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Jackson-Favard type problems for the weight $\exp(-|x|)$ on the real line

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ABSTRACT

It is known that the Jackson-Favard inequality on polynomial approximation with respect to the weight $\exp(-|x|)$ fails to hold [6]. We show that this phenomenon holds for higher derivatives as well, while a modified version suggested by Lubinsky can be established.

1. Introduction

In the theory of weighted polynomial approximation on the real line, the weight $w_\alpha(x) := \exp(-|x|^\alpha)$, $\alpha \ge 1$, plays a fundamental role. Let $C_\alpha(\mathbf{R})$ denote the set of continuous functions $f(x)$ on $\mathbf{R}$ such that

$$\lim_{|x|\to\infty} w_\alpha(x) f(x) = 0, \qquad \alpha \ge 1. \tag{1}$$

The rate of approximation is measured by the weighted modulus of continuity. It is formed as the sum of two quantities: the main part and the tail part. Let $f(x) \in C_\alpha(\mathbf{R})$, and for $0 < h \le 1$, let

$$\Delta_h^r f(x) := \sum_{i=0}^{r} \binom{r}{i} (-1)^i f(x + (r/2 - i)h), \quad r = 1, 2, \dots .$$

The r-th order weighted modulus of smoothness for the weight $w_\alpha(x)$ is defined as

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$$\begin{aligned}\omega_r(f,w_\alpha,t) := &\sup_{0<h\le t}\|w_\alpha(x)\Delta_h^r f(x)\|_{\{|x|\le h^{-1/(\alpha-1)}\}}\\ &+\inf_{p\in\Pi_{r-1}}\|w_\alpha(f-p)\|_{\{|x|\ge t^{-1/(\alpha-1)}\}}, \qquad \alpha>1\,,\end{aligned} \tag{2}$$

where Π_{r-1} is the set of polynomials of degree at most $r-1$ (see Ditzian-Totik [1], Ch. 11, Definition 11.2, where a more general situation is handled, for L_p, $1\le p\le\infty$). Here and in what follows, $\|\cdot\|_I$ or $\|\cdot\|$ means the supremum norm over a set I or $\mathbf{R}$, respectively. First we prove a basic property of this modulus.

Lemma 1. *If $f(x)\in C_\alpha(\mathbf{R})$, $\alpha>1$, then*

$$\lim_{t\to 0+}\omega_r(f,w_\alpha,t)=0\,.$$

Proof. Consider the K-functional

$$K_r(f,t^r)_{w_\alpha}=\inf_{g^{(r-1)}\in\mathrm{A.C.}}[\|w_\alpha(f-g)\|+t^r\|w_\alpha g^{(r)}\|]$$

of an $f\in C_\alpha(\mathbf{R})$ (cf. [1], (11.1.1)). It is known that this K-functional is equivalent to the modulus (2) (see [1], Theorem 11.2.3). Thus it is sufficient to prove that

$$\lim_{t\to 0+}K_r(f,t^r)_{w_\alpha}=0\,.$$

Given an arbitrary $\varepsilon>0$, we choose a polynomial $p=p(x,\varepsilon)$ such that $\|w_\alpha(f-p)\|<\varepsilon$. This is possible since according to the Akhiezer-Babenko condition, polynomials are dense in $C_\alpha(\mathbf{R})$. Next, choose $t>0$ so small that $t^r\|w_\alpha p^{(r)}\|<\varepsilon$. Thus $K_r(f,t^r)_{w_\alpha}\le 2\varepsilon$ for such t's, which proves the statement. $\square$

Working with such a modulus, one can state direct and converse approximation results, Bernstein type inequalities, etc. Among others, the following Jackson type theorem holds (stated also for L_p norms). Let

$$E_n(f,w_\alpha):=\inf_{p\in\Pi_n}\|w_\alpha(f-p)\|, \qquad \alpha\ge 1\,,$$

be the best weighted polynomial approximation for continuous functions satisfying (1).

