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A new result for global solvability and boundedness in the N-dimensional quasilinear chemotaxis model with logistic source and consumption of chemoattractant

Xinchao Song [a], Jiashan Zheng [a,b,*]

[a] School of Mathematics and Statistics Science, Ludong University, Yantai 264025, PR China
[b] School of Information, Renmin University of China, Beijing 100872, PR China

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ABSTRACT

We consider the following chemotaxis model

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(u\nabla v) + \mu(u-u^2), \quad x\in\Omega, t>0,\\ v_t - \Delta v = -uv, \quad x\in\Omega, t>0,\\ (D(u)\nabla u - \chi u\cdot\nabla v)\cdot\nu = \dfrac{\partial v}{\partial\nu} = 0, \quad x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\Omega \end{cases}$$

on a bounded domain $\Omega\subset\mathbb{R}^N (N\geq 1)$, with smooth boundary $\partial\Omega, \chi$ and μ are positive constants. Here, D is supposed to be smooth positive function satisfying $D(u)\geq C_D(u+1)^{m-1}$ for all $u\geq 0$ with some $C_D, m>0$. Besides appropriate smoothness assumptions, in this paper it is only required that

$$m > \begin{cases} 1 - \frac{\mu}{2\chi[1+\max_{1\leq s\leq 2}\lambda_0(s)\|v_0\|_{L^\infty(\Omega)}2^3]} & \text{if } N\leq 2,\\ 1 \quad \text{if } N\geq 3, \end{cases}$$

then for any sufficiently smooth initial data there exists a classical solution which is global in time and bounded, where λ_0 is a positive constant which is corresponding to the maximal Sobolev regularity. The results of this paper extends the results of Jin (J. Differential Equations 263 (9) (2017) 5759–5772), who proved the possibility of boundness of weak solutions, in the case $m>1$ and $N=3$. To the best of our knowledge, this is the first result which gives the relationship between m and $\frac{\mu}{\chi}$ that yields to the boundedness of the solution.

* Corresponding author at: School of Mathematics and Statistics Science, Ludong University, Yantai 264025, PR China.
E-mail addresses: lddxsxc@163.com (X. Song), zhengjiashan2008@163.com (J. Zheng).

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1. Introduction

Due to its important applications in biological and medical sciences, chemotaxis research has becomes one of the hottest topics in applied mathematics nowadays and tremendous theoretical progresses have been made in the past few decades. This paper is devoted to making further development for the following quasilinear chemotaxis systems with logistic source and consumption of chemoattractant, reading as

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u\nabla v) + \mu(u - u^2), \quad x \in \Omega, t > 0, \\ v_t = \Delta v - uv, \quad x \in \Omega, t > 0, \\ (D(u)\nabla u - \chi u \cdot \nabla v) \cdot \nu = \dfrac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta = \sum\limits_{i=1}^{N} \dfrac{\partial^2}{\partial x_i^2}$, $\dfrac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$, $\chi > 0$ is a parameter referred to as chemosensitivity, $\mu u(1 - u)(\mu > 0)$ and $-vu$ are the proliferation or death of bacteria according to a logistic law and the consumption of chemoattractant, respectively. Here $u = u(x, t)$ and $v = v(x, t)$ denotes the density of the cells population and the concentration of the chemoattractant, respectively. The nonlinear nonnegative function $D(u)$ satisfies

$$D \in C^2([0, \infty)) \tag{1.2}$$

and

$$D(u) \geq (u + 1)^{m-1} \text{ for all } u \geq 0 \tag{1.3}$$

with $m > 0$.

In order to better understand model (1.1), we can see some previous contributions in this direction. Assuming that $\mu \equiv 0$, the chemotaxis model (1.1) can be reduced to quasilinear chemotaxis model with consumption of chemoattractant

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u\nabla v), \quad x \in \Omega, t > 0, \\ v_t = \Delta v - uv, \quad x \in \Omega, t > 0. \end{cases} \tag{1.4}$$

When $D(u) \equiv 1$, Tao and Winkler ([20]) proved that problem (1.4) possesses global bounded smooth solutions in the spatially **two-dimensional** setting, whereas in the **three**-dimensional counterpart at least global weak solutions can be constructed which eventually become smooth and bounded. When $D(u) \geq C_D(u + 1)^{m-1}$ satisfies (1.2)–(1.3) with $m > \frac{1}{2}$ in the case $N = 1$ and $m > 2 - \frac{2}{N}$ in the case $N \geq 2$, it is shown that system (1.4) admits a unique global classical solution that is uniformly bounded ([27]), while if $m > \frac{3N}{2N+2}$ $(N \geq 3)$, (1.4) has a unique global classical solution (see Zheng and Wang [49]), which improves the results of [25] $(2 - \frac{6}{N+4} < \frac{3N}{2N+2}$ for $N \geq 3)$. Apart from the aforementioned system, a source of logistic type is included in (1.4) to describe the spontaneous growth of cells. The effect of preventing ultimate growth has been widely studied [14,18,50]. For instance, in **three**-dimensional case and $D(u) \equiv 1$, Zheng and Mu ([50]) proved that the system (1.1) admits a unique global classical solution if the initial data of v is small; while if μ is appropriately **large**, Lankeit and Wang ([18]) obtained the global boundedness of classical solutions of (1.1) for **arbitrarily large** initial data, and for any $\mu > 0$, they also established the existence of global weak solutions. Recently, if $N = 3$, Jin ([14]) showed that for any $m > 1, \mu > 0$ and for **arbitrarily large** initial datum, the problem (1.1) admits a global bounded solution. Note that the global existence and boundedness of solutions to (1.1) is still open in **higher dimensions** $(N \geq 4)$. It is the purpose of the present paper to clarify the issue of boundedness to solutions of (1.1) without any restriction on the **space** dimension.

We sketch here the main ideas and methods used in this article. One method of this paper is that we use the Maximal Sobolev regularity approach to prove the existence of bounded solutions. The Maximal Sobolev regularity approach has been widely used to obtain the existence of bounded solutions of the partial differential system (see e.g. [6,7,36,48,34]). Moreover, by careful analysis, one can develop **new** L^p-estimate techniques to raise the a priori estimate of solutions from

$$L^{\min\{1+\frac{\mu}{2\chi[1+\max_{1\le s\le 2}\lambda_0(s)\|v_0\|_{L^\infty(\Omega)}2^3]},2\}}(\Omega)(\text{which is a new a-priori estimate}) \to L^p(\Omega)(\forall\ p>1),$$

and then in view of the Moser iteration method (see e.g. Lemma A.1 of [19]), we finally established the L^∞ bound of u (see the proof of Theorem 1.1). Our main result is the following:

Theorem 1.1. *Assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\bar{\Omega})$ both are nonnegative, D satisfies (1.2)–(1.3) with $m>0$. For any positive constants χ and $\mu>0$, if*

$$m > \begin{cases} 1 - \frac{\mu}{2\chi[1+\max_{1\le s\le 2}\lambda_0(s)\|v_0\|_{L^\infty(\Omega)}2^3]} & \text{if } N\le 2,\\ 1 \quad \text{if } N\ge 3, \end{cases}$$

then there exists a pair $(u,v)\in (C^0(\bar{\Omega}\times[0,\infty))\cap C^{2,1}(\Omega\times(0,\infty))^2$ which solves (1.1) in the classical sense, where λ_0 is a positive constant (see Lemma 2.3). Moreover, both u and v are bounded in $\Omega\times(0,\infty)$.

By Theorem 1.1, we derive the following Corollary:

Corollary 1.1. *If $N\le 2$ and $m=1$, then for any $\mu>0$, then (1.1) possesses a unique global classical solution (u,v) which is bounded in $\Omega\times(0,\infty)$.*

Remark 1.1. (i) If $N=3$, then Theorem 1.1 is consistent with the result of Jin ([14]).

(ii) As far as I know that this is the first result which gives the relationship between m and $\frac{\mu}{\chi}$ that yields to boundedness of the solution.

(iii) The idea of the paper can also be solved other type of the models, e.g., chemotaxis-haptotaxis model (see [46]) and Keller–Segel system with logistic source (see [47]).

(iv) It concludes from Theorem 1.1 that large exponent m and $\frac{\mu}{\chi}$ benefit the boundedness of solutions.

If $D(u)\equiv 1$ and $\mu=0$, $-uv$ in the v-equation is replaced by $-v+u$, then (1.1) becomes the well-known Keller-Segel model introduced by Keller and Segel (see Keller and Segel [16,15]) in 1970:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & x\in\Omega, t>0,\\ v_t=\Delta v - v + u, & x\in\Omega, t>0. \end{cases} \tag{1.5}$$

Over the last decades, the Keller–Segel model has been extensively investigated; in particular, a large amount of work has been devoted to determining whether the solutions are global in time or blow up in finite time, see, for example, Cieślak et al. [9,8], Burger et al. [2], Horstmann and Winkler [12,30] and references therein. Additionally, recent studies have shown that the solution behavior can be also impacted by the volume-filling or prevention of overcrowding (see Calvez and Carrillo [3], Hillen and Painter [11]), the nonlinear diffusion (see Zheng [38,39,42], Ishida et al. [13], Tao and Winkler [19,35]]), and the logistic damping (see Wang et al. [24,26], Winkler and Tello [22,29], Zheng and Wang [44], Peng et al. [51,52]).

