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Sign-changing and nontrivial solutions for a class of Kirchhoff-type problems ☆

Bin Chen, Zeng-Qi Ou *

School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

A R T I C L E I N F O

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A B S T R A C T

In this paper, we study the existence of sign-changing (nodal) and nontrivial solutions for the nonlinear Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = \alpha u+\beta u^3 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha,\beta\in\mathbb{R}$ are two real parameters. With the help of nodal Nehari set, we first provide a description of a two-dimensional set in the (α,β) plane, which corresponds to the nonexistence and existence of sign-changing solutions for the above Kirchhoff-type equation. And then, we establish the existence result of nontrivial solutions via the minimax methods.

1. Introduction and main results

In this article, we discuss the existence of sign-changing and nontrivial solutions for the following Kirchhoff-type problem

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = \alpha u+\beta u^3 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^N(N=1,2,3)$ is a bounded domain with smooth boundary $\partial\Omega, a>0, b>0$, and $\alpha,\beta\in\mathbb{R}$ are two real parameters.

Let $L^p(\Omega)(1\le p<+\infty)$ be the Lebesgue space with the norm $|u|_p=(\int_\Omega |u|^pdt)^{1/p}$ and $H_0^1(\Omega)$ be the usual Hilbert space with the norm $\|u\|=(\int_\Omega |\nabla u|^2dt)^{1/2}$. From the Sobolev and Rellich embedding theorem,

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* Corresponding author.
E-mail addresses: 1083558093@qq.com (B. Chen), ouzengq707@sina.com (Z.-Q. Ou).

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the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $1 \le p \le 2^*$ and is compact for any $1 \le p < 2^*$, where $2^* := +\infty$ if $N = 1, 2$ and $2^* := \frac{2N}{N-2} = 6$ if $N = 3$. And then, there is a positive constant $\tau > 0$ such that

$$|u|_2 \le \tau\|u\|, \quad |u|_4 \le \tau\|u\| \quad \text{for any } u \in H_0^1(\Omega). \tag{1.2}$$

Problem (1.1) can be seen as a special form of the following Kirchhoff problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which is the stationary case of a nonlinear wave equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{1.4}$$

proposed by Kirchhoff [8] in 1883. (1.4) is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.4) have practical physical meaning: u denotes the displacement, f is the external force, b represents the initial tension, and a is related to the intrinsic properties of the string. Some early works related to problem (1.4) are [9,10,13,15].

In recent years, more and more researchers began to pay attention to the existence of the sign-changing solutions of Kirchhoff problem in bounded domain, see [1,4,7,11,12,14,16,18–20,23]. Specially, for the case that the nonlinearity f satisfies super-3-linear growth condition, Mao and Zhang [14], Shuai [18], Cheng and Tang [20] studied the existence of sign-changing solutions for problem (1.3). For the case that the nonlinearity f satisfies asymptotically 3-linear growth condition, Zhang and Perera [22], Mao and Luan [12] obtained the existence of sign-changing solutions for problem (1.3) via variational methods and invariant sets of descent flow. And in 2017, by the non-Nehari manifold method, Cheng and Tang [4] showed that if $f(x,t) = r(x,t) + \beta t^3$ satisfies

(f_1) $f(x,t) = o(t)$ as $|t| \to 0$ uniformly in $x \in \Omega$;
(f_2) (i) $\beta > b\mu_0$ and $r(x,t) = o(t^3)$ as $|t| \to \infty$ uniformly in $x \in \Omega$, and
 (ii) there exists a $\theta_0 \in (0,1)$ such that for any $t > 0$ and $\tau \in \mathbb{R}\backslash\{0\}$

$$\left[\frac{r(\tau)}{\tau^3} - \frac{r(t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + \frac{a\theta_0\lambda_1|1-t^2|}{(t\tau)^2} \ge 0,$$

where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and $\mu_0 := \inf\{\max\{\|u^+\|^4, \|u^-\|^4\} : |u^\pm|_4^4 = 1,\ u \in H_0^1(\Omega)\}$, then problem (1.3) has a sign-changing solution with positive energy and precisely two sign-changing domains. We must point out that the condition (f_2) implies $r(x,t)t \le a\theta_0\lambda_1 t^2 < a\lambda_1 t^2$ for any $(x,t) \in \Omega \times \mathbb{R}$. Especially, in the same year, Zhong and Tang (see [23]) studied the non-existence and existence of sign-changing solution for problem (1.1) and obtained the following result: there is a constant $\Lambda \ge 2\mu_1^2$ such that

(i) for any $\alpha < a\lambda_1$ and $0 < \beta \le b\Lambda$, problem (1.1) has no sign-changing solutions,

(ii) for any $\alpha < a\lambda_1$ and $\beta > b\Lambda$, problem (1.1) has at least a sign-changing solution with two sign-changing domains, where $\mu_1 > 0$ is the first eigenvalue of the following eigenvalue problem

$$\begin{cases} -\int_\Omega |\nabla u|^2 dx \Delta u = \mu u^3 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

and can be defined as follows:

$$\mu_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^4}{|u|_4^4}.$$

In [4] and [23], the methods of finding sign-changing solutions depended heavily on the conditions $r(x,t)t < a\lambda_1 t^2$ and $\alpha < a\lambda_1$, respectively. So an interesting question is whether these conditions can be relaxed to obtain a sign-changing solution for problem (1.1).

We first present some facts about the eigenvalue problem (1.5). According to [17], the eigenvalues of problem (1.5) can be defined as

$$\mu_k := \inf_{h \in \Sigma_k} \max_{z \in S^{k-1}} \|h(z)\|^4, \tag{1.6}$$

where S^{k-1} denotes the unit sphere in R^k, $\Sigma_k := \{h \in C(S^{k-1}, H) : h \text{ is odd}\}$ and $H = \{u \in H_0^1(\Omega) : |u|_4^4 = 1\}$, and we have $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots$ and $\mu_k \to +\infty$ as $n \to \infty$. Moreover, according to [22], μ_1 has a normalized eigenfunction $\psi_1 > 0$ in Ω, but we do not know whether μ_1 is simple and isolated. Finally, we denote by $\sigma(-\|\cdot\|^2\Delta)$ the set of eigenvalues of problem (1.5) defined by (1.6) and $\mathcal{B}_\mu$ the set of solutions for a given μ. With regard to the eigenvalues of problem (1.5), we want to mention [16]. Using the Yang index, Perera and Zhang also constructed an unbounded sequence of minimax eigenvalues of problem (1.5), denoted by $0 < \gamma_1 \le \gamma_2 \le \cdots \le \gamma_k \le \cdots$ and $\gamma_k \to \infty$ as $k \to \infty$. But it is not clear whether the sequences $\{\gamma_k\}$ and $\{\mu_k\}$ are coincident expect for $k = 1$.

Before stating our main results, for convenience, let us first give some notations. Define the C^1 energy functional $E_{\alpha,\beta}$: $H_0^1(\Omega) \to R$ as follows:

$$E_{\alpha,\beta}(u) = \frac{1}{2} H_\alpha(u) + \frac{1}{4} G_\beta(u) \quad \text{for all } u \in H_0^1(\Omega),$$

where

$$H_\alpha(u) := a\|u\|^2 - \alpha|u|_2^2, \qquad G_\beta(u) := b\|u\|^4 - \beta|u|_4^4.$$

For any $u, v \in H_0^1(\Omega)$, we have

$$\langle {E'}_{\alpha,\beta}(u), v\rangle = \left(a + b\|u\|^2\right) \int\limits_\Omega \nabla u \nabla v dx - \alpha \int\limits_\Omega uv dx - \beta \int\limits_\Omega u^3 v dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and from the variational view of point, the weak solutions of problem (1.1) correspond to the critical points of the functional $E_{\alpha,\beta}$. Furthermore, if $u \in H_0^1(\Omega)$ is a solution of problem (1.1) and $u^\pm \not\equiv 0$, then u is a nodal solution of problem (1.1), where $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \min\{u(x), 0\}$. By a simple calculation, we can obtain

$$\begin{gathered}
H_\alpha(u) = H_\alpha(u^+) + H_\alpha(u^-), \quad G_\beta(u) = G_\beta(u^+) + G_\beta(u^-) + 2b\|u^+\|^2\|u^-\|^2, \\
E_{\alpha,\beta}(u) = E_{\alpha,\beta}(u^+) + E_{\alpha,\beta}(u^-) + \frac{b}{2}\|u^+\|^2\|u^-\|^2, \\
\langle E'_{\alpha,\beta}(u), u^+\rangle = H_\alpha(u^+) + G_\beta(u^+) + b\|u^+\|^2\|u^-\|^2, \\
\langle E'_{\alpha,\beta}(u), u^-\rangle = H_\alpha(u^-) + G_\beta(u^-) + b\|u^+\|^2\|u^-\|^2.
\end{gathered}$$

Set

$$\mathcal{M}_{\alpha,\beta} = \left\{u \in H_0^1(\Omega) : \ u^\pm \not\equiv 0, \ \langle {E'}_{\alpha,\beta}(u), u^+\rangle = \langle {E'}_{\alpha,\beta}(u), u^-\rangle = 0\right\},$$

evidently, $\mathcal{M}_{\alpha,\beta}$ contains all the sign-changing solutions of problem (1.1), and $\mathcal{M}_{\alpha,\beta}$ is a subset of Nehari manifold $\mathcal{N}_{\alpha,\beta}$ related with the functional $E_{\alpha,\beta}$:

$$\mathcal{N}_{\alpha,\beta} = \left\{u \in H_0^1(\Omega) : \langle E'_{\alpha,\beta}(u), u\rangle = H_\alpha(u) + G_\beta(u) = 0\right\}.$$