Theorem A. (Lubinsky [7], (3.12)). *Let $r\ge 1$. If $f(x)\in C_\alpha(\mathbf{R})$, then*

$$E_n(f,w_\alpha)\le c_r\omega_r\left(f,w_\alpha,\frac{1}{n^{1-1/\alpha}}\right), \qquad \alpha>1\,.$$

Of course (2) makes no sense when $\alpha=1$, i.e. when

$$w(x):=w_1(x)=e^{-|x|}\,.$$

In this borderline case, it was Freud, Giroux and Rahman [3] who first established a polynomial approximation result in L^1 metric: they proved that if $f(x)\in L^1(\mathbf{R})$, then

$$\inf_{\deg p\le\Pi_n}\int\limits_{\mathbf{R}}w(x)|f(x)-p(x)|\,dx\le C\left[\omega\left(f,\frac{1}{\log n}\right)+\int\limits_{|x|\ge\sqrt{n}}w(x)|f(x)|\,dx\right]\,,$$

where

$$\omega(f,t) = \sup_{|h|\le t} \int_{\mathbf{R}} \Delta_h^1(fw)\,dx + t\int_{\mathbf{R}} w(x)|f(x)|\,dx\,.$$

In the supremum norm, the following definition works:

$$\begin{aligned}\omega_r(f,w,t) := &\sup_{0<h\le t} \|w(x)\Delta_h^r f(x)\|_{\{|x|\le e^{\varepsilon/t}\}}\\ &+ \inf_{p\in\Pi_{r-1}} \|w(x)(f(x)-p)\|_{\{|x|\ge e^{\varepsilon/t}-1\}}, \qquad 0<\varepsilon<1\end{aligned} \tag{3}$$

(see e.g. [7], (5.2), where again the definition is more general as mentioned above). Just like in case $\alpha > 1$, we prove the basic property of this modulus.

Lemma 2. *If $f(x) \in C_1(\mathbf{R})$, then*

$$\lim_{t\to 0+} \omega_r(f,w,t) = 0\,.$$

Proof. We cannot repeat the proof of Lemma 1, since in case $\alpha = 1$ we do not know the K-functional. The tail part in (3), even with $p(x) \equiv 0$, decreases to zero when $t \to 0+$, by (3). As for the main part, we get, even for $x \in \mathbf{R}$,

$$\begin{aligned} w(x)|\Delta_h^r f(x)| =& w(x)\left|\sum_{i=0}^{r}\binom{r}{i}(-1)^i[f(x+(r/2-i)h)-f(x)]\right|\\ \le& 2^r e^{-|x|} \max_{|y-x|\le rh/2} |f(y)-f(x)|\\ \le& 2^r \max_{|y-x|\le rh/2}\left[|e^{-|y|}f(y)-e^{-|x|}f(x)| + |e^{|y|-|x|}-1|e^{-|y|}|f(y)|\right]\\ \le& 2^r[\omega^\star(wf, rh/2) + rh\|wf\|]\,,\end{aligned}$$

where ω^* is the ordinary modulus of continuity of the function $g(x) := w(x)f(x)$ on $\mathbf{R}$. Thus we have to show that $\lim_{t\to 0+}\omega^\star(g,t) = 0$. Since by assumption $\lim_{|x|\to\infty} g(x) = 0$, the function $G(x) := g(\tan(x/2))$ is continuous and 2π periodic. We obtain

$$\begin{aligned}|g(x+h/2)-g(x-h/2)| =& |G(2\arctan(x+h/2)) - G(2\arctan(x-h/2))|\\ \le& \omega^\star(G, 2(\arctan(x+h/2)-\arctan(x-h/2)))\\ \le& \omega^\star(G,2h)\end{aligned}$$

which tends to zero as $h \to 0+$. $\square$

In (2) the presence of the infimum (called “tail part”) is necessary in order to achieve some results for this modulus, since in the first term (called “main part”) the norm cannot be extended to the whole real line. However this is not so obvious at all in case of (3); this will be clarified in Section 2. Lemma 1 ensures that the next theorem is a reasonable estimate for the best approximation.