Before stating our main results about the model (1.1), let us mention that Tuval et al. ([23]) proposed the following chemotaxis(-Navier)-Stokes model,

$$\begin{cases} u_t + w \cdot \nabla u = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v), \quad x \in \Omega, \ t > 0, \\ v_t + w \cdot \nabla v = \Delta v - uv, \quad x \in \Omega, \ t > 0, \\ w_t = \Delta w + \kappa (w \nabla \cdot w) + \nabla P + u \nabla \varphi, \quad x \in \Omega, \ t > 0, \\ \nabla \cdot w = 0, \quad x \in \partial\Omega, \ t > 0, \end{cases} \tag{1.6}$$

which describes the motion of oxygen-driven swimming the cells in an incompressible fluid, where w represents the velocity field of the fluid subject to an incompressible Navier-Stokes equation with pressure P and a gravitational force $\nabla\phi$. Here $\kappa \in \{0, 1\}$ is related to the strength of nonlinear fluid convection. The chemotaxis fluid system (1.6) and its closely related variant have extensively been studied during the past years (see e.g. Chae et al. [4,5], Tao and Winkler [21,31–33], Zheng [42,45] and references therein).

2. Preliminaries

In order to prove the main results, we first state several elementary lemmas which will be needed later.

Lemma 2.1. *([10,13,40,41]) Let $s \geq 1$ and $q \geq 1$. Assume that $p > 0$ and $a \in (0, 1)$ satisfy*

$$\frac{1}{2} - \frac{p}{N} = (1-a)\frac{q}{s} + a\left(\frac{1}{2} - \frac{1}{N}\right) \quad \text{and} \quad p \leq a.$$

Then there exist $c_0, c_0' > 0$ such that for all $u \in W^{1,2}(\Omega) \cap L^{\frac{s}{q}}(\Omega)$,

$$\|u\|_{W^{p,2}(\Omega)} \leq c_0 \|\nabla u\|^a_{L^2(\Omega)} \|u\|^{1-a}_{L^{\frac{s}{q}}(\Omega)} + c_0' \|u\|_{L^{\frac{s}{q}}(\Omega)}.$$

Lemma 2.2. *([21,44]) Let $T > 0$, $\tau \in (0, T)$, $A > 0$ and $B > 0$, and suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous such that*

$$y'(t) + Ay(t) \leq h(t) \quad \text{for a.e.} \quad t \in (0, T)$$

with some nonnegative function $h \in L^1_{loc}([0, T))$ satisfying

$$\int_t^{t+\tau} h(s)ds \leq B \quad \text{for all} \quad t \in (0, T - \tau).$$

Then

$$y(t) \leq \max\left\{y_0 + B, \frac{B}{A\tau} + 2B\right\} \quad \text{for all} \quad t \in (0, T).$$

Lemma 2.3. *([6,7,43]) Suppose $\gamma \in (1, +\infty)$, $g \in L^\gamma((0, T); L^\gamma(\Omega))$. Let v be a solution of the following initial boundary value*

$$\begin{cases} v_t - \Delta v + v = g, \quad (x, t) \in \Omega \times (0, T), \\ \dfrac{\partial v}{\partial \nu} = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\ v(x, 0) = v_0(x), \quad (x, t) \in \Omega. \end{cases}$$

Then there exists a positive constant $\lambda_0 := \lambda_0(\Omega, \gamma, N)$ such that if $s_0 \in [0, T)$, $v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N)$ with $\dfrac{\partial v(\cdot, s_0)}{\partial \nu} = 0$, then

$$\int_{s_0}^{T} e^{\gamma s}\|v(\cdot,t)\|^{\gamma}_{W^{2,\gamma}(\Omega)}ds \le \lambda_0\left(\int_{s_0}^{T} e^{\gamma s}\|g(\cdot,s)\|^{\gamma}_{L^{\gamma}(\Omega)}ds + e^{\gamma s_0}(\|v_0(\cdot,s_0)\|^{\gamma}_{W^{2,\gamma}(\Omega)})\right).$$

Lemma 2.4. *(Lemma 2.2 of [49]) Suppose that $\beta > \max\{1, \frac{N-2}{2}\}$ and $\Omega \subset \mathbb{R}^N (N \ge 3)$ is a bounded domain with smooth boundary. Moreover, assume that*

$$\lambda \in [2\beta + 2, \frac{N(2\beta+1) - 2(\beta+1)}{N-2}].$$

Then there exists $C > 0$ such that for all $\varphi \in C^2(\bar{\Omega})$ fulfilling $\varphi \cdot \frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ we have

$$\|\nabla\varphi\|_{L^{\lambda}(\Omega)} \le C\||\nabla\varphi|^{\beta-1}D^2\varphi\|^{\frac{2(\lambda-N)}{(2\beta-N+2)\lambda}}_{L^2(\Omega)}\|\varphi\|^{\frac{2N\beta-(N-2)\lambda}{(2\beta-N+2)\lambda}}_{L^{\infty}(\Omega)} + C\|\varphi\|_{L^{\infty}(\Omega)}.$$

The first lemma concerns the local solvability of problems (1.1). The proof is based on well-established methods involving the Schauder fixed point theorem, the standard regularity theory of parabolic equation (for details see Lemma 1.1 of [29] and [1,9,24,26,28,37]).

Lemma 2.5. *Suppose that $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a bounded domain with smooth boundary, D satisfies (1.2)–(1.3). Then for nonnegative triple $(u_0, v_0) \in C(\bar{\Omega}) \times W^{1,\infty}(\bar{\Omega})$, problem (1.1) has a unique local-in-time non-negative classical functions*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ v \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \cap L^{\infty}((0, T_{max}); W^{1,\infty}(\Omega)), \end{cases} \tag{2.1}$$

where T_{max} denotes the maximal existence time. Moreover, if $T_{max} < +\infty$, then

$$\|u(\cdot,t)\|_{L^{\infty}(\Omega)} \to \infty \quad as \quad t \nearrow T_{max} \tag{2.2}$$

is fulfilled.

Lemma 2.6. *(Lemma 3.2 of [14]) There exists $C > 0$ such that the solution (u, v) of (1.1) satisfies*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le C \quad for\ all \quad t \in (0, T_{max}), \tag{2.3}$$

$$\|v(\cdot,t)\|_{L^{\infty}(\Omega)} \le \|v_0\|_{L^{\infty}(\Omega)} \quad for\ all \quad t \in (0, T_{max}) \tag{2.4}$$

and

$$\int_t^{t+\tau}\int_{\Omega} u^2 \le C \quad for\ all \quad t \in (0, T_{max} - \tau), \tag{2.5}$$

where

$$\tau := \min\{1, \frac{1}{6}T_{max}\}. \tag{2.6}$$

Now, collecting Lemma 2.4 and Lemma 2.6, we derive that:

Lemma 2.7. *Let $N \ge 3$ and $\beta > \max\{1, \frac{N-2}{2}\}$. Then there exists a positive constant κ_0 such that the solution of (1.1) satisfies*

$$\|\nabla v\|^{2\beta+2}_{L^{2\beta+2}(\Omega)} \le \kappa_0(\||\nabla v|^{\beta-1}D^2 v\|^2_{L^2(\Omega)} + 1). \tag{2.7}$$

Proof. Let $\varphi = v$ and $\lambda = 2\beta + 2$ in Lemma 2.4, then by using $\frac{2(\lambda-N)}{(2\beta-N+2)\lambda}\lambda = 2$ and (2.4), we can obtain the result. □

3. A priori estimates

In this section, we are going to establish an iteration step to develop the main ingredient of our result. Firstly, employing almost exactly the same arguments as in the proof of Lemma 2.1 in [29] (see also Lemma 3.2 of [14]), we may derive the following Lemma:

Lemma 3.1. *Under the assumptions in Theorem 1.1, we derive that there exists a positive constant C such that the solution of (1.1) satisfies*

$$\int_\Omega |\nabla v(x,t)|^2 \leq C \quad \text{for all} \quad t \in (0, T_{max}) \tag{3.1}$$

and

$$\int_t^{t+\tau}\int_\Omega [|\nabla v|^2 + u^2 + |\Delta v|^2] \leq C \quad \text{for all} \quad t \in (0, T_{max} - \tau), \tag{3.2}$$

where τ is given by (2.6).

