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Optimal stopping of one-dimensional diffusions with integral criteria

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A B S T R A C T

This paper provides a full characterization of the value function and solution(s) of an optimal stopping problem for a one-dimensional diffusion with an integral criterion. The results hold under very weak assumptions, namely, the diffusion is assumed to be a weak solution of stochastic differential equation satisfying the Engelbert-Schmidt conditions, while the (stochastic) discount rate and the integrand are required to satisfy only general integrability conditions.

1. Introduction

Optimal stopping problems attracted generations of mathematicians due to both their interesting mathematical characteristics and their important applications. Early work was developed by Dynkin [12], Grigelionis and Shiryaev [19], Dynkin and Yushkevich [13]. A general theory can be found in books by Shiryaev [36] and Peskir and Shiryaev [31]. Several methods have been developed to deal with this type of problems.

Methods based on excessive functions date back to the pioneer work of Dynkin [12], and have been used by, among others, Dynkin and Yushkevich [13], Fakeev [14], Thompson [37], Shiryaev [36], Salminen [34], Alvarez [1], Dayanik and Karatzas [11], Lamberton and Zervos [24], among others. These methods are tightly connected with the concavity and monotonicity properties of the value function.

An alternative approach based on variational methods and inequalities was pioneered by Grigelionis and Shiryaev [19], and Bensoussan and Lions [8]. It was used in many works, namely Nagai [27], Friedman

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[16], Krylov [22], Bensoussan and Lions [9] Øksendal [28], Lamberton [23], Lamberton and Zervos [24], Rüschendorf and Urusov [33], Belomestny, Rüschendorf and Urusov [7], among others. Usually this approach requires some regularity assumptions on the problem's data and on the value function. Progress has been made in relaxing these assumptions, showing that the value function satisfies the appropriate variational inequality in various weak senses (see, for example, Friedman [16], Nagai [27], Zabczyk [39], Øksendal and Reikvam [29], Bassan and Ceci [4], Bensoussan and Lions [9], Lamberton [23], Lamberton and Zervos [24]). The variational approach allows for the development of some effective numerical methods (see, for example, Glowinski, Lions and Trémolières [17], or Zhang [40]).

A third approach, based on change of measure techniques and martingale theory, was introduced by Beibel and Lerche [5,6], and was further developed by several authors, namely Alvarez [1–3], Lerche and Urusov [26], Lempa [25], Christensen and Irle [10]. This approach proved successful in characterizing the optimal strategy at any given point of the state space.

In this paper we consider the optimal stopping problem of a general diffusion when the optimality criterion is an integral functional. More precisely, we seek the stopping time $\hat{\tau}$ maximizing the expected outcome

$$J(x,\tau) = \mathbb{E}_x\left[\int_0^\tau e^{-\rho_s}\Pi(X_s)ds\right], \tag{1}$$

where

$$\rho_t = \int_0^t r(X_s)ds \qquad 0 \le t < \tau_I, \tag{2}$$

and X solves the stochastic differential equation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t \tag{3}$$

up to the explosion time τ_I (possibly infinite). $\mathbb{E}_x$ means expected value conditional on $X_0 = x$, W is a standard Brownian motion and r, α, σ and Π are measurable real functions, satisfying minimal assumptions discussed in Section 2 below. In particular, the functions r, α, σ and Π may be discontinuous. As usual, τ is an admissible stopping time if and only if it is a stopping time with respect to the filtration generated by the process X and $\tau \le \tau_I$ almost certainly.

This class of optimal stopping problems has received little attention compared with optimal stopping problems where the functional being maximized is of type

$$\tilde{J}(x,\tau) = \mathbb{E}_x\left[e^{-\rho_\tau}\Pi(X_\tau)\chi_{\tau<\tau_I}\right]. \tag{4}$$

This is understandable, since the functional (4) arises naturally in many applications, particularly in the theory of American Options in mathematical finance. However, the problem (1)–(2)–(3) also has important applications, among others, in the theories of Asian Options and Real Options. Further, some known problems in the literature of optimal stopping and stochastic control can be reduced to the form (1)–(2)–(3) (see for example, Graversen, Peskir and Shiryaev [18], and Karatzas and Ocone [20]). It is known that, under an integrability assumption (see Remark 2.1 below), Problem (1)–(2)–(3) can be reduced to an equivalent problem of the form (4)–(2)–(3). We do not assume such integrability condition and therefore reduction of (1)–(2)–(3) to (4)–(2)–(3) is, in general, not possible. Similarly, it is common to assume that the instantaneous discount rate $r(\cdot)$ in (2) is non-negative (see e.g. Alvarez [3], Beibel and Lerche[6], Belomestny, Rüschendorf and Urusov [7], Christensen and Irle [10], Dayanik and Karatzas [11], Fakeev [14], Graversen, Peskir and Shiryaev [18], Grigelionis and Shiryaev [19], Lamberton [23], Lamberton and Zervos [24], Lempa

[25], Oksendal and Reikvam [29], Peskir [30], Ruschendorf and Urusov [33], Samee [35], Zabczyk [39]), but in this work we do not put any constraint in the sign of r, requiring it to satisfy only a local integrability condition.

Our approach is closely related to the works of Rüschendorf and Urusov [33], and Belomestny, Rüschendorf and Urusov [7]. We show that the value function solves a variational inequality in the Carathéodory sense. Thus, it is a continuously differentiable function with absolutely continuous first derivative, and it is not necessary to consider further weak solutions. The free boundary is fixed by a C_1 *fit condition*, coupled with a *global* non-negativity condition. Notice that the necessity (or not) of a *smooth fit principle* is a topic of current literature. For instance, works by Dayanik and Karatzas [11] (section 7), Villeneuve [38], Rüschendorf and Urusov [33], Belomestny, Rüschendorf and Urusov [7], and Lamberton and Zervos [24], prove that in certain cases, the *smooth fit principle* holds. This contrasts with works by Salminen [34], Peskir [30], and Samee [35], which find examples where the *smooth fit principle* fails.

Rüschendorf and Urusov [33] and Belomestny, Rüschendorf and Urusov [7] deal with the problem (1)–(2)–(3) assuming that the function Π is of so-called "two-sided form". The corresponding variational inequality is solved assuming a priori that the value function coincides on its support with the solution of an ordinary differential equation with two-sided zero boundary condition. Therefore, the method does not provide any information in cases when the value function is of some other form (e.g., a solution of the differential equation with only one-sided zero boundary condition), even if Π belongs to the restricted class of functions of "two-sided form". In this paper, we solve the variational inequality without assuming any particular behavior for Π or the value function, obtaining a characterization of the value function in terms of Π and the fundamental solution of a system of linear differential equations. As can be expected with this generality, the value function can assume many different forms, but it can always be found, at least on a given compact interval, by solving a finite-dimensional system of nonlinear equations. In particular, we address the issues raised in the remarks after Theorem 2.2 and in the remarks after Theorem 2.3 of Rüschendorf and Urusov [33], as well as in the remarks after Theorem 2.2 of Belomestny, Rüschendorf and Urusov [7].

Lamberton and Zervos [24] show that the value function for the problem (4)–(2)–(3) is the difference between two convex functions. Every function with absolutely continuous first derivative can be represented as the difference between two convex functions, but the converse is not true, since the derivative of a convex function can have countably many points of discontinuity. Thus our results show that the value function for the problem (1)–(2)–(3) is somewhat more regular than the solutions in [24].

Contrary to the results above, we do not assume any bound on the growth of the value function. This reflects the fact that in our approach the value function can be constructed as the upper envelope of a family of fundamental solutions, rather than as a lower envelope, as in the superhamonic characterization used in Dayanik and Karatzas [11], and Lamberton and Zervos [24]. Also, our proof is considerably shorter and much simpler, using only the Itō-Tanaka and occupation times formulae and some basic theory of linear ordinary differential equations. Our characterization of the value function reduces to some finite-dimensional equations, which is quite convenient for applications.

This paper is organized as follows. Section 2 contains the complete definition of problem (1)–(2)–(3), with the formulation of our working assumptions. Section 3 contains an outline of some elementary background material and sets some notation not introduced in Section 2. Section 4 contains the main results in the paper and some discussion on their usage to solve problems of type (1)–(2)–(3). Proofs of these results are postponed to Section 6. Section 5 contains some examples of solutions of optimal stopping problems.

2. Problem setting

Let $\alpha, r, \Pi : I \mapsto \mathbb{R}$, $\sigma : I \mapsto]0, +\infty[$ be Borel-measurable functions, where $I =]m, M[$ is an open interval with $-\infty \leq m < M \leq +\infty$. $\overline{I} = I \cup \{\infty\}$ denotes the one-point (Aleksandrov) compactification of I.

Assumption 2.1. *The functions $\frac{1}{\sigma^2}$, $\frac{\alpha}{\sigma^2}$ are locally integrable with respect to the Lebesgue measure in I.*

By Theorem 5.15 in Chapter 5 of Karatzas and Shreve [21], Assumption 2.1 guarantees existence and uniqueness (in law) of a weak solution for the stochastic differential equation (3), up to explosion time. In all the following, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P, X, W)$ denotes a given weak solution up to explosion time of equation (3). τ_I denotes the explosion time, and the process X is extended to the time interval $[0, +\infty[$ by setting $X_t = \infty$ for $t \geq \tau_I$. For every $t \geq 0$, $\mathcal{F}_t^X$ is the σ-algebra generated by $\{X_s\}_{0\leq s\leq t}$, augmented with all the P-null events. Due to continuity of X, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous. We denote by $\mathcal{T}$ the set of all stopping times τ adapted with respect to the filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$ such that $\tau \leq \tau_I$. The optimal stopping problem considered in this paper consists of finding the maximizers of (1) over the set $\mathcal{T}$.

For any real-valued function f, we set

$$f^+(x) = \max(f(x), 0), \qquad f^-(x) = \max(-f(x), 0).$$

Besides Assumption 2.1, we take the following assumptions concerning the functional (1):

Assumption 2.2. *The function $\frac{r}{\sigma^2}$ is locally integrable with respect to the Lebesgue measure in I.*

Assumption 2.3. *The function $\frac{\Pi}{\sigma^2}$ is locally integrable with respect to the Lebesgue measure in I, the sets $\{x \in I : \Pi(x) > 0\}$ and $\{x \in I : \Pi(x) < 0\}$ have both positive Lebesgue measure, and*

$$\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^+(X_t)dt\right] < +\infty \qquad \forall x \in I. \tag{5}$$

Remark 2.1. If (5) is replaced by the stronger

$$\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\left|\Pi(X_t)\right| dt\right] < +\infty \qquad \forall x \in I,$$

then Problem (1)–(2)–(3) can be reduced to a problem of type (4)–(2)–(3). However, such reduction is not, in general, possible under condition (5).

