Let us note that, by the Fubini theorem,

$$F_\infty^k F_{n_k}^{k-1} = \sup_{n_k<\infty} \int\limits_{X_k} \max_{n_{k-1}:n_{k-1}<n_k} \int\limits_{X_{k-1}} \geq$$

$$\sup_{n_k<\infty} \max_{n_{k-1}:n_{k-1}<n_k} \int\limits_{X_k} \int\limits_{X_{k-1}} = \sup_{n_{k-1}<\infty} \int\limits_{X_{k-1}} \int\limits_{X_k} = F_\infty^{k-1} \int\limits_{X_k} . \tag{2.10}$$

Therefore, by (2.8)

$$\tau_k(\mathcal{K}) = F_\infty^k F_{n_k}^{k-1} \prod_{i=k-2}^{1} F_{n_{i+1}}^i \prod_{j=k}^{1} \kappa_j(x_1x_2\ldots x_{j-1}dx_j) \geq$$

$$F_\infty^{k-1} \int\limits_{X_k} \prod_{i=k-2}^{1} F_{n_{i+1}}^i \prod_{j=k}^{1} \kappa_j(x_1x_2\ldots x_{j-1}dx_j).$$

Reasoning as in (2.10) we obtain: for each $1 \le i \le k-2$

$$\int\limits_{X_k} F^i_{n_{i+1}} \ge F^i_{n_{i+1}} \int\limits_{X_k} .$$

Applying this relation to $i = k-2, ..., 1$ consecutively we obtain:

$$\tau_k(\mathcal{K}) \ge F_\infty^{k-1} \prod_{i=k-2}^{1} F^i_{n_{i+1}} \int\limits_{X_k} \kappa_k(x_1 x_2 \dots x_{j-1} dx_k) \prod_{j=k-1}^{1} \kappa_j(x_1 x_2 \dots x_{j-1} dx_j) =$$

$$F_\infty^{k-1} \prod_{i=k-2}^{1} F^i_{n_{i+1}} \prod_{j=k-1}^{1} \kappa_j(x_1 x_2 \dots x_{j-1} dx_j) = \tau_{k-1}(\mathcal{K}). \quad \square$$

Proposition 2.6. $\tau_k(\Phi) \le [t(\Phi)]^k$ *for all* $k \in \mathbb{N}$*; hence a tempered sequence of densities is weakly tempered of any order* k*.*

Proof. Let $k \ge 2$. The definition of ψ_k implies

$$\varphi_k(ux)\varphi_l(u) \le \psi_k(x)\varphi_l(u) \quad \text{for all} \quad x, u \in X, l < k. \tag{2.11}$$

Therefore, for any increasing set of positive integers $\{n_i, 1 \le i \le k\}$

$$\prod_{i=1}^{k} \varphi_{n_i}(x_1 \dots x_i) = \varphi_{n_k}(x_1 x_2 \dots x_k)\varphi_{n_{k-1}}(x_1 x \dots x_{k-1}) \prod_{i=1}^{k-2} \varphi_{n_i}(x_1 x_2 \dots x_i) \le$$

$$\psi_{n_k}(x_k) \prod_{i=1}^{k-1} \varphi_{n_i}(x_1 x_2 \dots x_i)) \le ... \le \prod_{i=1}^{k} \psi_{n_i}(x_i).$$

Hence

$$\tau_k(\Phi) \le \sup_{n_k} \int\limits_X \mu(dx_k)\psi_{n_k}(x_k) \max_{n_{k-1}<n_k} \int\limits_X \mu(dx_{k-1})\psi_{n_{k-1}}(x_{k-1}) \dots$$

$$\max_{n_2<n_3} \int\limits_X \mu(dx_2)\psi_{n_2}(x_2) \max_{n_1<n_2} \int\limits_X \mu(dx_1)\psi_{n_1}(x_1) \le [t(\Phi)]^k. \quad \square$$

Remark 2.6. If $\Phi = \{\varphi_n\}$ and $\Theta = \{\theta_n\}$ are sequences of densities, $\varphi_n \le C\psi_n, n \in \mathbb{N}$, with some constant C and Θ is weakly tempered then $\tau_k(\Phi) \le C^k \tau_k(\Theta)$, so Φ is weakly tempered.

2.3. Ergodic and Følner sequences

Definition 2.5. We say that a sequence $\mathcal{K} = \{\kappa_n\}$ is *(left) ergodic* if for each $x \in X$ the total variation $\operatorname*{var}_{A\in\mathcal{B}}(\kappa_n(xA) - \kappa_n(A)) \to 0$ as $n \to \infty$.

In the case when the measures ν_n posses densities φ_n the definition can be rewritten as follows.

Definition 2.6. A sequence of densities $\Phi = \{\varphi_n\}$ is said to be (left) *ergodic* if

$$\lim_{n\to\infty} \int\limits_X |\varphi_n(xy) - \varphi_n(y)|\mu(dy) = 0, \quad x \in X.$$

If $\{A_n\}$ is a sequence of μ-integrable sets in X, $\varphi_n = \frac{1}{\mu(A_n)}1_{A_n}$ then ergodicity of $\{\varphi_n\}$ is equivalent to the *Følner property* of the sequence $\{A_n\}$:

$$\lim_{n\to\infty} \frac{\mu(A_n \triangle xA_n)}{\mu(A_n)} = 0, \ x \in X. \tag{2.12}$$

The next statement has been proved in [33,34].

Proposition 2.7. *Each amenable σ-compact locally compact group X possesses Følner tempered sequences of compact sets.*

Remark 2.7. It is clear that ergodicity of $\{\varphi_n\}$ coincides with the following property of the subgraphs: $\lim_{n\to\infty} \tilde{\mu}(G_n \triangle xG_n) = 0, x \in X$. Therefore, if for some positive sequence $\{c_n\}$ the sequence of subgraphs $\{G(c_n\varphi_n)\} = \{(e, c_n)G(\varphi_n)\}$ is a Følner sequence in the group $X \times \mathbb{R}$ then the sequence $\{\varphi_n\}$ is ergodic.

Remark 2.8. If $\Phi = \{\varphi_n\}, \varphi_n = \psi_n \cdot \chi_n$, where $\{\psi_n\}$ and $\{\chi_n\}$ are ergodic sequences and $\sup_{n\in\mathbb{N}} ||\psi_n||_\infty < \infty$, $\sup_{n\in\mathbb{N}} ||\chi_n||_\infty < \infty$, then Φ is ergodic too.

§5 contains examples of sequences of sets, densities and measures possessing properties considered above (see also §7 in [47]), and in §6.3 we provide counterexamples emphasizing the distinction between these properties.

3. Dominated ergodic theorem

As before, X is an amenable σ-compact locally compact group; μ and ν are its left and right Haar measures.

3.1. *Some Lamperti type representations in the "space–time" Banach spaces $\tilde{L}^{\alpha,\gamma}$*

3.1.1. *The spaces $\tilde{L}^{\alpha,\gamma}$ and the representations $\tilde{T}$*

We fix some mutually conjugate numbers $\alpha, \beta \in [1, \infty]$. We denote $L^\gamma := L^\gamma(X, \mathcal{B}, \nu), 1 \le \gamma \le \infty$.

T is a right Lamperti representation of X in $L^\alpha(\Omega, \mathcal{F}, m)$, $1 \le \alpha < \infty$, or a right L-representation in L^∞ (see Subsect. 6.2), S is the conjugate left Lamperti representation in $L^\beta(\Omega, \mathcal{F}, m)$ (see Proposition 6.2; for the case $\alpha = \infty$ see Subsect. 6.2). For simplicity, we temporary assume that these representations are positive (we can arrive to the general case by considering the modulus $|T|$ of the representation T). We consider the positive linear right representation $y \mapsto R_y$ and the left representation $y \mapsto R'_y$, in the linear space of all $\mathcal{B}$-measurable functions $M(X, \mathcal{B})$ where $R_y f(x) := f(xy^{-1}), R'_y f(x) := R_y^{-1} f(x) = R_{y^{-1}} f(x) = f(xy), f \in M(X, \mathcal{B})$; it is clear that the restrictions of the operators R_y and R'_y to each space $L^\gamma, 1 \le \gamma \le \infty$, are isometries in this space. Let $\tilde{\Omega} := \Omega \times X, \tilde{\mathcal{F}} := \mathcal{F} \times \mathcal{B}, \tilde{m} := m \times \nu$. Let us consider the linear spaces $\tilde{L}^{\alpha,\gamma}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m})$ of all $\tilde{\mathcal{F}}$-measurable functions $\tilde{f}(\omega, x)$ with the finite norm $||\tilde{f}||_{\tilde{L}^{\alpha,\gamma}} := ||||\tilde{f}||_{L^\alpha(\Omega)}||_{L^\gamma}, 1 \le \gamma \le \infty$. Lemma 16(b) and Theorem 17 in [18], Ch. III, §11 imply that there is a natural isomorphism between the linear normed space $\tilde{L}^{\alpha,\gamma}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m})$ and the Banach space $L^\gamma(X, \mathcal{B}, \nu; B)$ of B-valued functions where $B := L^\alpha(\Omega, \mathcal{F}, m)$; hence our space $\tilde{L}^{\alpha,\gamma}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m})$ is a Banach space. Note also that if $\tilde{f} \in \tilde{L}^{\alpha,\gamma}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m})$ then $\tilde{f}(\cdot, x) \in L^\alpha(\Omega, \mathcal{F}, m)$ for m-a.e. $x \in X$, and $\tilde{f}(\omega, \cdot) \in L^\gamma$ for m-a.e. ω in Ω. If $\tilde{f} \in \tilde{L}^{\alpha,\gamma}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m})$ we define $\left(\tilde{T}_y \tilde{f}\right)(\omega, x) := T_y R_y \tilde{f}(\omega, x) = R_y T_y \tilde{f}(\omega, x) = T_y \tilde{f}(\omega, xy^{-1})$. It is clear that $\tilde{T}_y$ is a linear operator in $\tilde{L}^{\alpha,\gamma}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m})$ and $||\tilde{T}_y||_{\tilde{L}^{\alpha,\gamma}} = ||T_y||_{L^\alpha(\Omega)}$. If $T_y = h_y \Psi_y$ is the canonical representation T_y, then $\tilde{T}_y = h_y \tilde{\Psi}_y$ where $\tilde{\Psi}_y = \Psi_y R_y$; hence $\tilde{T}_y$ is a L-type operator. Of course, if $\tilde{f} \in L^{\alpha,\gamma}$, the function $y \mapsto \tilde{f}\tilde{T}_y$ is measurable and bounded, hence integrable with respect to every probability measure on $\mathcal{B}$.

For each $y \in X$ by isometry of R_y as an operator in $L^\alpha, 1 \le \alpha < \infty$, the linear isometric operator $R'_y := R_y^{-1} = R_{y^{-1}}$ acting in L^β is the conjugate with respect to the operator R_y in $L^\beta(X,\mu)$. The spaces L^∞ and L^1 also form a dual pair with respect to the bounded bilinear functional $\langle f, g\rangle := \int_{\mathbb{R}^m} f\bar{g}\nu(dx)$, and $\langle R_y f, g\rangle := \int_{\mathbb{R}^m} f(xy^{-1})\overline{g(x)}\nu(dx) = \int_{\mathbb{R}^m} f(x)\overline{g(xy)}\nu(dx)$. Therefore $R'_y = R_y^{-1} = R_{y^{-1}}$ is the conjugate to R_y for this dual pair too (R'_y is the restriction of the conjugate operator in $(L^\infty)'$ to the invariant subspace L^1 - see also Subsect. 6.2).

If (α,β) and (γ,δ) are pairs of mutually conjugate numbers, $1 \le \alpha,\beta,\gamma,\delta \le \infty$, we also consider the duality relation between $L^{\alpha,\gamma}$ and $L^{\beta,\delta}$ specified by the bilinear functional $\langle \tilde{f}, \tilde{g}\rangle := \int_{\tilde{\Omega}}[\tilde{f}(\omega,x)\overline{\tilde{g}(\omega,x)}]\tilde{m}(d\omega dx)$ where $\tilde{f} \in \tilde{L}^{\alpha,\gamma}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{m}), g \in \tilde{L}^{\beta,\delta}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{m})$; this functional is bounded since the Hölder inequality implies that

$$\langle \tilde{f}, \tilde{g}\rangle \le ||\tilde{f}||_{\tilde{L}^{\alpha,\gamma}}||g||_{\tilde{L}^{\beta,\delta}}. \tag{3.1}$$

Since

$$\langle \tilde{T}_y\tilde{f}, \tilde{g}\rangle = \langle T_yR_y\tilde{f}, \tilde{g}\rangle = \int_{\tilde{\Omega}} T_yR_y\tilde{f}(\omega,x)\cdot\overline{\tilde{g}(\omega,x)}\tilde{m}(d\omega dx) =$$

$$\int_{\tilde{\Omega}} R_y\tilde{f}(\omega,x)\cdot\overline{\tilde{g}(\omega,x)}T'_y\tilde{m}(d\omega dx) = \int_{\tilde{\Omega}} \tilde{f}(\omega,x)\cdot\overline{\tilde{g}(\omega,x)}T'_yR_y^{-1}\tilde{m}(d\omega)dx$$

the operator $\tilde{T}'_y := T'_yR'_y = S_yR'_y$ is the conjugate operator with respect to $\tilde{T}_y$; we denote it also by $\tilde{S}_y$, so $\tilde{S}'_y = \tilde{T}_y$. The mapping $y \mapsto \tilde{S}_y$ is a positive left L-type representation in $\tilde{L}^{\beta,\delta}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{m})$ and $||\tilde{S}_y||_{\tilde{L}^{\beta,\delta}} = ||T_y||_\alpha$ (see Subsect. 6.2).

The construction of the auxiliary representation $y \mapsto \tilde{T}_y$ in the Banach space $\tilde{L}^{\alpha,\gamma}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{m})$ has been an intermediate step in our construction of an appropriate $L^1 - L^\infty$ Lamperti representation in another Banach space, which is needed for the proof of the DET. The idea of such "time–space" constructions and prototypes of Lemmas 3.1–3.4 below are due to Kan [29] who considered powers of (not necessarily invertible) operators with Lamperti conjugates. We assume $1 < \alpha < \infty$.

Lemma 3.1. *Let $\varphi(\omega) > 0$ a.e., $\varphi \in L^\beta(\Omega,\mathcal{F},m)$ and let $u(\omega,x) := [S_{x^{-1}}\varphi(\omega)]^{\beta-1}$. Then*

a) the function $(\omega,x) \mapsto u(\omega,x)$ is measurable and $u(\cdot,x) > 0$ m-a.e. for all $x \in X$; for each $x \in X$, $u(\cdot,x) \in L^\alpha(\Omega)$ and moreover $\int_\Omega (u(\omega,x))^\alpha dm \le \|T\|_\alpha^\beta\|\varphi\|_\beta^\beta$;

*b) $u\tilde{T}_y(\omega,x) \le ||T||_\alpha^\beta u(\omega,x)$ $m-$*a.e. for all $x, y \in X$.

Proof. a) First, it is clear that u is measurable and, for each $x \in X$, $u(\omega,x) > 0$ m-a.e., and the structure of the operators S_x (see Proposition 6.1) implies $u(\cdot,x) > 0$ m-a.e. for all $x \in X$; $\int_\Omega (u(\omega,x))^\alpha dm = \int_\Omega [S_{x^{-1}}\varphi(\omega)]^{(\beta-1)\alpha}\,dm = \int_\Omega [S_{x^{-1}}\varphi(\omega)]^\beta\,dm \le \|S\|_\beta^\beta\|\varphi\|_\beta^\beta = \|T\|_\alpha^\beta\|\varphi\|_\beta^\beta$.

b) Fix x and y. Since S is a left Lamperti representation in $L^\beta(\Omega,m)$, we have: $(S_yg)^{\beta-1}S'_y \le ||T_y||_\alpha^\beta g^{\beta-1}$ if $g \in L_+^\beta$ (see Property 3 in Proposition 6.5), and putting $g = S_{x^{-1}}\varphi$ we obtain:

$$u\tilde{T}_y(\omega,x) = [S_{x^{-1}}\varphi(\omega)]^{\beta-1}\tilde{T}_y =$$

$$S_{(xy^{-1})^{-1}}[\varphi(\omega)]^{\beta-1}S'_y = [S_yS_{x^{-1}}\varphi(\omega)]^{\beta-1}S'_y \le ||T_y||_\alpha^\beta(S_{x^{-1}}\varphi(\omega))^{\beta-1} = ||T||_\alpha^\beta u(\omega,x). \quad \square$$

Note that, since $(\alpha-1)(\beta-1) = 1$, we have

$$[u(\omega,x)]^{\alpha-1} = [u(\omega,x)]^{\frac{1}{\beta-1}} = S_{x^{-1}}\varphi(\omega). \tag{3.2}$$

Lemma 3.2. *If $1 < \alpha < \infty$, for each $y \in X$ the function $u^{\alpha-1}$ is invariant with respect to $\tilde{T}'_y = \tilde{S}_y$.*

Proof.
$$\tilde{T}'_y(u(\omega,x)^{\alpha-1}) = S_y(u(\omega,xy)^{\alpha-1}) = S_y(u(\omega,xy)^{\frac{1}{\beta-1}}) = S_y S_{(xy)^{-1}}\varphi(\omega)$$
$$= S_y S_{y^{-1}} S_{x^{-1}}\varphi(\omega) = S_{x^{-1}}\varphi(\omega) = [u(\omega,x)]^{\frac{1}{\beta-1}} = u^{\alpha-1}(\omega,x). \quad \square$$

3.1.2. The spaces $\tilde{L}^\gamma_u$ and the representations $\tilde{U}^\gamma$

In what follows we consider the version of u defined in Lemma 3.1; we denote $u^{-1} := \frac{1}{u}$.