Lemma 3.2. *Let $g(l) := \mu - (l-1)\chi - (l-1)\chi\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3$, where $l \geq 1$, χ, μ are positive constants, $\bar{\lambda}_0 = \max_{1\leq s\leq 2}\lambda_0(\Omega, s, N)$ and λ_0 is the same as in Lemma 2.3. Then there exists a positive constant $l_0 \in (1, 2]$ such that*

$$g(l_0) \geq \frac{\mu}{2}. \tag{3.3}$$

Proof. Obviously, $g(1 + \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]}) = \frac{\mu}{2}$ and g is monotonically decreasing with respect to l. Next, choose $l_0 = \min\{1 + \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]}, 2\}$, and then, $l_0 \leq 1 + \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]}$, so that, with the help of monotonicity of g, we also derive that

$$g(l_0) \geq g(1 + \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]}) = \frac{\mu}{2}.$$

The proof Lemma 3.2 is completed. □

Lemma 3.3. *Let (u, v) be a solution to (1.1) on $(0, T_{max})$ and D satisfy (1.2)–(1.3) with $m > 0$. Assume that $m > 1 - \frac{\mu}{2\chi[1+\max_{1\leq s\leq 2}\lambda_0(s)\|v_0\|_{L^\infty(\Omega)}2^3]}$ and $N \leq 2$. Then for all $p > 1$, there exists a positive constant $C := C(p, |\Omega|, \mu, \lambda_0, \chi)$ such that*

$$\int_\Omega u^p(x,t) \leq C \quad \text{for all} \quad t \in (0, T_{max}). \tag{3.4}$$

Proof. Firstly, let us pick any $s_0 \in (0, T_{max})$ and $s_0 \leq 1$. Then from the regularity principle asserted by Lemma 2.5, we derive that $(u(\cdot, s_0), v(\cdot, s_0)) \in C^2(\bar{\Omega})$ with $\frac{\partial v_{\cdot,s_0)}}{\partial\nu} = 0$ on $\partial\Omega$, so that in particular we can there exists a positive constant K such that

$$\|u(\cdot,\tau)\|_{L^\infty(\Omega)} \leq K, \quad \|v(\cdot,\tau)\|_{L^\infty(\Omega)} \leq K \quad \text{and} \quad \|\nabla v(\cdot,\tau)\|_{L^\infty(\Omega)} \leq K \quad \text{for all} \quad \tau \in (0, s_0]. \tag{3.5}$$

Assume that l_0 is the same as in Lemma 3.2. Multiplying the first equation of (1.1) by u^{l_0-1}, integrating over Ω and using (1.3), we get

$$\begin{array}{rl} & \frac{1}{l_0}\frac{d}{dt}\|u\|^{l_0}_{L^{l_0}(\Omega)} + (l_0-1)\int\limits_\Omega u^{m+l_0-3}|\nabla u|^2 \\ \le & -\chi\int\limits_\Omega \nabla\cdot(u\nabla v)u^{l_0-1} + \int\limits_\Omega u^{l_0-1}(\mu u - \mu u^2) \ \text{ for all } \ t\in(0,T_{max}), \end{array} \tag{3.6}$$

which implies that,

$$\begin{array}{rl} \frac{1}{l_0}\frac{d}{dt}\|u\|^{l_0}_{L^{l_0}(\Omega)} \le & -\frac{l_0+1}{l_0}\int\limits_\Omega u^{l_0} - \chi\int\limits_\Omega \nabla\cdot(u\nabla v)u^{l_0-1} \\ & +\int\limits_\Omega \left(\frac{l_0+1}{l_0}u^{l_0} + u^{l_0-1}(\mu u - \mu u^2)\right) \ \text{ for all } \ t\in(0,T_{max}). \end{array} \tag{3.7}$$

Next, for any positive constant $\varepsilon_1 > 0$, we derive from the Young inequality that

$$\begin{array}{rl} & \int\limits_\Omega \left(\frac{l_0+1}{l_0}u^{l_0} + u^{l_0-1}(\mu u - \mu u^2)\right) \\ = & \frac{l_0+1}{l_0}\int\limits_\Omega u^{l_0} + \mu\int\limits_\Omega u^{l_0} - \mu\int\limits_\Omega u^{l_0+1} \\ \le & (\varepsilon_1 - \mu)\int\limits_\Omega u^{l_0+1} + C_1(\varepsilon_1, l_0) \ \text{ for all } \ t\in(0,T_{max}), \end{array} \tag{3.8}$$

where

$$C_1(\varepsilon_1, l_0) = \frac{1}{l_0+1}\left(\varepsilon_1\frac{l_0+1}{l_0}\right)^{-l_0}\left(\frac{l_0+1}{l_0}+\mu\right)^{l_0+1}|\Omega|.$$

Now, integrating by parts to the first term on the right hand side of (3.6) and using the Young inequality, we conclude that

$$\begin{array}{rl} & -\chi\int\limits_\Omega \nabla\cdot(u\nabla v)u^{l_0-1} \\ = & (l_0-1)\chi\int\limits_\Omega u^{l_0-1}\nabla u\cdot\nabla v \\ \le & \frac{l_0-1}{l_0}\chi\int\limits_\Omega u^{l_0}|\Delta v| \\ \le & (l_0-1)\chi\int\limits_\Omega u^{l_0}|\Delta v| \\ \le & (l_0-1)\chi\int\limits_\Omega u^{l_0+1} + (l_0-1)\chi\int\limits_\Omega |\Delta v|^{l_0+1} \ \text{ for all } \ t\in(0,T_{max}). \end{array} \tag{3.9}$$

Thus, inserting (3.9) into (3.7), we conclude that

$$\begin{aligned}\frac{1}{l_0}\frac{d}{dt}\|u\|^{l_0}_{L^{l_0}(\Omega)} \leq\ & (\varepsilon_1+(l_0-1)\chi-\mu)\int_\Omega u^{l_0+1}dx-\frac{l_0+1}{l_0}\int_\Omega u^{l_0}dx\\ & +(l_0-1)\chi\int_\Omega|\Delta v|^{l_0+1}dx+C_1(\varepsilon_1,l_0).\end{aligned}$$

For any $t\in(s_0,T_{max})$, employing the variation-of-constants formula to the above inequality, we obtain

$$\begin{aligned}&\frac{1}{l_0}\|u(\cdot,t)\|^{l_0}_{L^{l_0}(\Omega)}\\ \leq\ & \frac{1}{l_0}e^{-(l_0+1)(t-s_0)}\|u(\cdot,s_0)\|^{l_0}_{L^{l_0}(\Omega)}+(\varepsilon_1+(l_0-1)\chi-\mu)\int_{s_0}^t e^{-(l_0+1)(t-s)}\int_\Omega u^{l_0+1}dxds\\ & +(l_0-1)\chi\int_{s_0}^t e^{-(l_0+1)(t-s)}\int_\Omega|\Delta v|^{l_0+1}dxds+C_1(\varepsilon_1,l_0)\int_{s_0}^t e^{-(l_0+1)(t-s)}ds\\ \leq\ & (\varepsilon_1+(l_0-1)\chi-\mu)\int_{s_0}^t e^{-(l_0+1)(t-s)}\int_\Omega u^{l_0+1}dxds\\ & +(l_0-1)\chi\int_{s_0}^t e^{-(l_0+1)(t-s)}\int_\Omega|\Delta v|^{l_0+1}dxds+C_2(l_0,\varepsilon_1),\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}C_2:=C_2(l_0,\varepsilon_1):=\ & \frac{1}{l_0}\|u(\cdot,s_0)\|^{l_0}_{L^{l_0}(\Omega)}+C_1(\varepsilon_1,l_0)\int_{s_0}^t e^{-(l_0+1)(t-s)}ds\\ \leq\ & \frac{1}{l_0}\|u(\cdot,s_0)\|^{l_0}_{L^{l_0}(\Omega)}+\frac{C_1(\varepsilon_1,l_0)}{l_0+1}.\end{aligned}$$

Let $t\in(s_0,T_{max})$ and rewrite the second equation as

$$v_t-\Delta v+v=-vu+v.$$

Now, by Lemma 2.3 and (2.4), we derive that for all $t\in(0,T_{max})$,

$$\begin{aligned}&(l_0-1)\chi\int_{s_0}^t e^{-(l_0+1)(t-s)}\int_\Omega|\Delta v|^{l_0+1}dxds\\ =\ & (l_0-1)\chi e^{-(l_0+1)t}\int_{s_0}^t e^{(l_0+1)s}\int_\Omega|\Delta v|^{l_0+1}dxds\\ \leq\ & (l_0-1)\chi e^{-(l_0+1)t}\lambda_0\left[\int_{s_0}^t\int_\Omega e^{(l_0+1)s}|-vu+v|^{l_0+1}dxds+e^{(l_0+1)s_0}\|v(s_0,t)\|^{l_0+1}_{W^{2,l_0+1}}\right]\\ \leq\ & (l_0-1)\chi e^{-(l_0+1)t}\bar{\lambda}_0\left[\|v_0\|_{L^\infty(\Omega)}2^{l_0+1}\int_{s_0}^t\int_\Omega e^{(l_0+1)s}(u^{l_0+1}+1)dxds+e^{(l_0+1)s_0}\|v(s_0,t)\|^{l_0+1}_{W^{2,l_0+1}}\right],\end{aligned}\tag{3.11}$$

where $\bar{\lambda}_0 = \max_{1\le s\le 2}\lambda_0(\Omega, s, N)$ and λ_0 is the same as in Lemma 2.3. By substituting (3.11) into (3.10), we get

$$\begin{aligned}
&\frac{1}{l_0}\|u(\cdot,t)\|^{l_0}_{L^{l_0}(\Omega)} \\
\le\ & (\varepsilon_1 + (l_0-1)\chi + (l_0-1)\chi\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^{l_0+1} - \mu)\int_{s_0}^{t} e^{-(l_0+1)(t-s)}\int_\Omega u^{l_0+1}dxds \\
&+(l_0-1)\chi e^{-(l_0+1)(t-s_0)}\bar{\lambda}_0\|v(s_0,t)\|^{l_0+1}_{W^{2,l_0+1}} \\
&+(l_0-1)\chi e^{-(l_0+1)t}\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^{l_0+1}\int_{s_0}^{t}\int_\Omega e^{(l_0+1)s}dxds + C_2(l_0,\varepsilon_1).
\end{aligned} \tag{3.12}$$

Now, using $1 < l_0 \le 2$, then we conclude that

$$\begin{aligned}
&(l_0-1)\chi + (l_0-1)\chi\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3 \\
\ge\ & (l_0-1)\chi + (l_0-1)\chi\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^{l_0+1}.
\end{aligned}$$