If $u \in \mathcal{N}_{\alpha,\beta}$, we have

$$E_{\alpha,\beta}(u) = \frac{1}{4}H_\alpha(u) = -\frac{1}{4}G_\beta(u).$$

Now we are ready to state our main results. We begin with the nonexistence of sign-changing solutions for problem (1.1). We need to introduce the following infimum:

$$\begin{aligned}\mu^* :&= \inf\left\{\max\left\{\frac{\|u^+\|^4 + \|u^+\|^2\|u^-\|^2}{|u^+|_4^4}, \frac{\|u^-\|^4 + \|u^+\|^2\|u^-\|^2}{|u^-|_4^4}\right\} : u \in H_0^1(\Omega), u^\pm \not\equiv 0\right\} \\ &= \inf\left\{\max\left\{\frac{\|u\|^2\|u^+\|^2}{|u^+|_4^4}, \frac{\|u\|^2\|u^-\|^2}{|u^-|_4^4}\right\} : u \in H_0^1(\Omega), u^\pm \not\equiv 0\right\}.\end{aligned}$$

Theorem 1.1. *If $(\alpha, \beta) \in (-\infty, a\lambda_2] \times (-\infty, b\mu_1) \cup (-\infty, a\lambda_1] \times (-\infty, b\mu^*]$, problem (1.1) has no sign-changing solutions.*

Remark 1.1. If μ_1 is simple, $(-\infty, a\lambda_2] \times (-\infty, b\mu_1)$ of Theorem 1.1 may be replaced with $(-\infty, a\lambda_2] \times (-\infty, b\mu_1]$. We will show that $\mu_1 \le \mu^* < +\infty$ in Lemma 2.1. If $\mu_1 = \mu^*$, we have $(-\infty, a\lambda_1] \times (-\infty, b\mu^*] \subset (-\infty, a\lambda_2] \times (-\infty, b\mu_1]$, but if $\mu_1 < \mu^*$, $(-\infty, a\lambda_1] \times (-\infty, b\mu^*]$ and $(-\infty, a\lambda_2] \times (-\infty, b\mu_1]$ are different. But, Zhong and Tang in [23] considered the nonexistence of sign-changing solutions for problem (1.1) with only for the case $\alpha < a\lambda_1$ and $0 < \beta \le b\Lambda$. Hence, Theorem 1.1 extends and supplements the result of Zhong and Tang in [23].

In order to find sign-changing solution with positive energy of problem (1.1), we need to divide $\mathcal{M}_{\alpha,\beta}$ into the following three sets:

$$\begin{aligned}\mathcal{M}^1_{\alpha,\beta} &= \left\{u \in \mathcal{M}_{\alpha,\beta} :\ G_\beta(u^\pm) + b\|u^+\|^2\|u^-\|^2 < 0\right\}, \\ \mathcal{M}^2_{\alpha,\beta} &= \left\{u \in \mathcal{M}_{\alpha,\beta} :\ G_\beta(u^\pm) + b\|u^+\|^2\|u^-\|^2 > 0\right\}, \\ \mathcal{M}^3_{\alpha,\beta} &= \left\{u \in \mathcal{M}_{\alpha,\beta} :\ \left(G_\beta(u^+) + b\|u^+\|^2\|u^-\|^2\right)\ \left(G_\beta(u^-) + b\|u^+\|^2\|u^-\|^2\right) < 0\right\}.\end{aligned}$$

We will minimize the energy functional $E_{\alpha,\beta}$ restricted to $\mathcal{M}^1_{\alpha,\beta}$, instead of $\mathcal{M}_{\alpha,\beta}$. Let

$$\mathcal{A}_L(\beta) = \left\{u \in H_0^1(\Omega) : u^\pm \not\equiv 0, G_\beta(u^\pm) + b\|u^+\|^2\|u^-\|^2 \le 0\right\},$$

we define

$$\alpha_L(\beta) = \inf\left\{\min\left\{\frac{\|u^+\|^2}{|u^+|_2^2}, \frac{\|u^-\|^2}{|u^-|_2^2}\right\} : u \in \mathcal{A}_L(\beta)\right\},$$

and we assume that $\alpha_L(\beta) = +\infty$ if $\mathcal{A}_L(\beta) = \varnothing$. Our main result is the following theorem:

Theorem 1.2. *Assume that $(\alpha, \beta) \in (-\infty, a\alpha_L(\beta)) \times (b\mu^*, +\infty)$, then problem (1.1) possesses at least a sign-changing solution $u_{\alpha,\beta}$ with positive energy.*

Remark 1.2. Comparing the definitions of $\alpha_L(\beta)$ and λ_1, we obtain $\lambda_1 < \alpha_L(\beta)$ for any $\beta > b\mu^*$ in Lemma 2.4. In addition, a couple of main properties of the function $\alpha_L(\beta)$ are collected in Lemma 2.4 below, although some of them are not directly related to the proof of Theorem 1.2, they can help us understand the graph of the function $\alpha_L(\beta)$ more intuitively.

We obtain the existence of sign-changing solution for problem (1.1) when $\alpha < a\alpha_L(\beta)$ and $\beta > b\mu^*$, while Zhong and Tang [23] only considered the case for $\alpha < a\lambda_1$ and $\beta > b\Lambda$. Theorem 1.2 expands the scope of the existence of sign-changing solution for problem (1.1), and hence, Theorem 1.2 can be regarded as the extension and supplementary work of Zhong and Tang. It is easy to see that the nonlinearity $\alpha t + \beta t^3$ does not satisfy the condition (f_1) of [4]. Therefore, our result is new and interesting.

Finally, we will consider the existence of nontrivial solution for problem (1.1) with negative energy. Let $\{\lambda_k\}$ be the eigenvalues of $(-\Delta, H_0^1(\Omega))$, and let $\beta \in R$, define

$$\alpha^*(\beta) := \sup\left\{\frac{\|u\|^2}{|u|_2^2} : \ u \in \mathcal{B}_{\beta/b}\backslash\{0\}\right\},$$

where $\mathcal{B}_\mu$ is the set of solutions for problem (1.5) for a given μ, and we assume that $\alpha^*(\beta) = -\infty$ if $\mathcal{B}_{\beta/b} = \{0\}$. Let μ_{k+1} be the $(k+1)$th eigenvalue of problem (1.5), define

$$k_\beta := \min\{k \in N : \ \beta < b\mu_{k+1}\}.$$

The third result of this paper is stated in the following theorem:

Theorem 1.3. *Suppose that $\frac{\beta}{b} \in \overline{R/\sigma(-\|\cdot\|^2\Delta)}$, for any $\frac{\alpha}{a} > \max\{\alpha^*(\beta), \lambda_{k_\beta+1}\}$, problem (1.1) possesses at least a nontrivial solution u with $E_{\alpha,\beta}(u) < 0$.*

Remark 1.3. As far as we know, Theorem 1.3 is the first result on the existence of nontrivial solution for problem (1.1) with negative energy.

We organize this paper as follows. For the rest of this section, we recall a variant of the deformation lemma (see [21], Theorem 2.3) which is very important for proving Theorem 1.2 and Theorem 1.3. In Section 2, we will be devoted to completing the proofs of Theorem 1.1 and Theorem 1.2. In Section 3, we present the proof of Theorem 1.3 via minimax method.

Lemma 1.1. *Let I be a C^1-functional on a Banach space X, $S \subset X, c \in R$, $\varepsilon, \delta > 0$ such that*

$$\|I'(u)\| \geq \frac{8\varepsilon}{\delta} \quad \textit{for any } u \in I^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}.$$

Then there exists a deformation $\eta \in C([0,1] \times X, X)$ such that

(i) *$\eta(t,u) = u$, if $t = 0$ or if $u \notin I^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$;*
(ii) *$\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$;*
(iii) *$I(\eta(t,u))$ is non increasing with respect to t for any $u \in X$;*
(iv) *$I(\eta(t,u)) < c$ for any $u \in I^c \cap S_{2\delta}$ and $t \in [0,1]$,*
(v) *if I is even, then $\eta(t,\cdot)$ is odd for any $t \in [0,1]$.*

2. The proofs of Theorem 1.1 and Theorem 1.2

In this section, we will first pay attention to the proof of Theorem 1.1. From the definition of $\mathcal{M}_{\alpha,\beta}$, to prove Theorem 1.1, we just need $\langle E'_{\alpha,\beta}(u), u^+\rangle \neq 0$ or $\langle E'_{\alpha,\beta}(u), u^-\rangle \neq 0$ for any $u \in H_0^1(\Omega)$ with $u^{\pm} \not\equiv 0$. In order to achieve this goal, we need to discuss the property of μ^* defined in Section 1.

Lemma 2.1. *$\mu_1 \leq \mu^* < +\infty$. $\mathcal{A}_1(\beta) \neq \varnothing$ if and only if $\beta > b\mu^*$, where*

$$\mathcal{A}_1(\beta) = \left\{u \in H_0^1(\Omega) : u^{\pm} \not\equiv 0, G_\beta(u^{\pm}) + b\|u^+\|^2\|u^-\|^2 < 0\right\}.$$

Proof. From the definitions of μ^* and μ_1, we first have

$$\begin{aligned}
\mu^* &= \inf\left\{\max\left\{\frac{\|u^+\|^4 + \|u^+\|^2\|u^-\|^2}{|u^+|_4^4}, \frac{\|u^-\|^4 + \|u^+\|^2\|u^-\|^2}{|u^-|_4^4}\right\} : u \in H_0^1(\Omega), u^{\pm} \not\equiv 0\right\} \\
&\geq \inf\left\{\frac{\|u^+\|^4 + \|u^+\|^2\|u^-\|^2}{|u^+|_4^4} : u \in H_0^1(\Omega), u^{\pm} \not\equiv 0\right\} \\
&\geq \inf\left\{\frac{\|u^+\|^4}{|u^+|_4^4} : u \in H_0^1(\Omega), u^{\pm} \not\equiv 0\right\} \\
&\geq \inf\left\{\frac{\|u\|^4}{|u|_4^4} : u \in H_0^1(\Omega) \setminus \{0\}\right\} \\
&= \mu_1.
\end{aligned}$$

Let $\beta > b\mu^*$, from the definition of μ^*, there is a $u_0 \in H_0^1(\Omega)$ with $u_0^{\pm} \not\equiv 0$ such that

$$\frac{\beta}{b} > \max\left\{\frac{\|u_0^+\|^4 + \|u_0^+\|^2\|u_0^-\|^2}{|u_0^+|_4^4}, \frac{\|u_0^-\|^4 + \|u_0^+\|^2\|u_0^-\|^2}{|u_0^-|_4^4}\right\} \geq \mu^*,$$

which shows that $b\|u_0^{\pm}\|^4 + b\|u_0^+\|^2\|u_0^-\|^2 - \beta|u_0^{\pm}|_4^4 < 0$. Hence $u_0 \in \mathcal{A}_1(\beta)$.