Theorem B. (Lubinsky [5], Theorem 1). *Let $f(x) \in C_1(\mathbf{R})$. Then*

$$E_n(f,w) \le c_r\omega_r\left(f,w,\frac{1}{\log n}\right)$$

where $c_r > 0$ does not depend on f and n.

2. Jackson-Favard type inequalities

For differentiable functions the so-called Jackson-Favard type inequalities provide estimates for the best (weighted) approximation. Let $f(x)$ be such that $f^{(r-1)}(x)$, $r \geq 1$, is absolutely continuous and $\|w_\alpha f^{(r)}\| < \infty$. Then we have

$$E_n(f, w_\alpha) \leq \frac{c_{r,\alpha}}{n^{r(1-1/\alpha)}} \|w_\alpha f^{(r)}\| < \infty, \qquad \alpha > 1 \tag{4}$$

(see Freud [2] for $\alpha \geq 2$, and Levin-Lubinsky [4] for $1 < \alpha < 2$). Obviously, the condition $\|w_\alpha f^{(r)}\| < \infty$ here implies $\lim_{|x|\to\infty} w_\alpha(x) f(x) = 0$, since otherwise no polynomial approximation would be possible for such functions. We mention that the latter implication can also be seen directly, namely by the l'Hospital rule we have

$$\begin{aligned}
\lim_{|x|\to\infty} w_\alpha(x)|f^{(r-1)}(x)| &= \lim_{|x|\to\infty} \frac{\left|\int_0^x f^{(r)}(t)\,dt + f^{(r-1)}(0)\right|}{\exp|x|^\alpha} \\
&\leq \lim_{|x|\to\infty} \frac{\int_0^x |f^{(r)}(t)|\,dt}{\exp|x|^\alpha} \\
&= \lim_{|x|\to\infty} \frac{w_\alpha(x)|f^{(r)}(x)|}{\alpha|x|^{\alpha-1}} \\
&\leq \frac{\|w_\alpha f^{(r)}\|}{\alpha} \lim_{|x|\to\infty} \frac{1}{|x|^{\alpha-1}} \\
&= 0, \qquad \alpha > 1\,.
\end{aligned}$$

Hence $\|w_\alpha f^{(r-1)}\| < \infty$, and we can repeat the above consideration with $r-1$ instead of r, etc., finally arriving at the relation $\lim_{|x|\to\infty} w_\alpha(x) f(x) = 0$. Note that this property does not hold for $\alpha = 1$: take $f(x) = e^x$, then $\|wf^{(r)}\| = \lim_{x\to\infty} w(x)f(x) = 1$, for any $r \geq 1$.

Lubinsky also proved that in case $\alpha = 1$ no similar result can hold for $r = 1$.

Theorem C. ([6], Corollary 1.3). *There does not exist a sequence $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that for all $f(x)$ absolutely continuous with $\|wf'\| < \infty$ we would have*

$$E_n(f, w) \leq \varepsilon_n \|wf'\|, \qquad n = 1, 2, \dots .$$

We now generalize this statement for $r > 1$. We will see that, while positive results like (4) are easily obtained from the statement for $r = 1$, the other direction is not so straightforward.