Therefore, in light of Lemma 3.2, we derive

$$(l_0-1)\chi + (l_0-1)\chi\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3 - \frac{\mu}{2} = [\mu - g(l_0)] - \frac{\mu}{2} = \frac{\mu}{2} - g(l_0) \le 0,$$

so that by the choice of $\varepsilon_1 = \frac{\mu}{4}$ in (3.12), we derive that there exists a positive constant C_3 such that

$$\int_\Omega u^{l_0}(x,t)dx \le C_3 \quad \text{for all} \quad t\in(s_0, T_{max}). \tag{3.13}$$

Case $1 + \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]} < 2$: then $l_0 = \min\{1 + \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]}, 2\} = 1 + \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]}$ and $1 < l_0 < 2$. Next, we fix $l_0 < q < \frac{2l_0}{2-l_0}$ and choose some $\alpha > \frac{1}{2}$ which is close to $\frac{1}{2}$ (e.g. $\alpha \in (\frac{1}{2}, \frac{1}{2} + \frac{\delta_0(2-l_0)^2}{2l_0[2l_0-(2-l_0)\delta_0]})$ with $0 < \delta_0 < \frac{1}{2}(\frac{2l_0}{2-l_0} - q)$) such that

$$q < \frac{1}{\frac{1}{l_0} - \frac{1}{2} + \frac{2}{2}(\alpha - \frac{1}{2})} \le \frac{2l_0}{2-l_0}. \tag{3.14}$$

Now, involving the variation-of-constants formula for v, we have

$$v(\cdot,t) = e^{-(t-s_0)(-\Delta+1)}v(\cdot,s_0) + \int_{s_0}^{t} e^{-(t-s)(-\Delta+1)}(-v(\cdot,s)u(\cdot,s) + v(\cdot,s))ds, \quad t\in(s_0,T_{max}). \tag{3.15}$$

Hence, in view of $N \le 2$, it follows from (3.5), (2.4) and (3.15) that there exist positive constants ρ_0, C_4 and C_5 such that

$$\begin{aligned}
&\|(-\Delta+1)^{\alpha}v(\cdot,t)\|_{L^q(\Omega)}\\
\le\;& C_4\int_{s_0}^{t}(t-s)^{-\alpha-\frac{N}{2}(\frac{1}{l_0}-\frac{1}{q})}e^{-\rho_0(t-s)}\|-v(\cdot,s)u(\cdot,s)+v(\cdot,s)\|_{L^{l_0}(\Omega)}ds\\
&+C_4(t-s_0)^{-\alpha-\frac{N}{2}(1-\frac{1}{q})}\|v(\cdot,s_0)\|_{L^1(\Omega)}\\
\le\;& C_5\int_{0}^{+\infty}\sigma^{-\alpha-\frac{N}{2}(\frac{1}{l_0}-\frac{1}{q})}e^{-\rho_0\sigma}d\sigma+C_5(t-s_0)^{-\alpha-\frac{N}{2}(1-\frac{1}{q})}K.
\end{aligned}\tag{3.16}$$

Hence, due to (3.14) and (3.16), we have

$$\int_\Omega|\nabla v(\cdot,t)|^q\le C_6 \text{ for all } t\in(s_0,T_{max}) \text{ and } q\in(l_0,\frac{2l_0}{2-l_0}).\tag{3.17}$$

Finally, in view of (3.5) and (3.17), we can get

$$\int_\Omega|\nabla v(\cdot,t)|^q\le C_7 \text{ for all } t\in(0,T_{max}) \text{ and } q\in(l_0,\frac{2l_0}{2-l_0})\tag{3.18}$$

with some positive constant C_7.

Since $m>1-\frac{\mu}{2\chi[1+\bar\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}$ yields to $\varepsilon=\frac{m-\left[1-\frac{\mu}{2\chi[1+\bar\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}\right]}{2}>0$, so that, by $l_0=1+\frac{\mu}{2\chi[1+\bar\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}$ (see Lemma 3.2), we derive that

$$q_0=\frac{1+\frac{\mu}{2\chi[1+\bar\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}}{(1-\frac{\mu}{2\chi[1+\bar\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]})_++\varepsilon}<\frac{1+\frac{\mu}{2\chi[1+\bar\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}}{(1-\frac{\mu}{2\chi[1+\bar\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]})_+}=\frac{2l_0}{2(2-l_0)}.\tag{3.19}$$

Next, for any $p>\max\{1,1-m,m+l_0-1-\frac{l_0}{q_0}\}$, multiplying both sides of the first equation in (1.1) by u^{p-1}, integrating over Ω, integrating by parts and using (1.3), we arrive at

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)}+(p-1)\int_\Omega u^{m+p-3}|\nabla u|^2dx\\
\le\;& -\chi\int_\Omega\nabla\cdot(u\nabla v)u^{p-1}dx+\int_\Omega u^{p-1}(\mu u-\mu u^2)dx\\
=\;& \chi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla vdx+\int_\Omega u^{p-1}(\mu u-\mu u^2)dx,
\end{aligned}\tag{3.20}$$

which together with the Young inequality implies that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)}+(p-1)\int_\Omega u^{m+p-3}|\nabla u|^2dx\\
\le\;& \frac{p-1}{2}\int_\Omega u^{m+p-3}|\nabla u|^2dx+\frac{\chi^2(p-1)}{2}\int_\Omega u^{p+1-m}|\nabla v|^2dx-\frac{\mu}{2}\int_\Omega u^{p+1}dx+C_8
\end{aligned}\tag{3.21}$$

for some positive constant C_8. In light of the Hölder inequality and (3.18), we derive at

$$\begin{aligned}\frac{\chi^2(p-1)}{2}\int_\Omega u^{p+1-m}|\nabla v|^2 \leq & \ \frac{\chi^2(p-1)}{2}\left(\int_\Omega u^{\frac{q_0}{q_0-1}(p+1-m)}\right)^{\frac{q_0-1}{q_0}}\left(\int_\Omega |\nabla v|^{2q_0}\right)^{\frac{1}{q_0}} \\ \leq & \ C_9\|u^{\frac{m+p-1}{2}}\|^{2\frac{p+1-m}{m+p-1}}_{L^{2\frac{q_0}{q_0-1}\frac{p+1-m}{m+p-1}}(\Omega)},\end{aligned} \tag{3.22}$$

where C_9 is a positive constant. Due to $q_0>1, p>\max\{1-m, m+l_0-1-\frac{l_0}{q_0}\}$, we have

$$\frac{l_0}{m+p-1}\leq\frac{q_0}{q_0-1}\frac{p+1-m}{m+p-1}<+\infty,$$

which together with the Gagliardo–Nirenberg inequality implies that

$$\begin{aligned}C_9\|u^{\frac{m+p-1}{2}}\|^{2\frac{p+1-m}{m+p-1}}_{L^{2\frac{q_0}{q_0-1}\frac{p+1-m}{m+p-1}}(\Omega)} \leq & \ C_{10}(\|\nabla u^{\frac{m+p-1}{2}}\|^{\mu_1}_{L^2(\Omega)}\|u^{\frac{m+p-1}{2}}\|^{1-\mu_1}_{L^{\frac{2l_0}{m+p-1}}(\Omega)}+\|u^{\frac{m+p-1}{2}}\|_{L^{\frac{2l_0}{m+p-1}}(\Omega)})^{2\frac{p+1-m}{m+p-1}} \\ \leq & \ C_{11}(\|\nabla u^{\frac{m+p-1}{2}}\|^{2\frac{p+1-m}{m+p-1}\mu_1}_{L^2(\Omega)}+1) \\ = & \ C_{11}(\|\nabla u^{\frac{m+p-1}{2}}\|^{\frac{2[q_0(p+1-m)-l_0(q_0-1)]}{q_0(m+p-1)}}_{L^2(\Omega)}+1)\end{aligned} \tag{3.23}$$

with some positive constants C_{10}, C_{11} and

$$\mu_1=\frac{\frac{2(m+p-1)}{2l_0}-\frac{2(m+p-1)(q_0-1)}{2q_0(p+1-m)}}{1-\frac{2}{2}+\frac{2(m+p-1)}{2l_0}}=(m+p-1)\frac{\frac{2}{2l_0}-\frac{2(q_0-1)}{2q_0(p+1-m)}}{1-\frac{2}{2}+\frac{2(m+p-1)}{2l_0}}\in(0,1).$$

On the other hand, in view of (3.19), by $l_0=1+\frac{\mu}{2\chi[1+\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}$ and $m>1-\frac{\mu}{2\chi[1+\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}$, we derive that

$$\frac{q_0(p+1-m)-l_0(q_0-1)}{q_0(m+p-1)}<1. \tag{3.24}$$

Hence, in view of the Young inequality, we have

$$\frac{\chi^2(p-1)}{2}\int_\Omega u^{p+1-m}|\nabla v|^2dx \ \leq \frac{p-1}{4}\int_\Omega u^{m+p-3}|\nabla u|^2dx+C_{12}. \tag{3.25}$$

Inserting (3.25) into (3.21), we conclude that

$$\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)}+\frac{p-1}{4}\int_\Omega u^{m+p-3}|\nabla u|^2dx+\frac{\mu}{2}\int_\Omega u^{p+1}dx\leq C_{13}.$$