On the other hand, if $\mathcal{A}_1(\beta) \neq \varnothing$, there exists a $u_0 \in H_0^1(\Omega)$ with $u_0^{\pm} \not\equiv 0$ such that

$$b\|u_0^{\pm}\|^4 + b\|u_0^+\|^2\|u_0^-\|^2 - \beta|u_0^{\pm}|_4^4 < 0.$$

Therefore, we have

$$\max\left\{\frac{\|u_0^+\|^4 + \|u_0^+\|^2\|u_0^-\|^2}{|u_0^+|_4^4}, \frac{\|u_0^-\|^4 + \|u_0^+\|^2\|u_0^-\|^2}{|u_0^-|_4^4}\right\} < \frac{\beta}{b},$$

and by the definition of μ^*, one has $\mu^* < \frac{\beta}{b}$, that is $\beta > b\mu^*$. $\square$

It follows from the proof of the above lemma that if $\beta \leq b\mu^*$, then $\mathcal{A}_1(\beta) = \varnothing$, that is, $G_\beta(u^+) + b\|u^+\|^2\|u^-\|^2 \geq 0$ or $G_\beta(u^-) + b\|u^+\|^2\|u^-\|^2 \geq 0$ for any $u \in H_0^1(\Omega)$ with $u^{\pm} \not\equiv 0$.

Proof of Theorem 1.1. (i) $\alpha \leq a\lambda_2$, $\beta < b\mu_1$. According to [2], the second eigenvalue λ_2 of the operator $-\Delta$ in $H_0^1(\Omega)$ can be characterized as

$$\lambda_2 = \inf\left\{\max\left\{\frac{\|u^+\|^2}{|u^+|_2^2}, \frac{\|u^-\|^2}{|u^-|_2^2}\right\} : u \in H_0^1(\Omega), u \not\equiv 0\right\},$$

hence for any $u \in H_0^1(\Omega)$ with $u^{\pm} \not\equiv 0$, it holds

$$\lambda_2 \leq \frac{\|u^+\|^2}{|u^+|_2^2} \quad \text{or} \quad \lambda_2 \leq \frac{\|u^-\|^2}{|u^-|_2^2}.$$

Let $\alpha \leq a\lambda_2$, for any $u \in H_0^1(\Omega)$ with $u^\pm \not\equiv 0$, we obtain

$$H_\alpha(u^+) = a\|u^+\|^2 - \alpha|u^+|_2^2 \geq a(\|u^+\|^2 - \lambda_2|u^+|_2^2) \geq 0$$

or

$$H_\alpha(u^-) = a\|u^-\|^2 - \alpha|u^-|_2^2 \geq a(\|u^-\|^2 - \lambda_2|u^-|_2^2) \geq 0.$$

If $\beta < b\mu_1$, from the definition of μ_1, we have

$$G_\beta(u^\pm) = b\|u^\pm\|^4 - \beta|u^\pm|_4^4 > b(\|u^\pm\|^4 - \mu_1|u^\pm|_4^4) \geq 0$$

for any $u \in H_0^1(\Omega)$ with $u^\pm \not\equiv 0$. Therefore, if $(\alpha, \beta) \in (-\infty, a\lambda_2] \times (-\infty, b\mu_1)$, we obtain that either

$$\langle E'_{\alpha,\beta}(u), u^+\rangle \neq 0 \quad \text{or} \quad \langle E'_{\alpha,\beta}(u), u^-\rangle \neq 0,$$

that is, $\mathcal{M}_{\alpha,\beta} = \varnothing$.

(ii) $\alpha \leq a\lambda_1$, $\beta \leq b\mu^*$. Let $\beta \leq b\mu^*$, from Lemma 2.1, we obtain for any $u \in H_0^1(\Omega)$ with $u^\pm \not\equiv 0$, it holds

$$G_\beta(u^+) + b\|u^+\|^2\|u^-\|^2 \geq 0 \quad \text{or} \quad G_\beta(u^-) + b\|u^+\|^2\|u^-\|^2 \geq 0.$$

And if $\alpha \leq a\lambda_1$, it follows that

$$H_\alpha(u^\pm) = a\|u^\pm\|^2 - \alpha|u^\pm|_2^2 \geq a(\|u^\pm\|^2 - \lambda_1|u^\pm|_2^2) > 0$$

for any $u \in H_0^1(\Omega)$ with $u^\pm \not\equiv 0$. Hence, if $(\alpha, \beta) \in (-\infty, a\lambda_1] \times (-\infty, b\mu^*]$, we can also see that either

$$\langle E'_{\alpha,\beta}(u), u^+\rangle \neq 0 \quad \text{or} \quad \langle E'_{\alpha,\beta}(u), u^-\rangle \neq 0,$$

namely, $\mathcal{M}_{\alpha,\beta} = \varnothing$. $\square$

In the next part of this section, our purpose is to prove theorem 1.2 by means of considering the minimization problem on $\mathcal{M}^1_{\alpha,\beta}$. It follows from Lemma 2.1 that $\mathcal{A}_1(\beta) \neq \varnothing$ when $\beta > b\mu^*$. Next, we will first prove that the set $\mathcal{M}^1_{\alpha,\beta} = \mathcal{A}_1(\beta) \cap \mathcal{M}_{\alpha,\beta} \neq \varnothing$ if (α, β) lies in a suitable set of R^2. And then, we present the properties of $\alpha_L(\beta)$. Finally, the infimum $m_{\alpha,\beta} := \inf\limits_{u \in \mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta}(u)$ can be achieved by some $u \in \mathcal{M}^1_{\alpha,\beta}$ for any $\alpha < a\alpha_L(\beta)$ and $\beta > b\mu^*$.

Lemma 2.2. *For every $u \in \mathcal{A}_1(\beta)$ with $H_\alpha(u^\pm) > 0$, there exists a unique pair $(s_u, t_u) \in R^+ \times R^+$ such that $s_u u^+ + t_u u^- \in \mathcal{M}^1_{\alpha,\beta}$ and $E_{\alpha,\beta}(s_u u^+ + t_u u^-) = \max\limits_{s,t>0} E_{\alpha,\beta}(su^+ + tu^-)$. Moreover, if $\langle E'_{\alpha,\beta}(u), u^\pm\rangle \leq 0$, then $(s_u, t_u) \in (0, 1] \times (0, 1]$.*

Proof. First of all, from the definition of $\mathcal{M}^1_{\alpha,\beta}$, it is easy to see that $s_u u^+ + t_u u^- \in \mathcal{M}^1_{\alpha,\beta} = \mathcal{A}_1(\beta) \cap \mathcal{M}_{\alpha,\beta}$ if and only if (s_u, t_u) is a unique solution of the following system:

$$\begin{cases} s^4 G_\beta(u^+) + bs^2t^2\|u^+\|^2\|u^-\|^2 = -s^2 H_\alpha(u^+), \\ t^4 G_\beta(u^-) + bs^2t^2\|u^+\|^2\|u^-\|^2 = -t^2 H_\alpha(u^-). \end{cases}$$

Therefore, it is sufficient to prove that the following system has a unique pair positive solution:

$$\begin{cases} SG_\beta(u^+) + bT\|u^+\|^2\|u^-\|^2 = -H_\alpha(u^+), \\ Sb\|u^+\|^2\|u^-\|^2 + TG_\beta(u^-) = -H_\alpha(u^-). \end{cases} \tag{2.1}$$

For any $u \in \mathcal{A}_1(\beta)$, that is $G_\beta(u^\pm) + b\|u^+\|^2\|u^-\|^2 < 0$, with $H_\alpha(u^\pm) > 0$, we have

$$D = \begin{vmatrix} G_\beta(u^+) & b\|u^+\|^2\|u^-\|^2 \\ b\|u^+\|^2\|u^-\|^2 & G_\beta(u^-) \end{vmatrix} = G_\beta(u^+)\ G_\beta(u^-) - (b\|u^+\|^2\ \|u^-\|^2)^2 > 0,$$

$$D_S = \begin{vmatrix} -H_\alpha(u^+) & b\|u^+\|^2\|u^-\|^2 \\ -H_\alpha(u^-) & G_\beta(u^-) \end{vmatrix} = -H_\alpha(u^+)G_\beta(u^-) + b\|u^+\|^2\|u^-\|^2 H_\alpha(u^-) > 0,$$

$$D_T = \begin{vmatrix} G_\beta(u^+) & -H_\alpha(u^+) \\ b\|u^+\|^2\|u^-\|^2 & -H_\alpha(u^-) \end{vmatrix} = -H_\alpha(u^-)G_\beta(u^+) + b\|u^+\|^2\|u^-\|^2 H_\alpha(u^+) > 0.$$

Hence, let $S = \frac{D_S}{D}$ and $T = \frac{D_T}{D}$, (S,T) is the unique solution of system (2.1). Consequently, let $s_u = S^{\frac{1}{2}}$ and $t_u = T^{\frac{1}{2}}$, we conclude that $s_u u^+ + t_u u^- \in \mathcal{M}^1_{\alpha,\beta}$.