We fix some $\alpha, 1 < \alpha < \infty$, and consider the measure $\tilde{m}_u(M) := \int_M u^\alpha d\tilde{m}, M \in \mathcal{F}\times\mathcal{B}$. For each $\gamma, 1 \le \gamma < \infty$, we consider the space $\tilde{L}^\gamma_u(\tilde{\Omega}) := \tilde{L}^{\gamma,\infty}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m}_u)$ of all $\mathcal{F}\times\mathcal{B}$-measurable functions $\tilde{f}(\omega,x)$ with the finite norm

$$||\tilde{f}||_{\tilde{L}^\gamma_u} = ||\left[\int_\Omega |\tilde{f}(\omega,\cdot)|^\gamma u^\alpha(\omega,\cdot)m(d\omega)\right]^{\frac{1}{\gamma}}||_{L^\infty}, 1 \le \gamma < \infty, \tag{3.3}$$

$$||\tilde{f}||_{\tilde{L}^\infty_u} = ||\tilde{f}||_{\tilde{L}^\infty}. \tag{3.4}$$

Since the mapping $i_\gamma : \tilde{f} \mapsto \tilde{f}u^{\frac{\alpha}{\gamma}}$ is an isomorphism from the space $\tilde{L}^{\gamma,\infty}(\tilde{\Omega}, \tilde{\mathcal{F}}, u^\alpha d\tilde{m}_u)$ onto $\tilde{L}^{\gamma,\infty}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{m})$, our new space is also a Banach space.

For each $x \in X$ we consider invertible positive "right" operators $\tilde{U}_y := i_\alpha \tilde{T}_y i_\alpha^{-1}$ defined on $\tilde{L}^\alpha_u(\tilde{\Omega})$, i.e. if $\tilde{f} \in \tilde{L}^\alpha_u(\tilde{\Omega})$ we put

$$\tilde{f}\tilde{U}_y(\omega,x) := u^{-1}(\omega,x)[u(\omega,x)\tilde{f}(\omega,x)]\tilde{T}_y, y \in X. \tag{3.5}$$

In the following statement we summarize some properties of the operators $\tilde{U}_y$.

Lemma 3.3. *For each $x \in X$ the mapping $\tilde{U} : y \mapsto \tilde{U}_y$ is a measurable bounded positive right L-representation of X in $\tilde{L}^\alpha_u$, and, moreover,*

$$\|\tilde{U}_y\|_{\tilde{L}^\alpha_u} = \|\tilde{T}_y\|_{\tilde{L}^\alpha} = \|T_y\|_\alpha. \tag{3.6}$$

Proof. If $fT_y = h^T_y f \Psi^T_y$ is the canonical representation for T_y (see Subsect. 6.1) then $\tilde{f}\tilde{U}_y = \tilde{h}_y \cdot \tilde{f}\tilde{\Psi}_y$ where $\tilde{h}_y(\omega,x) := u^{-1}(\omega,x)h^T_y(\omega)(u(\omega,x))\Psi^T_y R_y$ and $\tilde{\Psi}_y := \Psi^T_y R_y$; $\tilde{\Psi}_y$ is a multiplicative operator with Property (3) of Theorem 6.1. Statement (3.6) is evident. $\square$

We will consider the measures $\hat{m}^y_u$ on $\mathcal{F}\times\mathcal{B}$ defined as follows: $\hat{m}^y_u(\Lambda \times M) := (u^\alpha)\tilde{\Psi}_y \tilde{m}(\Lambda\Psi_y \times MR_y)$.

One can easily check that, for each $\gamma, 1 \le \gamma \le \infty$, the linear manifold $\tilde{L}^\alpha_u \cap \tilde{L}^\gamma_u$ is dense in $\tilde{L}^\gamma_u$. The rest of this subsection is devoted to the possibility of extension of the operators $\tilde{U}_y$ from this set to (bounded) operators in the whole space $\tilde{L}^\gamma_u$.

Lemma 3.4. *1. Each operator $\tilde{U}_y$ can be extended to an isometry in $\tilde{L}^1_u(\tilde{\Omega})$.*

2. Each operator $\tilde{U}_y$ can be extended to bounded operator in $\tilde{L}^\infty_u(\tilde{\Omega})$ and $\|\tilde{U}_y\|_{\tilde{L}^\infty_u} \le \|T\|^\alpha_\alpha$.

Proof. 1. By virtue of Lemmas 3.2 and 3.1, for each $f \in \tilde{L}^\alpha_u(\tilde{\Omega}) \cap \tilde{L}^1_u(\tilde{\Omega}), f(\omega) \ge 0$ m-a.e. we have:

$$||f\tilde{U}_y||_{\tilde{L}^1_u} = ||\int_\Omega f\tilde{U}_y(\omega,x)u(\omega,x)^\alpha dm||_{L^\infty} = ||\int_\Omega u^{-1}(uf)\tilde{T}_y(\omega,x)[u(\omega,x)]^\alpha dm||_{L^\infty} =$$
$$||\int_\Omega (uf)\tilde{T}_y u^{\alpha-1} dm||_{L^\infty} = ||\int_\Omega uf\tilde{T}^*_y(u^{\alpha-1})dm||_{L^\infty}$$
$$= ||\int_\Omega ufu^{\alpha-1}dm||_{L^\infty} = ||\int_\Omega fu^\alpha dm||_{L^\infty} = ||f||_{\tilde{L}^1_u}.$$

Hence $\|f\tilde{U}_y\|_{\tilde{L}^1_u} = ||f||_{\tilde{L}^1_u}$; since $\tilde{L}^\alpha_u \cap \tilde{L}^1_u$ is dense in $\tilde{L}^1_u$ and the operator $\tilde{U}_y$ is invertible, Statement 1 is proved.

2. Let $f \in \tilde{L}^\infty_u(\tilde{\Omega}) \cap \tilde{L}^\alpha_u(\tilde{\Omega})$, $f \geq 0$ $\tilde{m}$-a.e. Since the operators $\tilde{T}_y$ and $\tilde{U}_y$ are positive, by Lemma 3.1, $f\tilde{U}_y = u^{-1}(uf)\tilde{T}_y \leq \|f\|_{\tilde{L}^\infty_u} u^{-1}(u\tilde{T}_y) \leq \|f\|_{L^\infty_u} u^{-1}\|T\|^\alpha_\alpha = \|f\|_{\tilde{L}^\infty_u}\|T\|^\alpha_\alpha$, hence $||f\tilde{U}_y||_{\tilde{L}^\infty_u} \leq \|T\|^\alpha_p \|f\|_{\tilde{L}^\infty_u}$. □

As in Lemma 3.3 we can easily verify that $\tilde{U}$ is an L-representation in $\tilde{L}^\infty_u(\tilde{\Omega})$ and in $\tilde{L}^1_u(\tilde{\Omega})$.

Lemma 3.5. *Let V be an invertible L-operator in $\tilde{L}^\alpha_u$ that can be extended to L-operators in $\tilde{L}^1_u$ and $\tilde{L}^\infty_u$. Then for each $\gamma, 1 \leq \gamma \leq \infty$,*

$$||V||_{\tilde{L}^\gamma_u} \leq ||V||_{\tilde{L}^\infty_u} ||V^{-1}||^{\frac{1}{\gamma}}_{\tilde{L}^\infty_u} ||V||^{\frac{1}{\gamma}}_{\tilde{L}^1_u}. \tag{3.7}$$

Proof. Assume V and V^{-1} have the canonical representations: $V = h\Psi, V^{-1} = h_{V^{-1}}\Psi_{V^{-1}}$. By Propositions 6.1–6.3, we have: $||V||_{\tilde{L}^\infty_u} = |||h|||_{\tilde{L}^\infty_u}, ||V||_{\tilde{L}^\gamma_u} = ||\Psi^{-1}|h| \left(\frac{d\tilde{m}}{dm}\right)^{\frac{1}{\gamma}} ||_{\tilde{L}^\infty_u}, 1 \leq \gamma < \infty$; $\Psi = \Psi^{-1}_{V^{-1}}$ and

$$h = (\Psi h_{V^{-1}})^{-1} = \left(\Psi^{-1}_{V^{-1}} h_{V^{-1}}\right)^{-1}. \tag{3.8}$$

The condition of the Lemma implies: when $\gamma = 1$

$$\Psi^{-1}|h| \left(\frac{d\tilde{m}}{dm}\right) \leq ||V||_{\tilde{L}^1_u}, \tag{3.9}$$

and when $\gamma = \infty$

$$\Psi^{-1}|h| \leq ||V||_{\tilde{L}^\infty_u}, \Psi^{-1}_{V^{-1}}|h_{V^{-1}}| \leq |V^{-1}||_{\tilde{L}^\infty_u}. \tag{3.10}$$

Relations (3.8) and (3.10) imply: $|h| \geq (||V^{-1}||_{\tilde{L}^\infty_u})^{-1}$. Since Ψ is a positive operator and $\Psi c = c, c \in \mathbb{R}$, we have $\Psi^{-1}|h| \geq (||V^{-1}||_{\tilde{L}^\infty_u})^{-1}$, and by (3.9)

$$\frac{d\tilde{m}}{dm}(\omega, x) \leq ||V||_{\tilde{L}^1_u} ||V^{-1}||_{\tilde{L}^\infty_u}. \tag{3.11}$$

Therefore (3.7) holds. □

Lemmas 3.4 and 3.5 (with $V = \tilde{U}_y$) imply the following statement.

Corollary 3.1. *All operators $\tilde{U}_y$ are consistent with operators acting in $\tilde{L}^\gamma_u$ (denoted also $\tilde{U}_y$) and $||\tilde{U}_y||_{\tilde{L}^\gamma_u} \leq \|T\|^{2\alpha}_\alpha$ $(1 \leq \gamma \leq \infty)$.*

Proposition 3.1. *For each $x \in X$ the mapping $\tilde{U} : y \mapsto \tilde{U}_y$ is a measurable bounded positive right $\tilde{L}^1_u - \tilde{L}^\infty_u$ representation of X and, moreover,*

$$C_1 := \sup_{1 \leq \gamma \leq \infty} \|\tilde{U}\|_{\tilde{L}^\gamma_u} \leq \|T\|^{2\alpha}_\alpha. \tag{3.12}$$

3.2. Space $\tilde{L}^{\delta,1}_u$ and representation $\tilde{W}'$

As before, $1 < \alpha < \infty$. Consider the right representation $\tilde{U}$ acting in all spaces $L^\gamma_u = L^{\gamma,\infty}_u, 1 \leq \gamma \leq \infty$, defined in the previous subsection. For each pair of mutually conjugate numbers γ, δ we consider two duality relations. The first one is between the Banach spaces $\tilde{L}^{\gamma,\infty}_u$ and $\tilde{L}^{\delta,1}_u$ established by the bilinear form

$\langle f,g\rangle_u := \int_{\tilde{\Omega}} f\bar{g}u^\alpha d\tilde{m}$. It is easy to check that $|\langle f,g\rangle_u| \le ||f||_{\tilde{L}_u^{\gamma,\infty}}||g||_{\tilde{L}_u^{\delta,1}}$. We denote by $\tilde{U}_y'$ the operator in $\tilde{L}_{u,w}^{\beta,1}$ conjugate to $\tilde{U}_y$ with respect to this bilinear form. By Proposition 3.1, *$\tilde{U}'$ is a positive $\tilde{L}_u^{1,1} - \tilde{L}_u^{\infty,1}$ representation.*

Let us consider a strictly positive probability density w on X with respect to ν and the Banach spaces $\tilde{L}_{u,w}^{\gamma,\infty} = \tilde{L}_u^{(\gamma,\infty)}$ and the spaces $\tilde{L}_{u,w}^{\delta,1}$ of all $(\mathcal{F},\mathcal{B})$-measurable functions with the finite norms: $||g||_{u,w}^{\gamma,1} := \int_X[\int_\Omega |g(\omega,x)|^\gamma(u(\omega,x))^\alpha m(d\omega)]^{\frac{1}{\gamma}}]w(x)\nu(dx)$.

Our second dual pair is $\tilde{L}_u^{\alpha,\infty} = \tilde{L}_{u,w}^{\alpha,\infty}, \tilde{L}_{u,w}^{\beta,1}$ with respect to the bilinear form

$$\langle f,g\rangle_{u,w} := \int_{\tilde{\Omega}} f\bar{g}u^\alpha w(x)d\tilde{m}.$$

Note that the mapping $i_w : g \mapsto gw$ is a positive linear isometry from $\tilde{L}_{u,w}^{\alpha,\infty}$ onto $\tilde{L}_u^{\alpha,\infty}$ and from $\tilde{L}_{u,w}^{\beta,1}$ onto $\tilde{L}_u^{\beta,1}$. We denote $\tilde{W}_y := i_w\tilde{U}_y i_w^{-1}$ (acting in $\tilde{L}_{u,w}^{\alpha,\infty} = \tilde{L}_u^{\alpha,\infty}$) and $\tilde{W}_y' := i_w\tilde{U}_y' i_w^{-1}$; these are right (resp. left) positive operators in $\tilde{L}_{u,w}^{\alpha,\infty}$ (resp. $\tilde{L}_{u,w}^{\beta,1}$). The operator $\tilde{W}_y'$ is the conjugate to $\tilde{W}_y$.

It is easy to verify that Proposition 3.1 and the relation between the operators $\tilde{U}_y, \tilde{U}_y'$ and $\tilde{W}_y'$ imply the following property of the representation $\tilde{W}'$.

Proposition 3.2. *$\tilde{W}' : y \mapsto \tilde{W}_y'$ is a $\tilde{L}_{u,w}^{1,1} - \tilde{L}_{u,w}^{\infty,1}$ Lamperti positive representation of X, respectively, and*

$$\max_{1\le\gamma\le\infty 1}\{||\tilde{W}'||_{\tilde{L}_{u,w}^{\gamma,1}}\} = C_1 \le ||T||_\alpha^{2\alpha}.$$

3.3. Employing the conjugate representation $\tilde{W}'$

The following statement is a generalized version of a result due to Duncan [17]. Let $1 < \alpha < \infty$.

Lemma 3.6. *Let $\mathcal{K} = \{\kappa_n\}$ be a sequence of probability measures on X. Assume that there is a constant $K > 0$ such that for each $N \in \mathbb{N}, N \ge 1$, and for each collection of mod $\tilde{m}$ pairwise disjoint sets $\Lambda_1, ..., \Lambda_N \in \mathcal{F}\times\mathcal{B}$ with*

$$||\sum_{n=1}^N \int_X (\tilde{W}_y'(1_{\Lambda_n}))(\omega,x))\kappa_n(dy)||_{\tilde{L}_{u,w}^{\beta,1}} \le K. \tag{3.13}$$

Then for any non-negative function $f \in \tilde{L}_u^{\alpha,\infty}$

$$\int_{\tilde{\Omega}}[\sup_{n\le N}\int_X f(\omega,x)\tilde{W}_y\cdot\kappa_n(dy)](u(\omega,x))^\alpha m(d\omega)w(x)\nu(dx) \le K||f||_{\tilde{L}_u^{\alpha,\infty}}. \tag{3.14}$$

Proof. By Lemma 3.1, $1 \in \tilde{L}_{u,w}^{\beta,1}$. Denote $fV_n(\omega,x) := \int_X f(\omega,x)\tilde{W}_y\kappa_n(dy)$. Then $V_n'g(\omega,x) = \int_X \tilde{W}_y'g(\omega,x)\kappa_n(dy)$. Put $M_k = \{\omega : \sup_{n\le N} fV_n = fV_k\}, \Lambda_n = M_n\setminus\cup_{k<n}M_k, (n \le N)$. Then $\sup_{n\le N} fV_n = \sum_{n=1}^N fV_n\cdot 1_{\Lambda_n}$, and

$$\int_{\tilde{\Omega}}\sup_{n\le N} fV_n(u(\omega,x))^\alpha w(x)d\tilde{m} = \int_{\tilde{\Omega}}\sum_{n=1}^N fV_n 1_{\Lambda_n}(u(\omega,x))^\alpha w(x)d\tilde{m} =$$

$$\sum_{n=1}^N\langle fV_n, 1_{\Lambda_n}\rangle_{u,w} = \sum_{n=1}^N\langle f, V_n'1_{\Lambda_n}\rangle_{u,w} = \langle f, \sum_{n=1}^N V_n'1_{\Lambda_n}\rangle_{u,w} \le$$

$$||f||_{\tilde{L}_u^{\alpha,\infty}}||\sum_{n=1}^N V_n'1_{\Lambda_n}||_{\tilde{L}_{u,w}^{\beta,1}} \le K\|f\|_{\tilde{L}_{u,w}^{\alpha,\infty}}. \quad \square$$

We denote $d\tilde{m}_{u,w} := u^{\alpha} w dm\, d\nu$.