Therefore, integrating the above inequality with respect to t and employing the Hölder inequality, we derive that there exists a positive constant C_{14} such that

$$\|u(\cdot,t)\|_{L^p(\Omega)}\leq C_{14} \ \text{ for all } \ p>1 \ \text{ and } \ t\in(0,T_{max}). \tag{3.26}$$

Case $1+\frac{\mu}{2\chi[1+\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]}\geq 2$: then $l_0=\min\{1+\frac{\mu}{2\chi[1+\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]},2\}=2$: we fix $l_0<q<+\infty$ and choose some $\alpha>\frac{1}{2}$ which is close to $\frac{1}{2}$ (e.g. $\alpha\in(\frac{1}{2},\frac{1}{2}+\frac{1}{q})$) such that

$$q<\frac{1}{\frac{1}{l_0}-\frac{1}{2}+\frac{2}{2}(\alpha-\frac{1}{2})}<+\infty. \tag{3.27}$$

Now, applying the variation-of-constants formula for v, we conclude that

$$v(\cdot,t) = e^{-(t-s_0)(-\Delta+1)}v(\cdot,s_0) + \int_{s_0}^{t} e^{-(t-s)(-\Delta+1)}(-v(\cdot,s)u(\cdot,s) + v(\cdot,s))ds, \quad t \in (s_0, T_{max}). \tag{3.28}$$

Hence, in light of $N \leq 2$, we derive from (3.5), (2.4) and (3.28) that there exist positive constants $\tilde{\rho}_0, C_{15}$ and C_{16} such that

$$\begin{aligned}
&\|(-\Delta+1)^{\alpha}v(\cdot,t)\|_{L^q(\Omega)}\\
\leq\ & C_{15}\int_{s_0}^{t}(t-s)^{-\alpha-\frac{N}{2}(\frac{1}{l_0}-\frac{1}{q})}e^{-\tilde{\rho}_0(t-s)}\|-v(\cdot,s)u(\cdot,s)+v(\cdot,s)\|_{L^{l_0}(\Omega)}ds\\
&+C_{15}(t-s_0)^{-\alpha-\frac{N}{2}(1-\frac{1}{q})}\|v(\cdot,s_0)\|_{L^1(\Omega)}\\
\leq\ & C_{16}\int_{0}^{+\infty}\sigma^{-\alpha-\frac{N}{2}(\frac{1}{l_0}-\frac{1}{q})}e^{-\tilde{\rho}_0\sigma}d\sigma + C_{16}(t-s_0)^{-\alpha-\frac{N}{2}(1-\frac{1}{q})}K.
\end{aligned} \tag{3.29}$$

Hence, (3.27) and (3.29) imply that

$$\int_{\Omega}|\nabla v(\cdot,t)|^q \leq C_{17} \ \text{ for all } \ t \in (s_0, T_{max}) \ \text{ and } \ q \in (l_0, +\infty),$$

which combined with (3.5) yields to

$$\int_{\Omega}|\nabla v(\cdot,t)|^q \leq C_{18} \ \text{ for all } \ t \in (0, T_{max}) \ \text{ and } \ q \in (l_0, +\infty) \tag{3.30}$$

with some positive constant C_{18}.

Next, for any $p > m+1$, multiplying both sides of the first equation in (1.1) by u^{p-1}, integrating over Ω, integrating by parts, using (1.3) and employing the Young inequality, we conclude that there exists a positive constant C_{19} such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)} + (p-1)\int_{\Omega}u^{m+p-3}|\nabla u|^2dx\\
\leq\ & \frac{p-1}{2}\int_{\Omega}u^{m+p-3}|\nabla u|^2dx + \frac{\chi^2(p-1)}{2}\int_{\Omega}u^{p+1-m}|\nabla v|^2dx - \frac{\mu}{2}\int_{\Omega}u^{p+1}dx + C_{19}.
\end{aligned} \tag{3.31}$$

By $m > 0$, we derive that

$$p+1-m < p+1,$$

so that, the Young inequality and (3.30) yield to,

$$\frac{\chi^2(p-1)}{2}\int_{\Omega}u^{p+1-m}|\nabla v|^2dx \quad \leq \frac{\mu}{4}\int_{\Omega}u^{p+1}dx + C_{20},$$

which together with (3.31) implies that

$$\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)} + \frac{p-1}{2}\int_\Omega u^{m+p-3}|\nabla u|^2dx + \frac{\mu}{4}\int_\Omega u^{p+1}dx \le C_{21}. \tag{3.32}$$

Hence, integrating (3.32) with respect to t and using the Hölder inequality yields

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C_{22} \quad \text{for all} \quad p>1 \quad \text{and} \quad t\in(0,T_{max}) \tag{3.33}$$

for some positive constant C_{22}. The proof Lemma 3.3 is complete. □

Lemma 3.4. *Assume that $m>1$ and $N\ge 3$. Let (u,v) be a solution to (1.1) on $(0,T_{max})$. Then for all $p>1$, there exists a positive constant $C:=C(p,|\Omega|,\mu,\chi)$ such that*

$$\int_\Omega u^p(x,t) \le C \quad \textit{for all} \quad t\in(0,T_{max}). \tag{3.34}$$

Proof. Let $\beta>\max\{1,\frac{N-2}{2}\}$ and

$$\beta<p<\beta+(m-1)(\beta+1). \tag{3.35}$$

Observing that $\nabla v\cdot\nabla\Delta v = \frac{1}{2}\Delta|\nabla v|^2 - |D^2v|^2$, in light of a straightforward computation using the second equation in (1.1) and several integrations by parts, we conclude that

$$\begin{aligned}
\frac{1}{2\beta}\frac{d}{dt}\|\nabla v\|^{2\beta}_{L^{2\beta}(\Omega)} &= \int_\Omega |\nabla v|^{2\beta-2}\nabla v\cdot\nabla(\Delta v - uv)\\
&= \frac{1}{2}\int_\Omega |\nabla v|^{2\beta-2}\Delta|\nabla v|^2 - \int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2\\
&\quad +\int_\Omega uv\nabla\cdot(|\nabla v|^{2\beta-2}\nabla v)\\
&= -\frac{\beta-1}{2}\int_\Omega |\nabla v|^{2\beta-4}\left|\nabla|\nabla v|^2\right|^2 + \frac{1}{2}\int_{\partial\Omega}|\nabla v|^{2\beta-2}\frac{\partial|\nabla v|^2}{\partial\nu}\\
&\quad -\int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2 + \int_\Omega uv|\nabla v|^{2\beta-2}\Delta v + \int_\Omega uv\nabla v\cdot\nabla(|\nabla v|^{2\beta-2})\\
&= -\frac{2(\beta-1)}{\beta^2}\int_\Omega \left|\nabla|\nabla v|^\beta\right|^2 + \frac{1}{2}\int_{\partial\Omega}|\nabla v|^{2\beta-2}\frac{\partial|\nabla v|^2}{\partial\nu} - \int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2\\
&\quad +\int_\Omega uv|\nabla v|^{2\beta-2}\Delta v + \int_\Omega uv\nabla v\cdot\nabla(|\nabla v|^{2\beta-2})
\end{aligned} \tag{3.36}$$

for all $t\in(0,T_{max})$. Now, by using the idea of [13], we will estimate the right hand of (3.36). To this end, firstly, we observe that

$$\begin{aligned}
&\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2\beta-2}\\
\le\ & C_\Omega\int_{\partial\Omega}|\nabla v|^{2\beta}\\
=\ & C_\Omega\||\nabla v|^\beta\|^2_{L^2(\partial\Omega)}.
\end{aligned} \tag{3.37}$$

Let us take $r \in (0, \frac{1}{2})$. By compactness of the embedding $W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ (see e.g. [13]), we have

$$\||\nabla v|^\beta\|^2_{L^2(\partial\Omega)} \le C_1 \||\nabla v|^\beta\|^2_{W^{r+\frac{1}{2},2}(\Omega)}. \tag{3.38}$$

In order to apply Lemma 2.1 to estimate the right-hand side of (3.38), let us pick $a \in (0,1)$ satisfying

$$a = \frac{\frac{1}{4} + \frac{\beta}{2} + \frac{\gamma}{N} - \frac{1}{2}}{\frac{1}{N} + \frac{\beta}{2} - \frac{1}{2}}.$$

Noting that $\gamma \in (0, \frac{1}{2})$ and $\beta > 1$ imply that $\gamma + \frac{1}{2} \le a < 1$, we see from the fractional Gagliardo–Nirenberg inequality and boundedness of $|\nabla v|^2$ (see Lemma 3.1) that

$$\begin{aligned} & \||\nabla v|^\beta\|^2_{W^{r+\frac{1}{2},2}(\Omega)} \\ \le\ & c_0 \|\nabla |\nabla v|^\beta\|^a_{L^2(\Omega)} \||\nabla v|^\beta\|^{1-a}_{L^{\frac{2}{\beta}}(\Omega)} + c_0' \||\nabla v|^\beta\|_{L^{\frac{2}{\beta}}(\Omega)} \\ \le\ & C_2 \|\nabla |\nabla v|^\beta\|^a_{L^2(\Omega)} + C_2. \end{aligned} \tag{3.39}$$

Combining (3.37) and (3.38) with (3.39), we obtain

$$\int_{\partial\Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2\beta-2} \le C_3 \|\nabla |\nabla v|^\beta\|^a_{L^2(\Omega)} + C_3. \tag{3.40}$$