It turns out (see Proposition 6.2) that Assumption 2.3 is equivalent to the apparently weaker:

Assumption 2.4. *The function $\frac{\Pi}{\sigma^2}$ is locally integrable with respect to the Lebesgue measure in I, the sets $\{x \in I : \Pi(x) > 0\}$ and $\{x \in I : \Pi(x) < 0\}$ have both positive Lebesgue measure, and there is some $x \in I$ such that*

$$\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^+(X_t)dt\right] < +\infty.$$

We will see in Section 3 that local integrability of $\frac{\alpha}{\sigma^2}$, $\frac{r}{\sigma^2}$ and $\frac{\Pi}{\sigma^2}$ is necessary and sufficient for existence of solution for Equation (8) and therefore, it is necessary for existence of solution of the variational inequality (7). Further, if the set $\{x \in I : \Pi(x) > 0\}$ is negligible, then $\tau \equiv 0$ is trivially optimal. Conversely, when the set $\{x \in I : \Pi(x) < 0\}$ is negligible, then τ_I is trivially optimal. Taking into account the equivalence between Assumptions 2.3 and 2.4, if $\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^+(X_t)dt\right] = +\infty$ and $\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^-(X_t)dt\right] < +\infty$ then τ_I is trivially optimal. If $\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^+(X_t)dt\right] = \mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^-(X_t)dt\right] = +\infty$ then, the functional (1) is not well defined at least for some stopping times $\tau \in \mathcal{T}$. Thus, Assumption 2.3 excludes some trivial cases and cases where optimization cannot be carried over the set $\mathcal{T}$.

In Rüschendorf and Urusov [33], a similar problem is considered without requiring $\mathbb{E}_x \left[\int_0^{\tau_I} e^{-\rho_t}\Pi^+(X_t)dt\right] < \infty$, opting instead to optimize (1) over the set of all stopping times $\tau \leq \tau_I$ such that $\mathbb{E}_x \left[\int_0^{\tau} e^{-\rho_t}\Pi(X_t)dt\right]$ is well defined. It is shown that such formulation includes cases where the *value function*

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left[\int_0^{\tau} e^{-\rho_s}\Pi(X_s)ds\right]. \tag{6}$$

is well defined and finite but there is no optimal stopping time. We will show below (Theorem 4.1) that Assumptions 2.1, 2.2 and 2.3 exclude the occurrence of such phenomena: an optimal stopping time always exists of the form

$$\tau = \inf\{t \geq 0 : V(X_t) = 0\} \wedge \tau_I.$$

3. Background and notation

Taking into account the general results relating variational inequalities with optimal stopping (see, e.g. Peskir and Shiryaev [31] or Krylov [22]), it is expected that the value function (6) satisfies the Hamilton-Jacobi-Bellman equation

$$\min\left\{r(x)v(x) - \alpha(x)v'(x) - \frac{\sigma(x)^2}{2}v''(x) - \Pi(x), v(x)\right\} = 0. \tag{7}$$

Often, similar variational inequalities are presented in slightly different forms, as free boundary problems, as in Grigelionis and Shiryaev [19]. Obviously any solution v of (7) must coincide with a solution of the ordinary differential equation

$$r(x)v(x) - \alpha(x)v'(x) - \frac{\sigma(x)^2}{2}v''(x) - \Pi(x) = 0, \tag{8}$$

in any interval where $v(x) > 0$. Equation (8) is equivalent to the system of first-order differential equations

$$w'(x) = A(x)w(x) + b(x), \tag{9}$$

where

$$w(x) = \begin{pmatrix} v(x) \\ v'(x) \end{pmatrix}, \quad b(x) = \begin{pmatrix} 0 \\ -\frac{2\Pi(x)}{\sigma(x)^2} \end{pmatrix} \quad \text{and} \quad A(x) = \begin{pmatrix} 0 & 1 \\ \frac{2r(x)}{\sigma(x)^2} & -\frac{2\alpha(x)}{\sigma(x)^2} \end{pmatrix}.$$

Solutions for the system (9) are understood in the Carathéodory sense, that is, $w : I \mapsto \mathbb{R}^2$ is said to be a solution of (9) if it is absolutely continuous and satisfies

$$w(x) = w(a) + \int_a^x (A(z)w(z) + b(z))\,dz \qquad \forall x \in I,$$

where a is an arbitrary point of I. Thus, the solutions of equation (8) are continuously differentiable functions with absolutely continuous first derivatives. Similarly, we say that a function v is a solution of the Hamilton-Jacobi-Bellman equation (7) if and only if v is continuously differentiable, its first derivative is absolutely continuous, and v satisfies (7) almost everywhere with respect to the Lebesgue measure. In other words, any solution v of equations (7) or (8) can be written as the difference between two convex functions with absolutely continuous derivatives. This class of functions is a subset of the class used in Lamberton and Zervos [24], but we do not use this fact in this paper.

Let

$$\Phi(x,y) = \begin{pmatrix} \phi_{11}(x,y) & \phi_{12}(x,y) \\ \phi_{21}(x,y) & \phi_{22}(x,y) \end{pmatrix}$$

be the fundamental solution of the homogeneous system $w' = Aw$. That is, Φ the unique solution of the matrix differential equation

$$\frac{\partial}{\partial y}\Phi(x,y) = A(y)\Phi(x,y), \qquad \Phi(x,x) = Id$$

where Id represents the identity matrix.

Local integrability of $\frac{\alpha}{\sigma^2}$ and $\frac{r}{\sigma^2}$, as required in Assumptions 2.1 and 2.2, is necessary and sufficient for existence and uniqueness of $\Phi(x,y)$ for every $x, y \in I$ (see e.g. Theorem 1.3 in Filippov [15]). The additional Assumption 2.3 guarantees existence of one unique solution for the non-homogeneous system (9) defined in the whole interval I, for every initial condition $v(a) = \hat{v}_1$, $v'(a) = \hat{v}_2$ with $a \in I$, $\hat{v}_1, \hat{v}_2 \in \mathbb{R}$. Any solution of (9) can be written in the form

$$w(x) = \Phi(a,x)\left(w(a) + \int_a^x \Phi(a,z)^{-1} b(z)dz\right) = \Phi(a,x)w(a) + \int_a^x \Phi(z,x)b(z)dz, \tag{10}$$

where a is an arbitrary point of I. That is, any solution of (8) can be written in the form

$$v(x) = v(a)\phi_{11}(a,x) + v'(a)\phi_{12}(a,x) - \int_a^x \frac{2\Pi(z)}{\sigma(z)^2}\phi_{12}(z,x)dz, \qquad \forall x \in I. \tag{11}$$

For any $a, b \in I$, with $a < b$, and any $d \in \mathbb{R}$, we introduce the functions

$$v_{a,d}(x) = d\phi_{12}(a,x) - \int_a^x \frac{2\Pi(z)}{\sigma(z)^2}\phi_{12}(z,x)dz \qquad x \in I, \tag{12}$$

$$v^{[a,b]}(x) = \frac{\int_a^b \frac{2\Pi(z)}{\sigma(z)^2}\phi_{12}(z,b)dz}{\phi_{12}(a,b)}\phi_{12}(a,x) - \int_a^x \frac{2\Pi(z)}{\sigma(z)^2}\phi_{12}(z,x)dz \qquad x \in I. \tag{13}$$

These functions are, respectively, the solution of (8) with initial conditions $v(a) = 0$, $v'(a) = d$, and the solution of (8) with boundary conditions $v(a) = v(b) = 0$. We will show below (Proposition 6.1) that Assumption 2.3 implies $\phi_{12}(a,b) > 0$ for every $m < a < b < M$ and hence $v^{[a,b]}$ is well defined and is the unique solution of the corresponding boundary value problem. Belomestny, Rüschendorf and Urusov [7] proved a similar result using the probabilistic representation of such equation (8). We provide a shorter and more general proof using classical arguments from the theory of ordinary differential equations.

If $a = m$ or $b = M$ (or both), then we can pick monotonic sequences $a_n, b_n \in]a,b[$ such that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. If there is a function $v :]a,b[\mapsto \mathbb{R}$ such that

$$\lim_{n\to\infty} v^{[a_n,b_n]}(x) = v(x)$$

for every $x \in]a, b[$ and every sequences $a_n,\ b_n$ as above, then we denote that function by $v^{[a,b]}$. Existence of $v^{[m,b]}$, $v^{[a,M]}$, $v^{[m,M]}$ defined in this way is not in general guaranteed. Proposition 4.1(c) below shows that they exist in important cases. Notice that in the case $a = m$ (resp., $b = M$), the definition above does not imply that $\lim_{x \to a} v^{[a,b]}(x) = 0$ (resp., $\lim_{x \to b} v^{[a,b]}(x) = 0$). We will be specially interested in intervals such that

$$a < b \qquad \text{and} \qquad v^{[a,b]}(x) > 0 \ \ \forall x \in]a, b[. \tag{14}$$

Thus, we introduce the following definition.

Definition 3.1. We say that an interval $]a, b[$ with $m < a < b < M$, is *maximal for condition* (14) if it satisfies (14) and is not a proper subset of any other such interval.

If $a = m$ or $b = M$ (or both), we say that $]a, b[$ is maximal for condition (14) if there is a monotonically increasing sequence $]a_n, b_n[$ with $m < a_n < b_n < M$, such that every $]a_n, b_n[$ satisfies (14), $]a, b[= \bigcup_{n \in \mathbb{N}}]a_n, b_n[$, and $]a, b[$ is not a proper subset of any other such interval.

In the following, $\mathcal{L}^+$ denotes the set of all Lebesgue points of the function $x \mapsto \frac{\Pi(x)}{\sigma(x)^2}$ such that $\Pi(x) > 0$. $\mathcal{L}^-$ denotes the set of all Lebesgue points of the function $x \mapsto \frac{\Pi(x)}{\sigma(x)^2}$ such that $\Pi(x) < 0$.

4. Main results

In this section we state our main results without proofs. Full proofs are postponed to Section 6.

Throughout this section, Assumptions 2.1, 2.2 and 2.3 are supposed to hold.

Our characterization of the value function (Theorem 4.1) relies on maximal intervals for (14) and the corresponding functions $v^{[a,b]}$. Before stating the main result of the section, we give the following properties of maximal intervals.