Lemma 3.7. *Let l be a natural number, $k \geq 1, s_i > 0, i = 1, ..., l, \sum_{i=1}^{l} s_i = \beta$ and let f $f_1, \dots, f_l$ be non-negative functions in $\tilde{L}^{1,1}_{u,w}$. Then*

(i) for the non-negative values of $f_k^{s_k}$ we have $f_1^{s_1} \cdot \dots \cdot f_l^{s_l} \in \tilde{L}^{\beta,1}_{u,w}(\tilde{\Omega})$;

(ii) for each $y \in X$

$$\int_{\tilde{\Omega}} \prod_i^l [\tilde{W}'_y f_i]^{s_i} \tilde{m}_{u,w} \leq ||\tilde{W}'_y||^{\beta}_{\tilde{L}^{\beta,1}_{u,w}} \int_{\tilde{\Omega}} \prod_i^l f_i^{s_i} d\tilde{m}_{u,w} \leq C_1^{\beta} \int_{\tilde{\Omega}} \prod_i^l f_i^{s_i} d\tilde{m}_{u,w}.$$

Proof. Statement (i) follows readily from the Hölder inequality.

Proof of Statement (ii). By Lemma 3.1, $y \mapsto \tilde{W}'_y$ is a Lamperti left representation; let $\tilde{W}'_y f = h_y \cdot \Psi_y f$ be its canonical representation. We have: $\prod_{i=1}^{l} \tilde{W}'_y f_i^{s_i} = \prod_{i=1}^{l} (h_y \cdot \Psi_y f_i)^{s_i} = h^{\beta} \prod_{i=1}^{l} \Psi_y f_i^{s_i}$. Hence, if we denote $f = (\prod_{i=1}^{l} f_i^{s_i})^{\frac{1}{\beta}}$, we have for μ-a.e. $x \in X$:

$$\int_{\tilde{\Omega}} \prod_i^l [\tilde{W}'_y f_i]^{s_i} d\tilde{m}_{u,w} = \int_{\tilde{\Omega}} h^{\beta} \prod_{i=1}^{l} \Psi_y f_i^{s_i} d\tilde{m}_{u,w} = \int_{\tilde{\Omega}} h^{\beta} \Psi_y f^{\beta} d\tilde{m}_{u,w} =$$

$$\int_{\tilde{\Omega}} [\tilde{W}'_y f]^{\beta} d\tilde{m}_{u,w} \leq ||\tilde{W}'_y||^{\beta}_{L^{\beta,1}_{u,w}} \int_{\tilde{\Omega}} f^{\beta} d\tilde{m}_{u,w} = ||\tilde{W}'_y||^{\beta}_{L^{\beta,1}_{u,w}} \int_{\tilde{\Omega}} \prod_i^l f_i^{s_i} d\tilde{m}_{u,w}. \quad \square$$

The following key Lemma is a generalization and a refinement of a statement proved for measure preserving actions by Shulman [41,42] (see also [46] for the case $\alpha = 2$).

Lemma 3.8. *Let $1 < \beta < \infty, k \in \mathbb{N}, k \geq \beta$. Let $\{\Lambda_n\}$ be a sequence of disjoint $\mathcal{F} \times \mathcal{B}$-measurable subsets of Ω and let $\mathcal{K} = \{\kappa_n\}$ be a weakly tempered sequence of order k on X. Then for any $N \in \mathbb{N}$*

$$|| \sum_{n=1}^{N} \int_X \tilde{W}'_y (1_{\Lambda_n}(\omega, x)) \kappa_n(dy) ||_{\tilde{L}^{\beta,\infty}_{u,w}} \leq (k^k \tau_k(\mathcal{K}))^{\frac{1}{\beta}} \|T\|_{\alpha}^{2\alpha k+1} \|\varphi\|_{\beta}. \tag{3.15}$$

Proof. For $c_i > 0$ and $k \geq \beta$ we have $\sum_n c_n \leq (\sum_n c_n^{\frac{\beta}{k}})^{\frac{k}{\beta}}$ (see, e.g. Ch. I, §16 in [6]) and thus

$$(\sum_n c_n)^{\beta} \leq (\sum_n c_n^{\frac{\beta}{k}})^k; \tag{3.16}$$

Let $a_1 \geq 0, ..., a_N \geq 0$. We have

$$\left[\sum_{n=1}^{N} a_n\right]^k = \sum_{l=1}^{k} \sum_{\substack{\sum_{i=1}^{l} s_i = k \\ s_i > 0}} \frac{k!}{s_1! \dots s_l!} \sum_{n_l = l}^{N} \sum_{n_{l-1}=l-1}^{n_l - 1} \dots \sum_{n_2=2}^{n_3-1} \sum_{n_1=1}^{n_2-1} \prod_{i=1}^{l} a_{n_i}^{s_i}.$$

We use inequality (3.16) and the latter equality with $a_n = c_n^{\frac{\beta}{k}}$ where $c_1 > 0, \dots, c_N > 0$, and obtain:

$$\left[\sum_{n=1}^{N} c_n\right]^{\beta} \leq \sum_{l=1}^{k} \sum_{\substack{\sum_{i=1}^{l} s_i = k \\ s_i > 0}} \frac{k!}{s_1! \dots s_l!} \sum_{n_l = l}^{N} \sum_{n_{l-1}=l-1}^{n_l - 1} \dots \sum_{n_2=2}^{n_3-1} \sum_{n_1=1}^{n_2-1} \prod_{i=1}^{l} c_{n_i}^{t_i(s_i)}, \tag{3.17}$$

where $t_i(s_i) := \frac{s_i \beta}{k}$ (in what follows we usually drop the argument s_i). Note that $\sum_{i=1}^{k} t_i = \beta$.

Recall that $d\tilde{m}_{u,w} = u^\alpha w dm\, d\nu, W'$ is a left representation in $\tilde{L}^{\beta,1}_{u,w}$; see also Proposition 3.2. We denote $v_i := 1_{\Lambda_i}$. By virtue of (3.17) we obtain:

$$||\sum_{i=1}^N \int_X \tilde{W}'_y(1_{\Lambda_i}(\omega,x))\kappa_n(dy)||^\beta_{\tilde{L}^{\beta,1}_{u,w}} = \int_{\tilde{\Omega}} [\sum_{i=1}^N \int_X \tilde{W}'_y v_i(\omega,x)\kappa_n(dy)]^\beta \tilde{m}_{u,w}(d\omega dx) \leq$$

$$\int_{\tilde{\Omega}} \sum_{l=1}^k \sum_{\substack{\sum_{i=1}^l s_i = k \\ s_i > 0}} \frac{k!}{s_1! \ldots s_l!} \sum_{n_l=l}^N \sum_{n_{l-1}=l-1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \times$$

$$\prod_{i=1}^l \{\int_X [\tilde{W}'_{y_i} v_{n_i}(\omega,x)]\kappa_{n_i}(dy_i)\}^{t_i(s_i)} \tilde{m}_{u,w}(d\omega dx) =$$

$$\sum_{l=1}^k \sum_{\substack{\sum_{i=1}^l s_i = k \\ s_i > 0}} \frac{k!}{s_1! \ldots s_l!} D_{l,t_1(s_1),\ldots,t_l(s_l)} \tag{3.18}$$

where

$$D_{l,t_1,\ldots,t_l} :=$$
$$\sum_{n_l=l}^N \sum_{n_{l-1}=l-1}^{n_l-1} \cdots \sum_{n_1=1}^{n_2-1} \int_{\tilde{\Omega}} \prod_{i=1}^l [\int_X \tilde{W}'_{y_i} v_{n_i}(\omega,x)\kappa_{n_i}(dy_i)]^{t_i} \tilde{m}_{u,w}(d\omega dx),$$

$1 \leq l \leq k$ (when s_i is fixed we write t_i instead of $t_i(s_i)$). By Hölder's inequality

$$D_{l,t_1,\ldots,t_l} \leq$$
$$\sum_{n_l=l}^N \sum_{n_{l-1}=l-1}^{n_l-1} \cdots \sum_{n_1=1}^{n_2-1} \int_{\tilde{\Omega}} \{\prod_{i=1}^l \int_X [\tilde{W}'_{y_i} v_{n_i}(\omega,x)]^{t_i} \kappa_{n_i}(dy_i)\} \tilde{m}_{u,w}(d\omega dx).$$

Below we put $X_1 = X_2 = ... = X_l = X$ and denote:

$$\prod_{i=1}^l \int_{X_i} f(dz_\alpha, ..., dz_l) = \int_{X_l} (...(\int_{X_1} f(dz_1, ..., dz_l))...).$$

Using the Fubini theorem we obtain:

$$D_{l,t_1,\ldots,t_l} \leq$$
$$\sum_{n_l=l}^N \sum_{n_{l-1}=l-1}^{n_l-1} \cdots \sum_{n_1=1}^{n_2-1} \int_{\tilde{\Omega}} \{\prod_{i=1}^l \int_{X_i} \prod_{j=1}^l [\tilde{W}'_{y_j} v_{n_j}(\omega,x)]^{t_j} \kappa_{n_i}(dy_i)\} \tilde{m}_{u,w}(d\omega dx) =$$
$$\sum_{n_l=l}^N \sum_{n_{l-1}=l-1}^{n_l-1} \cdots \sum_{n_1=1}^{n_2-1} \prod_{i=1}^l \int_{X_i} \{\int_{\tilde{\Omega}} \prod_{j=1}^l [\tilde{W}'_{y_j} v_{n_j}(\omega,x)]^{t_j} \tilde{m}_{u,w}(d\omega dx)\} \kappa_{n_i}(dy_i).$$

We introduce new variables $z_1, z_2, ..., z_N$: $z_1 = y_1$, $z_i = y_{i-1}^{-1} \cdot y_i, 1 < i \leq N$, i.e.

$$y_i = z_1 \ldots z_{i-1} z_i, \quad i = 2, ..., N.$$

We obtain:

$$D_{l,t_1,\dots,t_l} \leq$$
$$\sum_{n_l=l}^{N} \sum_{n_{l-1}=l-1}^{n_l-1} \dots \sum_{n_1=1}^{n_2-1} \prod_{i=1}^{l} \int_{X_i} \{ \int_{\tilde{\Omega}} \prod_{j=1}^{l} \left[\tilde{W}'_{z_1 \dots z_{j-1} z_j} v_{n_j} \right]^{t_j} \tilde{m}_{u,w}(d\omega dx) \} \kappa_{n_i}(z_1 \dots z_{i-1} dz_i).$$

We have $\sum_{l=1}^{n} v_l \leq 1, n \in \mathbb{N}$, because the sets Λ_l are mutually disjoint. Let's fix a natural number $l \leq \min\{k, N\}$ and apply Lemma 3.7 and the well-known the property of the norm in the following evaluations:

$$D_{l,t_1,\dots,t_l} \leq \sum_{n_l=l}^{N} \sum_{n_{l-1}=l-1}^{n_l-1} \dots \sum_{n_1=1}^{n_2-1} \prod_{i=2}^{l} \times$$
$$\int_{X_i} \int_{X_1} \{ \int_{\tilde{\Omega}} [\tilde{W}'_{z_1} v_{n_1}]^{t_1} \prod_{j=2}^{l} \left[W'_{z_1} \tilde{W}'_{z_2 \dots z_{j-1} z_j} v_{n_j} \right]^{t_j} \tilde{m}_{u,w}(d\omega\, du) \} \kappa_{n_i}(z_1 \dots z_{i-1} dz_i) \leq$$
$$C_1^{\beta} \sum_{n_l=l}^{N} \sum_{n_{l-1}=l-1}^{n_l-1} \dots \sum_{n_1=1}^{n_2-1} \prod_{i=2}^{l} \int_{X_i} \int_{X_1} \{ \int_{\tilde{\Omega}} [v_{n_1}]^{t_1} \prod_{j=2}^{l} \left[\tilde{W}'_{z_2 \dots z_{j-1} z_j} v_{n_j} \right]^{t_j} \tilde{m}_{u,w}(d\omega\, dx) \} \times$$
$$\prod_{r=1}^{l} \kappa_{n_r}(z_1 \dots z_{r-1} dz_r) \leq$$
$$C_1^{\beta} \sum_{n_l=l}^{N} \sum_{n_{l-1}=l-1}^{n_l-1} \dots \sum_{n_2=2}^{n_3-1} \prod_{i=2}^{l} \int_{X_i} \{ \int_{\tilde{\Omega}} \sum_{n_1=1}^{n_2-1} v_{n_1} \prod_{j=2}^{l} \left[\hat{U}_{z_2 \dots z_{j-1} z_j} v_{n_j} \right]^{t_j} \tilde{m}_{u,w}(d\omega\, dx) \times$$
$$\max_{n_1 : n_1 < n_2} \int_{X_1} \prod_{r=1}^{l} \kappa_{n_r}(z_1 \dots z_{r-1} dz_r)$$

(the constant C_1 was defined in Lemma 3.1). Since $\sum_{n_1=1}^{n_2-1} v_{n_1} \leq 1$ we have

$$D_{l,t_1,\dots,t_l} \leq$$
$$C_1^{\beta} \sum_{n_l=l}^{N} \sum_{n_{l-1}=l-1}^{n_l-1} \dots \sum_{n_2=1}^{n_3-1} \prod_{i=2}^{l} \int_{X_i} \{ \int_{\tilde{\Omega}} \prod_{j=2}^{l} \left[\hat{U}_{z_2 \dots z_{j-1} z_j} v_{n_j} \right]^{t_j} \tilde{m}_{u,w}(d\omega\, dx) \} \times$$
$$\max_{n_1 : n_1 < n_2} \int_{X_1} \prod_{r=1}^{l} \kappa_{n_r}(z_1 \dots z_{r-1} z_r) \}.$$

We took the first step of our proof getting rid of the factors $\hat{U}_{z_1}$. Let us proceed this way and eliminate the factor $\hat{U}_{z_i}$ in each i-th step in which Lemma 3.7(1) is used for $L^{\sum_{j=i}^{l} t_i}(\tilde{\Omega}, \mathcal{F} \times B, \tilde{m}_{u,w})$. After $l - 1$ steps we obtain

$$D_{l,t_1,\dots,t_l} \leq C_1^{\beta(l-1)} \sum_{n_l=l}^{N} \int_{X_l} \int_{\tilde{\Omega}} \left[\hat{U}_{z_l} v_{n_l} \right]^{t_l} \tilde{m}_{u,w}(d\omega\, dx) \times$$
$$\{ \max_{n_{l-1} : n_{l-1} < n_l} \int_{X_{l-1}} \dots \max_{n_1 : n_1 < n_2} \int_{X_1} \prod_{r=1}^{l} \kappa_{n_r}(z_1 \dots z_{r-1} dz_r) \}.$$

At last, we perform the l^{th} step: using the previous estimate, Lemma 3.7 one more time and Lemma 3.1 obtain:

$$D_{l,t_1,\ldots,t_l} \leq C_1^{\beta l} \int\limits_{\tilde{\Omega}} \sum_{n_l=1}^{N} 1_{\Lambda_{n_l}} \tilde{m}_{u,w}(d\omega\, dx) \max_{n_l : n_l \leq N} \int\limits_X \max_{n_{l-1} : n_{l-1} < n_l} \int\limits_X \ldots$$

$$\max_{n_1 : n_1 < n_2} \int\limits_X \prod_{i=1}^{l} \kappa_{n_i}(z_1 \ldots z_{i-1} dz_i) \leq$$

$$C_1^{\beta l} r_k(\mathcal{K}) \int\limits_{\tilde{\Omega}} \tilde{m}_{u,w}(d\omega\, dx) \leq C_1^{\beta k} \tau_k(\mathcal{K}) \|T\|_\alpha^\beta \|\varphi\|_\beta^\beta.$$

By virtue of (3.18), this implies our statement since

$$\sum_{l=1}^{k} \sum_{\substack{\sum_{i=1}^{l} s_i = k \\ s_i > 0}} \frac{k!}{s_1! \ldots s_l!} \leq k^k. \quad \square$$

Recall that $\tilde{W} = \tilde{V}$. By virtue of Lemmas 3.6 and 3.8 we obtain the following statement.

