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On convergence of the Berezin transforms

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ABSTRACT

We prove approximation results about sequences of Berezin transforms of finite sums of finite product of Toeplitz operators (and bounded linear maps, in general) in the spirit of Ramadanov and Skwarczyński Theorems that are about convergence of Bergman kernels.

Let Ω be a domain in $\mathbb{C}^n$ and $A^2(\Omega)$ denote the Bergman space, the set of square integrable holomorphic functions, of Ω. Since the Bergman space $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, there exists a bounded orthogonal projection P_Ω from $L^2(\Omega)$ onto $A^2(\Omega)$. This is called the Bergman projection for Ω. We denote the Bergman kernel of Ω by K^Ω. The Berezin transform $B_\Omega T$ of a bounded linear operator T on $A^2(\Omega)$ is defined as

$$B_\Omega T(z) = \langle T k_z^\Omega, k_z^\Omega \rangle,$$

where $k_z^\Omega(\xi) = K^\Omega(\xi, z)/\sqrt{K^\Omega(z,z)}$ is the normalized Bergman kernel of Ω and $\langle ., . \rangle$ denotes the inner product on $A^2(\Omega)$.

Berezin transform is an important notion in operator theory. For instance, it is used to characterize compactness of operators in the Toeplitz algebra on the unit disc and the unit ball (see [1,16]) and in a subalgebra on more general domains in $\mathbb{C}^n$ (see [4,5]). Berezin transform is also an important tool in the characterization of compactness of the Hankel operators in [2].

There are different notions for convergence of operators on $A^2(\Omega)$. For instance, one can ask if a sequence of bounded operators defined on the same Bergman space converges to a bounded operator in the operator norm or in the weak sense. Now assume that, for each j, T_j is a bounded operator on $A^2(\Omega_j)$ and $\Omega_j \subset \Omega$

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(or $\Omega \subset \Omega_j$). Since the operators T_js are defined on different spaces it does not make sense to talk about convergence of $\{T_j\}$ in norm or weakly. However, we can compare Berezin transforms. That is, we can ask if $\{B_{\Omega_j} T_j\}$ converges to $B_\Omega T$ pointwise, locally uniformly, etc. This notion generalizes the weak convergence of operators because $B_\Omega T_j \to B_\Omega T$ pointwise on Ω whenever T_js are defined on $A^2(\Omega)$ and $T_j \to T$ weakly.

Let $\{\Omega_j\}$ be an increasing sequence of domains whose union is Ω. Ramadanov showed that (see [12,13]) the Bergman kernels $\{K^{\Omega_j}\}$ converge to K^Ω uniformly on compact subsets of $\Omega \times \Omega$. In this paper we prove results in the spirit of Ramadanov's result for Berezin transforms of bounded operators on the Bergman space.

The plan of the paper is as follows: In the next section we will state our main results. The proofs will be presented in the following section.

1. Main results

To state our results we need to define the restriction operator. Let $U \subset \Omega$ be domain in $\mathbb{C}^n$ and $R_U^\Omega : A^2(\Omega) \to A^2(U)$ denote the restriction operator. That is, $R_U^\Omega f = f|_U$. Then the adjoint $R_U^{\Omega *} : A^2(U) \to A^2(\Omega)$ of R_U^Ω is a bounded linear map and one can show that (see, for example, [3])

$$R_U^{\Omega *} f(z) = \int_U K^\Omega(z,w) f(w) dV(w),$$

where dV is the Lebesgue measure in $\mathbb{C}^n$. We note that if $\overline{U} \subset \Omega$, then Montel's Theorem implies that R_U^Ω is compact. Also $R_U^* T R_U$ is a bounded linear operator on $A^2(\Omega)$ whenever T is a bounded linear map on $A^2(U)$.

Throughout this paper Ef denotes the extension of f onto $\mathbb{C}^n$ trivially by zero and R_U will denote R_U^Ω when the domain Ω is clear from the context. Then the formula for $R_U^{\Omega *}$ above is $R_U^{\Omega *} = P_\Omega E$.

For $z,\, w \in \Omega$, let $K_z^\Omega(w) = K^\Omega(w,z)$. Notice that the normalized Bergman kernel k_z^Ω is well- defined whenever $K^\Omega(z,z) \neq 0$. In [8], Engliš observes that there are unbounded domains in $\mathbb{C}^n$ for which the zero set $\mathcal{Z}$ of the Bergman kernel on the diagonal $K^\Omega(z,z)$ is not empty. Namely, we denote

$$\mathcal{Z} = \left\{ z \in \Omega : K^\Omega(z,z) = 0 \right\}.$$

Definition 1. A domain Ω in $\mathbb{C}^n$ is called a non-trivial Bergman domain if $A^2(\Omega) \neq \{0\}$.

We note that Ω is a non-trivial Bergman domain if and only if $\mathcal{Z} \neq \Omega$. If Ω is bounded, then $\mathcal{Z}$ is empty because the constant functions belong to $A^2(\Omega)$ and $K^\Omega(z,z) \geq 1/\|1\|^2 > 0$ for all $z \in \Omega$. Therefore, bounded domains are non-trivial Bergman domains as well. The set $\mathcal{Z}$, if not empty and not equal to Ω, is a real-analytic variety in Ω with zero Lebesgue measure and it is a relatively closed subset of Ω. The normalized Bergman kernel k_z^Ω is a well defined function in $A^2(\Omega)$ for $z \in \Omega \setminus \mathcal{Z}$. In this paper we will always assume that Ω is a non-trivial Bergman domain.