Moreover, since $s_u u^+ + t_u u^- \in \mathcal{M}^1_{\alpha,\beta} \subset \mathcal{M}_{\alpha,\beta}$ and $G_\beta(u^\pm) < 0$, by a direct computation, we obtain

$$A = \frac{\partial^2 E_{\alpha,\beta}(su^+ + tu^-)}{\partial s^2}\bigg|_{(s_u,t_u)} = H_\alpha(u^+) + 3s_u^2 G_\beta(u^+) + bt_u^2\|u^+\|^2\|u^-\|^2$$
$$= 2s_u^2 G_\beta(u^+) < 0, \tag{2.2}$$

$$C = \frac{\partial^2 E_{\alpha,\beta}(su^+ + tu^-)}{\partial t^2}\bigg|_{(s_u,t_u)} = H_\alpha(u^-) + 3t_u^2 G_\beta(u^-) + bs_u^2\|u^+\|^2\|u^-\|^2$$
$$= 2t_u^2 G_\beta(u^-) < 0, \tag{2.3}$$

$$B = \frac{\partial^2 E_{\alpha,\beta}(su^+ + tu^-)}{\partial s \partial t}\bigg|_{(s_u,t_u)} = 2bs_u t_u\|u^+\|^2\|u^-\|^2 > 0, \tag{2.4}$$

$$B^2 - AC < 0. \tag{2.5}$$

Hence by (2.2)-(2.5), we have $E_{\alpha,\beta}(s_u u^+ + t_u u^-) = \max_{s,t>0} E_{\alpha,\beta}(su^+ + tu^-)$.

Finally, if $H_\alpha(u^\pm) + G_\beta(u^\pm) + b\|u^+\|^2\|u^-\|^2 = \langle E'_{\alpha,\beta}(u), u^\pm\rangle \le 0$, thus $H_\alpha(u^\pm) \le -G_\beta(u^\pm) - b\|u^+\|^2\|u^-\|^2$. Notice that $G_\beta(u^\pm) < 0$ and $H_\alpha(u^\pm) > 0$, one has

$$\begin{aligned} D_S &= -H_\alpha(u^+)G_\beta(u^-) + b\|u^+\|^2\|u^-\|^2 H_\alpha(u^-) \\ &\le \left(G_\beta(u^+) + b\|u^+\|^2\|u^-\|^2\right)\ G_\beta(u^-) \\ &\quad -b\|u^+\|^2\|u^-\|^2\ \left(G_\beta(u^-) + b\|u^+\|^2\|u^-\|^2\right) \\ &= G_\beta(u^+)G_\beta(u^-) - b^2\|u^+\|^4\|u^-\|^4 \\ &= D. \end{aligned}$$

Similarly, we can see that $D_T \le D$. Hence, $(s_u, t_u) \in (0,1] \times (0,1]$. $\square$

Lemma 2.3. *$\mathcal{M}^1_{\alpha,\beta} \neq \varnothing$ for any $(\alpha, \beta) \in (-\infty, a\alpha_1(\beta)) \times (b\mu^*, +\infty)$, where*

$$\alpha_1(\beta) = \sup\left\{\min\left\{\frac{\|u^+\|^2}{|u^+|_2^2}, \frac{\|u^+\|^2}{|u^-|_2^2}\right\} : u \in \mathcal{A}_1(\beta)\right\}.$$

Proof. Let $\beta > b\mu^*$, it follows from Lemma 2.1 that $\mathcal{A}_1(\beta) \neq \varnothing$, hence $\alpha_1(\beta) > -\infty$. Suppose that $\alpha < a\alpha_1(\beta)$, from the definition of $\alpha_1(\beta)$, there is a $u_0 \in \mathcal{A}_1(\beta)$ such that

$$\frac{\alpha}{a} < \min\left\{\frac{\|u_0^+\|^2}{|u_0^+|_2^2}, \frac{\|u_0^+\|^2}{|u_0^-|_2^2}\right\} \leq \alpha_1(\beta).$$

Therefore, we conclude that $u_0 \in \mathcal{A}_1(\beta)$ and $H_\alpha(u_0^\pm) > 0$. Combining with Lemma 2.2, we have $u_0 \in \mathcal{M}^1_{\alpha,\beta}$. □

In the following, let us recall the definition of $\alpha_L(\beta)$, and then we will discuss the main properties of $\alpha_L(\beta)$. Let

$$\mathcal{A}_L(\beta) = \left\{u \in H_0^1(\Omega) : u^\pm \not\equiv 0, G_\beta(u^\pm) + b\|u^+\|^2\|u^-\|^2 \leq 0\right\},$$

we define

$$\alpha_L(\beta) = \inf\left\{\min\left\{\frac{\|u^+\|^2}{|u^+|_2^2}, \frac{\|u^-\|^2}{|u^-|_2^2}\right\} : u \in \mathcal{A}_L(\beta)\right\}.$$

We assume that $\alpha_L(\beta) = +\infty$ if $\mathcal{A}_L(\beta) = \varnothing$. Notice that $\mathcal{A}_1(\beta) \subset \mathcal{A}_L(\beta)$, we have $\alpha_L(\beta) \leq \alpha_1(\beta)$. The following lemma contains the main properties of $\alpha_L(\beta)$.

Lemma 2.4. *The following assertions hold:*

(i) *$\alpha_L(\beta) = +\infty$ for any $\beta < b\mu^*$, and $\lambda_1 < \alpha_L(\beta)$ for any $\beta > b\mu^*$;*
(ii) *$\alpha_L(\beta)$ is decreasing for any $\beta > b\mu^*$;*
(iii) *$\alpha_L(\beta)$ is right-continuous for any $\beta > b\mu^*$;*
(iv) *Let $\mathcal{K}_{\alpha,\beta} = \mathcal{A}_L(\beta) \cap \left\{u \in H_0^1(\Omega) :\ H_\alpha(u^+) \leq 0\right\}$, if $\mathcal{K}_{\alpha,\beta} \neq \varnothing$, then $\alpha \geq a\alpha_L(\beta)$ and $\beta \geq b\mu^*$.*

Proof. (i) Similar to the proof of Lemma 2.1, it is easy to obtain that $\mathcal{A}_L(\beta) = \varnothing$ for any $\beta < b\mu^*$, and thus $\alpha_L(\beta) = +\infty$.

If $\beta > b\mu^*$, then $\varnothing \neq \mathcal{A}_1(\beta) \subset \mathcal{A}_L(\beta)$. From the definition of λ_1, for any sign-changing function $u \in H_0^1(\Omega)$, we have $\min\left\{\frac{\|u^+\|^2}{|u^+|_2^2}, \frac{\|u^-\|^2}{|u^-|_2^2}\right\} > \lambda_1$. Hence, in order to prove $\lambda_1 < \alpha_L(\beta)$, it is sufficient to show that for any $\beta > b\mu^*$, there exists a minimizer $u_\beta \in \mathcal{A}_L(\beta)$ of $\alpha_L(\beta)$. Set $\{u_n\} \subset \mathcal{A}_L(\beta)$ be a minimizing sequence for $\alpha_L(\beta)$, namely

$$\alpha_L(\beta) = \liminf_{n\to\infty} \min\left\{\frac{\|u_n^+\|^2}{|u_n^+|_2^2}, \frac{\|u_n^-\|^2}{|u_n^-|_2^2}\right\}.$$

Denoted by $v_n := \frac{u_n^+}{\|u_n\|} - \frac{u_n^-}{\|u_n\|}$, therefore $\{v_n\}$ is bounded, and then there is a subsequence, still denoted by $\{v_n\}$, and $v_0, w_1, w_2 \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_0, \quad v_n^+ \rightharpoonup w_1, \quad v_n^- \rightharpoonup w_2 \ \text{ in } H_0^1(\Omega), \tag{2.6}$$

$$v_n \to v_0, \quad v_n^+ \to w_1, \quad v_n^- \to w_2 \ \text{ in } L^2(\Omega) \text{ and } L^4(\Omega). \tag{2.7}$$

Since the maps $u \to u^+$ and $u \to u^-$ are continuous from $L^p(\Omega)$ to $L^p(\Omega)$ (see [3], Lemma 2.3), we have $v_0^+ = w_1 \geq 0$ and $v_0^- = w_2 \leq 0$ in Ω. Since $\{u_n\} \subset \mathcal{A}_L(\beta)$, one has

$$b\|u_n^\pm\|^4 - \beta|u_n^\pm|_4^4 + b\|u_n^+\|^2\|u_n^-\|^2 = b\|u_n\|^2\|u_n^\pm\|^2 - \beta|u_n^\pm|_4^4 \leq 0. \tag{2.8}$$

Hence, from the weak lower semicontinuity of norm, (2.6) and (2.7), we obtain

$$\begin{aligned} G_\beta(v_0^\pm) + b\|v_0^+\|^2\|v_0^-\|^2 &\leq \liminf_{n\to\infty} \left(G_\beta(v_n^\pm) + b\|v_n^+\|^2\|v_n^-\|^2\right) \\ &= \liminf_{n\to\infty} \frac{1}{\|u_n\|^4}(b\|u_n^\pm\|^4 - \beta|u_n^\pm|_4^4 + b\|u_n^+\|^2\|u_n^-\|^2) \leq 0. \end{aligned} \tag{2.9}$$

From (1.2), (2.8) and $v_n^\pm \not\equiv 0$, for any n, we have

$$b\|v_n^\pm\|^2 = \frac{b\|u_n\|^2\|u_n^\pm\|^2}{\|u_n\|^4} \leq \frac{\beta|u_n^\pm|_4^4}{\|u_n\|^4} = \beta|v_n^\pm|_4^4 \leq \beta\tau^4\|v_n^\pm\|^4,$$

which implies that $\|v_n^\pm\|^2 \geq \frac{b}{\beta\tau^4}$, and then $|v_n^\pm|_4^4 \geq \frac{b^2}{\beta^2\tau^4}$ for any n. Combining with (2.7), we obtain $v_0^\pm \not\equiv 0$. Hence together with (2.9), we have $v_0 \in \mathcal{A}_L(\beta)$.