Lemma 3.9. *Let $1 < \alpha < \infty$, $\beta = \frac{\alpha}{\alpha-1}$, $k \in \mathbb{N}, k \geq \beta$. Let $\mathcal{K} = \{\kappa_n\}$ be a weakly tempered sequence of order k on X. Then for any non-negative function $\tilde{g} \in \tilde{L}_u^{\alpha,\infty}$*

$$\int\limits_\Omega \int\limits_X \sup_{n \geq k} \int\limits_X (\tilde{g}\tilde{U}_y)(\omega, x)\kappa_n(dy)(u(\omega, x))^\alpha w(x)\nu(dx)m(d\omega)$$

$$\leq ||T||_\alpha^{2\alpha k+1}[k^k \tau_k(\mathcal{K})]^{\frac{1}{\beta}} \|\varphi\|_\beta ||g||_{\tilde{L}_u^\alpha}. \tag{3.19}$$

3.4. Back from the "space–time" $\tilde{\Omega}$ to the space Ω

One can readily rewrite inequality (3.19) in terms of $\tilde{T}$ and $\tilde{f}(\omega, x) \in \tilde{L}_u^{\alpha,\infty}$. For each $\tilde{f} \in \tilde{L}_u^{\alpha,\infty}$ we choose $\tilde{g} = u^{-1}\tilde{f}$. Recall that $\tilde{g}\tilde{W}_y = (u\tilde{g})\tilde{T}_y u^{-1} = \tilde{f}\tilde{T}_y u^{-1}$. And then we put $\tilde{f}(\omega, x) = f(\omega)1_X(x)$; now $||g||_{\tilde{L}_u^\alpha} = ||f||_\alpha$ and $\tilde{f}\tilde{T}_y(\omega, x) = fT_y(\omega)$. Recall: $(u(\omega, x))^{\alpha-1} = S_{x^{-1}}\varphi$.

Proposition 3.3. *For any $f \in L^\alpha(\Omega, \mathcal{F}, m), f \geq 0$,*

$$\int\limits_\Omega | \int\limits_X S_{x^{-1}}\varphi(\omega)w(x)\nu(dx) \cdot \sup_{n \geq k}[\int\limits_X]fT_y(\omega)\kappa_n(dy)]m(d\omega)| \leq$$

$$[k^k r_k(\mathcal{K}]^{\frac{1}{\beta}} \|T\|_\alpha^{2\alpha k+1} \|\varphi\|_\beta ||||f||_\alpha.$$

Proof.

$$\int\limits_\Omega \int\limits_X \sup_{n \geq k} \int\limits_X (\tilde{g}\tilde{U}_y)(\omega, x)\kappa_n(dy)(u(\omega, x))^\alpha w(x)\nu(dx)m(d\omega) =$$

$$\int\limits_\Omega \int\limits_X [\sup_{n \geq k} \int\limits_X fT_y(\omega)\kappa_n(dy)](u(\omega, x))^{\alpha-1}\nu(dx)w(x)m(d\omega) =$$

$$\int\limits_\Omega [\int\limits_X S_{x^{-1}}\varphi(\omega)w(x)\nu(dx)]\sup_{n \geq k}[\int\limits_X fT_y(\omega)\kappa_n(dy)]m(d\omega). \quad \square$$

In order to derive the DET we have to eliminate the disturbing factor $\int_X S_{x^{-1}}\varphi(\omega)w(x)\nu(dx)$. We start with a rather weak version of the "local" PET.

Lemma 3.10. *Let $\beta \geq 1$ and let S be a left L^β-representation of X, $\varphi \in L^\beta(\Omega, \mathcal{F}, m)$. There is a sequence of densities $\{w_i\}$ with support X such that*

$$\lim_{i\to\infty} \int\limits_X S_{x^{-1}}\varphi(\omega)w_i(x)\nu(dx) = \varphi(\omega)\ m\text{−}a.e. \tag{3.20}$$

Proof. a) *Construction of a sequence w_i with compact supports.* If X is discrete we may simply choose $w_i(x) = 1$ if $x = e$ and $w_i(x) = 0$ elsewhere $(i = 1, 2, ...)$. So let us assume that X is non-discrete. Equation (3.20) is equivalent to $\lim_{i\to\infty} \int_X S_x\varphi(\omega)\hat{w}_i(x)\mu(dx) = \varphi(\omega)$ m−a.e. where $\hat{w}_i(x) = w_i(x^{-1})$ and μ is the left Haar measure. Let $v \in L_1(m), ||v||_1 = 1$, $0 < v(\omega) \leq C < \infty$ for all $\omega \in \Omega$. Denote $m_v(\Lambda) := \int_\Lambda v\,dm$. Evidently $m_v(X) = 1, m_v \sim m, S_x\varphi \in L^\beta_{m_v}$. To simplify the notation, in this proof we write $||\cdot||_\beta$ instead of $||\cdot||_{L^\beta_{m_v}}$. Let U, V be bounded symmetric neighborhoods of the identity e with closures $[U], [V]$ and $V^2 \subset U$; $g(x) = 1_{[U]}(x)||S_x\varphi - \varphi||_\beta$. It is clear that $g \in L^1(X\,\mathcal{B}, \mu)$.

For each density λ $\int_X g(xy)\lambda(x)\mu(dx) = \int_X 1_{[U]}(xy)||S_{xy}\varphi - \varphi||_\beta\lambda(x)\mu(dx)$. By Theorem 20.12 in [24], there is a decreasing sequence of symmetric neighborhoods $U_i \subset V$ such that for each sequence $\{\lambda_i\}$ of symmetric densities supported by $[U_i]$ we have

$$(L^1(\mu)) \lim_{i\to\infty} \int\limits_X g(xy)\lambda_i(x)\mu(dx) = 0,$$

hence there is a subsequence $\{\eta_i\}$ of $\{\lambda_i\}$ such that

$$\lim_{i\to\infty} \int\limits_X 1_{[U]}(xy)||S_{xy}\varphi - \varphi||_\beta\eta_i(x)\mu(dx) = 0\ \mu\text{−a.e.}$$

Note that $xy \in U$, if $x, y \in V$. Let $y_0 \in V$ be such that

$$\lim_{i\to\infty} \int\limits_X ||S_{xy_0}\varphi - \varphi||_\beta\eta_i(x)\mu(dx) = 0.$$

By changing the argument and replacing the densities η_i by $\sigma_i(x) = \eta_i(xy_0^{-1})$ we obtain

$$\lim_{i\to\infty} \int\limits_X ||S_x\varphi - \varphi||_\beta\sigma_i(x)\mu(dx) = 0$$

b) *Construction of a sequence $\{w_i\}$ of densities supported by X.* Denote $\mathrm{supp}(\sigma_i) = U_iy_0$ by $\hat{U}_i$. Let χ_i be a sequence of strictly positive measurable functions on X such that $\int_{\hat{U}_i^c} \chi_i d\mu = \frac{1}{i}$. We have: $\int_{\hat{U}_i^c} \chi_i(x)||S_x\varphi - \varphi||_\beta\mu(dx) \leq \frac{1}{i}(||S||_\beta + 1)|||\varphi||_\beta \to 0$ as $i \to \infty$. It is clear that the sequence of densities $\pi_i = 1_{(\hat{U}_i)^c}\chi_i + (1 - \frac{1}{i})1_{(\hat{U}_i)}\sigma_i$ has the property:

$$\lim_{i\to\infty} \int\limits_X [\int\limits_\Omega |S_x\varphi - \varphi|^\beta dm_v]^{\frac{1}{\beta}}\pi_i(x)\mu(dx) = 0.$$

By the Fubini theorem and the Hölder inequality

$$\int_\Omega \int_X |S_x\varphi - \varphi|\pi_i(x)\mu(dx)dm_v \leq \int_X (\int_\Omega |S_x\varphi - \varphi|^\beta dm_v)^{\frac{1}{\beta}} \pi_i(x)\mu(dx)$$

It follows from the last two relations that $\lim_{i\to\infty} \int_\Omega \int_X |S_x\varphi - \varphi|\pi_i(x)\mu(dx)dm_v = 0$, hence for some subsequence $\{\hat{w}_l\}$ of the sequence $\{\pi_i\}$ we have $\lim_{i\to\infty} \int_X S_x\varphi\hat{w}_l(x)\mu(dx) = \varphi$ m-a.e., so the sequence $w_l = \hat{w}_l(x^{-1})$ meets the required property (3.20). $\square$

Recall that $|fT| = |f||T|$ for each $f \in L_u^\alpha$. The Fatou lemma, relation (3.20), and Proposition 3.3 bring us to the following result.

Theorem 3.1. *Let $1 < \alpha < \infty$, let $\mathcal{K} = \{\kappa_n\}$ be a sequence of probability measures on X, weakly tempered of order $k \geq \beta := \frac{\alpha}{\alpha-1}$. Assume that $T : x \mapsto T_x$ is a Lamperti right L^α-representation. Then*

$$\int_\Omega \sup_{n\geq 1}[\int_X |fT_y(\omega)|\kappa_n(dy)]\varphi(\omega)m(d\omega) \leq (k^k\tau_k(\mathcal{K}))^{\frac{1}{\beta}} \|T\|_\alpha^{2\alpha k+1}||\varphi||_\beta||f||_\alpha. \tag{3.21}$$

Let us remind that φ is an arbitrary strictly positive function in $L^\beta(\Omega)$, and functions $g \in L^\beta(\Omega), g \neq 0$ a.e., form a dense subset of this space. Let us consider the functional $l(g) = \int_\Omega \sup_{n\geq 1}[\int_X |fT_y)(\omega)|\kappa_n(dy)]g(\omega)$ $m(d\omega),\ g \in L^\beta(\Omega)$. We have:

$$||l||_\beta = ||\sup_{n\geq 1}[\int_X |fT_y)(\omega)|\kappa_n(dy)||_\alpha.$$

On the other hand, by (3.21),

$$||l||_\beta = |\sup_{g\in L^\beta, ||g||_\beta\leq 1} \int_\Omega \sup_{n\geq 1}[\int_X |fT_y)(\omega)|\kappa_n(dy))]g(\omega)m(d\omega)| \leq$$

$$\sup_{||g||_\beta\leq 1} \int_\Omega \sup_{n\geq 1}[\int_X |fT_y)(\omega)|\kappa_n(dy)]|g(\omega)|m(d\omega) \leq (k^k\tau_k(\kappa))^{\frac{1}{\beta}} \|T\|_\alpha^{2\alpha k+1}||f||_\alpha.$$

So, at last, we have arrived to the Dominated Ergodic Theorem.

Theorem 3.2. *Let $1 < \alpha < \infty$, let $\mathcal{K} = \{\kappa_n\}$ be a sequence of probability measures on X, weakly tempered of order $k \geq \beta := \frac{\alpha}{\alpha-1}$. Assume that $T : x \mapsto T_x$ is a Lamperti right L^α-representation, $f \in L^\alpha$. Then*

$$\int_\Omega \left(\sup_{n\geq 1}[\int_X |fT_y(\omega)|\kappa_n(dy)]\right)^\alpha m(d\omega) \leq (k^k\tau_k(\mathcal{K}))^{\alpha-1}||T||_\alpha^{\alpha(2\alpha k+1)}||f||_\alpha^\alpha. \tag{3.22}$$

Fix $\gamma > 1$. Consider a Lamperti right L^α-representation T. We apply Theorem 3.2 with γ instead of α to the representation $T^{(\frac{\alpha}{\gamma})}$ of X in $L^\gamma(\Omega)$, defined in Subsect. 6.2.2, and to the function $f^{\frac{\alpha}{\gamma}} \in L^\gamma(\Omega)$ (recall that $||f^{\frac{\alpha}{\gamma}}||_\gamma^\gamma = ||f||_\alpha^\alpha$ and $||T^{(\frac{\alpha}{\gamma})}||_\gamma^\gamma = ||T||_\alpha^\alpha$).

We obtain the following generalized version of Theorem 3.2.

Theorem 3.3. *Let $\alpha > 1, \gamma > 1$, let $\mathcal{K} = \{\kappa_n\}$ be a sequence of probability measures on X, weakly tempered of order $k \geq \frac{\gamma}{\gamma-1}$. Assume that $T : x \mapsto T_x$ is a Lamperti right L^α-representation, $f \in L^\alpha$. Then*

$$\int_\Omega \left(\sup_{n\geq 1}[\int_X |fT_y(\omega)|^{\frac{\alpha}{\gamma}}\kappa_n(y)\mu(dy)]\right)^\gamma m(d\omega) \leq (k^k\tau_k(\mathcal{K}))^{\gamma-1}||T||_\alpha^{\alpha(2\gamma k+1)}||f||_\alpha^\alpha. \tag{3.23}$$

Remark 3.1. When $\gamma > \alpha$, (3.23) with a different upper bound follows from Theorem 3.2 and the Hölder inequality.

Remark 3.2. If we choose $\gamma \geq 2$ then $\frac{\gamma}{\gamma-1} \leq 2$ and (3.23) holds for all $\alpha > 1$ when $\mathcal{K}$ is a weakly tempered sequence of order 2.

Corollary 3.2. *Let $\alpha > 1, \gamma > 1$, let $\mathcal{K} = \{\kappa_n\}$ be a sequence of probability measures on X, weakly tempered of order $k \geq \frac{\gamma}{\gamma-1}$. Assume that $T : x \mapsto T_x$ is a Lamperti right L^α-representation, $f \in L^\alpha$. Then* $\sup\limits_{n\geq 1} \int_X |fT_y(\omega)|^{\frac{\alpha}{\gamma}} \kappa_n(dy) < \infty$ *m-a.e.*

Remark 3.3. Relation (3.21) remains true if K is a "piecewise weakly tempered sequence": $\kappa_n := \sum_{i=1}^{\infty} a_i \kappa_n^i, n = 1, 2, ...,$ where $a_i \geq 0, i = 1, 2, ..., \sum_{i=1}^{\infty} a_i = 1$, and $\tau(\mathcal{K}) := \sum_{i=1}^{\infty} a_i \tau(\mathcal{K}^i) < \infty$. Of course, Theorems 3.2, 3.3 and Corollary 3.2 hold, too.

4. Pointwise ergodic theorems

Let us fix a right representation T in $L^\alpha(\Omega, \mathcal{F}, m), 1 < \alpha < \infty$. For any simple function g and each $x \in X$ we denote $g_x := gT_{x^{-1}} - g$. Let D_0^α be the linear manifold generated by such functions and let D^α be its closure in L^α. We denote by I^α the set of all T-invariant functions. Both spaces, D^α and I^α are T_x-invariant ($x \in X$). By reflexivity of L^α, the "ergodic decomposition" holds: $L^\alpha(\Omega, \mathcal{F}, m) = D^\alpha \oplus I^\alpha$ (see [46], Theorem 1.5.9 and Proposition 1.3.7). The projection of f onto I^α (along D^α), denoted by $f\mathbf{M}^{(\alpha)}$, is called the *mean value* of the orbit $\{fT_x,\ x \in X\}$. Of course, the mean value is T_x invariant: $f\mathbf{M}^{(\alpha)}T_x = f\mathbf{M}^{(\alpha)}$ for all $x \in X, f \in L^\alpha$. For each $f \in L^\alpha$ the ergodic decomposition can be written as

$$f = f_0 + f\mathbf{M}^{(\alpha)} \quad \text{where} \quad f_0 \in D^\alpha. \tag{4.1}$$

Theorem 4.1. *Fix $\alpha, 1 < \alpha < \infty$, and let $\mathcal{K} = \{\kappa_n\}$ be weakly tempered of order $k > \frac{\alpha}{\alpha-1}$ and ergodic. If $T : x \mapsto T_x$ is a Lamperti right L^α–representation then for each $f \in L^\alpha(\Omega, \mathcal{F}, m)$*

$$\lim_{n\to\infty} \int\limits_X fT_x \kappa_n(dx) = f\mathbf{M}^{(\alpha)} \ \ m\text{–a.e.} \tag{4.2}$$

Proof. We denote $f\mathbf{M}_n := \int_X fT_x\kappa_n(dx)$. It is clear that $\mathbf{M_n}$ are bounded linear operators in $L^\alpha(\Omega, \mathcal{F}, m)$. As above, for any simple function g on Ω and any $y \in X$ consider $g_y := gT_{y^{-1}} - g$. We will prove that for each y and each simple g

$$\lim_{n\to\infty} g_y\mathbf{M}_n = 0 \ m\text{–a.e.} \tag{4.3}$$

We have: $gT_{y^{-1}}\mathbf{M}_n = \int_G gT_{y^{-1}}T_x\kappa_n(dx) = \int_G gT_{y^{-1}x}\kappa_n(dx) = \int_G gT_z\kappa_n(ydz)$ and $g_y\mathbf{M}_n = \int_X gT_x(\omega)$ $(\kappa_n(ydx) - \kappa_n(dx))$. Since $k > \frac{\alpha}{\alpha-1}$ and hence $\alpha > \frac{k}{k-1}$, we may chose some $\gamma \in [\frac{k}{k-1}, \alpha)$; then $k \geq \beta_\gamma := \frac{\gamma}{\gamma-1}$. Let $b := \frac{\alpha}{\gamma}$ (note: $b > 1$). Using Hölder's inequality we obtain:

$$|g_y\mathbf{M}_n|^b \leq (\int\limits_X |gT_x(\omega)| var(\kappa_n(ydx) - \kappa_n(dx)))^b \leq$$

$$\int\limits_X |gT_x(\omega)|^b var(\kappa_n(ydx) - \kappa_n(dx))[\int\limits_X var(\kappa_n(ydx) - \kappa_n(dx))]^{b-1} \leq$$

$$\int\limits_X |gT_x(\omega)|^b(\kappa_n(ydx) + \kappa_n(dx))[\int\limits_X var(\kappa_n(ydx) - \kappa_n(dx))]^{b-1}.$$

Corollary 3.2 (applied to the functions g and $gT_{y^{-1}}$) shows that, in the last term of the above relation, the first factors form an m-a.e. bounded sequence, while the second factors tend to 0 since $\{\kappa_n\}$ is ergodic. This implies (4.2) for each $f \in D_0^\alpha$ (note that $f\mathbf{M}^{(\alpha)} = 0$ if $f \in D^\alpha$). Now, by Corollary 3.2 with $\gamma = \alpha$, $\sup_{n<\infty} |f\mathbf{M}_n(\omega)| < \infty$ m-a.e. for each $f \in L^\alpha$. By virtue of the Banach convergence principle (see, e.g., [18]) we obtain: $\lim_{n\to\infty} f\mathbf{M}_n = 0$ for any $f \in D^\alpha$. Since $(f\mathbf{M}^{(\alpha)})\mathbf{M}_n = f\mathbf{M}^{(\alpha)}, f \in L^\alpha$, it remains to employ the ergodic decomposition (4.1). $\square$

Remark 4.1. In the above theorem, if T acts also in L^∞ (e.g., if T is generated by a bounded left action τ), then (4.2) holds when $\mathcal{K} = \{\kappa_n\}$ is weakly tempered of order $k \geq \frac{\alpha}{\alpha-1}$ and is ergodic. Indeed, in this case for each $g \in L^\infty$ for m-almost all $\omega \in \Omega$

$$|g_y\mathbf{M}_n(\omega)| \leq \int_X |gT_x(\omega)| var[\kappa_n(ydx) - \kappa_n(dx)| \leq$$

$$||T||_\infty ||g||_\infty \int_X var(\kappa_n(ydx) - \kappa_n(dx)),$$

which implies (4.3); the rest of the proof is the same as in Theorem 4.2.