On the other hand, by $|\Delta v| \le \sqrt{N}|D^2 v|$ and the Young inequality, we can get

$$\begin{aligned} \int_\Omega uv|\nabla v|^{2\beta-2}\Delta v \le\ & \sqrt{N}\|v_0\|_{L^\infty(\Omega)} \int_\Omega u|\nabla v|^{2\beta-2}|D^2 v| \\ \le\ & \frac{1}{4}\int_\Omega |\nabla v|^{2\beta-2}|D^2 v|^2 + N\|v_0\|^2_{L^\infty(\Omega)} \int_\Omega u^2 |\nabla v|^{2\beta-2}. \end{aligned} \tag{3.41}$$

Next, due to the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \int_\Omega uv\nabla v \cdot \nabla(|\nabla v|^{2\beta-2}) =\ & (\beta-1)\int_\Omega uv|\nabla v|^{2(\beta-2)}\nabla v \cdot \nabla |\nabla v|^2 \\ \le\ & \frac{\beta-1}{8}\int_\Omega |\nabla v|^{2\beta-4}\left|\nabla |\nabla v|^2\right|^2 + 2(\beta-1)\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega |u|^2|\nabla v|^{2\beta-2} \\ \le\ & \frac{(\beta-1)}{2\beta^2}\int_\Omega \left|\nabla |\nabla v|^\beta\right|^2 + 2(\beta-1)\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega |u|^2|\nabla v|^{2\beta-2}. \end{aligned} \tag{3.42}$$

Now, collecting (3.36), (3.40)–(3.42) and using the Young inequality yields

$$\begin{aligned} & \frac{1}{2\beta}\frac{d}{dt}\|\nabla v\|^{2\beta}_{L^{2\beta}(\Omega)} + \frac{(\beta-1)}{\beta}\int_\Omega \left|\nabla |\nabla v|^\beta\right|^2 + \frac{3}{4}\int_\Omega |\nabla v|^{2\beta-2}|D^2 v|^2 \\ \le\ & C_4 \int_\Omega u^2|\nabla v|^{2\beta-2} + C_4 \quad \text{for all } \ t \in (0, T_{max}). \end{aligned} \tag{3.43}$$

Next, in light of the Young inequality and using Lemma 2.7, we derive

$$\begin{aligned} C_4\int_\Omega u^2|\nabla v|^{2\beta-2} \le &\ \frac{1}{8\kappa_0}\int_\Omega |\nabla v|^{2\beta+2}+C_5\int_\Omega u^{\beta+1} \\ \le &\ \frac{1}{8}\int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2+C_5\int_\Omega u^{\beta+1}+C_6 \ \text{ for all } \ t\in(0,T_{max}), \end{aligned} \tag{3.44}$$

where κ_0 is the same as (2.7). Inserting (3.44) into (3.43), we conclude that

$$\begin{aligned} &\frac{1}{2\beta}\frac{d}{dt}\|\nabla v\|^{2\beta}_{L^{2\beta}(\Omega)}+\frac{(\beta-1)}{\beta}\int_\Omega \left|\nabla|\nabla v|^\beta\right|^2+\frac{5}{8}\int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2 \\ \le &\ C_5\int_\Omega u^{\beta+1}+C_7 \ \text{ for all } \ t\in(0,T_{max}). \end{aligned} \tag{3.45}$$

Let $p>1$. Now, testing the first equation in (1.1) with u^{p-1} and integrating over Ω and using (1.3), we derive

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)}+(p-1)\int_\Omega u^{m+p-1}|\nabla u|^2 \\ \le &\ -\chi\int_\Omega \nabla\cdot(u\nabla v)u^{p-1}+\mu\int_\Omega u^{p-1}(u-u^2) \ \text{ for all } \ t\in(0,T_{max}). \end{aligned} \tag{3.46}$$

Next, integrating by parts to the first term on the right hand side of (3.46), using the Young inequality and Lemma 2.7, we obtain

$$\begin{aligned} &-\chi\int_\Omega \nabla\cdot(u\nabla v)u^{p-1} \\ \le &\ (p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\ \le &\ \frac{p-1}{4}\int_\Omega u^{m+p-3}|\nabla u|^2+(p-1)\chi^2\int_\Omega u^{p+1-m}|\nabla v|^2 \\ \le &\ \frac{p-1}{4}\int_\Omega u^{m+p-3}|\nabla u|^2 \\ &+\frac{\beta}{\beta+1}\left(\frac{1}{8\kappa_0}(\beta+1)\right)^{-\frac{1}{\beta}}\left[(p-1)\chi^2\right]^{\frac{\beta+1}{\beta}}\int_\Omega u^{(p+1-m)\frac{\beta+1}{\beta}}+\frac{1}{8\kappa_0}\int_\Omega |\nabla v|^{2\beta+2} \\ \le &\ \frac{p-1}{4}\int_\Omega u^{m+p-3}|\nabla u|^2+\frac{1}{8}\int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2 \\ &+\frac{\beta}{\beta+1}\left(\frac{1}{8\kappa_0}(\beta+1)\right)^{-\frac{1}{\beta}}\left[(p-1)\chi^2\right]^{\frac{\beta+1}{\beta}}\int_\Omega u^{(p+1-m)\frac{\beta+1}{\beta}}+C_8 \ \text{ for all } \ t\in(0,T_{max}), \end{aligned} \tag{3.47}$$

which together with (3.46) and the Young inequality implies that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)} + \frac{3(p-1)}{4}\int_\Omega u^{m+p-1}|\nabla u|^2 + \frac{(\beta-1)}{\beta}\int_\Omega \left|\nabla|\nabla v|^\beta\right|^2 + \frac{5}{8}\int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2 \\
\leq\ & \frac{1}{8}\int_\Omega |\nabla v|^{2\beta-2}|D^2 v|^2 + \frac{\beta}{\beta+1}\left(\frac{1}{8\kappa_0}(\beta+1)\right)^{-\frac{1}{\beta}}\left[(p-1)\chi^2\right]^{\frac{\beta+1}{\beta}}\int_\Omega u^{(p+1-m)\frac{\beta+1}{\beta}} \\
&+\mu\int_\Omega u^{p-1}(u-u^2) + C_8 \\
\leq\ & \frac{1}{8}\int_\Omega |\nabla v|^{2\beta-2}|D^2 v|^2 + \frac{\beta}{\beta+1}\left(\frac{1}{8\kappa_0}(\beta+1)\right)^{-\frac{1}{\beta}}\left[(p-1)\chi^2\right]^{\frac{\beta+1}{\beta}}\int_\Omega u^{(p+1-m)\frac{\beta+1}{\beta}} \\
&-\frac{\mu}{2}\int_\Omega u^{p+1} + C_9 \quad \text{for all} \quad t\in(0,T_{max}).
\end{aligned}
\tag{3.48}
$$

Collecting (3.45) and (3.48) yield to

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)} + \frac{1}{2\beta}\frac{d}{dt}\|\nabla v\|^{2\beta}_{L^{2\beta}(\Omega)} + \frac{3(p-1)}{4}\int_\Omega u^{m+p-1}|\nabla u|^2 \\
&+\frac{(\beta-1)}{\beta}\int_\Omega \left|\nabla|\nabla v|^\beta\right|^2 + \frac{1}{2}\int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2 + \frac{\mu}{2}\int_\Omega u^{p+1} \\
\leq\ & \frac{\beta}{\beta+1}\left(\frac{1}{8\kappa_0}(\beta+1)\right)^{-\frac{1}{\beta}}\left[(p-1)\chi^2\right]^{\frac{\beta+1}{\beta}}\int_\Omega u^{(p+1-m)\frac{\beta+1}{\beta}} \\
&+C_5\int_\Omega u^{\beta+1} + C_{10} \quad \text{for all} \quad t\in(0,T_{max}).
\end{aligned}
\tag{3.49}
$$

On the other hand, by (3.35), we derive that

$$
(p+1-m)\frac{\beta+1}{\beta} < p+1 \quad \text{and} \quad \beta+1 < p+1.
$$

Thus, with the help of the Young inequality, we conclude that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u\|^p_{L^p(\Omega)} + \frac{1}{2\beta}\frac{d}{dt}\|\nabla v\|^{2\beta}_{L^{2\beta}(\Omega)} + \frac{3(p-1)}{4}\int_\Omega u^{m+p-1}|\nabla u|^2 \\
&+\frac{(\beta-1)}{\beta}\int_\Omega \left|\nabla|\nabla v|^\beta\right|^2 + \frac{1}{2}\int_\Omega |\nabla v|^{2\beta-2}|D^2v|^2 + \frac{\mu}{4}\int_\Omega u^{p+1} \\
\leq\ & C_{11} \quad \text{for all} \quad t\in(0,T_{max}).
\end{aligned}
\tag{3.50}
$$

Therefore, letting $y := \int_\Omega u^p + \int_\Omega |\nabla v|^{2\beta}$ in (3.50) yield to

$$
\frac{d}{dt}y(t) + C_{13}y(t) \leq C_{12} \quad \text{for all} \quad t\in(0,T_{max}).
\tag{3.51}
$$

Thus a standard ODE comparison argument implies boundedness of $y(t)$ for all $t\in(0,T_{max})$. Clearly, $\|u(\cdot,t)\|_{L^p(\Omega)}$ and $\|\nabla v(\cdot,t)\|_{L^{2\beta}(\Omega)}$ are bounded for all $t\in(0,T_{max})$. Obviously, by $m>1$, we derive $\lim_{\beta\to+\infty}\beta = \lim_{\beta\to+\infty}\beta+(m-1)(\beta+1) = +\infty$, and hence, the boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ and the Hölder inequality imply the results. The proof Lemma 3.4 is complete. $\square$

Our main result on global existence and boundedness thereby becomes a straightforward consequence of Lemmas 3.3–3.4 and Lemma 2.5.