On the other hand, from the weak lower semicontinuity of norm, (2.6) and (2.7) again, we have

$$\begin{aligned} \alpha_L(\beta) &= \liminf_{n\to\infty} \min\left\{\frac{\|u_n^+\|^2}{|u_n^+|_2^2}, \frac{\|u_n^-\|^2}{|u_n^-|_2^2}\right\} \\ &= \liminf_{n\to\infty} \min\left\{\frac{\|u_n^+\|^2/\|u_n\|^2}{|u_n^+|_2^2/\|u_n\|^2}, \frac{\|u_n^-\|^2/\|u_n\|^2}{|u_n^-|_2^2/\|u_n\|^2}\right\} \\ &= \liminf_{n\to\infty} \min\left\{\frac{\|v_n^+\|^2}{|v_n^+|_2^2}, \frac{\|v_n^-\|^2}{|v_n^-|_2^2}\right\} \\ &\geq \min\left\{\frac{\|v_0^+\|^2}{|v_0^+|_2^2}, \frac{\|v_0^-\|^2}{|v_0^-|_2^2}\right\} \geq \alpha_L(\beta). \end{aligned} \tag{2.10}$$

By $v_0 \in \mathcal{A}_L(\beta)$ and (2.10), one gets that $u_\beta := v_0$ is a minimizer of the function $\alpha_L(\beta)$. Consequently, we have $\alpha_L(\beta) = \min\left\{\frac{\|u_\beta^+\|^2}{|u_\beta^+|_2^2}, \frac{\|u_\beta^-\|^2}{|u_\beta^-|_2^2}\right\} > \lambda_1$ for any $\beta > b\mu^*$.

(ii) Set $b\mu^* < \beta_1 \leq \beta_2$, then $\mathcal{A}_L(\beta_1) \subset \mathcal{A}_L(\beta_2)$, which implies $\alpha_L(\beta_2) \leq \alpha_L(\beta_1)$, that is, $\alpha_L(\beta)$ is decreasing for any $\beta > b\mu^*$.

(iii) Since $\alpha_L(\beta)$ is decreasing for any $\beta > b\mu^*$, it is sufficient to show that $\alpha_L(\beta_0) \leq \lim\limits_{\beta\to\beta_0+0} \alpha_L(\beta)$ for any $\beta_0 > b\mu^*$. Due to the function $\alpha_L(\beta)$ is monotone and bounded in some right neighborhood of β_0, we have $\lim\limits_{n\to\infty} \alpha_L(\beta_n) = \lim\limits_{\beta\to\beta_0+0} \alpha_L(\beta)$ for any decreasing sequence $\{\beta_n\}$ with $\beta_n \to \beta_0 + 0$ as $n \to \infty$. By the proof of assertion (i), there exists a minimizer $u_{\beta_n} \in \mathcal{A}_L(\beta_n)$ of $\alpha_L(\beta_n)$ for any $n \in N^+$, that is,

$$\lim_{n\to\infty} \min\left\{\frac{\|u_{\beta_n}^+\|^2}{|u_{\beta_n}^+|_2^2}, \frac{\|u_{\beta_n}^-\|^2}{|u_{\beta_n}^-|_2^2}\right\} = \lim_{n\to\infty} \alpha_L(\beta_n) = \lim_{\beta\to\beta_0+0} \alpha_L(\beta). \tag{2.11}$$

Moreover, without loss of generality, we suppose that $\|u_{\beta_n}^\pm\| = 1$, then passing to an appropriate subsequence if necessary, $u_{\beta_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$ and $u_{\beta_n} \to u_*$ in $L^2(\Omega)$ and $L^4(\Omega)$. Now let's prove that $u_* \in \mathcal{A}_L(\beta_0)$. First of all, since $\{u_{\beta_n}^\pm\}$ converges to $u_*^\pm$ strongly in $L^4(\Omega)$ and $2b - \beta_n|u_{\beta_n}^\pm|_4^4 = G_{\beta_n}(u_{\beta_n}^\pm) + b\|u_{\beta_n}^+\|^2\|u_{\beta_n}^-\|^2 \leq 0$, we have $u_*^\pm \not\equiv 0$. Moreover, by the weak lower semicontinuity of norm, it holds

$$G_{\beta_0}(u_*^\pm) + b\|u_*^+\|^2\|u_*^-\|^2 \leq \liminf_{n\to\infty}(G_{\beta_n}(u_{\beta_n}^\pm) + b\|u_{\beta_n}^+\|^2\|u_{\beta_n}^-\|^2) \leq 0.$$

Therefore, $u_* \in \mathcal{A}_L(\beta_0)$. Moreover, we have

$$\alpha_L(\beta_0) \le \min\left\{\frac{\|u_*^+\|^2}{|u_*^+|_2^2}, \frac{\|u_*^-\|^2}{|u_*^-|_2^2}\right\} \le \liminf_{n\to\infty} \min\left\{\frac{\|u_{\beta_n}^+\|^2}{|u_{\beta_n}^+|_2^2}, \frac{\|u_{\beta_n}^-\|^2}{|u_{\beta_n}^-|_2^2}\right\}. \tag{2.12}$$

Hence, it follows from (2.11), (2.12) that $\alpha_L(\beta_0) \le \lim_{\beta\to\beta_0+0} \alpha_L(\beta)$.

(iv) In fact, by the proof of Lemma 2.1 and $\varnothing \ne \mathcal{K}_{\alpha,\beta} \subset \mathcal{A}_L(\beta)$, we obtain $\beta \ge b\mu^*$. Suppose that $u \in \mathcal{K}_{\alpha,\beta}$, then $u \in \mathcal{A}_L(\beta)$ and $\frac{\|u^+\|^2}{|u^+|_2^2} \le \frac{\alpha}{a}$, and from the definition of $\alpha_L(\beta)$, we obtain $\alpha_L(\beta) \le \frac{\|u^+\|^2}{|u^+|_2^2}$, thus $\alpha \ge a\alpha_L(\beta)$ and $\beta \ge b\mu^*$. $\square$

Lemma 2.5. *For any $\alpha < a\alpha_L(\beta)$ and $\beta > b\mu^*$, there is $u_0 \in \mathcal{M}^1_{\alpha,\beta}$ such that*

$$E_{\alpha,\beta}(u_0) = m_{\alpha,\beta} = \inf_{u\in\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta}(u) \quad \text{and} \quad m_{\alpha,\beta} > 0.$$

Proof. It follows from Lemma 2.3 and $\alpha_1(\beta) \ge \alpha_L(\beta)$ that $\mathcal{M}^1_{\alpha,\beta} \ne \varnothing$ for any $(\alpha,\beta) \in (-\infty, a\alpha_L(\beta)) \times (b\mu^*, +\infty)$. Let $\{u_n\} \subset \mathcal{M}^1_{\alpha,\beta}$ be a minimizing sequence for the functional $E_{\alpha,\beta}$, namely $E_{\alpha,\beta}(u_n) \to m_{\alpha,\beta}$ as $n \to \infty$. From the definition of $\alpha_L(\beta)$, we have

$$\|u^\pm\|^2 \ge \alpha_L(\beta)|u^\pm|_2^2 \quad \text{for any } u \in \mathcal{M}^1_{\alpha,\beta}.$$

Hence, we get

$$\begin{aligned} E_{\alpha,\beta}(u_n) = \frac{1}{4}H_\alpha(u_n) &= \frac{1}{4}(a\|u_n^+\|^2 - \frac{\alpha}{\alpha_L(\beta)}\alpha_L(\beta)|u_n^+|_2^2) \\ &\quad + \frac{1}{4}(a\|u_n^-\|^2 - \frac{\alpha}{\alpha_L(\beta)}\alpha_L(\beta)|u_n^-|_2^2) \\ &\ge \frac{1}{4}(a\|u_n^+\|^2 - \frac{\alpha}{\alpha_L(\beta)}\|u_n^+\|^2) + \frac{1}{4}(a\|u_n^-\|^2 - \frac{\alpha}{\alpha_L(\beta)}\|u_n^-\|^2) \\ &= \frac{a\alpha_L(\beta) - \alpha}{4\alpha_L(\beta)}\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded from $\alpha < a\alpha_L(\beta)$. Hence, we can assume that, up to subsequences, there exists $u_0 \in H_0^1(\Omega)$ such that

$$u_n^\pm \rightharpoonup u_0^\pm \ \text{ in } H_0^1(\Omega), \qquad u_n^\pm \to u_0^\pm \ \text{ in } L^2(\Omega) \text{ and } L^4(\Omega)$$

We first claim $u_0^\pm \not\equiv 0$. Indeed, by (1.2), the definitions of $\mathcal{M}^1_{\alpha,\beta}$ and $\alpha_L(\beta)$, we have

$$\begin{aligned} a\|u_n^\pm\|^2 &< a\|u_n^\pm\|^2 + b\|u_n^\pm\|^4 + b\|u_n^+\|^2\|u_n^-\|^2 \\ &= \alpha|u_n^\pm|_2^2 + \beta|u_n^\pm|_4^4 < t_0 a\alpha_L(\beta)|u_n^\pm|_2^2 + \beta|u_n^\pm|_4^4 \\ &\le at_0\|u_n^\pm\|^2 + \beta|u_n^\pm|_4^4 \le at_0\|u_n^\pm\|^2 + \beta\tau^4\|u_n^\pm\|^4, \end{aligned} \tag{2.13}$$

where $t_0 \in (0,1)$ satisfies $\alpha < t_0 a\alpha_L(\beta) < a\alpha_L(\beta)$, which implies that $\|u_n^\pm\|^2 \ge \frac{a - at_0}{\beta\tau^4} > 0$. From (2.13), we have

$$\beta|u_n^\pm|_4^4 \ge a(1-t_0)\|u_n^\pm\|^2 \ge \frac{a^2(1-t_0)^2}{\beta\tau^4},$$

which implies that $u_0^\pm \not\equiv 0$ by $u_n^\pm \to u_0^\pm$ strongly in $L^4(\Omega)$.