Remark 4.2. If $1 < \alpha < \infty$ the Mean Ergodic Theorem: $(L^\alpha) \lim_{n\to\infty} \int_X fT_x\kappa_n(x)\mu(dx) = f\mathbf{M}^{(\alpha)}$ holds for every right L^α-representation T, every $f \in L^\alpha$, and each ergodic sequence of probability measures $\mathcal{K}$ (see Theorem 1.5.9 and Corollary 3.1.2 in [46]).

Corollary 4.1. *[47] Fix $1 < \alpha < \infty$, and let $\mathcal{A} = \{A_n\}$ be Følner and tempered. If $T : x \mapsto T_x$ is a Lamperti right L^α-representation then for each $f \in L^\alpha(\Omega, \mathcal{F}, m)$*

$$\lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} fT_x\mu(dx) = f\mathbf{M}^{(\alpha)} \ \ m\text{−}a.e. \tag{4.4}$$

Remark 4.1 implies the following statement.

Corollary 4.2. *If the sequence of probability measures $\mathcal{K} = \{\kappa_n\}$ is ergodic and weakly tempered of order $k \geq \frac{\alpha}{\alpha-1}$, then for each bounded left action $x \mapsto \tau_x$ of X in $(\Omega, \mathcal{F}, m)$ and for each $f \in L^\alpha(\Omega, \mathcal{F}, m)$ with $1 < \alpha < \infty$*

$$\lim_{n\to\infty} \int_X f(\tau_x\omega)\kappa_n(dx) = \mathbf{M}(f) \quad m\text{−}a.e. \tag{4.5}$$

In the case when $\alpha = \frac{k}{k-1}, k \in \mathbb{N}, k > 1$, the following statement has been proved by Shulman [41] (see also [46]) using a method based on Dunkan's "dual space" approach (see Dunkan [17]).

Corollary 4.3. *If $\{A_n\}$ is a Følner weakly tempered sequence of order $k \geq \frac{\alpha}{\alpha-1}$ then for each measure preserving left action $x \mapsto \tau_x$ of X in $(\Omega, \mathcal{F}, m)$ and $f \in L^\alpha(\Omega, \mathcal{F}, m)$ with $1 < \alpha < \infty$*

$$\lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} f(\tau_x\omega)\mu(dx) = \mathbf{M}(f) \quad m\text{−}a.e.$$

Remark 4.3. Using the "coverage" method Lindenstrauss [33,34] extended Shulman's theorem to the case $\alpha = 1$ (Shulman's "dual space" approach cannot work if $\alpha = 1$ since it leads to the PET for general Lamperti representations which fails when $\alpha = 1$).

Corollary 4.4. *Fix $1 < \alpha < \infty$ and let $\mathcal{K} = \{\kappa_n\}$ be ergodic and weakly tempered of order $k > \frac{\alpha}{\alpha-1}$. Let $T : x \mapsto T_x$ be a positive right L^α–representation. Then for each $f \in L^\alpha(\Omega, \mathcal{F}, m)$ relation (4.2) holds.*

Corollary 4.5. *Fix $1 < \alpha < \infty$, $\alpha \neq 2$, and let $\mathcal{K} = \{\kappa_n\}$ be ergodic and weakly tempered of order $k > \frac{\alpha}{\alpha-1}$. Let $T : x \mapsto T_x$ be an isometric right L^α–representation. Then for each $f \in L^\alpha(\Omega, \mathcal{F}, m)$ relation (4.2) holds.*

Remark 4.4. Theorem 4.1 and Corollary 4.2 show that the “left” notions of weakly tempered and of ergodic and of tempered sequences introduced in §2 are sufficient for averaging of right representations and left group actions; for left representations and right group actions the obvious “right” versions of these notions should be considered.

Remark 4.5. Theorem 4.1 remains true if $\mathcal{K}$ is an ergodic piecewise weakly tempered sequence of measures (see Remark 3.3).

Theorem 4.1 implies generalized versions of PETs for cocycles proved in [47].

In [44,45] simple counterexamples are provided which show that Theorem 4.1 and other results of this section may fail for a sequence of compact sets if the Følner condition or the regularity condition is not fulfilled (see also §6.3.2 in [46]).

5. Some classes of pointwise averaging sequences

We say that a sequence of densities $\Phi = \{\varphi_n\}$ is *pointwise averaging* if for each $\alpha, 1 < \alpha < \infty$, for each Lamperti representation T in L^α and each $f \in L^\alpha$ we have:

$$\lim_{n\to\infty} \int\limits_X fT_x\varphi_n(dx) = f\mathbf{M}^{(\alpha)} \;\; m\text{–}a.e. \tag{5.1}$$

Similarly, we can define *pointwise averaging sequences of integrable sets.*

In this section we consider examples of Følner regular sequences of sets, ergodic regular sequences of densities, and ergodic weakly tempered (of each order) sequences of densities: all such sequences are pointwise averaging by Section 4 (some of these examples are well-known (see [45,46])).

5.1. Pointwise averaging in groups of polynomial growth

We remind that a compactly generated locally compact group X is said to be of polynomial growth if there is a compact generating set U and constants $C, d > 0$ such that $\mu(U^n) \leq Cn^d, n \in \mathbb{N}$. Tessera [49] has proved that in such a group, *if U is a symmetric compact generating set, then the sequence $\{U^n\}$ is both regular and Følner*; a similar result for an increasing sequence of balls B_{r_n} of radii $r, r \to \infty$ (with respect to a periodic quasi-metric) was obtained by Breuillard [9].

From these results and from the PET in Tempelman [44,45], they deduced the PETs with $\{U^n\}$ and $\{B_{r_n}\}$ for measure preserving actions. Corollary 4.1 shows that *these sequences are pointwise averaging for all Lamperti representations.*

5.2. Sequences of bounded convex bodies in $\mathbb{R}^m$ and in $\mathbb{Z}^m$

Each increasing sequence of bounded convex bodies $\{A_n\}$ in $\mathbb{R}^m$ is regular (see [46] and the references therein). Let d_n denote the intrinsic diameters of the sets A_n; if $d_n \to \infty$, $\{A_n\}$ is also ergodic (see [15]).

Let $E_n := A_n \cap \mathbb{Z}^m, n = 1, 2, \ldots$. We will show that the sequence $\{E_n\}$ possesses the same properties in $\mathbb{Z}^m$.

Let B_r be the ball of radius r centered at $\mathbf{0}$. Without loss of generality we may assume that $\mathbf{0} \in A_n$ and that $B_{d_n/2} \in \bar{A}_n$; we also assume that $d_n > 2m^{\frac{1}{2}}$. Denote $K := [-\frac{1}{2}, \frac{1}{2}]^m$; of course, $K \subset B_{\frac{1}{2}\sqrt{m}}$; $[E_n] := E_n + K$. By Lemma 4 in [15], $A_n + B_{\frac{1}{2}\sqrt{m}} \subseteq \frac{d_n+m^{\frac{1}{2}}}{d_n} A_n$. Applying this Lemma one more time with $\frac{d_n-m^{\frac{1}{2}}}{d_n} A_n$ instead of A_n we obtain: $\frac{d_n-m^{\frac{1}{2}}}{d_n} A_n + B_{\frac{1}{2}\sqrt{m}} \subseteq A_n$. Let us note that for each $x \in \frac{d_n-m^{\frac{1}{2}}}{d_n} A_n$ there is a $z \in \mathbb{Z}^m$ such that $z \in x + K$; by the above inclusion, $z \in A_n$; hence $z \in E_n$ and $x \in z + K \subset [E_n]$. Thus

$$\frac{d_n - 2m^{\frac{1}{2}}}{d_n} A_n \subset [E_n] \subseteqq \frac{d_n + m^{\frac{1}{2}}}{d_n} A_n, \tag{5.2}$$

and, certainly,

$$\frac{d_n - 2m^{\frac{1}{2}}}{d_n} A_n \subset A_n \subseteqq \frac{d_n + m^{\frac{1}{2}}}{d_n} A_n;$$

hence

$$\frac{\mu(A_n \triangle [E_n])}{\mu(A_n)} \to 0, \frac{\mu((A_n + x) \triangle ([E_n] + x))}{\mu(A_n)} \to 0, \frac{\mu([E_n])}{\mu(A_n)} \to 1.$$

Assume $d_n \to \infty$. Since $\{A_n\}$ is Følner this implies that $\{[E_n]\}$ is Følner, too. If $z \in \mathbb{Z}^m$ then $([E_n] + z) \triangle [E_n] = [(E_n + z) \triangle E_n]$, and $\dfrac{|(E_n + z) \triangle E_n|}{|E_n|} = \dfrac{\mu(([E_n] + z) \triangle [E_n])}{\mu([E_n]}$ $\to 0$. So $\{E_n\}$ is Følner in $\mathbb{Z}^m$.

Now we will prove that $\{E_n\}$ is regular. Note that $A_n - A_n$ are convex bodies. Therefore (5.2) implies:

$$|E_n| = \mu([E_n]) > (\frac{d_n - 2m^{\frac{1}{2}}}{d_n})^m \mu(A_n)$$

and

$$|E_n - E_n| = \mu([E_n - E_n]) \leq \left(\frac{d(A_n - A_n) + m^{\frac{1}{2}}}{d(A_n - A_n)} \right)^m \mu(A_n - A_n)$$

where $d(A_n - A_n)$ is the intrinsic diameter of $A_n - A_n$. Hence $\{E_n\}$ is regular in $\mathbb{Z}^m$.

5.3. Sequences of concave densities on $\mathbb{R}^m$

Let $\{\varphi_n\}$ be a sequence of densities on $\mathbb{R}^m$ concave on their supports S_n. Then S_n are compact and convex. Assume that the sequence $\{S_n\}$ is increasing and the intrinsic diameters $d(S_n) \to \infty$. Put $b_k := \max\limits_{x \in S_{k-1}} \frac{\varphi_{k-1}(x)}{\varphi_k(x)}, k = 2, 3..., c_n = 2^n \prod_{k=2}^n b_k$. Then the sequence $\{c_n \varphi_n\}$ is increasing and $\max\limits_{x \in \mathbb{R}^m} [c_n \varphi_n(x)] \to \infty$. Section 5.2 and Remarks 2.4 and 2.7 show that the sequence $\{\varphi_n\}$ is ergodic and regular.

5.4. Averaging over sequences of homothetic sets and of rescaled densities

In this subsection we consider sequences Φ of "rescaled densities", i.e. sequences of densities of the form $\varphi_n(x) := t_n^{-m} \varphi(t_n^{-1} x)$, where φ is a density on $\mathbb{R}^m$, $t_1, t_2, ...$ are positive numbers, and $t_n \uparrow \infty$.

Example 5.1. Let A be an integrable set with positive measure. If $\varphi := \frac{1}{\mu(A)} 1_A$ then $\mu(A)\varphi(t_n^{-1}x)$ are indicators of homothetic sets $A_n = t_n A$, and $\varphi_n(x) = \frac{1}{\mu(A_n)} 1_{A_n}(x)$.

Theorem 4.1 claims that the combination of ergodicity and weak tempering is sufficient for pointwise averaging (of course weak tempering can be substituted by any stronger condition: weak regularity, regularity, or tempering). The following statement shows that for rescaled sequences ergodicity holds.

Proposition 5.1. *Each sequence $\{\varphi_n\}$ of rescaled densities Φ is ergodic and, in particular, a sequence of homothetic sets $\{A_n\}$ is Følner.*

Proof. For each $y \in \mathbb{R}^m$

$$\lim_{n\to\infty} t_n^{-m} \int\limits_{\mathbb{R}^m} |\varphi(t_n^{-1}(x+y)) - \varphi(t_n^{-1}x)|dx = \lim_{n\to\infty} \int\limits_{\mathbb{R}^m} |\varphi(x + t_n^{-1}y) - \varphi(x)|dx = 0,$$

since the mapping $z \mapsto f(z + \cdot)$ from $\mathbb{R}^m$ into $L^1(\mathbb{R}^m, \mathcal{B}, \mu)$ is continuous (see, e.g., Ch. 5, Theorem 20.4 in [24]). $\square$

5.4.1. Averaging by homothetic sets

Let A be a measurable set in $\mathbb{R}^m$ with finite non-zero Lebesgue measure $l_m(A) > 0$, $\hat{A} := \{x = \lambda y, 0 \le \lambda \le 1, y \in A\}$ (the "star-shaped hull" of A). The sequence of homothetic sets $\mathcal{A} = \{t_n A\}$ is ergodic by Proposition 5.1; if, moreover, A is compact, then $r(\mathcal{A}) \le \frac{l_m(A-\hat{A})}{l_m(A)}$; indeed $r(\mathcal{A}) \le \sup_n \frac{l_m(t_nA - t_n\hat{A})}{l_m(t_nA)} = \sup_n \frac{l_m(t_n(A-\hat{A}))}{l_m(t_nA)} = \frac{\mu(A-\hat{A})}{\mu(A)}$ (see also Example 5.2.9 in [46]) so $\{t_nA\}$ is regular (note that $r(\mathcal{A}) = 2^m$ if A is convex and symmetric about 0).

5.4.2. Averaging over sequences of rescaled densities on $\mathbb{R}^m$

Proposition 5.2. *If φ is a bounded density on $\mathbb{R}^m$ with compact support S, then the sequence $\varphi_n(x) := t_n^{-m}\varphi(t_n^{-1}x)$ is ergodic and regular, hence pointwise averaging.*

Proof. Let S_n and S be the supports of φ_n and φ, resp. Note that $S_n = t_nS$. Therefore the sequence $\{\varphi_n\}$ is regular by Subsect. 5.4.1 and Proposition 2.4; it is ergodic by Proposition 5.1. $\square$

Our next statements show that, under some conditions, the DET and PET hold for a sequence of rescaled densities with *unbounded* supports (of course such sequences are not tempered; the proof is somewhat more tedious then the one of Proposition 5.2).

Let $||\cdot||$ be an arbitrary norm in $\mathbb{R}^m$.

Proposition 5.3. *Let $\varphi(0) = \max\limits_{x\in\mathbb{R}^m} \varphi(x) < \infty$. Assume that there is a non-negative function $\tilde{\varphi}$ on $\mathbb{R}^+$ such that*

1. *$\varphi(x) \equiv \tilde{\varphi}(||x||), x \in \mathbb{R}^m$,*
2. *$\tilde{\varphi}$ is non-increasing in $[0, \infty)$,*
3. *$\tilde{\varphi}(u)$ convex on some interval $[a, \infty), a > 0$.*

Let $\varphi_n(x) := t_n^{-m}\varphi(t_n^{-1}x)$ $x \in \mathbb{R}^m$.

Then $A_k := \sup_{n_k<\infty} \int dx_k \ldots \int dx_2 \sup_{n_1:n_1\le n_2} \int dx_1 \prod_{i=1}^k \varphi_{n_i}(\sum_{j=1}^i x_j) < \infty$ for each $k \ge 2$. Thus the sequence $\{\varphi_n\}$ is weakly tempered of every order, and therefore, by Proposition 5.1, it is pointwise averaging.