Proposition 4.1. *The following statements hold true:*

a) *Different maximal intervals for* (14) *have empty intersection.*
b) *Every $x \in \mathcal{L}^+$ lies in some maximal interval for condition* (14)*. Conversely, if $]a, b[$ is maximal for* (14)*, then $]a, b[\cap \mathcal{L}^+ \neq \emptyset$.*
c) *If $]a, b[$ is maximal for* (14)*, then $v^{[a,b]}$ is well defined even if $a = m$ and/or $b = M$, and $v^{[a,b]}(x) \geq 0$ for every $x \in I$. Conversely, if v is a solution of* (8) *such that $v(x) \geq 0$ for every $x \in I$, and $a < b$ are two consecutive zeroes of v, then $]a, b[$ is maximal for* (14)*.*

By definition, maximal intervals have positive length. Since they are pairwise disjoint, this implies that there are at most countably many different maximal intervals for condition (14). Consequently, we have the following characterization of the value function.

Theorem 4.1. *Let $\{]a_k, b_k[,\ k = 1, 2, \ldots\}$ be the collection of all maximal intervals for condition* (14)*.*

The value function (6) *is*

$$V(x) = \begin{cases} v^{[a_k,b_k]}(x) & \text{for } x \in]a_k, b_k[, \quad k = 1, 2, \ldots, \\ 0 & \text{for } x \in I \setminus \bigcup_k]a_k, b_k[. \end{cases} \tag{15}$$

The random time $\tau = \inf\{t \geq 0 : V(X_t) = 0\} \wedge \tau_I$ is an optimal stopping time.

Theorem 4.1 begs for some practical way to identify the maximal intervals for (14). Proposition 4.1 gives some important information. We complete it with the following:

Proposition 4.2. *For any* $a \in I$, $b \in]a, M]$, $]a, b[$ *is maximal for* (14) *if and only if:*

a) $v_{a,0}(x) \geq 0$ *for every* $x \in I$, *and*
b) *there is a sequence* $a_n \in]a, M[\cap \mathcal{L}^-$ *such that* $\{x > a_n : v_{a_n,0}(x) \leq 0\} \neq \emptyset$ *for every* n *and*

$$\lim_{n\to\infty} a_n = a, \qquad \textit{and} \qquad \lim_{n\to\infty} \left(\inf\left\{x > a_n : v_{a_n,0}(x) \leq 0\right\}\right) = b.$$

In that case, $v^{[a,b]} = v_{a,0}$.

For any $b \in I$, $a \in [m, b[$, $]a, b[$ *is maximal for* (14) *if and only if:*

c) $v_{b,0}(x) \geq 0$ *for every* $x \in I$, *and*
d) *there is a sequence* $b_n \in]m, b[\cap \mathcal{L}^-$ *such that* $\{x < b_n : v_{b_n,0}(x) \leq 0\} \neq \emptyset$ *for every* n *and*

$$\lim_{n\to\infty} b_n = b, \qquad \textit{and} \qquad \lim_{n\to\infty} \left(\sup\left\{x < b_n : v_{b_n,0}(x) \leq 0\right\}\right) = a.$$

In that case, $v^{[a,b]} = v_{b,0}$.

Fix a interval $]a, b[$ with $m < a < b < M$, maximal for (14). Due to the propositions above, we have $v^{[a,b]} = v_{a,0} = v_{b,0}$. By (12), $v'_{a,0}(x) = -\int_a^x \frac{2\Pi(z)}{\sigma(z)^2}\phi_{22}(z, x)dz$. Hence, the points a, b solve the following set of nonlinear equations

$$\int_a^b \frac{\Pi(z)}{\sigma(z)^2}\phi_{12}(z, b)dz = 0, \qquad \int_a^b \frac{\Pi(z)}{\sigma(z)^2}\phi_{22}(z, b)dz = 0, \qquad a < b. \tag{16}$$

If $]a, M[$ is maximal for (14) and $a \in I$, then for any sequence $\{b_n \in I\}_{n\in\mathbb{N}}$ converging to M, a solves the equation:

$$\lim_{n\to\infty} \int_a^{b_n} \frac{\Pi(z)}{\sigma(z)^2}\phi_{12}(z, b_n)dz = 0. \tag{17}$$

Similarly, if $]m, b[$ is maximal for (14) and $b \in I$, then for any sequence $\{a_n \in I\}_{n\in\mathbb{N}}$ converging to m, b solves the equation:

$$\lim_{n\to\infty} \int_{a_n}^{b} \frac{\Pi(z)}{\sigma(z)^2}\phi_{12}(z, a_n) = 0. \tag{18}$$

In Section 5 we will see that equations (16), (17), (18) simplify considerably when X is a geometric Brownian motion.

Theoretically, the value function can be found through the following steps:

(I) Find the solutions of (16). Discard any solutions (a, b) such that $\int_a^x \frac{\Pi(z)}{\sigma(z)^2}\phi_{12}(z, x)dz > 0$ for some $x \in I$.

This yields at most countably many solutions (a_k, b_k), $k = 1, 2, \ldots$, and the collection of all the intervals between consecutive zeroes of some $v_{a_k,0}$ is the collection of all maximal intervals for (14), with $a > m$ and $b < M$.

(II) If there is some $a \in I$ such that $v_{a,0}(x) \geq 0$ for every $x \in I$, then find

$$\hat{a} = \inf \{a \in I : v_{a,0}(x) \geq 0 \text{ for every } x \in I\}, \qquad \hat{b} = \sup \{b \in I : v_{b,0}(x) \geq 0 \text{ for every } x \in I\}.$$

If $\hat{a} > m$, then $]m, \hat{a}[$ is maximal for (14). If $\hat{b} < M$, then $]\hat{b}, M[$ is maximal for (14).
This yields all maximal intervals of type $]m, a[$ or $]b, M[$, if such intervals exist.

(III) If for every $a \in I$ there is some $x \in I$ such that $v_{a,0}(x) < 0$, then I is maximal for (14).

5. Examples

Rüschendorf and Urusov [33], and Belomestny, Rüschendorf and Urusov [7] characterize the value function (6) as the solution of a free boundary problem, assuming that the function Π is of “two sided form” and the support of the value function is an interval $[a, b]$, with $m < a < b < M$. The results in the previous section do not require any particular structure neither for Π nor for the value function.

In Example 1 we discuss a case where Π is of “two sided form” but the value function may fail to satisfy the assumption in [33,7], depending on parameters. Example 2 deals with a simple case where Π is not of “two sided form”. In both Examples, we assume that the process X is a geometric Brownian motion and the discount rate is constant. This means that $\alpha(x) = \alpha x$, $\sigma(x) = \sigma x$, $r(x) = r$, with α, σ, r constants and $I =]0, +\infty[$. Moreover, $P_x\{\tau_I = +\infty\} = 1$, for every $x \in]0, +\infty[$, where P_x denotes the conditional probability in $X_0 = x$. The matrix $A(x)$ is

$$A(x) = \begin{pmatrix} 0 & 1 \\ \frac{2r}{\sigma^2 x^2} & -\frac{2\alpha}{\sigma^2 x} \end{pmatrix}.$$

Before presenting the examples, we will discuss the fundamental solution Φ associated with this matrix.

The ordinary differential equation (8) takes the form

$$rv(x) - \alpha x v'(x) - \frac{\sigma^2}{2} x^2 v''(x) - \Pi(x) = 0. \tag{19}$$

Using the change of variable $x = e^z$ and $y(z) = v(e^z)$, this reduces to the equation with constant coefficients:

$$ry(z) - \left(\alpha - \frac{\sigma^2}{2}\right) y'(z) - \frac{\sigma^2}{2} y''(z) - \Pi(e^z) = 0. \tag{20}$$

The fundamental matrix Φ is characterized by the roots of the characteristic polynomial of (20)

$$P(d) = -\frac{\sigma^2}{2} d^2 - \left(\alpha - \frac{\sigma^2}{2}\right) d + r.$$

Let d_1 and d_2 be the roots of P. The model’s data, (r, α, σ) may be parametrized by (d_1, d_2, σ) through the relations

$$\alpha = \frac{\sigma^2}{2}(1 - d_1 - d_2) \quad \text{and} \quad r = -\frac{\sigma^2}{2} d_1 d_2.$$

Three different cases must be considered: (i) $d_1 = \overline{d_2} \in \mathbb{C} \setminus \mathbb{R}$, (ii) $d_1 = d_2 \in \mathbb{R}$ and (iii) $d_1, d_2 \in \mathbb{R}$ with $d_1 \neq d_2$.

Case (i): Let $d_1 = a + ib$, $d_2 = a - ib$. The fundamental matrix associated to Equation (19) is

$$\Phi(x,y) = \left(\frac{y}{x}\right)^a \begin{pmatrix} \frac{b\cos(b\log(\frac{y}{x})) - a\sin(b\log(\frac{y}{x}))}{b} & \frac{x\sin(b\log(\frac{y}{x}))}{b} \\ -\frac{(a^2+b^2)\sin(b\log(\frac{y}{x}))}{by} & \frac{x}{y}\frac{b\cos(b\log(\frac{y}{x}))+a\sin(b\log(\frac{y}{x}))}{b} \end{pmatrix}.$$

Thus, the function $y \to \phi_{1,2}(x,y)$ has infinitely many zeroes. Therefore, in light of Proposition 6.1

$$\mathbb{E}_x\left[\int_0^{+\infty} e^{-rt}\Pi^+(X_t)dt\right] = +\infty \tag{21}$$

for every $x \in]0,+\infty[$ and every measurable Π such that the set $\{x > 0 : \Pi(x) > 0\}$ has strictly positive Lebesgue measure. Thus, Assumption 2.3/2.4 fails.

Case (ii): Let $d = d_1 = d_2$. In this case, the fundamental matrix is

$$\Phi(x,y) = \left(\frac{y}{x}\right)^d \begin{pmatrix} \left(1 - d\log\left(\frac{y}{x}\right)\right) & x\log\left(\frac{y}{x}\right) \\ -\frac{d^2}{y}\log\left(\frac{y}{x}\right) & \frac{x}{y}\left(1 + d\log\left(\frac{y}{x}\right)\right) \end{pmatrix}.$$

For every $x \in]0,+\infty[$, the function $y \mapsto \phi_{12}(x,y)$ has one unique zero. However, a tedious but trivial computation shows that

$$\lim_{n\to+\infty} v^{[\frac{1}{n},n]}(x) = +\infty \quad \text{for every } x > 0$$

whenever Π is non-negative and the set $\{x > 0 : \Pi(x) > 0\}$ has strictly positive Lebesgue measure. Thus, (21) holds also in this case.