Using the weak lower semicontinuity of norm and $\{u_n\} \subset \mathcal{M}^1_{\alpha,\beta} \subset \mathcal{A}_1(\beta)$, we obtain

$$G_\beta(u_0^\pm) + b\|u_0^+\|^2\|u_0^-\|^2 \le \liminf_{n\to\infty} \left(G_\beta(u_n^\pm) + b\|u_n^+\|^2\|u_n^-\|^2\right) \le 0,$$

and hence $u_0 \in \mathcal{A}_L(\beta)$. We now claim that $H_\alpha(u_0^\pm) > 0$. It follows from (iii) of Lemma 2.4 that $\mathcal{A}_L(\beta) \cap \left\{u \in H_0^1(\Omega) :\ H_\alpha(u^+) \le 0\right\} = \mathcal{K}_{\alpha,\beta} = \varnothing$ for any $\alpha < a\alpha_L(\beta)$, and hence $H_\alpha(u_0^+) > 0$. Similarly, $H_\alpha(u_0^-) > 0$ due to $\alpha < a\alpha_L(\beta)$ and $-u_0 \in \mathcal{A}_L(\beta)$. The claim is proved.

Moreover, from the weak lower semicontinuity of norm again, one has

$$\begin{aligned} &H_\alpha(u_0^\pm) + G_\beta(u_0^\pm) + b\|u_0^+\|^2\|u_0^-\|^2 \\ \le &\liminf_{n\to\infty} \left(H_\alpha(u_n^\pm) + G_\beta(u_n^\pm) + b\|u_n^+\|^2\|u_n^-\|^2\right) \le 0. \end{aligned} \tag{2.14}$$

Then, together with $H_\alpha(u_0^\pm) > 0$, we obtain

$$G_\beta(u_0^\pm) + b\|u_0^+\|^2\|u_0^-\|^2 < 0. \tag{2.15}$$

(2.14) together with (2.15) shows $u_0 \in \mathcal{A}_1(\beta)$ and $\langle E'_{\alpha,\beta}(u_0), u_0^\pm\rangle \le 0$.

Consequently, from Lemma 2.2, it follows that there exists a unique pair $(s_{u_0}, t_{u_0}) \in (0,1] \times (0,1]$ such that $s_{u_0}u_0^+ + t_{u_0}u_0^- \in \mathcal{M}^1_{\alpha,\beta}$. We have

$$\begin{aligned} m_{\alpha,\beta} &\le E_{\alpha,\beta}(s_{u_0}u_0^+ + t_{u_0}u_0^-) \\ &= E_{\alpha,\beta}(s_{u_0}u_0^+ + t_{u_0}u_0^-) - \frac{1}{4}\langle E'_{\alpha,\beta}(s_{u_0}u_0^+ + t_{u_0}u_0^-), s_{u_0}u_0^+ + t_{u_0}u_0^-\rangle \\ &= \frac{1}{4}\left(a\|s_{u_0}u_0^+ + t_{u_0}u_0^-\|^2 - \alpha|s_{u_0}u_0^+ + t_{u_0}u_0^-|_2^2\right) \\ &= \frac{1}{4}s_{u_0}^2(a\|u_0^+\|^2 - \alpha|u_0^+|_2^2) + \frac{1}{4}t_{u_0}^2(a\|u_0^-\|^2 - \alpha|u_0^-|_2^2) \\ &\le \frac{1}{4}(a\|u_0^+\|^2 - \alpha|u_0^+|_2^2) + \frac{1}{4}(a\|u_0^-\|^2 - \alpha|u_0^-|_2^2) \\ &= \frac{1}{4}(a\|u_0\|^2 - \alpha|u_0|_2^2) \\ &\le \liminf_{n\to\infty}\left(E_{\alpha,\beta}(u_n) - \frac{1}{4}\langle E'_{\alpha,\beta}(u_n), u_n\rangle\right) = m_{\alpha,\beta}, \end{aligned}$$

which leads to $(s_{u_0}, t_{u_0}) = (1,1)$. Hence, $u_0 = u_0^+ + u_0^- \in \mathcal{M}^1_{\alpha,\beta}$ and $E_{\alpha,\beta}(u_0) = m_{\alpha,\beta}$. Finally, notice that $H_\alpha(u_0^\pm) > 0$, we have

$$m_{\alpha,\beta} = \inf_{u\in\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta}(u) = E_{\alpha,\beta}(u_0) = \frac{1}{4}H_\alpha(u_0) > 0.$$

Therefore, we complete the proof. □

Proof of Theorem 1.2. Let $\alpha < a\alpha_L(\beta)$ and $\beta > b\mu^*$. Assume that $u_0 \in \mathcal{M}^1_{\alpha,\beta}$ is the minimizer obtained in Lemma 2.5, we now prove that u_0 is a critical point of the functional $E_{\alpha,\beta}$, namely $E'_{\alpha,\beta}(u_0) = 0$. The main idea of the proof comes from [2].

First of all, since $E_{\alpha,\beta}(u_0) = \max_{s,t>0} E_{\alpha,\beta}(su_0^+ + tu_0^-) > 0$ and $H_\alpha(u_0^\pm) > 0$, from the continuities of $E_{\alpha,\beta}(su_0^+ + tu_0^-)$ and $H_\alpha(tu_0^\pm)$ with respect to s and t, there is a constant $0 < \sigma < 1$ such that

$$\min_{t\in[1-\sigma,1+\sigma]} H_\alpha(tu_0^\pm) > 0 \text{ and } \ 0 < m := \max_{\partial D} E_{\alpha,\beta}(su_0^+ + tu_0^-) < m_{\alpha,\beta}, \tag{2.16}$$

where $D = (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$. For convenience, define the function $g : D \to H_0^1(\Omega)$ by $g(s,t) = su_0^+ + tu_0^-$ for any $(s,t) \in D$.

Let us suppose now that $E'_{\alpha,\beta}(u_0) \neq 0$, there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|E'_{\alpha,\beta}(u)\| \geq \varrho \quad \text{for all } \ u \in H_0^1(\Omega) \text{ and } \|u - u_0\| < 3\delta.$$

Let us take $\varepsilon = \min\{\frac{m_{\alpha,\beta}-m}{3}, \frac{\varrho\delta}{8}\}$ and $S_\delta = \{u \in H_0^1(\Omega) : \|u - u_0\| \leq 2\delta\}$, Lemma 1.1 implies that there exists a deformation $\eta \in C([0,1] \times H_0^1(\Omega), H_0^1(\Omega))$ such that

(i) $\eta(r,v) = v$ if $r = 0$ or if $E_{\alpha,\beta}(v) < m_{\alpha,\beta} - 2\varepsilon$;

(ii) $E_{\alpha,\beta}(\eta(r,v)) \leq E_{\alpha,\beta}(v)$ for all $v \in H_0^1(\Omega)$ and $r \in [0,1]$;

(iii) $E_{\alpha,\beta}(\eta(r,v)) < m_{\alpha,\beta}$ for any $v \in S_\delta$ with $E_{\alpha,\beta}(v) \leq m_{\alpha,\beta}$ and $r \in [0,1]$.

From Lemma 2.2 and (ii), one has

$$\max_{\{(s,t)\in D: g(s,t)\notin S_\delta\}} E_{\alpha,\beta}(\eta(r, g(s,t))) \leq \max_{\{(s,t)\in D: g(s,t)\notin S_\delta\}} E_{\alpha,\beta}(g(s,t)) < m_{\alpha,\beta},$$

and Lemma 2.2, (iii) and $(s_{u_0}, t_{u_0}) = (1,1)$ implies that

$$\max_{\{(s,t)\in D: g(s,t)\in S_\delta\}} E_{\alpha,\beta}\left(\eta(r, g(s,t))\right) < m_{\alpha,\beta}.$$

Therefore, we have

$$\max_{(s,t)\in D} E_{\alpha,\beta}(\eta(r, g(s,t))) < m_{\alpha,\beta}. \tag{2.17}$$

From the continuities of η and H_α and (2.16), there is a constant $r_0 \in (0,1]$ such that

$$H_\alpha(\eta^\pm(r_0, g(s,t))) > 0 \ \text{ for any } \ (s,t) \in D. \tag{2.18}$$

We now show that $\eta(r_0, g(D)) \cap \mathcal{M}^1_{\alpha,\beta} \neq \varnothing$. Let us define $\varphi_0, \varphi_1 : D \to R^2$ as follows:

$$\varphi_0(s,t) = \left(\langle E'_{\alpha,\beta}(su_0^+ + tu_0^-), su_0^+\rangle, \langle E'_{\alpha,\beta}(su_0^+ + tu_0^-), tu_0^-\rangle\right),$$
$$\varphi_1(s,t) = \left(\langle E'_{\alpha,\beta}(\eta(r_0, g(s,t))), \eta^+(r_0, g(s,t))\rangle, \langle E'_{\alpha,\beta}(\eta(r_0, g(s,t))), \eta^-(r_0, g(s,t))\rangle\right).$$

From $\eta(0, g(s,t)) = g(s,t) = su_0^+ + tu_0^-$ for any $(s,t) \in D$ and $u_0 \in \mathcal{M}^1_{\alpha,\beta}$, Lemma 2.2 and the degree theory now yields $\deg(\varphi_0, D, 0) = 1$. On the other hand, since $\varepsilon \leq \frac{m_{\alpha,\beta}-m}{3}$, $m < m_{\alpha,\beta} - 2\varepsilon$. Hence, from (i), for any $r \in [0,1]$ and $(s,t) \in \partial D$, we have $\eta(r, g(s,t)) = g(s,t)$, and it follows that

$$\varphi_0(s,t) = \varphi_1(s,t) \ \text{ for any } \ (s,t) \in \partial D.$$

From the homotopy invariance property of the degree, we have $\deg(\varphi_1, D, 0) = \deg(\varphi_0, D, 0) = 1$, that is, there exists $(s_0, t_0) \in D$ such that $\varphi_1(s_0, t_0) = 0$. Moreover, from (2.18), we have $\eta^\pm(r_0, g(s_0, t_0)) \not\equiv 0$, which implies that $\eta(r_0, g(s_0, t_0)) \in \mathcal{M}^1_{\alpha,\beta}$.