Lemma 3.5. *Suppose that the conditions of Theorem 1.1 hold. Let $T \in (0, T_{max})$ and (u, v) be the solution of (1.1). Then there exists a constant $C > 0$ independent of T such that the component v of (u, v) satisfies*

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } \ t \in (0, T). \tag{3.52}$$

Proof. Due to $\|u(\cdot, t)\|_{L^p(\Omega)}$ is bounded for any large p, we infer from the fundamental estimates for Neumann semigroup (see Lemma 4.1 of [12]) or the standard regularity theory of parabolic equation (see e.g. Ladyženskaja et al. [17]) that (3.52) holds. □

Lemma 3.6. *Assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\bar{\Omega})$ both are nonnegative. Let D satisfy (1.2)–(1.3) with $m > 0$. If*

$$m > \begin{cases} 1 - \frac{\mu}{2\chi[1+\lambda_0\|v_0\|_{L^\infty(\Omega)}2^3]} & \text{if } \ N \le 2, \\ 1 \quad \text{if } \ N \ge 3. \end{cases}$$

There exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } \ t \in (0, T_{max}). \tag{3.53}$$

Proof. Throughout the proof of Lemma 3.6, we use C_i $(i \in \mathbb{N})$ to denote the different positive constants independent of p and k $(k \in \mathbb{N})$.

Case $m \ge 1$: For any $p > 1$, multiplying both sides of the first equation in (1.1) by $(u+1)^{p-1}$, integrating over Ω, integrating by parts and using the Young inequality and (3.52), we derive that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u+1\|^p_{L^p(\Omega)} + C_D(p-1)\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\
\le\ & -\chi\int_\Omega \nabla\cdot(u\nabla v)(u+1)^{p-1} + \int_\Omega (u+1)^{p-1}(\mu u - \mu u^2) \\
=\ & \chi(p-1)\int_\Omega (u+1)^{p-1}\nabla u\cdot\nabla v + \int_\Omega (u+1)^{p-1}(\mu u - \mu u^2) \\
\le\ & \chi^2(p-1)C_1\int_\Omega (u+1)^{p-1}|\nabla u| + \int_\Omega (u+1)^{p-1}(\mu u - \mu u^2) \\
\le\ & \frac{(p-1)}{4}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 + \chi^2(p-1)C_1^2\int_\Omega (u+1)^{p+1-m} + \int_\Omega (u+1)^{p-1}(\mu u - \mu u^2) \\
\le\ & \frac{(p-1)}{4}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 + \chi^2(p-1)C_1^2\int_\Omega (u+1)^{p} + \int_\Omega (u+1)^{p-1}(\mu u - \mu u^2) \\
\le\ & \frac{(p-1)}{4}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 + C_2 p\int_\Omega (u+1)^{p} - \int_\Omega (u+1)^p - \mu\int_\Omega u^{p+1} \quad \text{for all} \quad t \in (0, T),
\end{aligned} \tag{3.54}$$

with $C_2 = C_1^2\chi^2 + \mu + 1$, where in the last inequality we have used the fact that $\int_\Omega \mu u(u+1)^{p-1} \le \int_\Omega \mu(u+1)^p$ and $-\int_\Omega \mu(u+1)^{p-1}u^2 \ge \int_\Omega \mu u^{p+1}$. Due to (3.54), we conclude that

$$\frac{d}{dt}\|u+1\|^p_{L^p(\Omega)} + \int_\Omega (u+1)^p + C_3\int_\Omega |\nabla(u+1)^{\frac{m+p-1}{2}}|^2 \le C_2 p^2\int_\Omega (u+1)^p \quad \text{for all} \quad t \in (0, T). \tag{3.55}$$

Now, we let $l_0 > \max\{1, m-1\}, p := p_k = 2^k(l_0+1-m)+m-1$ and

$$M_k = \max\{1, \sup_{t\in(0,T)} \int_\Omega (u+1)^{p_k}\} \quad \text{for} \quad k \in \mathbb{N}. \tag{3.56}$$

Hence, by the Gagliardo–Nirenberg inequality,

$$\begin{aligned} & C_2 p_k^2 \int_\Omega (u+1)^{p_k} \\ = \;& C_2 p_k^2 \|(u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k}{m+p_k-1}}_{L^{\frac{2p_k}{m+p_k-1}}(\Omega)} \\ \le \;& C_3 p_k^2 (\|\nabla (u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k}{m+p_k-1}\varsigma_1}_{L^2(\Omega)} \|(u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k}{m+p_k-1}(1-\varsigma_1)}_{L^1(\Omega)} + \|(u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k}{m+p_k-1}}_{L^1(\Omega)}), \end{aligned} \tag{3.57}$$

where

$$\frac{2p_k}{m+p_k-1}\varsigma_1 = \frac{2p_k}{m+p_k-1} \frac{N - \frac{N(m+p_k-1)}{2p_k}}{1-\frac{N}{2}+N} = \frac{2N(p_k+1-m)}{(N+2)(m+p_k-1)} < 2$$

and

$$\frac{2p_k}{m+p_k-1}(1-\varsigma_1) = \frac{2p_k}{m+p_k-1}(1 - \frac{N - \frac{N(m+p_k-1)}{2p_k}}{1-\frac{N}{2}+N}) = 2\frac{2p_k+N(m-1)}{(N+2)(m+p_k-1)}.$$

Therefore, an application of the Young inequality yields

$$\begin{aligned} C_2 p_k^2 \int_\Omega (u+1)^{p_k} \le \;& C_4 \|\nabla (u+1)^{\frac{m+p_k-1}{2}}\|^2_{L^2(\Omega)} + C_5 p_k^{\frac{(N+2)(m+p_k-1)}{p_k+(N+1)(m-1)}} \|(u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k+N(m-1)}{N(m-1)+m+p_k-1}}_{L^1(\Omega)} \\ & + C_6 p_k^2 \|(u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k}{m+p_k-1}}_{L^1(\Omega)} \\ \le \;& C_3 \|\nabla (u+1)^{\frac{m+p_k-1}{2}}\|^2_{L^2(\Omega)} + C_7 p_k^{\frac{(N+2)(m+p_k-1)}{p_k+(N+1)(m-1)}} \|(u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k}{m+p_k-1}}_{L^1(\Omega)}. \end{aligned} \tag{3.58}$$

Here we have used the fact that $\frac{2p_k+N(m-1)}{N(m-1)+m+p_k-1} \le \frac{2p_k}{m+p_k-1}$. Thus, in light of $m \ge 1$, by means of (3.56)–(3.58),

$$\begin{aligned} \frac{d}{dt}\|u+1\|^{p_k}_{L^{p_k}(\Omega)} + \int_\Omega (u+1)^{p_k} \le \;& C_7 p_k^{\frac{(N+2)(m+p_k-1)}{p_k+(N+1)(m-1)}} \|(u+1)^{\frac{m+p_k-1}{2}}\|^{\frac{2p_k}{m+p_k-1}}_{L^1(\Omega)} \\ \le \;& \bar{\lambda}^k M_{k-1}^{\frac{2p_k}{m+p_k-1}} \\ \le \;& \bar{\lambda}^k M_{k-1}^2 \quad \text{for all} \quad t \in (0,T) \end{aligned} \tag{3.59}$$

with some $\bar{\lambda} > 1$. Here we have used the fact that

$$\frac{(N+2)(m+p_k-1)}{p_k+(N+1)(m-1)} = \frac{2^k(l_0+1-m)(N+2)+2(N+2)(m-1)}{2^k(l_0+1-m)+(N+2)(m-1)} \le N+2$$

and

$$\frac{2p_k}{m+p_k-1} \le \frac{2(p_k+m-1)}{m+p_k-1} = 2.$$

Integrating (3.59) over $(0,t)$ with $t \in (0,T)$, we derive

$$\int_\Omega (u+1)^{p_k}(x,t) \le \max\{\int_\Omega (u+1)_0^{p_k}, \bar{\lambda}^k M_{k-1}^2\} \quad \text{for all} \quad t \in (0,T). \tag{3.60}$$

If $\int_\Omega (u+1)^{p_k}(x,t) \le \int_\Omega (u_0+1)^{p_k}$ for any large $k \in \mathbb{N}$, then we obtain (3.53) directly. Otherwise, by a straightforward induction, we have

$$\begin{aligned} \int_\Omega (u+1)^{p_k} &\le \bar{\lambda}^k(\bar{\lambda}^{k-1}M_{k-2}^2)^2 \\ &= \bar{\lambda}^{k+2(k-1)}M_{k-2}^{2^2} \\ &\le \bar{\lambda}^{k+\Sigma_{j=2}^k(j-1)}M_0^{2^k}. \end{aligned} \tag{3.61}$$

In light of $\ln(1+z) \le z$ for all $z \ge 0$, so that, taking p_k-th roots on both sides of (3.61), we can easily get (3.53).