Case (iii): Without loss of generality, we assume that $d_1 < d_2$. The fundamental matrix is

$$\Phi(x,y) = \begin{pmatrix} \frac{d_2\left(\frac{y}{x}\right)^{d_1} - d_1\left(\frac{y}{x}\right)^{d_2}}{d_2 - d_1} & x\frac{\left(\frac{y}{x}\right)^{d_2} - \left(\frac{y}{x}\right)^{d_1}}{d_2 - d_1} \\ d_1 d_2 \frac{\left(\frac{y}{x}\right)^{d_1-1} - \left(\frac{y}{x}\right)^{d_2-1}}{(d_2-d_1)x} & \frac{d_2\left(\frac{y}{x}\right)^{d_2-1} - d_1\left(\frac{y}{x}\right)^{d_1-1}}{d_2-d_1} \end{pmatrix}. \tag{22}$$

Like in case (ii), for every $x > 0$ the function $y \mapsto \phi_{12}(x,y)$ has one unique zero. Thus, the discussion above leaves this as the only interesting case. For this reason, in both examples below we will assume that $d_1, d_2 \in \mathbb{R}$, with $d_1 < d_2$.

Notice that in case (iii), substitution of (22) in (12), yields

$$v_{a,0}(x) = \frac{-2}{(d_2 - d_1)\sigma^2}\int_a^x \frac{\left(\frac{x}{z}\right)^{d_2} - \left(\frac{x}{z}\right)^{d_1}}{z}\Pi(z)dz. \tag{23}$$

Equations (16) reduce to

$$\int_a^b z^{-d_2-1}\Pi(z)dz = 0, \qquad \int_a^b z^{-d_1-1}\Pi(z)dz = 0, \qquad a < b, \tag{24}$$

and Equations (17), (18) become

$$\int_a^{+\infty} z^{-d_2-1}\Pi(z)dz = 0, \qquad \int_0^b z^{-d_1-1}\Pi(z)dz = 0, \tag{25}$$

respectively. Notice that (23)–(24)–(25) show that the inverse volatility $\frac{1}{\sigma^2}$ acts as multiplicative parameter in the value function.

Example 1. Fix $0 < x_1 < x_2 < +\infty$, and let Π be the piecewise constant function

$$\Pi(x) = 2\chi_{[x_1,x_2]}(x) - 1, \quad \text{for all } x > 0.$$

This function is of "two sided form" in the sense of Belomestny, Rüschendorf and Urusov [7].

Due to Proposition 4.1, $]0,+\infty[$ contains one unique maximal interval for (14), and it contains the interval $]x_1, x_2[$. Due to (23), for any $a \in]0, x_1[$,

$$v_{a,0}(x) = \frac{2}{(d_2 - d_1)\sigma^2}\left(x^{d_1}G_1(x) - x^{d_2}G_2(x)\right)$$

with

$$G_i(x) = \begin{cases} \frac{-1}{d_i}\left(a^{-d_i} - x^{-d_i}\right) & \text{for } x < x_1 \\ \frac{-1}{d_i}\left(a^{-d_i} - 2x_1^{-d_i} + x^{-d_i}\right) & \text{for } x_1 \le x \le x_2 \\ \frac{-1}{d_i}\left(a^{-d_i} - 2x_1^{-d_i} + 2x_2^{-d_i} - x^{-d_i}\right) & \text{for } x > x_2, \qquad i = 1, 2. \end{cases}$$

From this, it can be checked that if $d_2 > 0$, then for every sufficiently small $a > 0$ we have $v_{a,0}(x) > 0$ for every $x \neq a$. Therefore, the interval $]0, x_1]$ is not contained in the maximal interval for (14). A similar argument applied to the function $v_{b,0}$ with $b > x_2$ shows that if $d_1 < 0$ then the interval $[x_2, +\infty[$ is not contained in the maximal interval for (14). Therefore, for any $d_1 < d_2$, $]0, +\infty[$ cannot be the maximal interval. If $d_1 < 0 < d_2$ then the maximal interval $]a, b[$ must be such that $0 < a < x_1 < x_2 < b < +\infty$.

To see that in the case $d_1 < d_2 < 0$ the maximal interval can be either $]a, b[$ with $0 < a < x_1 < x_2 < b < +\infty$ or $]0, b[$ with $x_2 < b < +\infty$, we consider the case $d_1 = -2$, $d_2 = -1$, where explicit computations are trivial. Notice that for Π of "two sided form" and for $0 < a < b < +\infty$, $]a, b[$ is maximal if and only if (a, b) solves (24). For $d_1 = -2$, $d_2 = -1$, it is easy to check that (24) admits a solution with $0 < a < b < +\infty$ if and only if $x_1 > \frac{x_2}{3}$, and in that case

$$a = \frac{3x_1 - x_2}{2}, \qquad b = \frac{3x_2 - x_1}{2}.$$

If $x_1 < \frac{x_2}{3}$, the maximal interval is $]0, b[$, with b satisfying the second equality in (25), that is

$$b = \sqrt{2\left(x_2^2 - x_1^2\right)}.$$

Therefore, the value function is

$$V(x) = \begin{cases} v_{a,0}(x) = v_{b,0}(x), & \text{for } x \in [a, b] \\ 0, & \text{for } x \notin [a, b] \end{cases} \qquad \text{if } x_1 > \frac{x_2}{3},$$

$$V(x) = \begin{cases} v_{b,0}(x), & \text{for } x \le b \\ 0, & \text{for } x > b \end{cases} \qquad \text{if } x_1 \le \frac{x_2}{3},$$

with a, b given by the expressions above. In the second case, the value function is not supported in a compact subinterval of $]0, +\infty[$. Thus, this is an example of a problem that is not solved by the results in [33,7]. Graphs of the value function for both cases are shown in Fig. 1. Notice that the case $d_1 < d_2 < 0$ corresponds to a negative discount rate and the value function is unbounded.

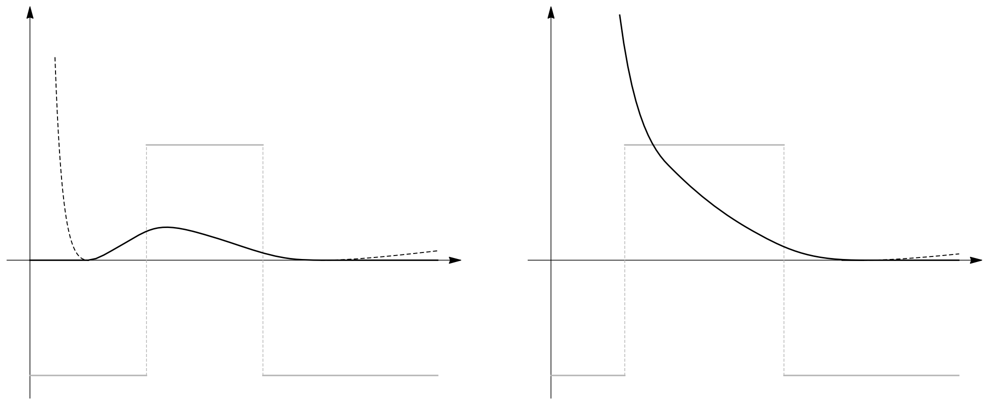

Fig. 1. Value functions for Example 1. The gray lines represent the functions Π. The black lines represent the value functions V. The dashed lines represent the functions $v_{b,0}$. Left-hand picture: $x_1 = 1$, $x_2 = 2$. Right-hand picture: $x_1 = \frac{19}{30}$, $x_2 = 2$. In both cases, $\sigma = 1$, $d_1 = -2$, $d_2 = -1$. Figures drawn to the same scale.

Similar examples with $0 < d_1 < d_2$ showing that the maximal interval can be either $]a, b[$, with $0 < a < x_1 < x_2 < b < +\infty$, or $]a, +\infty[$, with $0 < a < x_1$ can easily be constructed.

Example 2. Fix $0 < x_1 < x_2 < x_3 < x_4 < +\infty$, and let Π be the piecewise constant function

$$\Pi(x) = 2\chi_{[x_1,x_2]}(x) + 2\chi_{[x_3,x_4]}(x) - 1.$$

Thus, Π is positive in two separate intervals. This is the case discussed in the remarks following Theorem 2.3 of Rüschendorf and Urusov [33], and Theorem 2.2 of Belomestny, Rüschendorf and Urusov [7]. To discuss this case, we introduce the functions

$$\Pi_1(x) = 2\chi_{[x_1,x_2]}(x) - 1, \qquad \Pi_2(x) = 2\chi_{[x_3,x_4]}(x) - 1.$$

Let V, V_1, V_2 be the value functions corresponding to Π, Π_1, Π_2, respectively, and let $v_{a,0}$, $v^1_{a,0}$, $v^2_{a,0}$ be the corresponding functions defined by (23).

In [33,7] it is remarked that if the support of V_1 is an interval $[a, b]$ with $0 < a < b < +\infty$, then V_1 solves both the free-boundary problem corresponding to Π_1 and the free-boundary problem corresponding to Π, but V_1 may coincide or not with V in $[a, b]$. We will show that the results in Section 4 above easily distinguish these cases.

Suppose that $d_2 > 0$ (the case $d_1 < 0$ is analogous). From Example 1, there are constants $0 < a_1 < x_1 < x_2 < b_1 \leq +\infty$ such that:

$$V_1(x) = \begin{cases} v^1_{a_1,0}(x), & \text{for } x \in]a_1, b_1[, \\ 0, & \text{for } x \notin]a_1, b_1[. \end{cases}$$

Since $]a_1, b_1[$ is maximal for (14) with respect to Π_1, $v^1_{a_1,0}$ is non-negative in $]0, +\infty[$. It is easy to check that $v_{a_1,0}$ coincides with $v^1_{a_1,0}$ in the interval $]0, x_3]$ but these functions are distinct in the interval $]x_3, +\infty[$. Thus, it may happen that $v_{a_1,0}(x) < 0$ for some $x > x_3$. In that case, Proposition 4.1 shows that $]a_1, b_1[$ is not maximal with respect to Π and therefore $v_{a_1,0}$ does not coincide with the value function V in $[a_1, b_1]$. The Fig. 2 shows an example of this configuration. Conversely, if $v_{a_1,0}(x) \geq 0$ for every $x \in]0, +\infty[$, then $]a_1, b_1[$ is maximal with respect to Π and V coincides with $v^1_{a_1,0}$ in the interval $[a_1, b_1]$. The right-hand picture in Fig. 3 shows an example of this configuration.

Another way to see the same phenomenon is as follows. Let $]a_1, b_1[$, $]a_2, b_2[$ be the maximal intervals with respect to Π_1 and Π_2, respectively (by Example 1, these intervals exist and are unique, with $a_1, a_2 > 0$).