The step-by-step proof of Proposition 5.3 is performed in the following lemmas where it is assumed that φ is a density on $\mathbb{R}^m$ satisfying the conditions of the Proposition 5.3.

Lemma 5.1. *For each x with $||x|| \le a$ and each $y \in \mathbb{R}^m$ $\varphi(y) \le c\varphi(x)$ $(c := \frac{\varphi(0)}{\tilde{\varphi}(a)})$.*

Proof. For each pair x, y satisfying the above conditions $\varphi(y) < \varphi(0) = \frac{\varphi(0)}{\tilde{\varphi}(||x||)}\varphi(x) \leq \frac{\varphi(0)}{\tilde{\varphi}(a)}\varphi(x)$. $\square$

Lemma 5.2. *$\varphi(y + \alpha x) \leq c\varphi(y) + c\varphi(y + x)$ for all $x, y \in \mathbb{R}^m, 0 \leq \alpha \leq 1$.*

Proof. It follows from Lemma 5.1 that

if $||y|| < a$ then $\varphi(y + \alpha x) \leq c\varphi(y)$;

if $||y + x|| < a$ then $\varphi(y + \alpha x) \leq c\varphi(y + x)$.

Let's both $||y|| \geq a$ and $||y + x|| \geq a$. Then using convexity of $\tilde{\varphi}$

$$\varphi(y + \alpha x) = \varphi\left((1 - \alpha)y + \alpha(y + x)\right) \leq (1 - \alpha)\varphi(y) + \alpha\varphi(y + x).$$

In all cases $\varphi(y + \alpha x) \leq c\varphi(y) + c\varphi(y + x)$. $\square$

Lemma 5.3. *Let $s_1 \leq s_2 \leq ... \leq s_i$. For each $i \in \mathbb{N}$*

$$\varphi(\sum_{j=1}^{i} s_i^{-1} s_j x_j) \leq c^{i-1} \sum_{b \in \{0,1\}^{i-1}} \varphi(x_i + \sum_{j=1}^{i-1} b_j x_j). \tag{5.3}$$

Proof. If $i = 2$ then, by Lemma 5.2, $\varphi(x_2 + s_2^{-1} s_1 x_1) \leq c\varphi(x_2) + c\varphi(x_2 + x_1)$.

Let inequality (5.3) hold for $i - 1$. By Lemma 5.2 with $x = x_1, y = \sum_{j=2}^{i} s_i^{-1} s_j x_j, \alpha = s_i^{-1} s_1$,

$$\varphi(\sum_{j=1}^{i} s_i^{-1} s_j x_j) \leq c\varphi(\sum_{j=2}^{i} s_i^{-1} s_j x_j) + c\varphi(x_1 + \sum_{j=2}^{i} s_i^{-1} s_j x_j).$$

Using this inequality and inequality (5.3) for $i - 1$ we get:

$$\varphi(\sum_{j=1}^{i} s_i^{-1} s_j x_j) \leq c \cdot c^{i-2} \sum_{b \in \{0,1\}^{i-2}} \varphi(x_i + \sum_{j=2}^{i-1} b_j x_j) +$$

$$c \cdot c^{i-2} \sum_{b \in \{0,1\}^{i-2}} \varphi(x_i + x_1 + \sum_{j=2}^{i-1} b_j x_j) = c^{i-1} \sum_{b \in \{0,1\}^{i-1}} \varphi(x_i + \sum_{j=1}^{i-1} b_j x_j). \quad \square$$

Define $B_k := \{0,1\} \times \{0,1\}^2 \times ... \times \{0,1\}^k$, i.e. if $b \in B_k$, then $b = (b^{(1)}, b^{(2)}, ..., b^{(k)})$ where $b^{(i)} \in \{0,1\}^i$; $b_j^{(i)}$ denotes the j-th coordinate of $b^{(i)}, j = 1, 2, ..., i$.

Lemma 5.4. *Let $s_1 \leq s_2 \leq ... \leq s_k$. Then*

$$\prod_{i=1}^{k} \varphi(\sum_{j=1}^{i} s_i^{-1} s_j x_j) \leq c^{\frac{(k-1)k}{2}} \sum_{b \in B_k} \prod_{i=1}^{k} \varphi(x_i + \sum_{j=1}^{i-1} b_j^{(i)} x_j).$$

Proof. Lemma 5.3 implies

$$\prod_{i=1}^{k} \varphi(\sum_{j=1}^{i} s_i^{-1} s_j x_j) \leq \prod_{i=1}^{k} [c^{i-1} \sum_{b \in \{0,1\}^{i-1}} \varphi(x_i + \sum_{j=1}^{i-1} b_j^{(i)} x_j)] =$$

$$c^{\frac{(k-1)k}{2}} \sum_{b \in B_k} \prod_{i=1}^{k} \varphi(x_i + \sum_{j=1}^{i-1} b_j^{(i)} x_j). \quad \square$$

Lemma 5.5. $A_k \le (2c)^{\frac{(k-1)k}{2}}$.

Proof. We put $s_l = t_{n_l}, l = 1, ..., k$, in Lemma 5.4. By using this Lemma and the definition of A_k we readily get:

$$A_k \le c^{\frac{(k-1)k}{2}} \sum_{b \in B_k} \prod_{i=1}^{k} \int_{\mathbb{R}^m} \varphi(x_i + \sum_{j=1}^{i-1} b_j^{(i)} x_j) dx_i.$$

Put $y_i = x_i + \sum_{j=1}^{i-1} b_j^{(i)} x_j$ in the integral. We have:

$$A_k \le c^{\frac{(k-1)k}{2}} \sum_{b \in B_k} \prod_{i=1}^{k} \int_{\mathbb{R}^m} \varphi(y_i) dy_i = c^{\frac{(k-1)k}{2}} |B_k|.$$

It remains to note that $|B_k| = \prod_{i=1}^{k} 2^{i-1} = 2^{\frac{(k-1)k}{2}}$. $\square$

Example 5.2. a) The Cauchy density on $\mathbb{R}^m$

$$\psi(x) := \frac{\Gamma(\frac{m+1}{2})}{\pi^{\frac{m+1}{2}}} ([1 + ||x||^2)]^{-\frac{m+1}{2}} \tag{5.4}$$

satisfies the conditions of Proposition 5.3 with $a = (2+m)^{-\frac{1}{2}}$, the inflection point of $\tilde{\varphi}$.

b) Let V be a non-degenerate positive definite symmetric "covariance" matrix and $||x|| = \left(\langle V^{-1}x, x\rangle\right)^{\frac{1}{2}})$. The normal density

$$\theta_V(x) := \frac{1}{(2\pi \det(V))^{\frac{m}{2}}} \exp[-\frac{1}{2}\langle V^{-1}x, x\rangle] \tag{5.5}$$

satisfies the conditions of Proposition 5.3 with $a = 1$, the inflection point of $\tilde{\varphi}$.

c) Let $\alpha > 0$. The density $\tau(x) := \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)\Gamma(m])}(1 + ||x||)^{-(m+\alpha)}, x \in \mathbb{R}^m$, also satisfies the conditions of Proposition 5.3 (the function $\tilde{\varphi}$ is convex on $\mathbb{R}_+$).

So the rescaled sequences $\{\psi_n\}, \{\theta_n\}$ and $\{\tau_n\}$ are weakly tempered of any order, hence they are pointwise averaging.

Proposition 5.3 and Remark 2.6 imply the following, more general statement.

Corollary 5.1. *Let $\varphi(x) \le C\tilde{\varphi}(||x||)$ for all $x \in \mathbb{R}^m$, where C is a constant, $\tilde{\varphi}$ possesses properties 2) and 3) stated in Proposition 5.3, and $\int_0^\infty u^{m-1}\tilde{\varphi}(u)du < \infty$. Then the rescaled sequence $\{\varphi_n\}$ is pointwise averaging. In particular, this is the case when*

$$\varphi(x) \le \frac{C}{(1 + ||x||)^{m+\alpha}}, x \in \mathbb{R}^m \tag{5.6}$$

with some $\alpha > 0$.

Not that the above densities ψ and θ_V satisfy condition (5.6).

We state another application of Proposition 5.3 and Remark 2.6.

Corollary 5.2. *Assume that there is a sequence $t_n > 0, t_n \uparrow \infty$ such that $\varphi_n(x) \le \frac{Ct_n^\alpha}{(t_n+||x||)^{m+\alpha}}, x \in \mathbb{R}^m, n \in \mathbb{N}$, where $\alpha > 0$. Then $\{\varphi_n\}$ is weakly tempered of every order.*

5.5. Averaging by sequences of convolution powers of densities

In this Subsection we put $G = \mathbb{R}^m, m \geq 1$. If φ be a probability density on $\mathbb{R}^m$ we put $\varphi_n := \varphi^{*n}$, the nth power φ with respect to the convolution. First let us quote the following result (see [30]).

Proposition 5.4. *A sequence $\{\kappa^{*n}\}$ of convolution powers of a probability measure κ on a commutative group X is ergodic if and only if κ is strictly aperiodic, i.e. the support $s(\kappa)$ is not contained in a coset of a proper normal subgroup of X.*

Let φ be the density of a symmetric stable distribution with parameter $\hat{\alpha}$ $(0 < \hat{\alpha} \leq 2)$; this means that $\varphi^{*n}(x) = n^{-\frac{m}{\hat{\alpha}}}\varphi(n^{-\frac{1}{\hat{\alpha}}}x)$. Simple examples of stable densities are the Cauchy density ψ (see (5.4)) with $\tilde{\alpha} = 1$ and the normal density θ_V (see (5.5)) with $\tilde{\alpha} = 2$.

By Proposition 5.3, the densities ψ and θ_V are pointwise averaging.

Proposition 5.5. *Let φ be a stable density with parameter $0 < \hat{\alpha} \leq 2$; assume φ satisfies the conditions of Proposition 5.3 and $\Phi := \{\varphi^{*n}\}$. Then it is weakly tempered of each order and ergodic, and thus it is pointwise averaging.*

Remark 5.1. Let φ be a stable density constant on spheres, $0 < \hat{\alpha} \leq 2$. It is easy to prove that sequence Φ is weakly tempered of order 2 and the index $\rho_2(\Phi) < 2^{\frac{m}{\hat{\alpha}}}$. Indeed, $\varphi(x) = \hat{\varphi}(||x||)$ where $\hat{\varphi}$ is a decreasing function on $[0, \infty)$, and $\varphi_n(x) = n^{-\frac{m}{\hat{\alpha}}}\varphi\left(||x||/n^{\frac{1}{\hat{\alpha}}}\right)$, $\int_{\mathbb{R}^m} \varphi_{n_1}(y)\varphi_{n_2}(y + x)dy = \varphi_{n_1} * \varphi_{n_2}(x) = \varphi_{n_1+n_2}(x)$, and therefore

$$r_2(\Phi) = \int\limits_{\mathbb{R}^m} \max_{n_1 \leq n_2} \int\limits_{\mathbb{R}^m} \varphi_{n_1}(y)\varphi_{n_2}(y + x)dydx = \int\limits_{\mathbb{R}^m} \max_{n_1 \leq n_2} \varphi_{n_1+n_2}(x)dx =$$

$$\int\limits_{\mathbb{R}^m} \max_{n_1 \leq n_2} \frac{1}{(n_1 + n_2)^{\frac{m}{\hat{\alpha}}}}\varphi\left(||x||/(n_1 + n_2)^{\frac{1}{\hat{\alpha}}}\right) dx < \int\limits_{\mathbb{R}^m} \frac{1}{(n_2)^{\frac{m}{\hat{\alpha}}}}\varphi\left(||x||/(2n_2)^{\frac{1}{\hat{\alpha}}}\right) dx = 2^{\frac{m}{\hat{\alpha}}}.$$

Remark 5.2. Proposition 5.5 is a special case of Proposition 3.5 in [32] where it was proved, in particular, that if a density φ on a commutative group is symmetric and strictly aperiodic and $\varphi_n := \varphi^{*n}$ then the conclusion of Proposition 5.2 holds; we include Proposition 5.5 and Remark 5.1 as simple applications of our general results.

We turn now to averaging by convolutions of nonsymmetric densities.

Proposition 5.6. *Let $1 < \alpha < \infty$. Assume that φ is a density on $\mathbb{R}^m$ such that*

1) its characteristic function belongs $L^\alpha(\mathbb{R}^m)$ for some $\alpha \geq 1$ (this is the case when the density φ is bounded),

2) $\int_{\mathbb{R}^m} ||x||^{m+1}\varphi(x)dx < \infty$, $\int_{\mathbb{R}^m} x\varphi(x)dx = 0$, and

3) its covariance matrix V (i.e. the matrix with the components $V_{ij} = \int x_i x_j \varphi(x_1, \dots, x_m)dx, i, j = 1, ...m$) is non-degenerate.

Then the sequence $\{\varphi^{*n}\}$ is ergodic and weakly tempered of each order, hence pointwise averaging.

Proof. Our sequence is ergodic by Example 5.4. We will prove that it is weakly tempered of any order k. Along with the m-dimensional Cauchy density ψ and the normal density θ_V, we will consider the functions $\lambda_n(x) := n^{-\frac{m}{2}} \sum_{r=1}^{m-1} n^{-\frac{r}{2}} Q_r(n^{-\frac{1}{2}}x)\theta_V(n^{-\frac{1}{2}}x)$ where Q_r, $r \geq 1$ are polynomials of degree $d \leq m + 1$ which does not depend on n and, up to a constant, are specified in [8]. We apply Proposition 19.2 in [8] with $s = m + 1$ if $m \geq 2$; in the case when $m = 1$ we apply a theorem in [43] (see also the Note in §19 in [8]). For each $m \geq 1$ we have:

$$|\varphi^{*n}(x) - \theta_V^{*n}(x) - \lambda_n(x)| = o(n^{-\frac{m-1}{2}})\psi^{*n}(x),\ x \in \mathbb{R}^m. \tag{5.7}$$

It is clear $\theta_V^{*n}(x) \le C\psi^{*n}(x)$ and, if $\gamma > 1$,

$$\lambda_n(x) < Dn^{-\frac{1}{2}}n^{-\frac{m}{2}}\theta_{\gamma V}(n^{-1/2}x) < Fn^{-\frac{1}{2}}\psi^{*n}(x) \tag{5.8}$$

where C, D, F are positive constants. Therefore formula (5.7) implies for all $x \in \mathbb{R}^m$:

$$\varphi^{*n}(x) \le G\psi^{*n}(x),$$

and Remark 2.6 and Proposition 5.5 show that $\{\varphi^{*n}\}$ is weakly tempered of any order k. By virtue of relations (5.7) and (5.8), we have:

$$\int_{\mathbb{R}^m} |\varphi^{*n}(x) - \theta_V^{*n}(x)|\, dx = o(1). \tag{5.9}$$

By Proposition 5.5 the sequence $\{\theta^{*n}\}$ is ergodic, and (5.9) readily implies that $\{\varphi^{*n}\}$ possesses this property too. It remains to use Proposition 4.1. $\square$

In [7] the following theorem was proved: let φ be a strictly aperiodic density on $\mathbb{Z}$ such that $\sum_{z=1}^{\infty} z^2\varphi(z) < \infty$, $\sum_{z=1}^{\infty} z\varphi(z) = 0$; then for each invertible measure preserving transformation τ of a σ-finite measure space $(\Omega, \mathcal{F}, m)$ the sequence $\sum_{z=-\infty}^{\infty} f(\tau^z\omega)\varphi^{*n}(z)$ converges m-almost everywhere for each $f \in L^\alpha(\Omega, \mathcal{F}, m), p > 1$. The following statement provides a generalization of this theorem in several directions.

Proposition 5.7. *Assume that φ is a strictly aperiodic density on $\mathbb{Z}^m, m \ge 1$, $\sum_{z\in\mathbb{R}^m} ||z||^{m+1}\varphi(z) < \infty$, $\sum_{z\in\mathbb{R}^m} z\varphi(z) = 0$. The sequence $\{\varphi^{*n}\}$ is pointwise averaging.*

Proof. The proof is similar to that in Proposition 5.6: Corollary 22.3 in [8] presents an analog of formula (5.7), and this gives us analogs of formulas (5.5) and (5.9). $\square$

Remark 5.3. Theorems 5.6 and 5.7 follow from Proposition 5.2 and Theorem 5.7 in [13] and Theorem 1.2 in [19]. Our goal was to provide two more examples of weakly tempered, but not regular, sequences.