Theorem 1. *Let $r \geq 1$. Then there does not exist a sequence $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that for all $f(x)$ with absolutely continuous $f^{(r-1)}(x)$ and $\|wf^{(r)}\| < \infty$, we would have*

$$E_n(f, w) \leq \varepsilon_n^r \|wf^{(r)}\|, \qquad n = 1, 2, \dots . \tag{5}$$

Proof. Since the statement is true for $r = 1$, we may assume that $r \geq 2$. Suppose indirectly that the statement is not true for an $r \geq 2$, i.e. there exists a sequence $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that for all $f(x)$ with absolutely continuous $f^{(r-1)}(x)$ and $\|wf^{(r)}\| < \infty$ we have (5). We will deduce a contradiction from this inequality. Let $g(x)$ be an arbitrary absolutely continuous function such that $\|wg'\| < \infty$, and consider the Steklov transform

$$g_{n,r}(x) := \frac{1}{2\varepsilon_n} \int_{x-\varepsilon_n}^{x+\varepsilon_n} g_{n,r-1}(y)\,dy, \quad g_{n,1}(x) := g(x) \tag{6}$$

of $g(x)$. By induction it is readily seen that

$$g_{n,r}(x) = \frac{1}{(2\varepsilon_n)^{r-1}} \int_{-\varepsilon_n}^{\varepsilon_n} \cdots \int_{-\varepsilon_n}^{\varepsilon_n} g(x + t_1 + \cdots + t_{r-1})\,dt_1 \dots dt_{r-1}\,. \tag{7}$$

Hence

$$\begin{aligned} &w(x)|g_{n,r}(x) - g(x)| \\ &\le \frac{w(x)}{(2\varepsilon_n)^{r-1}} \left| \int_{-\varepsilon_n}^{\varepsilon_n} \cdots \int_{-\varepsilon_n}^{\varepsilon_n} \left(\int_x^{x+t_1+\cdots+t_{r-1}} g'(t)\,dt \right) dt_1 \dots dt_{r-1} \right| \\ &\le \frac{e^{(r-1)\varepsilon_n}}{(2\varepsilon_n)^{r-1}} \int_{-\varepsilon_n}^{\varepsilon_n} \cdots \int_{-\varepsilon_n}^{\varepsilon_n} \left| \int_x^{x+t_1+\cdots+t_{r-1}} w(t)|g'(t)|\,dt \right| dt_1 \dots dt_{r-1} \\ &\le \frac{(r-1)e^{(r-1)\varepsilon_n}\varepsilon_n}{2^{r-1}} \|wg'\|, \qquad x \in \mathbf{R}\,. \end{aligned} \tag{8}$$

On the other hand, we obtain from (6)

$$g_{n,r}^{(r)}(x) = \frac{g_{n,r-1}^{(r-1)}(x+\varepsilon_n) - g_{n,r-1}^{(r-1)}(x-\varepsilon_n)}{2\varepsilon_n}$$

whence

$$\|wg_{n,r}^{(r)}\| \le \frac{e^{\varepsilon_n}}{\varepsilon_n} \|wg_{n,r-1}^{(r-1)}\| \le \cdots \le \frac{e^{(r-1)\varepsilon_n}}{\varepsilon_n^{r-1}} \|wg'\| < \infty\,. \tag{9}$$

Now we wish to apply (5) with $g_{n,r}(x)$ in place of $f(x)$. For this purpose we have to show that $g_{n,r}(x) \in C_1^r(\mathbf{R})$. We obtain, using (7),

$$\begin{aligned} &w(x)|g_{n,r}(x)| \\ &\le \frac{e^{(r-1)\varepsilon_n}}{(2\varepsilon_n)^{r-1}} \int_{-\varepsilon_n}^{\varepsilon_n} \cdots \int_{-\varepsilon_n}^{\varepsilon_n} e^{-|x+t_1+\cdots+t_{r-1}|} |g(x + t_1 + \cdots + t_{r-1})|\,dt_1 \dots dt_{r-1} \\ &\le e^{(r-1)\varepsilon_n} \max_{|y| \ge |x| - (r-1)\varepsilon_n} (w(y)|g(y)|) \to 0 \quad \text{as} \quad |x| \to \infty, \end{aligned}$$

since $g(x) \in C_1^1(\mathbf{R})$. Thus by (5) we get

$$E_n(g_{n,r}, w) \le \varepsilon_n^r \|wg_{n,r}^{(r)}\| \le e^{(r-1)\varepsilon_n} \varepsilon_n \|wg'\|\,.$$