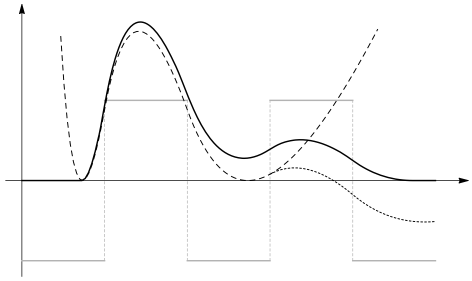

Fig. 2. Example 2: Value functions for Π_1 and Π. The gray line represents the function Π. The dashed line represents the function $v^1_{a_1,0}$. The dotted line represents the function $v_{a_1,0}$. The black line represents the value function for Π. In its support, the value function coincides with a function $v_{a,0}$ with $a < a_1$, but this is not apparent in the graph due to scale. Parameters: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, $\sigma = \frac{1}{3}$, $d_1 = -1$, $d_2 = 1$.

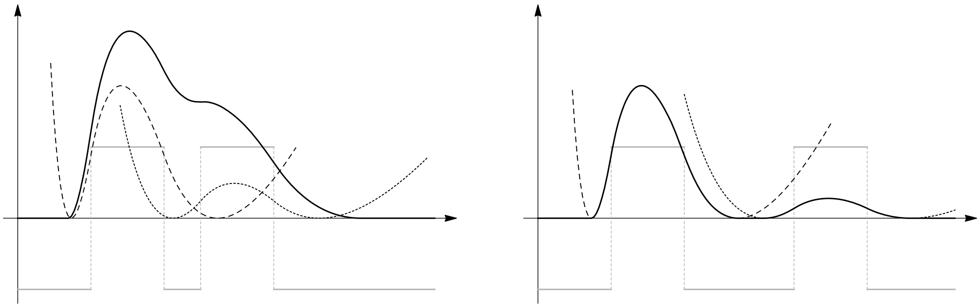

Fig. 3. Example 2: Value functions for Π_1, Π_2, and Π. Grey lines represent the function Π. Dashed lines represent the functions $v^1_{a_1,0}$. Dotted lines represent the functions $v^2_{a_2,0}$. Black lines represent the value function for Π. Left-hand picture: $x_1 = 1$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3.5$, $b_1 \approx 2.73 > a_2 \approx 2.12$. Right-hand picture: $x_1 = 1$, $x_2 = 2$, $x_3 = 3.5$, $x_4 = 4.5$, $b_1 \approx 2.73 < a_2 \approx 3.09$. In both cases, $\sigma = \frac{1}{3}$, $d_1 = -1$, $d_2 = 1$. Figures drawn to the same scale.

If $a_2 < b_1$, then Proposition 4.1 states that these intervals cannot be maximal with respect to Π. Hence, the maximal interval for Π must be a larger interval $]a, b[$ containing $]a_1, b_1[\cup]a_2, b_2[$. Conversely, if $a_2 \geq b_1$, then $]a_1, b_1[$, $]a_2, b_2[$ are both maximal with respect to Π, and therefore the value function is

$$V(x) = \begin{cases} v^1_{a_1,0}(x) = v_{a_1,0}(x), & \text{for } x \in [a_1, b_1], \\ v^2_{a_2,0}(x) = v_{a_2,0}(x), & \text{for } x \in [a_2, b_2], \\ 0, & \text{for } x \notin [a_1, b_1] \cup [a_2, b_2]. \end{cases}$$

The Fig. 3 shows an example with $a_2 < b_1$ and an example with $a_2 > b_1$.

6. Proofs

6.1. Some preliminary results

The results in Section 4 depend critically on the following Proposition.

Proposition 6.1. *Suppose Assumptions 2.1 and 2.2 hold. If there is some function Π satisfying Assumption 2.3, then $\phi_{12}(a, b) > 0$ for every $a, b \in I$ with $a < b$.*

The proof of this Proposition requires several intermediate lemmata, which we formulate and prove below. As a corollary, we will prove the following.

Proposition 6.2. *Under Assumptions 2.1 and 2.2, Assumptions 2.3 and 2.4 are equivalent.*

Another easy corollary of Proposition 6.1 is the following Lemma, that will be useful to several arguments in the next subsections.

Lemma 6.1. *Suppose Assumptions 2.1, 2.2, and 2.3 hold. If u, v are solutions of (8), and there are two points $a, b \in I$ such that*

$$u(a) = v(a), \qquad u(b) = v(b), \qquad a \neq b$$

then $u \equiv v$.

Proof. Follows immediately from Proposition 6.1 and equality (11). □

To prove Proposition 6.1, we start with Lemmata 6.2 and 6.3, which contain some simple properties of the fundamental solution Φ.

Lemma 6.2. *Under Assumptions 2.1, 2.2, the following statements are true for every $a \in I$:*

a) *There is some $b \in]a, M[$ such that $\phi_{12}(a, x) > 0$ for every $x \in]a, b[$.*
b) *If there is some $x \in]a, M[$ such that $\phi_{12}(a, x) = 0$, then $\phi_{11}(a, b) < 0$ and $\phi_{22}(a, b) < 0$ for $b = \min\{x > a : \phi_{12}(a, x) = 0\}$.*
c) *If the function $x \mapsto \phi_{12}(a, x)$ is strictly positive in the interval $]a, b[$, then the function $x \mapsto \phi_{12}(x, b)$ is strictly positive in the interval $]a, b[$.*

Proof. Statement *(a)* follows immediately from the fact that

$$\frac{\partial}{\partial x}\phi_{12}(a, x) = \phi_{22}(a, x) \qquad \forall x \in I,$$

and $\phi_{12}(a, a) = 0$, $\phi_{22}(a, a) = 1$.

To prove statement *(b)*, notice that $\phi_{22}(a, b) = \frac{\partial}{\partial x}\phi_{1,2}(a, b) \leq 0$. Since $\det\Phi(a, x) > 0$ for every $x \in I$, $\phi_{12}(a, b) = 0$ implies $\phi_{11}(a, b)\phi_{22}(a, b) > 0$, and the statement follows.

Finally, to prove statement *(c)*, we start by recalling that $\Phi(a, b) = \Phi(x, b)\Phi(a, x)$. Therefore:

$$\phi_{12}(x, b) = \frac{1}{\det\Phi(a, x)}\left(\phi_{12}(a, b)\phi_{11}(a, x) - \phi_{11}(a, b)\phi_{12}(a, x)\right). \tag{26}$$

If $\phi_{12}(a, b) > 0$, this reduces to

$$\phi_{12}(x, b) = \frac{\phi_{12}(a, b)\phi_{12}(a, x)}{\det\Phi(a, x)}\left(\frac{\phi_{11}(a, x)}{\phi_{12}(a, x)} - \frac{\phi_{11}(a, b)}{\phi_{12}(a, b)}\right).$$

A simple computation shows that

$$\frac{\partial}{\partial x}\frac{\phi_{11}(a, x)}{\phi_{12}(a, x)} = -\frac{\det\Phi(a, x)}{\phi_{12}(a, x)^2} < 0.$$

Hence, the function $x \mapsto \frac{\phi_{11}(a,x)}{\phi_{12}(a,x)}$ is strictly decreasing in $]a, b]$ and therefore $\phi_{12}(x, b) > 0$ for every $x \in]a, b[$. If $\phi_{12}(a, b) = 0$, then the equality (26) reduces to

$$\phi_{12}(x, b) = -\frac{\phi_{11}(a, b)\phi_{12}(a, x)}{\det\Phi(a, x)}.$$

By statement *(b)*, $\phi_{11}(a, b) < 0$ and therefore, $\phi_{12}(x, b) > 0$ for every $x \in]a, b[$. □

Lemma 6.3. *Let Assumptions 2.1, 2.2 hold, and suppose that there are some $a, b \in I$ such that $a < b$ and $\phi_{12}(a, b) = 0$. Then, for every $a' \in]m, a[$ there is some $b' \in [a, b[$ such that $\phi_{12}(a', b') = 0$. Similarly, for every $b' \in]b, M[$ there is some $a' \in]a, b]$ such that $\phi_{12}(a', b') = 0$.*

Proof. Fix $a, b \in I$ such that $a < b$ and $\phi_{12}(a, b) = 0$. Without loss of generality, we may assume that $\phi_{12}(a, x) > 0$ for every $x \in]a, b[$ (take a subinterval, if necessary).

Fix $a' < a$. Since $x \geq a$, $\Phi(a', x) = \Phi(a, x)\Phi(a', a)$, we have

$$\phi_{12}(a', x) = \phi_{11}(a, x)\phi_{12}(a', a) + \phi_{12}(a, x)\phi_{22}(a', a) \qquad \forall x \in]a, b[.$$

By statement *(b)* of Lemma 6.2, this must be negative for every x sufficiently close to b if $\phi_{12}(a', a) > 0$. Thus, $\phi_{12}(a', x)$ must have a zero in $[a, b[$.

Now, fix $b' > b$. Since $\Phi(a, b') = \Phi(b, b')\Phi(a, b)$, $\phi_{12}(a, b) = 0$ implies $\phi_{12}(a, b') = \phi_{12}(b, b')\phi_{22}(a, b)$. By statement *(b)* of Lemma 6.2, this must be negative if $\phi_{12}(b, b') > 0$. Hence the function $x \mapsto \phi_{12}(x, b')$ must have a zero in $]a, b]$. $\square$

Lemma 6.4 relates the sign of ϕ_{12} with the sign of solutions of equations of type (8). To prove Proposition 6.1, we need to consider such equations with different functions instead of Π. That is, we consider variants of equation (8) of the type:

$$r(x)v(x) - \alpha(x)v'(x) - \frac{\sigma(x)^2}{2}v''(x) - g(x) = 0, \tag{27}$$

where $g : I \mapsto \mathbb{R}$ is a measurable function such that $\frac{g}{\sigma^2}$ is locally integrable in I with respect to the Lebesgue measure.

Lemma 6.4. *Suppose Assumptions 2.1, 2.2 hold, and let $g : I \mapsto [0, +\infty[$ be a measurable function such that $\frac{g}{\sigma^2}$ is locally integrable, and $\int_a^b \frac{g(z)}{\sigma(z)^2}dz > 0$. Equation (27) admits a non-negative solution in the interval $[a, b] \subset I$ if and only if $\phi_{12}(a, x) > 0$ for every $x \in]a, b]$.*

Proof. The function

$$v(x) = K\phi_{12}(a, x) - \int_a^x \frac{2g(z)}{\sigma(z)^2}\phi_{12}(z, x)dz$$

is a solution of (27). For sufficiently large $K \in]0, +\infty[$, it is non-negative in $[a, b]$, provided ϕ_{12} is strictly positive in $]a, b]$.