Finally, from (2.17), we have

$$E_{\alpha,\beta}(\eta(r_0, g(s_0, t_0))) < m_{\alpha,\beta} = \inf_{u\in\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta}(u),$$

which is a contradiction. Hence, we have $E'_{\alpha,\beta}(u_0) = 0$. $\square$

3. The proof of Theorem 1.3

In this section, our purpose is to prove Theorem 1.3 with the aid of minimax method and Lemma 1.1, here we need the functional $E_{\alpha,\beta}$ satisfies the $(PS)_c$ condition, that is, if any sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $E_{\alpha,\beta}(u_n) \to c \in R, E'_{\alpha,\beta}(u_n) \to 0$ as $n \to \infty$, $\{u_n\}$ has a convergent subsequence. Let S_+^k denote the closed unit upper hemisphere in R^{k+1} with the boundary S^{k-1} and

$$\Lambda_{k+1} = \{u \in H_0^1(\Omega) : \ \|u\|^4 \geq \mu_{k+1}|u|_4^4\}.$$

We have the following useful fact:

Lemma 3.1. *$h(S_+^k) \cap \Lambda_{k+1} \neq \varnothing$ for any $h \in \{h \in C(S_+^k, H_0^1(\Omega)) : h|_{S^{k-1}}$ is odd$\}$.*

Proof. Let $h \in \{h \in C(S_+^k, H_0^1(\Omega)) : h|_{S^{k-1}}$ is odd$\}$, if there exists $u_0 \in h(S_+^k)$ such that $|u_0|_4 = 0$, we get $u_0 \in \Lambda_{k+1}$. Without loss of generality, we suppose that $|u|_4 > 0$ for any $u \in h(S_+^k)$. Define $\bar{h} : S^k \to H_0^1(\Omega)$ as

$$\bar{h}(z) = \begin{cases} h(z)/|h(z)|_4 & \text{if } z \in S_+^k, \\ -h(-z)/|h(-z)|_4 & \text{if } z \in S_-^k, \end{cases}$$

it is not difficult to verify that $\bar{h} \in \Sigma_{k+1}$. Therefore, from the definition of μ_{k+1}, see (1.6), there is $z_0 \in S^k$ such that $\|\bar{h}(z_0)\|^4 \geq \mu_{k+1}|\bar{h}(z_0)|_4^4 = \mu_{k+1}$. Since $\bar{h}(z)$ is odd, we can choose $z_0 \in S_+^k$. Consequently, $h(S_+^k) \cap \Lambda_{k+1} \neq \varnothing$. □

Lemma 3.2. *Suppose that $\frac{\beta}{b} \notin \sigma(-\|\cdot\|^2\Delta)$, the functional $E_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for every $c \in R$.*

Proof. Assume that $\{u_n\}$ is a Palais-Smale sequence at the level c, namely

$$E_{\alpha,\beta}(u_n) \to c, \quad E'_{\alpha,\beta}(u_n) \to 0 \ \text{ as } n \to \infty.$$

We first show that $\{u_n\}$ is bounded. Indeed if not, we suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n := \frac{u_n}{\|u_n\|}$, therefore $\|v_n\| = 1$ and there is a subsequence, still denoted by $\{v_n\}$, and $v_0 \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_0 \ \text{ in } H_0^1(\Omega), \quad v_n \to v_0 \ \text{ in } L^2(\Omega) \text{ and } L^4(\Omega).$$

This shows that

$$\frac{|\langle E'_{\alpha,\beta}(u_n), v_n - v_0\rangle|}{\|u_n\|^3} \leq \frac{\|E'_{\alpha,\beta}(u_n)\|_{(H_0^1(\Omega))^*}}{\|u_n\|^3}\|v_n - v_0\| \leq C_1 \frac{\|E'_{\alpha,\beta}(u_n)\|_{(H_0^1(\Omega))^*}}{\|u_n\|^3} \to 0$$

as $n \to \infty$, where C_1 is a positive constant, and we obtain

$$\begin{aligned} o_n(1) &= \frac{1}{\|u_n\|^3}\langle E'_{\alpha,\beta}(u_n), v_n - v_0\rangle \\ &= \frac{1}{\|u_n\|^3}\Bigg((a + b\|u_n\|^2)\int_\Omega \nabla u_n \nabla(v_n - v_0)dx \\ &\quad -\alpha\int_\Omega u_n(v_n - v_0)dx - \beta\int_\Omega u_n^3(v_n - v_0)dx\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \frac{a}{\|u_n\|^2} \int_\Omega \nabla v_n \nabla (v_n - v_0) dx + b \int_\Omega \nabla v_n \nabla (v_n - v_0) dx \\
&- \frac{\alpha}{\|u_n\|^2} \int_\Omega v_n (v_n - v_0) dx - \beta \int_\Omega v_n^3 (v_n - v_0) dx \\
&= b \int_\Omega \nabla v_n \nabla (v_n - v_0) dx + o_n(1) \\
&= b\langle -\Delta v_n, v_n - v_0 \rangle + o_n(1)
\end{aligned} \tag{3.1}
$$

as $n \to \infty$, which implies $\limsup_{n\to\infty} \langle -\Delta v_n, v_n - v_0\rangle \leq 0$. Due to the (S_+) property for the operator $-\Delta$ (see [5, Theorem 10]) and $\|v_n\| = 1$ for any n, we have $v_n \to v_0 \neq 0$ in $H_0^1(\Omega)$.

On the other hand, we have

$$
o_n(1) = \langle \frac{E'_{\alpha,\beta}(u_n)}{\|u_n\|^3}, \xi \rangle = o_n(1) + b\|v_n\|^2 \int_\Omega \nabla v_n \nabla \xi dx - \beta \int_\Omega v_n^3 \xi dx \tag{3.2}
$$

for any $\xi \in H_0^1(\Omega)$, and hence letting $n \to \infty$, we see that

$$
b\|v_0\|^2 \int_\Omega \nabla v_0 \nabla \xi dx - \beta \int_\Omega v_0^3 \xi dx = 0, \tag{3.3}
$$

which implies that $\frac{\beta}{b} \in \sigma(-\|\cdot\|^2\Delta)$ and v_0 is the associated eigenfunction corresponding to $\frac{\beta}{b}$, and we reach a contradiction. Hence, $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Next, we will show that $\{u_n\}$ has a convergent subsequence. Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, without loss of generality, we suppose that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_n \to u_0$ in $L^2(\Omega)$ and $L^4(\Omega)$. Therefore, from the Hölder's inequality, we have

$$
\begin{aligned}
o_n(1) &= \langle E'_{\alpha,\beta}(u_n) - E'_{\alpha,\beta}(u_0), u_n - u_0 \rangle \\
&= a \int_\Omega \nabla u_n \nabla (u_n - u_0) dx + b\|u_n\|^2 \int_\Omega \nabla u_n \nabla (u_n - u_0) dx \\
&\quad - a \int_\Omega \nabla u_0 \nabla (u_n - u_0) dx - b\|u_0\|^2 \int_\Omega \nabla u_0 \nabla (u_n - u_0) dx + o_n(1) \\
&\geq a\|u_n - u_0\|^2 + b\|u_n\|^2 (\|u_n\|^2 - \|u_n\|\|u_0\|) \\
&\quad + b\|u_0\|^2 (\|u_0\|^2 - \|u_n\|\|u_0\|) + o_n(1) \\
&= a\|u_n - u_0\|^2 + b(\|u_n\| - \|u_0\|)(\|u_n\|^3 - \|u_0\|^3) + o_n(1),
\end{aligned} \tag{3.4}
$$

which shows that $u_n \to u_0$ in $H_0^1(\Omega)$ as $n \to \infty$. Hence the functional $E_{\alpha,\beta}$ satisfies the $(PS)_c$ condition. □

Proof of Theorem 1.3. The proof can be divided into two cases according to the value range of β.