This together with (8) yields

$$E_n(g, w) \le \|w(g - g_{n,r})\| + E_n(g_{n,r}, w) \le c_r \varepsilon_n \|wg'\|\,,$$

which means that the Favard inequality holds in case $r = 1$, which is impossible. This contradiction proves the theorem. □

3. A modified Jackson-Favard inequality

In [6], Remark (d), Lubinsky mentions the possibility of a slightly modified Jackson-Favard inequality in the form

$$E_n(f,w) \le \eta_n \|wf'\| + \|wf\|_{\{|x|\ge\xi_n\}}$$

where $\{\eta_n\} \searrow 0$ and $\{\xi_n\} \nearrow \infty$ are two sequences independent of f. In fact, such an estimate can be given for higher derivatives, even with a restricted norm for the Favard part.

Theorem 2. *Let $r \ge 1$. Then for all $f(x) \in C_1(\mathbf{R})$ such that $f^{(r-1)}(x)$ is absolutely continuous we have*

$$E_n(f,w) \le c_r \left(\frac{\|wf^{(r)}\|_{\{|x|\le n^\varepsilon+r\}}}{\log^r n} + \|wf\|_{\{|x|\ge n^\varepsilon-1\}} \right),$$

where $0 < \varepsilon < 1$ is arbitrary fixed, and $c_r > 0$ depends only on r.

Remark. Notice that we did not require the boundedness of $\|wf^{(r)}\|$, as long as we have $\|wf^{(r)}\|_{\{|x|\le n^\varepsilon+r\}} = o(\log^r n)$ as $n \to \infty$.

The following corollary shows that the tail part in the estimate of Theorem 2 may be larger than the main part.

Corollary. *There exists an $f(x) \in C_1^r(\mathbf{R})$ such that*

$$\limsup_{n\to\infty} \frac{\|wf\|_{\{|x|\ge n^\varepsilon-1\}} \log^r n}{\|wf^{(r)}\|_{\{|x|\le n^\varepsilon+r\}}} = \infty\,.$$

Namely, otherwise we would have for all such $f(x)$

$$E_n(f,w) \le c_r \frac{\|wf^{(r)}\|_{\{|x|\le n^\varepsilon+r\}}}{\log^r n}$$

by Theorem 2, which contradicts Theorem 1 with $\varepsilon_n = 1/\log^r n$.

Proof of Theorem 2. This is an easy consequence of the result of Lubinsky (see Theorem B above). We have to estimate the first part of the modulus of smoothness (3). We have

$$\begin{aligned}
\Delta_h^r f(x) =& \Delta_h^{r-1} f(x+h/2) - \Delta_h^{r-1} f(x-h/2)\\
=& h^{r-1}(f^{(r-1)}(u) - f^{(r-1)}(v))\\
=& h^{r-1}\int_v^u f^{(r)}(t)\,dt, \qquad x - rh/2 < u,\ v < x + rh/2\,.
\end{aligned}$$

Hence

$$\begin{aligned}
w(x)|\Delta_h^r f(x)| \le& e^{rh/2} h^{r-1} \left| \int_v^u w(t)|f^{(r)}(t)|\,dt \right|\\
\le& r e^{rh/2} h^r \|wf^{(r)}\|_{\{|x|\le n^\varepsilon+rh/2\}}, \qquad |x| \le n^\varepsilon\,.
\end{aligned}$$

Thus (3) yields with $0 < h \le t = 1/\log n$,

$$\omega_r(f, w, 1/\log n) \le c_r \frac{\|wf^{(r)}\|_{\{|x| \le n^\varepsilon + r\}}}{\log^r n} + \|wf\|_{\{|x| \ge n^\varepsilon - 1\}}, \qquad n \ge 2\,,$$

which proves the theorem. □

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