Now, suppose that there is some $x_0 \in]a, b]$ such that $\phi_{12}(a, x_0) \leq 0$. Without loss of generality, we may assume that $x_0 = b = \min\{x > a : \phi_{12}(a, x) = 0\}$ (take a subinterval on $[a, b]$, if necessary). Fix v, a solution of (27). By (11),

$$v(b) = v(a)\phi_{11}(a, b) - \int_a^b \frac{2g(z)}{\sigma(z)^2}\phi_{12}(z, b)dz.$$

Lemma 6.2 states that $\phi_{11}(a, b) < 0$ and $\phi_{12}(z, b) > 0$ for every $z \in]a, b[$. Therefore, $v(b) < 0$. $\square$

For any $a \in I$, we define the stopping time

$$\tau_a = \inf\{t \geq 0 : X_t = a\} \wedge \tau_I.$$

It is clear that τ_a in an admissible stopping time, as defined in Section 2.

The following Lemmata 6.5 and 6.6 relate the solutions of equation (27) with the value of a functional of type (1). The results and the arguments in the proofs are similar to several published results (see, e.g. Dayanik and Karatzas [11], Rüschendorf and Urusov [33], Belomestny, Rüschendorf and Urusov [7], Lamberton and Zervos [24], and references therein). However, since similar arguments are used to prove other results below, we outline the argument in the proof of Lemma 6.5.

Lemma 6.5. *Suppose Assumptions 2.1, 2.2 hold, and let $g : I \mapsto [0, +\infty[$ be a measurable function such that $\frac{g}{\sigma^2}$ is locally integrable. Let v be a solution of equation (27), non-negative in a compact interval $[a, b] \subset I$. Then*

$$\mathbb{E}_x \left[\int_0^{\tau_a \wedge \tau_b} e^{-\rho_s} g(X_s) ds \right] \leq v(x) \qquad \forall x \in]a, b[.$$

Proof. Consider the sequence of stopping times

$$\theta_n = \min \left\{ n, \inf \left\{ t \geq 0 : \int_0^t \sigma^2(X_s) ds = n \text{ or } \rho_t = -n \right\} \right\}.$$

Using the Itō-Tanaka formula and the occupation times formula (see for example theorem VI.1.5 and corollary VI.1.6 in Revuz and Yor [32]), we obtain

$$\begin{aligned}
&e^{-\rho_{\tau_a \wedge \tau_b \wedge \theta_n}} v(X_{\tau_a \wedge \tau_b \wedge \theta_n}) = \\
=&v(x) + \int_0^{\tau_a \wedge \tau_b \wedge \theta_n} e^{-\rho_s} \left(-rv + \alpha v' + \frac{\sigma^2}{2} v'' \right) \circ X_s ds + \int_0^{\tau_a \wedge \tau_b \wedge \theta_n} e^{-\rho_s} (\sigma v') \circ X_s dW_s = \\
=&v(x) - \int_0^{\tau_a \wedge \tau_b \wedge \theta_n} e^{-\rho_s} g(X_s) ds + \int_0^{\tau_a \wedge \tau_b \wedge \theta_n} e^{-\rho_s} (\sigma v') \circ X_s dW_s.
\end{aligned}$$

It is easy to check that

$$\mathbb{E}_x \left[\int_0^{\tau_a \wedge \tau_b \wedge \theta_n} \left(e^{-\rho_s} (\sigma v') \circ X_s \right)^2 ds \right] \leq n e^{2n} \max\{ v'(x)^2 : x \in [a, b] \} < \infty.$$

Therefore, $\mathbb{E}_x \left[\int_0^{\tau_a \wedge \tau_b \wedge \theta_n} e^{-\rho_s} (\sigma v') \circ X_s dW_s \right] = 0$ and

$$0 \leq \mathbb{E}_x \left[e^{-\rho_{\tau_a \wedge \tau_b \wedge \theta_n}} v \left(X_{\tau_a \wedge \tau_b \wedge \theta_n} \right) \right] = v(x) - \mathbb{E}_x \left[\int_0^{\tau_a \wedge \tau_b \wedge \theta_n} e^{-\rho_s} g(X_s) ds \right].$$

Making $n \to \infty$, the result follows from the Lebesgue monotone convergence theorem. $\square$

Lemma 6.6. *Suppose Assumptions 2.1, 2.2 hold, and fix a compact interval $[a, b] \subset I$ such that $\phi_{12}(a, x) > 0$ for every $x \in]a, b]$. Let $g : I \mapsto \mathbb{R}$ be a measurable function such that $\frac{g}{\sigma^2}$ is locally integrable with respect to the Lebesgue measure.*

There is one unique solution of equation (27) with boundary conditions $v(a) = v(b) = 0$. If v is such a solution, then

$$\mathbb{E}_x\left[\int_0^{\tau_a\wedge\tau_b} e^{-\rho_s} g(X_s)ds\right] = v(x) \qquad \forall x \in]a,b[.$$

Proof. Existence and uniqueness of v follows directly from equality (11).

Fix $[a,b]$ as above, and let θ_n be the sequence of stopping times introduced in the proof of Lemma 6.5. Thus,

$$\mathbb{E}_x\left[e^{-\rho_{\tau_a\wedge\tau_b\wedge\theta_n}} v(X_{\tau_a\wedge\tau_b\wedge\theta_n})\right] = v(x) - \mathbb{E}_x\left[\int_0^{\tau_a\wedge\tau_b\wedge\theta_n} e^{-\rho_s} g(X_s)ds\right]. \tag{28}$$

For every stopping time $\theta \le \tau_a \wedge \tau_b$, we have

$$0 \le e^{-\rho_\theta} = 1 + \int_0^{\theta} -e^{-\rho_s} r(X_s)ds = 1 + \int_0^{\theta} e^{-\rho_s}\left(r^-(X_s) - r^+(X_s)\right)ds \le 1 + \int_0^{\tau_a\wedge\tau_b} e^{-\rho_s} r^-(X_s)ds.$$

Substituting r^- for g in Lemmata 6.4 and 6.5, we see that $\mathbb{E}_x\left[\int_0^{\tau_a\wedge\tau_b} e^{-\rho_s} r^-(X_s)ds\right] < +\infty$. Since v is bounded in $[a,b]$ and $v(X_{\tau_a\wedge\tau_b}) = 0$, the Lebesgue dominated convergence theorem states that

$$\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\rho_{\tau_a\wedge\tau_b\wedge\theta_n}} v(X_{\tau_a\wedge\tau_b\wedge\theta_n})\right] = 0.$$

Using the Lebesgue monotone convergence theorem on the right-hand side of (28), we obtain the lemma in the case $g \ge 0$. In the general case $g : [a,b] \mapsto \mathbb{R}$, the lemma holds for the positive function $|g|$. Hence, we can apply the Lebesgue dominated convergence theorem to both sides of (28) to finish the proof. □

The result provided in the next lemma is already known and its proof can be found in Karatzas and Shereve [21], Chapter 5.5.C.

Lemma 6.7. *Under Assumption 2.1, for every $m < a < b < M$ and every $x \in]a,b[$:*

$$P_x\{\tau_b < \tau_a\} = \frac{\int_a^x e^{-\int_a^{z_1} \frac{2\alpha}{\sigma^2} dz_2} dz_1}{\int_a^b e^{-\int_a^{z_1} \frac{2\alpha}{\sigma^2} dz_2} dz_1}.$$

In particular, $0 < P_x\{\tau_b < \tau_a\} < 1$ for every $x \in]a,b[$.

The preceding lemmas allow us to obtain Lemma 6.8, from which Proposition 6.1 follows.

Lemma 6.8. *Let Assumptions 2.1, 2.2 hold, and fix $a, b \in I$ such that $a < b$ and $\phi_{12}(a,b) = 0$, then*

$$\mathbb{E}_x\left[\int_0^{\tau_{a'}\wedge\tau_{b'}} e^{-\rho_s} g(X_s)ds\right] = +\infty$$

for every $a' \in]m,a]$, $b' \in [b,M[$, $x \in]a',b'[$, and every measurable function $g \ge 0$, such that $\{x \in [a',b'] : g(x) > 0\}$ has positive Lebesgue measure.

Proof. Fix $[a,b]$ as above. Without loss of generality, we may assume that $\phi_{12}(a,x) > 0$ for every $x \in]a,b[$ (take a subinterval if necessary). Fix $a' \in]m,a]$, $b' \in [b,M[$, and a measurable function $g \geq 0$ such that $\{x \in [a',b'] : g(x) > 0\}$ has positive Lebesgue measure. Due to Lemma 6.3, we may assume that $\{x \in [a,b] : g(x) > 0\}$ has positive Lebesgue measure (shift the interval, if necessary).

For every constant $\varepsilon \in]0, b-a[$, we have $\phi_{12}(a,x) > 0$ for every $x \in]a, b-\varepsilon[$. By equality (11),

$$v_\varepsilon(x) = \frac{\int_a^{b-\varepsilon} \frac{2g(z)}{\sigma(z)^2}\phi_{12}(z,b-\varepsilon)dz}{\phi_{12}(a,b-\varepsilon)}\phi_{12}(a,x) - \int_a^x \frac{2g(z)}{\sigma(z)^2}\phi_{12}(z,x)dz$$

is the unique solution of (27) with boundary conditions $v(a) = v(b-\varepsilon) = 0$. Since ϕ_{12} is continuous on both arguments, it follows that $\phi_{12}(z, b-\varepsilon)$ converges to $\phi_{12}(z,b)$ uniformly with respect to $z \in [a,b]$ when $\varepsilon \to 0^+$. Therefore, it follows from statement (c) in Lemma 6.2 that $\lim\limits_{\varepsilon\to 0^+} v_\varepsilon(x) = +\infty$ for every $x \in]a,b[$. By Lemma 6.6, for every $x \in]a,b[$ and every $\varepsilon \in]0, b-x[$, we have

$$\mathbb{E}_x\left[\int_0^{\tau_{a'}\wedge\tau_{b'}} e^{-\rho_s} g(X_s)ds\right] \geq \mathbb{E}_x\left[\int_0^{\tau_a\wedge\tau_{b-\epsilon}} e^{-\rho_s} g(X_s)ds\right] = v_\varepsilon(x),$$

and therefore, $\mathbb{E}_x\left[\int_0^{\tau_{a'}\wedge\tau_{b'}} e^{-\rho_s} g(X_s)ds\right] = +\infty$ for every $x \in]a,b[$.