Case $\frac{\beta}{b} \notin \sigma(-\|\cdot\|^2\Delta)$. First of all, from the definition of $\alpha^*(\beta)$, we have $\frac{\alpha}{a} > \lambda_{k_\beta+1} = \max\{\alpha^*(\beta), \lambda_{k_\beta+1}\}$. According to [6], λ_k can be defined as

$$
\lambda_k := \inf_{h\in\Gamma_k} \sup_{z\in S^{k-1}} \|h(z)\|^2,
$$

where $\Gamma_k := \{h \in C(S^{k-1}, S) : \ h \text{ is odd}\}$ and $S := \{u \in H_0^1(\Omega) : \ |u|_2 = 1\}$. There exist a $\varepsilon_0 > 0$ and $h_0 \in \Gamma_{k_\beta+1} \subset C(S^{k_\beta}, H_0^1(\Omega))$ such that $\frac{\alpha}{a} > \lambda_{k_\beta+1} + \varepsilon_0$ and

$$\max_{z\in S^{k_\beta}} \|h_0(z)\|^2 < \lambda_{k_\beta+1} + \frac{\varepsilon_0}{2}$$

via the definition of $\lambda_{k_\beta+1}$. Therefore we can choose $t_0 > 0$ small enough such that

$$\begin{aligned}\rho &= \max_{z\in S^{k_\beta}} E_{\alpha,\beta}(t_0 h_0(z)) \\ &= \max_{z\in S^{k_\beta}} \left(\frac{1}{2}t_0^2(a\|h_0(z)\|^2 - \alpha) + \frac{1}{4}t_0^4(b\|h_0(z)\|^4 - \beta|h_0(z)|_4^4)\right) \\ &\le \frac{at_0^2}{2}(\max_{z\in S^{k_\beta}} (\|h_0(z)\|^2 - \frac{\alpha}{a}) + \frac{t_0^4}{4}\max_{z\in S^{k_\beta}} (b\|h_0(z)\|^4 - \beta|h_0(z)|_4^4) \\ &\le -\frac{a\varepsilon_0}{4}t_0^2 + \frac{t_0^4}{4}\max_{z\in S^{k_\beta}} (b\|h_0(z)\|^4 - \beta|h_0(z)|_4^4) < 0.\end{aligned} \tag{3.5}$$

On the other hand, using $b\mu_{k_\beta+1} > \beta$ and the Hölder's inequality, we get

$$\begin{aligned}\delta := \inf_{u\in\Lambda_{k_\beta+1}} E_{\alpha,\beta}(u) &= \inf_{u\in\Lambda_{k_\beta+1}} \left(\frac{1}{2}a\|u\|^2 - \frac{1}{2}\alpha|u|_2^2 + \frac{1}{4}b\|u\|^4 - \frac{1}{4}\beta|u|_4^4\right) \\ &\ge \inf_{u\in\Lambda_{k_\beta+1}} \left(-\frac{\alpha}{2}|\Omega|^{\frac{1}{2}}|u|_4^2 + \frac{b\mu_{k_\beta+1} - \beta}{4\mu_{k_\beta+1}}\|u\|^4\right) \\ &\ge \inf_{u\in\Lambda_{k_\beta+1}} \left(-\frac{\alpha}{2}|\Omega|^{\frac{1}{2}}\mu_1^{\frac{-1}{2}}\|u\|^2 + \frac{b\mu_{k_\beta+1} - \beta}{4\mu_{k_\beta+1}}\|u\|^4\right) > -\infty.\end{aligned} \tag{3.6}$$

From $h_0 \in \Gamma_{k_\beta+1}$ and Lemma 3.1 with $k = k_\beta$, that is, $h_0(S_+^{k_\beta}) \cap \Lambda_{k_\beta+1} \neq \varnothing$, we have $\rho \ge \delta$.

Finally, we will show that the functional $E_{\alpha,\beta}$ has at least one critical value in $[\delta - 1, \rho]$. Suppose that the conclusion is false, since the functional $E_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for every $c \in R$, Lemma 1.1 shows that there exists a deformation $\eta \in C([0,1] \times H_0^1(\Omega), H_0^1(\Omega))$ such that $\eta(t, \cdot)$ is odd for any $t \in [0,1]$ and

$$E_{\alpha,\beta}(\eta(1, t_0 h_0(z))) < \delta - 1 \quad \text{for all} \ \ z \in S^{k_\beta}. \tag{3.7}$$

But notice that $h_0(z)$ is odd in S^{k_β} and $\eta(1,u)$ is odd in $H_0^1(\Omega)$, we have $\eta(1, t_0 h_0(z))$ is odd in S^{k_β}. From Lemma 3.1 with $k = k_\beta$, it follows that $\eta(1, t_0 h_0(S_+^{k_\beta})) \cap \Lambda_{k_\beta+1} \neq \varnothing$. Hence, there exists $z_1 \in S^{k_\beta}$ such that $\eta(1, (t_0 h_0(z_1))) \in \Lambda_{k_\beta+1}$, whence $\delta \le E_{\alpha,\beta}(\eta(1, t_0 h_0(z_1)))$ form (3.6), which is a contradiction with (3.7). Hence the hypothesis can not hold and the proof is completed.

Case $\frac{\beta}{b} \in \sigma(-\|\cdot\|^2\Delta)$**.** Set $\frac{\alpha}{a} > \max\{\alpha^*(\beta), \lambda_{k_\beta+1}\}$, (3.5) holds, that is, there exist a $t_0 > 0$ and $h_0 \in \Gamma_{k_\beta+1}$ such that

$$\rho = \max_{z\in S^{k_\beta}} E_{\alpha,\beta}(t_0 h_0(z)) < 0. \tag{3.8}$$

Choose $\{\frac{\beta_n}{b}\} \subset R\backslash\sigma(-\|\cdot\|^2\Delta)$ with $\beta_n \to \beta$ as $n \to \infty$, by a simple calculation, we get

$$\begin{aligned}\rho_n :&= \max_{z\in S^{k_\beta}} E_{\alpha,\beta_n}(t_0 h_0(z)) \\ &= \max_{z\in S^{k_\beta}} \left(E_{\alpha,\beta}(t_0 h_0(z)) + \frac{1}{4}\beta|t_0 h_0(z)|_4^4 - \frac{1}{4}\beta_n|t_0 h_0(z)|_4^4\right)\end{aligned}$$

$$\leq \rho + \frac{|\beta - \beta_n| t_0^4}{4} \max_{z \in S^{k_\beta}} |h_0(z)|_4^4 < 0 \tag{3.9}$$

for large n. Notice that $\beta < b\mu_{k_\beta+1}$, we have $\beta_n < b\mu_{k_\beta+1}$ for sufficiently large $n \in N$, and hence similar to the proof of the case $\frac{\beta}{b} \notin \sigma(-\|\cdot\|^2\Delta)$, we see that there exists a critical point u_n of E_{α,β_n} in $H_0^1(\Omega)$ with $c_n := E_{\alpha,\beta_n}(u_n) \in [\delta_n - 1, \rho_n]$, where $\delta_n := \inf_{u \in \Lambda_{k_\beta+1}} E_{\alpha,\beta_n}(u) > -\infty$. From the Hölder's inequality and (1.2), for any $\xi \in H_0^1(\Omega)$, we have

$$\begin{aligned} |\langle E'_{\alpha,\beta}(u_n), \xi\rangle| = |\langle E'_{\alpha,\beta}(u_n) - E'_{\alpha,\beta_n}(u_n), \xi\rangle| = \left| (\beta_n - \beta) \int_\Omega u_n^3 \xi dx \right| \\ \leq |\beta_n - \beta|\ |u_n|_4^3\ |\xi|_4 \leq \tau^4 |\beta_n - \beta|\ \|u_n\|^3\ \|\xi\|, \end{aligned} \tag{3.10}$$

which implies $\frac{E'_{\alpha,\beta}(u_n)}{\|u_n\|^3} \to 0$ as $n \to \infty$.

Next we will show that $\{u_n\}$ converges strongly in $H_0^1(\Omega)$. It is sufficient to show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. And then, combining (3.10) with the boundedness of $\{u_n\}$, we obtain that $\{u_n\}$ is a Palais-Smale sequence for the functional $E_{\alpha,\beta}$, thus $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, such that (3.4) holds and then $u_n \to u_0$ in $H_0^1(\Omega)$.

We now prove that $\{u_n\}$ is bounded. Indeed, if $\|u_n\| \to \infty$ as $n \to \infty$, let $v_n := \frac{u_n}{\|u_n\|}$, from (3.1), (3.2) and (3.3), we can assume $v_n \to v_0 \in \mathcal{B}_{\beta/b}\backslash\{0\}$. Since

$$\frac{\delta_n - 1}{\|u_n\|^2} \leq \frac{c_n}{\|u_n\|^2} = H_\alpha(v_n) = \frac{1}{\|u_n\|^2}\left(4E_{\alpha,\beta_n}(u_n) - \langle E'_{\alpha,\beta_n}(u_n), u_n\rangle\right) \leq \frac{\rho_n}{\|u_n\|^2} < 0,$$

if $\{\delta_n\}$ is bounded, then $H_\alpha(v_n) \to H_\alpha(v_0) = 0$ as $n \to \infty$. But from $\frac{\alpha}{a} > \alpha^*(\beta) = \sup\{\frac{\|u\|^2}{|u|_2^2} :\ u \in \mathcal{B}_{\beta/b}\backslash\{0\}\}$, we see

$$H_\alpha(u) = a\|u\|^2 - \alpha|u|_2^2 < 0$$

for any $u \in \mathcal{B}_{\beta/b}\backslash\{0\}$, and hence $H_\alpha(v_0) < 0$, which is a contradiction with $H_\alpha(v_0) = 0$. Consequently, $u_n \to u_0$ in $H_0^1(\Omega)$ and u_0 is a critical point of $E_{\alpha,\beta}$.

We now claim that $\{\delta_n\}$ is bounded. Taking $\beta_0 \in R$ such that $\beta_n < \beta_0 < b\mu_{k_\beta+1}$ for sufficiently large $n \in N$, then (3.6) holds, that is, $\inf_{u \in \Lambda_{k_\beta+1}} E_{\alpha,\beta_0}(u) > -\infty$. On the other hand, $E_{\alpha,\beta_n}(u) > E_{\alpha,\beta_0}(u)$ for any $u \in H_0^1(\Omega)$, hence $\{\delta_n\}$ is bounded.

Furthermore, from (3.8) and (3.9), we have

$$E_{\alpha,\beta}(u_0) = \limsup_{n\to\infty} E_{\alpha,\beta_n}(u_n) = \limsup_{n\to\infty} c_n \leq \limsup_{n\to\infty} \rho_n \leq \rho + o(1) < 0,$$

which implies that u_0 is nontrivial solution and its energy is negative, and the proof is completed. □

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