6. Appendix

6.1. Some properties of invertible Lamperti operators and Lamperti group representations

6.1.1. Basic properties of Lamperti operators in $L^\alpha, 1 \le \alpha < \infty$

Let α, β be mutually conjugate numbers, $1 \le \alpha \le \infty$, $L^\alpha = L^\alpha(\Omega, \mathcal{F}, m)$, and let $M(\Omega, \mathcal{F})$ denote the linear space of all $\mathcal{F}$-measurable functions on Ω. An operator Ψ in $M(\Omega, \mathcal{F})$ is said to be *strictly positive* if $Tf > 0$ a.e. whenever $f > 0$ a.e. In this section the term *"invertible Lamperti operator" means that the operator and its inverse are Lamperti operators.*

Definition 6.1. A mapping Ψ in $\mathcal{F}$ is called a σ-automorphism if (1) it maps $\mathcal{F}$ onto $\mathcal{F}$ and is invertible, (2) $\Psi(\cup_{n=1}^{\infty}\Lambda_n) = \cup_{n=1}^{\infty}\Psi\Lambda_n$ if Λ_n are disjoint, (3) $\Psi(\Lambda^c) = (\Psi\Lambda)^c$ for each $\Lambda \in \mathcal{F}$, (4) $m(\Psi\Lambda) = 0$ if and only if $m(\Lambda) = 0$.

Proposition 6.1. *Each invertible Lamperti operator T in $L^\alpha, 1 \le \alpha < \infty$, has the following form:*

$$Tf(\omega) = h(\omega)\Psi f(\omega); \tag{6.1}$$

here $1_\Lambda h \in L^\alpha(\Omega, \mathcal{F}, m)$ when $m(\Lambda) < \infty$, $h \ne 0$ m-a.e., and Ψ is an invertible strictly positive linear transformation in $M(\Omega, \mathcal{F})$ with the following properties:

(1) For each $\Lambda \in \mathcal{F}, \Psi 1_\Lambda$ is an indicator of a set in $\mathcal{F}$, denoted ΨL; the mapping $\Lambda \mapsto \Psi\Lambda$ is an σ-automorphism of the σ-field $\mathcal{F}$ (hence $\Psi 1 = 1$ and $\Lambda \mapsto \hat{m}(\Lambda) := m(\Psi\Lambda)$ is a measure on $\mathcal{F}$ equivalent to m).

(2) $|\Psi f| = \Psi(|f|), f \in M(\Omega, \mathcal{F})$.

(3) $(\Psi f)^\gamma = \Psi f^\gamma, \gamma \in \mathbb{R}_+, f \in M(\Omega, \mathcal{F})$.

(4) $\Psi(f \cdot g) = \Psi f \cdot \Psi g, f, g \in M(\Omega, \mathcal{F})$.

(5) If $f, g \in M(\Omega, \mathcal{F})$ and $g\Psi f \in L^1(\Omega, \mathcal{F}, m)$, then $\int_\Omega g \cdot \Psi f dm = \int_\Omega \Psi^{-1} g \cdot f \frac{d\hat{m}}{dm} dm$; in particular, if $\Psi f \in L^1(\Omega, \mathcal{F}, m)$ then $\int_\Omega \Psi f dm = \int_\Omega f \frac{d\hat{m}}{dm} dm$.

(6) $\Psi \frac{d\hat{m}_\Psi}{dm} = \left(\frac{d\hat{m}_{\Psi^{-1}}}{dm}\right)^{-1}$ m-a.e.

Moreover,

(7) $||T||_\alpha = ||\Psi^{-1}(|h|)(\frac{d\hat{m}}{dm})^{\frac{1}{\alpha}}||_\infty < \infty$.

Vice versa, each operator T of the form (6.1), where $h(\omega) \ne 0$ m-a.e. and Ψ is an invertible strictly positive operator in $M(\Omega)$ with the above properties (1)–(6), is an invertible Lamperti operator in L^α.

Proof. Properties (1)–(4) are well-known (see [28]).

Property (5) follows from the “leading” special case when $f = 1_\Lambda, g = 1_\Gamma, \Lambda, \Gamma \in \mathcal{F}, m(\Lambda) < \infty, m(\Gamma) < \infty$: $\int_\Omega 1_\Gamma \cdot \Psi 1_\Lambda dm = m(\Gamma \cap \Psi\Lambda) = m(\Psi(\Psi^{-1}\Gamma \cap \Lambda)) = \hat{m}(\Psi^{-1}\Gamma \cap \Lambda) = \int_\Omega \Psi^{-1} 1_\Gamma \cdot 1_\Lambda \frac{d\hat{m}}{dm} dm$.

Let us prove Statement (6). Using Property (5) with Ψ^{-1} instead of Ψ we obtain: for each $\Lambda \in \mathcal{F}$ we have:

$$m(\Lambda) = m(\Psi\Psi^{-1}\Lambda) = \hat{m}_\Psi(\Psi^{-1}\Lambda) = \int\limits_\Omega \frac{d\hat{m}_\Psi}{dm} \cdot \Psi^{-1} 1_\Lambda dm =$$

$$\int\limits_\Omega \Psi \frac{d\hat{m}_\Psi}{dm} \cdot 1_\Lambda \cdot \frac{d\hat{m}_{\Psi^{-1}}}{dm} dm$$

which implies Statement (6).

Property (7): $||T||_\alpha^\alpha = \sup\limits_{f:||f^\alpha||_1 \le 1} |\int_\Omega |Tf^\alpha| dm|$; on the other hand, by Property (5), $\int_\Omega Tf^\alpha dm = \int_\Omega h^\alpha \Psi(f^\alpha) dm = \int_\Omega \Psi^{-1} h^\alpha \frac{d\hat{m}}{dm} f^\alpha dm =: l(f^\alpha)$ where $l(u)$ is a linear functional on L^1; hence $\mathrm{esup}_x[\Psi^{-1}|h|^\alpha \frac{d\hat{m}}{dm}] = ||l||_1 = ||T||_\alpha^\alpha$. $\square$

Definition 6.2. We will refer to formula (6.1) as the *canonical representation* of the Lamperti operator T.

Proposition 6.2. *(1) If T_1, T_2 are invertible Lamperti operators then T_1T_2 is also an invertible Lamperti operator and $h_{T_1T_2} = h_{T_1} \cdot \Psi_{T_1} h_{T_2}, \Psi_{T_1T_2} = \Psi_{T_1}\Psi_{T_2}$. Hence $\Psi_{T^{-1}} = \Psi_T^{-1}, h_{T^{-1}} = (\Psi_T^{-1} h)^{-1}$.*

(2) An invertible Lamperti operator is positive if and only if $h > 0$ m-a.e., so each positive invertible Lamperti operator is strictly positive.

(3) The modulus of an invertible Lamperti operator T is defined as follows: $|T|f(\omega) = |h(\omega)|\Psi f(\omega)$. It is clear that $|T|$ is an invertible strictly positive operator, $|Tf| = |T||f|, f \in L^\alpha(\Omega, \mathcal{F}, m), |T_1T_2| = |T_1| \cdot |T_2|$ and $|T^{-1}| = |T|^{-1}$.

(4) If T is an invertible Lamperti operator in $L^\alpha (1 \le \alpha < \infty)$ then the conjugate operator T^* is an invertible Lamperti operator in L^β. Moreover $T^* g = \Psi^{-1}\bar{h} \cdot \frac{d\hat{m}}{dm} \cdot \Psi^{-1} g$, $g \in L^\beta$; hence $\Psi_{T^*} = \Psi^{-1}; h_{T^*} = \Psi^{-1}\bar{h} \cdot \frac{d\hat{m}}{dm}$.

Proof. 1. Follows from (6.1).

2–3. Each of these statements readily follows from the previous ones.

4. See Proposition 3.1 in [28].

5. If $f \in L^\alpha, g \in L^\beta$, then by Proposition 6.1(5), $\langle Tf, g\rangle = \int_\Omega Tf \cdot \bar{g}\, dm = \int_\Omega \Psi f \cdot \bar{h}\bar{g} dm = \int_\Omega f \cdot \overline{\Psi^{-1}h} \cdot \overline{\Psi^{-1}g} \cdot \frac{d\hat{m}}{dm} dm = \langle f, \Psi^{-1}h \cdot \Psi^{-1}g \cdot \frac{d\hat{m}}{dm}\rangle$. $\square$

6.2. Lamperti type operators (L-operators) and Lamperti type representations in Banach function spaces

Let $(\Omega, \mathcal{F}, m)$ be a σ-finite measure space, B a Banach space contained in $\mathcal{M}(\Omega, \mathcal{F})$.

Definition 6.3. An operator in B is *an invertible Lamperti type operator*, or in short *an invertible L-operator*, if it has a representation (6.1) where 1) $h \neq 0$ m-a.e., 2) Ψ is an invertible strictly positive linear transformation in $M(\Omega, \mathcal{F})$ 3) for each $\Lambda \in \mathcal{F}$, $\Psi 1_\Lambda$ is an indicator of a set in $\mathcal{F}$ denoted $\Psi\Lambda$; 4) the mapping $\Lambda \mapsto \Psi\Lambda$ is an σ-automorphism of the σ-field $\mathcal{F}$, hence $\Psi 1 = 1$ and $\Lambda \mapsto \hat{m}(\Lambda) := m(\Psi\Lambda)$ is a measure on $\mathcal{F}$ equivalent to m.

Of course Statements (1)–(6) of Proposition 6.1 and Proposition 6.2 are fulfilled.

Example 6.1. Let T be an operator in $L^\infty(\Omega, \mathcal{F}, m)$ "preserving support separation": $Tf \cdot Tg = 0$ whenever $f \cdot g = 0$ and $f, g \in L^\infty(\Omega, \mathcal{F}, m)$. Then it has the canonical representation (6.1) where Ψ is a positive linear transformation in set $\mathcal{SM}(\Omega, \mathcal{F})$ of all measurable simple functions on Ω and the mapping $\Lambda \mapsto \Psi(\Lambda) := \Psi_T(1_\Lambda)$ is an automorphism of the field $\mathcal{F}$ (see, e.g. [28]). In general, Ψ does not need to be a σ-automorphism, so in this case T is not a L-type operator. But, of course $Tf = h \cdot R_y f$ is a L-type operator in $L^\infty(X, \mathcal{B}, \nu)$ if $h \in \mathcal{M}(X, \mathcal{B})$ and $h \neq 0$ ν-a.e., and R is the right shift: $R_y f(x) = f(xy^{-1})$.

Proposition 6.3. *Let T be an invertible L-operator in $L^\infty(\Omega, \mathcal{F}, m)$. Then*

(1) Ψ is an invertible linear isometry in $L^\infty(\Omega, \mathcal{F}, m)$;

(2) $||T||_\infty = ||h||_\infty = ||\Psi^{-1}h||_\infty$, so Proposition 6.1(7) is valid here too if we assume $\frac{1}{\infty} = 0$.

Proof. 1) Since $\Psi(c) = c, c \in \mathbb{R}$, and Ψ is positive, $|f(\omega)| \leq ||f||_\infty$ m-a.e. implies $\Psi|f(\omega)| \leq \Psi||f||_\infty = ||f||_\infty, f \in L^\infty(\Omega, \mathcal{F}, m)$. But T^{-1} is also a L-operator in $L^\infty(\Omega, \mathcal{F}, m)$ and $\Psi_{T^{-1}} = \Psi^{-1}$, so we have: $\Psi^{-1}|g(\omega)| \leq ||g||_\infty, g \in L^\infty(\Omega, \mathcal{F}, m)$. Put $g = \Psi f$ in the last inequality. We obtain: $\Psi^{-1}\Psi f(\omega)| = |f| \leq ||\Psi f||_\infty$. Thus $||\Psi f||_\infty = ||f||_\infty, f \in L^\infty(\Omega, \mathcal{F}, m)$.

2) $||T||_\infty = \sup\limits_{f:||f||_\infty \leq 1} ||h\Psi f||_\infty \leq ||h||_\infty \sup\limits_{f:||f||_\infty \leq 1} ||f||_\infty = ||h||_\infty$ and $||T1||_\infty = ||h||_\infty$. $\square$

Definition 6.4. Consider a dual pair of Banach spaces B and B' with respect to a bilinear form $\langle f, g\rangle$, $f \in B, g \in B'$ such for a finite constant C we have: $|\langle f, g\rangle| \leq C||f||_B||g||_{B'}, f \in B, g \in B'$. Assume T is an operator in B, and for some $g \in B'$ there is a point $g' \in B'$ such that $\langle Tf, g\rangle = \langle f, g'\rangle, f \in B$; in this case we say that this is the value of the *conjugate linear transformation* at $g \in B'$ and denote $T'g := g'$.

It is clear that the domain D of T' is a linear manifold in B'.

Remark 6.1. Let T^* be the conjugate to T in B^*. If $D = B'$ then it is a T^*-invariant subset of B^* and T' coincides with the restriction of T^* onto B'; $T'' = T$.

Proposition 6.4. *Let B and B' are Banach spaces contained in $\mathcal{M}(\Omega, \mathcal{F})$ and $|\int_\Omega f\bar{g}dm| \leq C||f||_B||g||_{B'}$, $f \in B, g \in B'$ where $C < \infty$. If B, B' is a dual pair of Banach spaces with respect to the bilinear form $\langle f, g\rangle = \int f\bar{g}dm$ on $B \times B'$, and T is an invertible L-operator in B, then its conjugate T' is also* *an invertible L-operator in B', and, as in Proposition 6.2(5), $T'g = \Psi^{-1}\bar{h} \cdot \frac{d\hat{m}}{dm} \cdot \Psi^{-1}g,\ g \in B'$, i.e. $\Psi_{T'} = \Psi^{-1}; h_{T'} = \Psi^{-1}\bar{h} \cdot \frac{d\hat{m}}{dm}$.*

Example 6.2. $L^\infty(\Omega, \mathcal{F}, m)$ and $L^1(\Omega, \mathcal{F}, m)$ form a dual pair with respect to $\langle f, g\rangle = \int f\bar{g}dm$. Therefore for each L-operator T in $L^\infty(\Omega, \mathcal{F}, m)$ the conjugate L-operator T' is well defined in $L^1(\Omega, \mathcal{F}, m)$.

Lamperti-type representations in Banach function spaces are representation by Lamperti type operators; in §3 such representations are generated by Lamperti representations in the spaces L^α where $1 \le \alpha < \infty$.

6.2.1. Some properties of Lamperti group representations

Propositions 6.1 and 6.2 imply useful properties of Lamperti representations.

If $\Psi_x, x \in X$, are invertible strictly positive linear operators in $M(\Omega)$ each satisfying Property (1) of Proposition 6.1, we consider the measures $\hat{m}_x(\Lambda) := m(\Psi_x\Lambda), \Lambda \in \mathcal{F}$, equivalent to our basic measure m.

Proposition 6.5. *(1) Let $T_x, x \in X$ be a collection of operators in L^α. $T : x \mapsto T_x$ is a Lamperti right representation if and only if*

$$fT_x(\omega) = h_x(\omega) \cdot f\Psi_x(\omega), x \in X \tag{6.2}$$

where all operators Ψ_x in $M(\Omega)$ are invertible, strictly positive, and possess properties (1)–(6) stated in Proposition 6.1, $\Psi : x \mapsto \Psi_x$ is a right representation of X in $M(\Omega)$ and $(\omega, x) \mapsto h_x(\omega)$ is a multiplicative right cocycle with respect to the right representation Ψ, i.e. for each $x, x_1, x_2 \in X$ we have: m-a.e. $h_e(\omega) = 1$, $h_{x_1x_2}(\omega) = h_{x_1}\Psi_{x_2}(\omega) \cdot h_{x_2}(\omega)$.

(2) If T is a Lamperti right representation of X in L^α then the mapping $T^* : x \mapsto T^*_x$ is a Lamperti left representation of X in L^β.

(3) Let T be a positive right representation of X in L^α. Then for each $f \in L^\alpha_+$ we have: $T^*_x(fT_x)^{\alpha-1} \le ||T_x||^\alpha_\alpha f^{\alpha-1}$.

Proof. 1. See Propositions 6.1 and 6.2(2).

2. $T^*_xT^*_y = (T_yT_x)^* = T^*_{xy}$, so T^* is a left representation. By Proposition 6.2(7), the operators T^*_x are Lamperti.

3. By (6.2) and Proposition 6.1(5,6), for each $g \in L^\beta$

$$\int_\Omega g \cdot T^*_x(fT_x)^{\alpha-1}dm = \int_\Omega gT_x \cdot (fT_x)^{\alpha-1}dm = \int_\Omega h_x \cdot g\Psi_x \cdot [h_x \cdot f\Psi_x]^{\alpha-1}dm =$$

$$\int_\Omega h^\alpha_x \cdot (gf^{\alpha-1})\Psi_x dm = \int_\Omega gf^{\alpha-1} \cdot \Psi^{-1}_x h^\alpha_x \cdot \frac{d\hat{m}_x}{dm}dm.$$

Since g is an arbitrary function in L^β, $T^*_x(fT_x)^{\alpha-1} = f^{\alpha-1} \cdot \Psi^{-1}_x h^\alpha_x \cdot \frac{d\hat{m}_x}{dm} \le ||T||^\alpha_\alpha f^{\alpha-1}$. □

Proposition 6.6. *Let T be a measurable invertible Lamperti right representation.*

(1) The representation Ψ and the functions $(\omega, x) \mapsto h_x(\omega), (\omega, x) \mapsto \Psi^{-1}_x h_x(\omega)$ and $(\omega, x) \mapsto \frac{d\hat{m}_x}{dm}(\omega)$ are measurable.

(2) The representation $|T|$ is measurable.