Now, fix $c \in]a,b[$ and $x \in]a',b'[\setminus]a,b[$. Assume that $x \in]a',c[$ (the case $x \in]c,b'[$ is analogous), and let $\theta_n = \inf\{t \geq 0 : \rho_t = n\}$. By Lemma 6.7, there is some $n \in \mathbb{N}$ such that $P_x\{\tau_c < \tau_{a'} \wedge \theta_n\} > 0$. Therefore,

$$\begin{aligned}
\mathbb{E}_x\left[\int_0^{\tau_{a'}\wedge\tau_{b'}} e^{-\rho_s} g(X_s)ds\right] \geq & \mathbb{E}_x\left[\int_{\tau_c}^{\tau_{a'}\wedge\tau_{b'}} e^{-\rho_s} g(X_s)ds \chi_{\{\tau_c<\tau_{a'}\}}\right] \\
\geq & \mathbb{E}_x\left[\int_{\tau_c}^{\tau_{a'}\wedge\tau_{b'}} e^{-(\rho_s-\rho_{\tau_c})} g(X_s)ds\ e^{-\rho_{\tau_c}} \chi_{\{\tau_c<\tau_{a'}\wedge\theta_n\}}\right] \\
\geq & \mathbb{E}_c\left[\int_0^{\tau_{a'}\wedge\tau_{b'}} e^{-\rho_s} g(X_s)ds\right] e^{-n} P_x\{\tau_c < \tau_{a'} \wedge \theta_n\} = +\infty. \quad \square
\end{aligned}$$

Concerning the proof of Proposition 6.2, notice that the final argument in the proof of Lemma 6.8 shows that existence of some $x \in I$ such that $\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^+(X_t)dt\right] = \infty$ implies that $\mathbb{E}_x\left[\int_0^{\tau_I} e^{-\rho_t}\Pi^+(X_t)dt\right] = \infty$ for every $x \in I$.

6.2. Proof of Proposition 4.1

The following Lemma is an easy consequence of Proposition 6.1.

Lemma 6.9. *Suppose Assumptions 2.1, 2.2, and 2.3 hold. For any point $x_0 \in I$ such that $v_{x_0,0}(x) < 0$ for some $x \in I$, there is a compact interval $[a,b] \subset I$ satisfying (14) such that $x_0 \in]a,b[$. Conversely, if $[a,b] \subset I$ satisfies (14) and there is some $x \in I$ such that $v^{[a,b]}(x) < 0$, then there is a compact interval $[a',b'] \subset I$ satisfying (14) such that $[a,b] \subset]a',b'[$.*

Proof. Due to Proposition 6.1, equality (12) implies that the mapping $d \mapsto v_{x_0,d}(x_1)$ is strictly increasing for fixed $x_0 < x_1$, and strictly decreasing for fixed $x_1 < x_0$.

Fix $x_0, x_1 \in I$ such that $v_{x_0,0}(x_1) < 0$, with $x_0 < x_1$ (the case $x_1 < x_0$ is analogous). Fix $d > 0$ sufficiently small such that $v_{x_0,d}(x_1) < 0$. Since $d > 0$, there is some $\varepsilon > 0$ such that $v_{x_0,d}(x) > 0$ for every $x \in]x_0, x_0+\varepsilon]$ and $v_{x_0,d}(x) < 0$ for every $x \in [x_0 - \varepsilon, x_0[$. Set $b = \min\{x > x_0 : v_{x_0,d}(x) \leq 0\}$. It is clear that $b \in]x_0, x_1[$. Then, there is some $d_1 < v'_{x_0,d}(b)$, such that $v_{b,d_1}(x_0 - \varepsilon) < 0$. Let $a = \max\{x \leq x_0 : v_{b,d_1}(x) \leq 0\}$. Since $v_{b,d_1}(x) > v_{x_0,d}(x)$ for every $x < b$, it follows that $a \in]x_0 - \varepsilon, x_0[$. Thus, $x_0 \in]a, b[$ and $]a, b[$ satisfies (14).

If there is some $x \in]b, M[$ such that $v^{[a,b]}(x) < 0$, then, we can use the argument above taking $v_{a,d}$ with $d > \left(v^{[a,b]}\right)'(a)$. If there is some $x \in]m, a[$ such that $v^{[a,b]}(x) < 0$, then, we can take $v_{b,d}$ with $d < \left(v^{[a,b]}\right)'(b)$. $\square$

The argument used to prove Lemma 6.9 can be adapted to prove the following Lemma.

Lemma 6.10. *Suppose Assumptions 2.1, 2.2, and 2.3 hold. For any compact intervals* $[a, b]$, $[a', b'] \subset I$ *satisfying condition* (14)*, such that* $a < a' < b < b'$,

$$v^{[a,b']}(x) > \max\left(v^{[a,b]}(x), v^{[a',b']}(x)\right) \qquad \forall x \in]a, b'[.$$

Hence, $[a, b']$ *satisfies* (14).

Proof. Let

$$\hat{d} = \max\left\{d \geq 0 : v_{a,d}(x) = v^{[a',b']}(x) \text{ for some } x \in [a', b']\right\}.$$

Notice that $\hat{d} > \left(v^{[a,b]}\right)'(a)$, and therefore $v_{a,\hat{d}}(x) > v^{[a,b]}(x)$ for every $x > a$.

By continuity, there is some $\hat{x} \in [a', b']$ such that $v_{a,\hat{d}}(\hat{x}) = v^{[a',b']}(\hat{x})$. If $\hat{x} \in]a', b'[$, then the maximality of $\hat{d}$ implies that $v'_{a,\hat{d}}(\hat{x}) = \left(v^{[a',b']}\right)'(\hat{x})$. Thus, by uniqueness of the solution of the ODE (8) with given initial value and derivative, $v_{a,\hat{d}} = v^{[a',b']}$. Since this is a contradiction, we conclude that $\hat{x} = b'$ and $v'_{a,\hat{d}}(b') < \left(v^{[a',b']}\right)'(b')$. Therefore $v^{[a,b']} = v_{a,\hat{d}}$ and $v_{a,\hat{d}}(x) > v^{[a',b']}(x)$ for every $x < b'$. $\square$

Proposition 4.1 follows from the lemmata above.

Lemma 6.10 shows that if $\hat{x}$ lies in some interval satisfying (14), then the union of all intervals containing $\hat{x}$ and satisfying (14) is a maximal interval for (14). The fact that maximal intervals are pairwise disjoint is also an immediate consequence of Lemma 6.10.

Fix $\hat{x} \in \mathcal{L}^+$. Then, $v'_{\hat{x},0}(x) = -\int_{\hat{x}}^{x} \frac{2\Pi(z)}{\sigma(z)^2}\phi_{22}(z)dz < 0$ for every $x > \hat{x}$, sufficiently close to $\hat{x}$. Therefore, $v_{\hat{x},0}(x) < 0$ for every $x > \hat{x}$, sufficiently close to $\hat{x}$, and Lemma 6.9 shows that $\hat{x}$ lies in some interval satisfying (14). Conversely, if $[a, b] \subset I$ and $v^{[a,b]}(x) > 0$ for every $x \in]a, b[$, then the equality (11) implies that $\int_x^b \frac{\Pi(z)}{\sigma(z)^2}\phi_{12}(z, b)dz > 0$ for some $x \in [a, b[$. Due to Proposition 6.1, this implies $]a, b[\cap \mathcal{L}^+ \neq \emptyset$.

If $]a, b[\subset I$ is maximal for (14) then Lemma 6.9 states that $v^{[a,b]}(x) \geq 0$ for every $x \in I$. Conversely, any $[a, b] \subset I$ such that $v^{[a,b]}(x) \geq 0$ for every $x \in I$ must be maximal, since any non-negative $v^{[a',b']}$, with $a' \leq a$ and $b' \geq b$, must coincide with $v^{[a,b]}$ in at least two points and therefore, by Lemma 6.1, it must coincide with $v^{[a,b]}$.

It only remains to prove that if $]a, b[$ is maximal and $a = m$ or $b = M$, then $v^{[a,b]}$ is well defined and non-negative. Let $]a, b[$ be maximal for (14). For any compact intervals $[a_1, b_1]$, $[a_2, b_2]$ satisfying (14), such that $[a_1, b_1] \subset]a_2, b_2[$ and $[a_2, b_2] \subset]a, b[$, Lemma 6.1 implies that $v^{[a_1,b_1]}(x) < v^{[a_2,b_2]}(x)$ for every $x \in]a_1, b_1[$. Hence, for any monotonically increasing sequence of compact intervals $[a_n, b_n] \subset]a, b[$ satisfying (14), such that $]a, b[= \bigcup_{n\in\mathbb{N}} [a_n, b_n]$, the function $v(x) = \lim_{n\to\infty} v^{[a_n,b_n]}(x)$ is well defined, it is strictly positive in the interval $]a, b[$ and does not depend on the particular sequence $[a_n, b_n]$. Further, $v^{[a_n,b_n]}(x)$ and $\left(v^{[a_n,b_n]}\right)'(x)$

converge uniformly on compact intervals. Hence, v must be a solution of Equation (8) and $v(x) \geq 0$ for every $x \in I$.

6.3. Proof of Theorem 4.1

Consider a compact interval $I' = [m', M']$ with $m < m' < M' < M$. We will start by proving a version of Theorem 4.1 for the problem of maximizing (1) over the subset $\mathcal{T}' = \{\tau \in \mathcal{T} : \tau \leq \tau_{m'} \wedge \tau_{M'}\}$.

Theorem 6.1. *Suppose that Assumptions 2.1, 2.2 and 2.3 hold. If $v : [m', M'] \mapsto [0, \infty[$ is a Carathéodory solution of the Hamilton-Jacobi-Bellman equation (7) with boundary conditions $v(m') = v(M') = 0$, then v coincides with the value function*

$$V_{I'}(x) = \sup_{\tau \in \mathcal{T}'} \mathbb{E}_x \left[\int_0^\tau e^{-\rho_t} \Pi(X_s) ds \right] \qquad x \in I', \tag{29}$$

and $\tau = \inf\{t \geq 0 : v(X_t) = 0\} \wedge \tau_{m'} \wedge \tau_{M'}$ is a maximizer of (1) over the set $\mathcal{T}'$.

Proof. Let $v : [m', M'] \mapsto [0, \infty[$ be a Carathéodory solution of (7) such that $v(m') = v(M') = 0$.