Case $N \le 2$ and $1 - \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]} < m < 1$: Due to Lemma 3.3, we may choose

$$\tilde{p}_0 := 1 + 30\frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]} \tag{3.62}$$

such that

$$\int_\Omega (u+1)^{\tilde{p}_0}(x,t) \le C_9 \quad \text{for all} \quad t \in (0,T_{max}). \tag{3.63}$$

Next, testing the first equation in (1.1) by $(u+1)^{p-1}$, integrating over Ω, integrating by parts and applying the Young inequality and (3.52), we derive that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u+1\|_{L^p(\Omega)}^p + C_D(p-1)\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\ \le\ & -\chi\int_\Omega \nabla\cdot(u\nabla v)(u+1)^{p-1} + \int_\Omega (u+1)^{p-1}(\mu u - \mu u^2) \\ \le\ & \chi(p-1)C_1\int_\Omega (u+1)^{p-1}|\nabla u| + \int_\Omega (u+1)^{p-1}(\mu u-\mu u^2) \\ \le\ & \frac{(p-1)}{4}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 + \chi^2(p-1)C_1^2\int_\Omega (u+1)^{p+1-m} + \int_\Omega (u+1)^{p-1}(\mu u - \mu u^2) \\ \le\ & \frac{(p-1)}{4}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\ & + C_{10}p\int_\Omega (u+1)^{p+1-m} - \int_\Omega (u+1)^p - \mu\int_\Omega u^{p+1} \quad \text{for all} \quad t \in (0,T), \end{aligned} \tag{3.64}$$

where $C_{10} = C_1^2\chi^2 + \mu + 1$. Here we have used the fact that $\int_\Omega u(u+1)^{p-1} \le \int_\Omega (u+1)^p \le \int_\Omega (u+1)^{p+1-m}$ and $-\int_\Omega (u+1)^{p-1}u^2 \ge \int_\Omega u^{p+1}$. Therefore, (3.64) yields to

$$\frac{d}{dt}\|u+1\|_{L^p(\Omega)}^p + \int_\Omega (u+1)^p + C_{11}\int_\Omega |\nabla(u+1)^{\frac{m+p-1}{2}}|^2 \le C_{12}p^2\int_\Omega (u+1)^{p+1-m} \quad \text{for all} \quad t \in (0,T). \tag{3.65}$$

Let $p := \tilde{p}_k = 2^k(\tilde{p}_0 + 1 - m) + m - 1$ and

$$\tilde{M}_k = \max\{1, \sup_{t\in(0,T)} \int_\Omega (u+1)^{\tilde{p}_k}\} \quad \text{for} \quad k \in \mathbb{N}, \tag{3.66}$$

where $\tilde{p}_0$ is given by (3.62). Thus, Gagliardo–Nirenberg inequality yields to

$$\begin{aligned} & C_{12}\tilde{p}_k^2 \int_\Omega (u+1)^{\tilde{p}_k+1-m} \\ = \; & C_{12}\tilde{p}_k^2 \|(u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}}_{L^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}}(\Omega)} \\ \le \; & C_{13}\tilde{p}_k^2 (\|\nabla (u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}\varsigma_2}_{L^2(\Omega)} \|(u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}(1-\varsigma_2)}_{L^1(\Omega)} + \|(u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}}_{L^1(\Omega)}), \end{aligned} \tag{3.67}$$

where

$$\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}\varsigma_2 = \frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1} \frac{2 - \frac{2(m+\tilde{p}_k-1)}{2(\tilde{p}_k+1-m)}}{1-\frac{2}{2}+2} = \frac{\tilde{p}_k + 3(1-m)}{m+\tilde{p}_k-1} < 2$$

and

$$\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}(1-\varsigma_2) = \frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}(1 - \frac{2 - \frac{2(m+\tilde{p}_k-1)}{2(\tilde{p}_k+1-m)}}{1-\frac{2}{2}+2}) = 1.$$

Therefore, in light of the Young inequality, we conclude that

$$\begin{aligned} C_{12}\tilde{p}_k^2 \int_\Omega (u+1)^{\tilde{p}_k+1-m} \le \; & C_{11}\|\nabla (u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^2_{L^2(\Omega)} + C_{14}\tilde{p}_k^{\frac{4(m+\tilde{p}_k-1)}{\tilde{p}_k+5(m-1)}} \|(u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{\tilde{p}_k+5(m-1)}{2(\tilde{p}_k+m-1)}}_{L^1(\Omega)} \\ & + C_{15}\tilde{p}_k^2 \|(u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}}_{L^1(\Omega)} \\ \le \; & C_{11}\|\nabla (u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^2_{L^2(\Omega)} + C_{16}\tilde{p}_k^{\frac{4(m+\tilde{p}_k-1)}{\tilde{p}_k+5(m-1)}} \|(u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}}_{L^1(\Omega)}. \end{aligned} \tag{3.68}$$

Here we have used the fact that $\frac{\tilde{p}_k+5(m-1)}{2(\tilde{p}_k+m-1)} \le \frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}$ and $\frac{4(m+\tilde{p}_k-1)}{\tilde{p}_k+5(m-1)} \ge 2$. Therefore, in light of $m > 1 - \frac{\mu}{2\chi[1+\bar{\lambda}_0\|v_0\|_{L^\infty(\Omega)}2^3]}$, by means of (3.62), (3.66)–(3.68),

$$\begin{aligned} \frac{d}{dt}\|u+1\|^{\tilde{p}_k}_{L^{\tilde{p}_k}(\Omega)} + \int_\Omega (u+1)^{\tilde{p}_k} \le \; & C_{16}\tilde{p}_k^{\frac{4(m+\tilde{p}_k-1)}{\tilde{p}_k+5(m-1)}} \|(u+1)^{\frac{m+\tilde{p}_k-1}{2}}\|^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}}_{L^1(\Omega)} \\ \le \; & \tilde{\tilde{\lambda}}^k M_{k-1}^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}} \quad \text{for all} \quad t \in (0,T) \end{aligned} \tag{3.69}$$

with some $\tilde{\tilde{\lambda}} > 1$. Here we have used the fact that

$$\frac{4(m+\tilde{p}_k-1)}{\tilde{p}_k+5(m-1)} = 4\frac{2^k(l_0+1-m)+2(m-1)}{2^k(l_0+1-m)+6(m-1)} \le 4\frac{l_0+1-m+2(m-1)}{l_0+1-m+6(m-1)} \le 6$$

and

$$\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1} = 2\frac{2^k(\tilde{p}_0+1-m)}{2^k(\tilde{p}_0+1-m)+2(m-1)} = 2(1+\frac{1-m}{2^k(\tilde{p}_0+1-m)+2(m-1)}) := \kappa_k.$$

Here we note that $\kappa_k = 2(1+\varepsilon_k)$ for $k \geq 1$, where ε_k satisfies $\varepsilon_k \leq \frac{C_{17}}{2^k}$ for all k with some $C_{17} > 0$. Next, we integrate (3.69) over $(0,t)$ with $t \in (0,T)$, then yields to

$$\int_\Omega (u+1)^{\tilde{p}_k}(x,t) \leq \max\{\int_\Omega (u+1)_0^{\tilde{p}_k}, \tilde{\tilde{\lambda}}^k M_{k-1}^{\frac{2(\tilde{p}_k+1-m)}{m+\tilde{p}_k-1}}\} \text{ for all } t \in (0,T). \tag{3.70}$$

If $\int_\Omega (u+1)^{\tilde{p}_k}(x,t) \leq \int_\Omega (u_0+1)^{\tilde{p}_k}$ for any large $k \in \mathbb{N}$, then we derive (3.53) holds. Otherwise, by a straightforward induction, we have

$$\int_\Omega (u+1)^{\tilde{p}_k} \leq \tilde{\tilde{\lambda}}^{k+\sum_{j=2}^k (j-1)\cdot\prod_{i=j}^k \kappa_i} \tilde{M}_0^{\prod_{i=1}^k \kappa_i} \text{ for all } k \geq 1. \tag{3.71}$$

On the other hand, due to the fact that $\ln(1+x) \leq x$ (for all $x \geq 0$),

$$\begin{aligned}\prod_{i=j}^k \kappa_i = & \; 2^{k+1-j} e^{\Sigma_{i=j}^k \ln(1+\varepsilon_j)} \\ \leq & \; 2^{k+1-j} e^{\Sigma_{i=j}^k \varepsilon_j} \\ \leq & \; 2^{k+1-j} e^{C_{17}} \quad \text{for all } k \geq 1 \text{ and } j = \{1,\ldots,k\}.\end{aligned}$$

In light of the above inequality, with the help of (3.71), we conclude that

$$\left(\int_\Omega (u+1)^{\tilde{p}_k}\right)^{\frac{1}{\tilde{p}_k}} \leq \tilde{\tilde{\lambda}}^{\frac{k}{\tilde{p}_k}+\frac{\sum_{j=2}^k (j-1)\cdot\prod_{i=j}^k \kappa_i}{\tilde{p}_k}} \tilde{M}_0^{\frac{\prod_{i=1}^k \kappa_i}{\tilde{p}_k}} \text{ for all } k \geq 1, \tag{3.72}$$

which after taking $k \to \infty$ readily implies that (3.53) holds. □

Proof of Theorem 1.1. Theorem 1.1 will be proved if we can show $T_{max} = \infty$. Suppose on contrary that $T_{max} < \infty$. In view of (3.53), we apply Lemma 2.5 to reach a contradiction. Hence the classical solution (u,v) of (1.1) is global in time and bounded. □

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