(3) The representation T^* is measurable.

Proof. 1. a) For each x and $\Lambda \in \mathcal{F}$ we have $1_\Lambda\Psi_x = 1_{\{\omega: 1_\Lambda T_x(\omega) > 0\}}$, so $\{(\omega, x) : 1_\Lambda\Psi_x(\omega) = 1\} = \{(\omega, x) : 1_\Lambda T_x(\omega) > 0\} \in \mathcal{F} \times \mathcal{B}$ and $\{(\omega, x) : 1_\Lambda\Psi_x(\omega) = 0\} \in \mathcal{F} \times \mathcal{B}$. This implies that for all simple functions

$f : (\omega, x) \mapsto f\Psi_x(\omega)$ is measurable. Since Ψ_x is a positive operator, $f_n \to f$ in measure implies $f_n\Psi_x \to f\Psi_x$ in measure, so the function $(\omega, x) \to f\Psi_x$ is measurable for each $f \in M(\Omega, \mathcal{F})$.

b) If $m(\Omega) < \infty$, then $h_x = 1_\Omega T_x$, and if $m(\Omega) = \infty$ then $h_x = \sum_{i=1}^{\infty} I_{\Lambda_i} T_x$ where $\{\Lambda_i\}$ is a cover of Ω by disjoint sets of finite measure; hence $(x, \omega) \mapsto h_x(\omega)$ is measurable.

c) By Proposition 6.2(3), $\Psi_x^{-1} h_x = (h_{x^{-1}})^{-1}$, so $(x, \omega) \mapsto \Psi_x^{-1} h_x(\omega)$ is measurable, too.

d) Let $\{L_n^{(i)}, n \in \mathbb{N}\}_{i=1}^{\infty}$ be a sequence of refining covers of Ω by sets of finite measure m and let the collection of sets $\cup_{i.n\in\mathbb{N}} L_n^{(i)}$ generate the σ-field $\mathcal{F}$; put

$$g^{(i)}(\omega, x) := \sum_{n=1}^{\infty} \frac{\hat{m}_x(\Lambda_n^{(i)})}{m(\Lambda_n^{(i)})} 1_{\Lambda_n^{(i)}}(\omega).$$

Then $\dfrac{d\hat{m}_x}{dm}(\omega) = \lim_{i\to\infty} g^{(i)}(\omega)$ m-a.e. (cf. [16], §VII.8), so $(\omega, x) \to \dfrac{d\hat{m}_x}{dm}(\omega)$, is measurable.

2. The definition of $|T|$ and Statement (1) imply that T is measurable.

3. Since $fT_x = h_x \cdot f\Psi_x$ we have, by Proposition 6.2(7), $T_x^* g = \Psi_x^{-1} h_x \cdot \dfrac{d\hat{m}_x}{dm} \cdot \Psi_{x^{-1}} g$ where $g \in L^\beta(\Omega, \mathcal{F}, m)$; thus, by Statement (1), T^* is measurable. □

6.2.2. Lamperti representations $T^{(b)}$

Let $1 \le \alpha < \infty, 0 < b < \alpha$, and let T be a Lamperti right representation of X in L^α. We will consider "right" operators $T_x^{(b)}$ in $L^{\frac{\alpha}{b}}$ defined for each $f \in L^\alpha$ as follows: $f^b T_x^{(b)}(\omega) = (fT_x(\omega))^b$.

The following proposition clarifies and justifies this definition.

Proposition 6.7. *Let $1 < \alpha < \infty, 0 < b < \alpha, \gamma := \frac{\alpha}{b}$ (so $1 < \gamma < \infty$).*

(1) $f \in L^\alpha(\Omega, \mathcal{F}, m)$ if and only if $f^b \in L^\gamma(\Omega, \mathcal{F}, m)$; moreover, $\|f^b\|_\gamma^\gamma = \|f\|_\alpha^\alpha$, i.e. the mapping $\iota : f \mapsto f^b$ is an isometric mapping from L^α onto L^γ.

(2) Let T be a measurable invertible Lamperti right representation in L^α and let

$$f^b T_x^{(b)}(\cdot) := (fT_x(\cdot))^b \quad \textit{where} \quad f \in L^\alpha. \tag{6.3}$$

Then $T^{(b)} : x \mapsto T_x^{(b)}$ is a measurable invertible Lamperti right representation in L^γ and $\|T_x^{(b)}\|_\gamma^\gamma = \|T_x\|_\alpha^\alpha$, $x \in X$.

Proof. (1) $\|f^b\|_\gamma^\gamma = \left(\int_\Omega (|f|^b)^\gamma dm\right) = \left(\int_\Omega |f|^\alpha dm\right) = \|f\|_\alpha^\alpha$.

(2) Let $fT_x(\cdot) = h_x(\cdot) \cdot f\Psi_x(\cdot)$ be canonical representation of the operator T_x (see Proposition 6.5). Then

$$f^b T_x^{(b)} = h_x^b \cdot (f\Psi_x)^b = h_x^b \cdot f^b \Psi_x \tag{6.4}$$

is canonical representation for the operator $T_x^{(b)}$. It is clear that $T_x^{(b)}$ is an invertible linear Lamperti operator in L^γ, $x \in X$. We have $T_x^{(b)} = \iota T_x \iota^{-1}$, and hence $\|T_x^{(b)}\|_\gamma^\gamma = \|T_x\|_\alpha^\alpha$. The equivalence of the operators $T_x^{(b)}$ and T_x also implies that $T^{(b)} : x \mapsto T_x^{(b)}$ is an invertible Lamperti representation in L^γ. Measurability of $T^{(b)}$ follows from (6.4) and Proposition 6.6(1). □

6.3. Examples: distinctions between properties of sequences defined in §2

6.3.1. A regular but not Følner sequence of sets in $\mathbb{R}^2$

Example 6.3. The sequence of rectangles $A_n := [0, n] \times [0, 1]$ is regular but not Følner.

6.3.2. A Følner and tempered but non-regular sequence of sets in $\mathbb{R}^2$

Example 6.4. Let $\{a_n\}$ and $\{b_n\}$ be increasing sequences of positive numbers such that $b_n < a_n$, $b_n \to \infty$, $\sup \frac{a_n}{b_n} = \infty$. Let $A_n := [-a_n, a_n] \times [-b_n, b_n] \cup [-b_n, b_n] \times [-a_n, a_n]$. Then $[-a_n, a_n] \times [-a_n, a_n] \subset A_n - A_n$, $\sup_n [\mu(A_n)]^{-1} \mu(A_n - A_n) > \sup_n (8a_n b_n)^{-1}(4a_n^2) = \sup_n \frac{a_n}{2b_n} = \infty$, so the sequence $\{A_n\}$ is not regular. In addition to the previous assumptions, let $a_{n-1} = b_n, n > 1$. Then

$$A_n - A_{n-1} \subset ([-(a_n + a_{n-1}), a_n + a_{n-1}] \times [-(b_n + a_{n-1}), b_n + a_{n-1}])$$
$$\cup \alpha ([-(b_n + a_{n-1}), b_n + a_{n-1}] \times [-(a_{n-1} + a_n), a_{n-1} + a_n])$$

and

$$[\mu(A_n)]^{-1} \mu[A_n - A_{n-1}) \leq (8a_n b_n - b_n^2)^{-1} 8(a_n + a_{n-1})(b_n + a_{n-1}) =$$
$$16 \cdot \frac{a_n + b_n}{8a_n - b_n} \to 2.$$

Hence the sequence $\{A_n\}$ is tempered; it easy to verify that it is Følner.

6.3.3. A Følner and weakly tempered but non-tempered sequence of densities with bounded supports on $\mathbb{R}$

Example 6.5. Let φ be a bounded density on $\mathbb{R}$ with $\int_{-\infty}^{\infty} x\varphi(x)dx = 0$, $\int_{-\infty}^{\infty} x^2\varphi(x)dx = \sigma^2 < \infty$ and let $s(\varphi)$ contain an interval $(-a, a), a > 0$. Put $\varphi_n := \varphi^{*n}$. By the local central limit theorem (see [43]) for each $0 < \varepsilon < 1$ and each $y \in \mathbb{R}$

$$\varphi^{*n}(y) \geq \frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{y^2}{2\sigma^2 n}} - \frac{\varepsilon}{\sqrt{2\pi n}\sigma} =: \xi_n(y)$$

if n is large enough. Thus if n is large and $-a(n-1) < x < a(n-1)$, we have: $0 \in (-a(n-1)+x, a(n-1)+x)$; therefore

$$\psi_n(x) = \sup\{\varphi^{*n}(y), y \in [-an, an] \cap (-a(n-1)+x, a(n-1)+x)\} >$$
$$\xi_n(0) \geq \frac{1-\varepsilon}{\sqrt{2\pi n}\sigma}.$$

Hence, if n is large, $\int \psi_n(x)dx \geq \int_{-a(n-1)}^{a(n-1)} \psi_n(x)dx \geq \dfrac{1-\varepsilon}{\sqrt{2\pi n}\sigma} 2a(n-1)$, and the sequence $\{\varphi^{*n}\}$ is *not tempered*. But, by virtue of Proposition 5.6, it is *weakly tempered of each order $k \geq 1$ and ergodic.*

Acknowledgment

The authors are grateful to Michael Lin for very useful discussions; his comments and suggestions considerably improved the paper. We are also grateful to the referee for his attention and useful recommendations.

References

[1] M.A. Akcoglu, A pointwise ergodic theorem in L^p-spaces, Canad. J. Math. 27 (1975) 1075–1082.
[2] N. Asmar, E. Berkson, T.A. Gillespie, Transference of strong type maximal inequalities by separation preserving representations, Amer. J. Math. 113 (1991) 47–74.
[3] N. Asmar, E. Berkson, T.A. Gillespie, Transference of almost everywhere convergence, J. Operator Theory 27 (1992) 283–308.

[4] I. Assani, Sur les opérateurs a puissances bornées et la thérème ergodique ponctuel dans $L^\alpha[0,1], 1 < \alpha < \infty$, Canad. J. Math. 38 (1986) 937–946.
[5] I. Assani, On the punctual and local ergodic theorem for nonpositive power bounded operators in $L^p_{\mathbf{C}}[0;1]; 1 < p < +\infty$, Proc. Amer. Math. Soc. 96 (1986) 306–310.
[6] E.F. Beckenbach, R. Bellman, Inequalities, Springer, Berlin–Göttingen–Heidelberg, 1961.
[7] A. Bellow, R. Jones, J. Rosenblatt, Almost everywhere convergence of convolution powers, Ergodic Theory Dynam. Systems 14 (1994) 415–432.
[8] R.N. Bhattacharya, R. Ranga Rao, Normal Approximation and Asymptotic Expansions, Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1986.
[9] E. Breuillard, Geometry of locally compact groups of polynomial growth and shape of large balls, Groups Geom. Dyn. 8 (2014) 669–732.
[10] A.P. Calderon, A general ergodic theorem, Ann. of Math. (2) 58 (1953) 182–191.
[11] R.V. Chacon, A class of linear transformations, Proc. Amer. Math. Soc. 15 (1964) 560–564.
[12] G. Cohen, Doubly power-bounded operators on L^p, $2 \neq p > 1$, J. Math. Anal. Appl. 466 (2018) 1327–1336.
[13] G. Cohen, C. Cuny, M. Lin, Almost everywhere convergence of powers of some positive L_α contractions, J. Math. Anal. Appl. 420 (2014) 1129–1153.
[14] M. Cotlar, A unified theory of Hilbert transforms and ergodic theorems, Rev. Mat. Cuyana 1 (1956) 105–167.
[15] M.M. Day, Ergodic theorems for Abelian semi-groups, Trans. Amer. Math. Soc. 51 (1942) 339–412.
[16] J.L. Doob, Stochastic Processes, John Wiley & Sons Inc., Chapman & Hall, New York, London, 1953.
[17] R. Duncan, Pointwise convergence theorems for self-adjoint and unitary contractions, Ann. Probab. 5 (1977) 622–626.
[18] N. Dunford, J.T. Schwartz, Linear Operators, Part I, John Wiley & Sons Inc., New York, 1988.
[19] N. Dungey, Time regularity for random walks on locally compact groups, Probab. Theory Related Fields 137 (2007) 429–442.
[20] W. Emerson, The pointwise ergodic theorem for amenable groups, Amer. J. Math. 96 (1974) 472–478.
[21] M. Feder, On power bounded operators and the pointwise ergodic theorem, Proc. Amer. Math. Soc. 83 (1981) 349–353.
[22] F. Greenleaf, Invariant Means on Topological Groups and Their Applications, Van-Nostrand, New York, 1969.
[23] F. Greenleaf, W. Emerson, Group structure and the pointwise ergodic theorem for connected amenable groups, Adv. Math. 14 (1974) 153–172.
[24] E. Hewitt, K.N. Ross, Abstract Harmonic Analysis, 1, Springer, Berlin–Götingen–Heidelberg, 1963.
[25] A. Ionescu Tulcea, Ergodic properties of isometries in L^α spaces, Bull. Amer. Math. Soc. 70 (1964) 366–371.
[26] A. Ionescu Tulcea, On the category of certain classes of transformations in ergodic theory, Trans. Amer. Math. Soc. 114 (1965) 261–279.
[27] R. Jones, J. Rosenblatt, A. Tempelman, Ergodic theorems for convolutions of a measure on a group, Illinois J. Math. 38 (1994) 521–553.
[28] C.-H. Kan, Ergodic properties of Lamperti operators, Canad. J. Math. 30 (1978) 1206–1214.
[29] C.-H. Kan, Ergodic properties of Lamperti operators. II, Canad. J. Math. 35 (1983) 577–588.
[30] J. Kerstan, K. Matthes, Gleichverteilungseigenschaften von Faltungpotenzen auf lokalkompakten Abelschen Gruppen, Wiss. Z. Univ Jena 14 (1965) 457–462.
[31] J. Lamperti, On the isometries of certain function-spaces, Pacific J. Math. 8 (1958) 459–466.
[32] M. Lin, R. Wittmann, Ergodic sequences of averages of group representations, Ergodic Theory Dynam. Systems 14 (1994) 181–196.
[33] E. Lindenstrauss, Pointwise theorems for amenable groups, Electron. Res. Announc. Am. Math. Soc. 5 (1999) 82–90.
[34] E. Lindenstrauss, Pointwise theorems for amenable groups, Invent. Math. 146 (2001) 259–295.
[35] J.-C. Lootgieter, On power-subadditive positive operators on the L_p spaces ($1 < p < \infty$), Proc. Amer. Math. Soc. 145 (2017) 4299–4311.
[36] F.J. Martin-Reyes, A. de la Torre, The dominated ergodic theorem for invertible, positive operators, in: Semesterbericht Funktionalanalysis Tübingen, Sommersemester, 1985, pp. 143–150.
[37] F.J. Martin-Reyes, A. de la Torre, The dominated ergodic estimate for mean bounded, invertible, positive operators, Proc. Amer. Math. Soc. 104 (1988) 69–75.
[38] S.A. McGrath, Some ergodic theorems for commuting L_1 contractions, Studia Math. 70 (1981) 153–160.
[39] A. Nevo, Pointwise Ergodic Theorms for Actions of Groups, Handbook of Dynamical Systems, vol. 1B, 2006, pp. 871–982.
[40] H.R. Pitt, Some generalizations of the ergodic theorem, Proc. Camb. Philos. Soc. 38 (1942) 325–343.
[41] A.L. Shulman, Pointwise Averaging Sequences on Groups, PhD Thesis, Vilnius, 1988 (in Russian). Advisor: A. Tempelman.
[42] A. Shulman, Maximal ergodic theorems on groups, 1988. This article was archived in the Lithuanian Institute of Science and Technical Information (copies were available upon requests); in the 1990s the Institute and its archive were abolished.
[43] P. Survila, A local limit theorem for densities, Lith. Math. J. 3 (1963) 535–540.
[44] A.A. Tempelman, Ergodic theorems for general dynamical systems, Dokl. Akad. Nauk SSSR 176 (1967) 790–793 (in Russian); English translation: Sov. Math., Dokl. 8 (1967) 1213–1216.
[45] A.A. Tempelman, Ergodic theorems for general dynamical systems, Tr. Mosk. Mat. Obs. 26 (1972) 95–132 (in Russian; English translation: pp. 94–132).
[46] A. Tempelman, Ergodic Theorems for Group Actions, Kluwer Academic Publishers Group, Dordrecht, 1992.
[47] A. Tempelman, Pointwise ergodic theorems for bounded Lamperti representations of amenable groups, Proc. Amer. Math. Soc. 143 (2015) 4989–5004.
[48] A.A. Tempelman, A.L. Shulman, Pointwise ergodic theorems on groups, 1987. This article was archived in the Lithuanian Institute of Science and Technical Information (copies were available upon requests); in the 1990s the Institute and its archive were abolished.
[49] R. Tessera, Volume of spheres in doubling metric measured spaces and in groups of polynomial growth, Bull. Soc. Math. France 135 (2007) 47–64.
[50] N. Wiener, The ergodic theorem, Duke Math. J. 5 (1939) 1–18.