Fix $\theta \in \mathcal{T}'$, and consider the sequence

$$\theta_n = \min \left\{ n, \tau_{I'}, \inf \left\{ t \geq 0 : \int_0^t \sigma^2(X_s) ds = n \text{ or } \rho_t = -n \right\} \right\}$$

Notice that, due to Lemma 6.8, Assumption 2.3 implies $\phi_{12}(m', x) > 0$ for every $x \in]m', M']$. Therefore, Lemmata 6.4, 6.5 imply that $\mathbb{E}_x \left[\int_0^{\tau_{m'} \wedge \tau_{M'}} e^{-\rho_s} |\Pi(X_s)| ds \right] < \infty$. Therefore, the argument used to prove Lemma 6.5 yields

$$\mathbb{E}_x \left[e^{-\rho_{\theta \wedge \theta_n}} v(X_{\theta \wedge \theta_n}) \right] = v(x) + \mathbb{E}_x \left[\int_0^{\theta \wedge \theta_n} e^{-\rho_s} \left(-rv + \alpha v' + \frac{\sigma^2}{2} v'' \right) \circ X_s ds \right] =$$

$$= v(x) - \mathbb{E}_x \left[\int_0^{\theta \wedge \theta_n} e^{-\rho_s} \left(rv - \alpha v' - \frac{\sigma^2}{2} v'' - \Pi \right) \circ X_s ds \right] - \mathbb{E}_x \left[\int_0^{\theta \wedge \theta_n} e^{-\rho_s} \Pi(X_s) ds \right]$$

for every $x \in I'$. By assumption, $rv - \alpha v' - \frac{\sigma^2}{2} v'' - \Pi \geq 0$ and $v \geq 0$. Hence,

$$0 \leq \mathbb{E}_x \left[e^{-\rho_{\theta \wedge \theta_n}} v(X_{\tau \wedge \theta_n}) \right] \leq v(x) - \mathbb{E}_x \left[\int_0^{\theta \wedge \theta_n} e^{-\rho_s} \Pi(X_s) ds \right].$$

The Lebesgue monotone convergence theorem states that

$$\lim_{n \to \infty} \mathbb{E}_x \left[\int_0^{\theta \wedge \theta_n} e^{-\rho_s} \Pi^{\pm}(X_s) ds \right] = \mathbb{E}_x \left[\int_0^{\theta} e^{-\rho_s} \Pi^{\pm}(X_s) ds \right],$$

and therefore

$$\mathbb{E}_x\left[\int_0^\theta e^{-\rho_s}\Pi(X_s)ds\right] \leq v(x).$$

Since θ is arbitrary, this proves that $V_{I'} \leq v$.

The random variable $\inf\{t \geq 0 : v(X_t) = 0\}$ is a stopping time, since it is the first hitting time of a closed set by a continuous process adapted to a complete filtration. Fix x such that $v(x) > 0$, and let $a = \max\{y \in [m', x[: v(y) = 0\}$, $b = \min\{y \in]x, M'] : v(y) = 0\}$. Then, v coincides with a solution of equation (8) in the interval $[a, b]$ and, since $\phi_{12}(m', y) > 0$ for every $y \in]m', M']$, Lemma 6.2 shows that $\phi_{12}(a, y) > 0$ for every $y \in]a, b]$. Therefore, Lemma 6.6 states that

$$\mathbb{E}_x\left[\int_0^\tau e^{-\rho_s}\Pi(X_s)ds\right] = v(x),$$

and thus $V_{I'} = v$. □

The next theorem shows that under Assumptions 2.1, 2.2 and 2.3, a function v satisfying the assumptions of Theorem 6.1 exists.

Theorem 6.2. *If Assumptions 2.1, 2.2 and 2.3 hold, then the Hamilton-Jacobi-Bellman equation (7) admits a solution with boundary conditions $v(m') = v(M') = 0$. This solution is given by the right-hand side of (15), with I' instead of I.*

Proof. Let $\{]a_k, b_k[\subset I',\ k = 1, 2, \ldots\}$ be the collection of all maximal intervals for (14), and let $v : I' \mapsto [0, +\infty[$ be the function defined by the right-hand side of (15).

It can be checked that v is continuously differentiable with absolutely continuous first derivative, and $\lim\limits_{x\to m^+} v(x) = \lim\limits_{x\to M^-} v(x) = 0$. For almost every $z \in \bigcup\limits_k]a_k, b_k[$, v satisfies the differential equation (8). By Proposition 4.1, $\mathcal{L}^+ \subset \bigcup\limits_k]a_k, b_k[$. Therefore, for almost every $z \in I' \setminus \bigcup\limits_k]a_k, b_k[$:

$$r(z)v(z) - \alpha(z)v'(z) - \frac{\sigma(z)^2}{2}v''(z) - \Pi(z) = -\Pi(z) \geq 0.$$

Hence, v is a solution of the Hamilton-Jacobi-Bellman equation (7). □

Theorems 6.1 and 6.2 show that Theorem 4.1 holds for any compact interval $I' \subset I$. We now proceed to prove that it holds for I.

Suppose that Assumptions 2.1, 2.2 and 2.3 hold, pick a monotonically increasing sequence of compact intervals $I_n = [a_n, b_n] \subset I$ such that $I = \bigcup\limits_{n\in\mathbb{N}} I_n$, and let

$$V_n(x) = \sup_{\tau\in\mathcal{T}_n} \mathbb{E}_x\left[\int_0^\tau e^{-\rho_t}\Pi(X_t)dt\right],$$

where $\mathcal{T}_n = \{\tau \in \mathcal{T} : \tau \leq \tau_{a_n} \wedge \tau_{b_n}\}$.

Theorems 6.1 and 6.2 state that V_n is the Carathéodory solution of (7) with boundary conditions $v(a_n) = v(b_n) = 0$, and a corresponding optimal stopping time is

$$\tau_n = \inf\{t \geq 0 : V_n(X_t) = 0\} \wedge \tau_{a_n} \wedge \tau_{b_n}.$$

Since $\mathcal{T}_n \subset \mathcal{T}_{n+1} \subset \mathcal{T}$, it is clear that

$$V_n(x) \leq V_{n+1}(x) \leq V(x), \qquad \forall x \in [a_n, b_n],\ n \in \mathbb{N}. \tag{30}$$

For every stopping time $\theta \in \mathcal{T}$ and every $x \in I$, the Lebesgue monotone convergence theorem states that

$$\lim_{n\to\infty} \mathbb{E}_x \left[\int_0^{\theta\wedge\tau_{a_n}\wedge\tau_{b_n}} e^{-\rho t}\Pi^-(X_t)dt \right] = \mathbb{E}_x \left[\int_0^{\theta} e^{-\rho t}\Pi^-(X_t)dt \right],$$

$$\lim_{n\to\infty} \mathbb{E}_x \left[\int_0^{\theta\wedge\tau_{a_n}\wedge\tau_{b_n}} e^{-\rho t}\Pi^+(X_t)dt \right] = \mathbb{E}_x \left[\int_0^{\theta} e^{-\rho t}\Pi^+(X_t)dt \right].$$

Therefore,

$$\mathbb{E}_x \left[\int_0^{\theta} e^{-\rho t}\Pi(X_t)dt \right] = \lim \mathbb{E}_x \left[\int_0^{\theta\wedge\tau_{a_n}\wedge\tau_{b_n}} e^{-\rho t}\Pi(X_t)dt \right] \leq \lim V_n(x).$$

Since this inequality holds for every stopping time $\theta \in \mathcal{T}$ and $\theta \wedge \tau_{a_n} \wedge \tau_{b_n} \in \mathcal{T}_n$, it follows that

$$V(x) = \lim_{n\to\infty} V_n(x) \qquad \forall x \in I.$$

By Definition 3.1 and Theorems 6.1 and 6.2, it follows that the value function V satisfies (15).

From the considerations above and (30), it follows that the sequence τ_n is monotonically increasing and converges to $\tau = \inf\{t \geq 0 : V(X_t) = 0\} \wedge \tau_I$. Therefore, the Lebesgue monotone convergence theorem states that

$$\lim_{n\to\infty} \mathbb{E}_x \left[\int_0^{\tau_n} e^{-\rho t}\Pi^+(X_t)dt \right] = \mathbb{E}_x \left[\int_0^{\tau} e^{-\rho t}\Pi^+(X_t)dt \right],$$

$$\lim_{n\to\infty} \mathbb{E}_x \left[\int_0^{\tau_n} e^{-\rho t}\Pi^-(X_t)dt \right] = \mathbb{E}_x \left[\int_0^{\tau} e^{-\rho t}\Pi^-(X_t)dt \right].$$

Hence,

$$V(x) = \lim_{n\to\infty} V_n(x) = \lim_{n\to\infty} \mathbb{E}_x \left[\int_0^{\tau_n} e^{-\rho t}\Pi(X_t)dt \right] = \mathbb{E}_x \left[\int_0^{\tau} e^{-\rho t}\Pi(X_t)dt \right].$$

That is, τ is an optimal stopping time.

6.4. Proof of Proposition 4.2

Let Assumptions 2.1, 2.2, 2.3 hold.

Fix $a \in I$, $b \in]a, M]$, and suppose that $]a, b[$ is maximal for (14). By Proposition 4.1, $v^{[a,b]} \geq 0$. The proof of Proposition 4.1 shows that $v^{[a,b]}$ is a solution of the differential equation 8, even in the case $b = M$. Hence $v^{[a,b]} = v_{a,0}$ and *(a)* holds. Fix $[a_1, b_1] \subset]a, b[$, a compact interval satisfying (14). Then, there is an interval $]a_2, b_1[$, maximal for (14) when we consider the interval $]m, b_1[$ instead of I. By Proposition 4.1, $v^{[a_2,b_1]}$ must be non-negative in $]m, b_1[$. Hence, $v^{[a_2,b_1]} = v_{a_2,0}$. By the considerations preceding Theorem 6.1, $v_{a_2,0}(b_1) = 0$. Since $v_{a_2,0}(x) > 0$ for every $x > a_2$ sufficiently close to a_2, it follows that there is some $a_3 \in \mathcal{L}^-$ arbitrarily close to a_2. Thus, *(b)* also holds.

Now, fix $a \in I$, $b \in]a, M]$, and suppose that *(a)* and *(b)* hold. Let a_n be a sequence as in *(b)*, and let $b_n = \inf\{x > a_n : v_{a_n,0}(x) \le 0\}$. Since $]a, b[= \bigcup_{n\in\mathbb{N}}]a_n, b_n[$, Lemma 6.10 guarantees that $]a, b[$ satisfies (14). Due to Lemma 6.1, non-negativity of $v_{a,0}$ implies that $]a, b[$ is maximal for (14).

The proof for the case $b \in I$, $a \in [m, b[$ is analogous.

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