In the example given in [8], there exists a bounded function ϕ on an unbounded pseudoconvex complete Reinhardt domain Ω such that the Berezin transform $B_\Omega T_\phi$ of the (bounded) Toeplitz operator on Ω has a singularity at a point in $\mathcal{Z}$. However, the map $z \mapsto k_z^\Omega$ is continuous from $\Omega \backslash \mathcal{Z}$ to $L^2(\Omega)$ since

$$\left\| k_z^\Omega - k_w^\Omega \right\|_{L^2(\Omega)}^2 = 2 - 2\mathrm{Re} \left\langle k_z^\Omega, k_w^\Omega \right\rangle = 2 - 2\mathrm{Re} \frac{K^\Omega(w,z)}{\sqrt{K^\Omega(z,z)}\sqrt{K^\Omega(w,w)}} \tag{1}$$

and both $K^\Omega(w,z)$ and $K^\Omega(w,w)$ converge to $K^\Omega(z,z)$ as w converges to z in $\Omega \backslash \mathcal{Z}$. Hence, the Berezin transform $B_\Omega T$ of a bounded operator T on $A^2(\Omega)$ is always a well-defined, bounded and continuous function, on $\Omega \backslash \mathcal{Z}$. This can be seen from the inequality $|B_\Omega T(z)| = |\langle T k_z^\Omega, k_z^\Omega \rangle| \leq \|T\|$ and

$$
\begin{aligned}
|B_\Omega T(z) - B_\Omega T(w)| &\le \left|\langle Tk_z, k_z^\Omega - k_w^\Omega\rangle\right| + \left|\langle T(k_z^\Omega - k_w^\Omega), k_w^\Omega\rangle\right| \\
&\le \|T\| \left\|k_z^\Omega - k_w^\Omega\right\|_{L^2(\Omega)} + \left\|T(k_z^\Omega - k_w^\Omega)\right\|_{L^2(\Omega)}
\end{aligned}
$$

for every $z, w \in \Omega\backslash\mathcal{Z}$.

Our first two results below can be seen as analogues of Ramadanov's and Skwarczyński's Theorems.

Theorem 1. *Let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega_j \subset \Omega_{j+1}$ for all j and $\Omega = \cup_{j=1}^\infty \Omega_j$ be a non-trivial Bergman domain. Let T be a bounded linear map on $A^2(\Omega)$. Then $B_{\Omega_j} R_{\Omega_j} T R_{\Omega_j}^* \to B_\Omega T$ uniformly on compact subsets of $\Omega\backslash\mathcal{Z}$ as $j \to \infty$. Furthermore, if Ω is bounded, then $EB_{\Omega_j} R_{\Omega_j} T R_{\Omega_j}^* \to B_\Omega T$ in $L^p(\Omega)$ as $j \to \infty$ for all $0 < p < \infty$.*

Theorem 2. *Let Ω be a non-trivial Bergman domain and $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ for all j. Assume $K^{\Omega_j}(z,z) \to K^\Omega(z,z)$ as $j \to \infty$ for every $z \in \Omega$. Let T be a bounded linear map on $A^2(\Omega)$. Then $B_{\Omega_j}(R_\Omega^{\Omega_j})^* T R_\Omega^{\Omega_j} \to B_\Omega T$ uniformly on compact subsets of $\Omega\backslash\mathcal{Z}$ as $j \to \infty$. Furthermore, if Ω is bounded, then $B_{\Omega_j}(R_\Omega^{\Omega_j})^* T R_\Omega^{\Omega_j} \to B_\Omega T$ in $L^p(\Omega)$ as $j \to \infty$ for all $0 < p < \infty$.*

The next result describes the convergence of the Berezin transforms when the symbols of Toeplitz operators are restricted onto the subdomains. To clarify the notation below, $\phi|_U$ denotes the restriction of ϕ onto $U, R_U\phi$.

Theorem 3. *Let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega_j \subset \Omega_{j+1}$ for all j and $\Omega = \cup_{j=1}^\infty \Omega_j$ be a non-trivial Bergman domain. Assume that $T = \sum_{m=1}^l T_{\phi_{m,1}} \cdots T_{\phi_{m,k_m}}$ is a finite sum of finite products of Toeplitz operators with bounded symbols on Ω and $T^{\Omega_j} = \sum_{m=1}^l T_{\phi_{m,1}|_{\Omega_j}} \cdots T_{\phi_{m,k_m}|_{\Omega_j}}$ for each j. Then $B_{\Omega_j} T^{\Omega_j} \to B_\Omega T$ uniformly on compact subsets of $\Omega\backslash\mathcal{Z}$ as $j \to \infty$. Furthermore, if Ω is bounded, then $EB_{\Omega_j} T^{\Omega_j} \to B_\Omega T$ in $L^p(\Omega)$ as $j \to \infty$ for all $0 < p < \infty$.*

Remark 1. We note that the T^{Ω_j} in the theorem above depends on the symbols and hence representation of T. However, representation of products of Toeplitz operators is not unique. For instance, Çelik and Zeytuncu in [6] showed that there exists a Reinhardt domain Ω in $\mathbb{C}^2$ such that there exists non-trivial nilpotent Toeplitz operators on $A^2(\Omega)$. Hence the zero operator has multiple representations. However, since the Berezin transform of T is independent of its representation, the Berezin transforms of T^{Ω_j} converge to the same limit for any representation of T.

For a function $\phi \in L^q(\Omega)$, assuming the Toeplitz operator T_ϕ is bounded on $A^2(\Omega)$, we define the Berezin transform $B_\Omega\phi$ of ϕ as $B_\Omega\phi(z) = B_\Omega T_\phi(z)$ for $z \in \Omega$. Hence

$$
B_\Omega\phi(z) = \langle T_\phi k_z^\Omega, k_z^\Omega\rangle = \langle P_\Omega \phi k_z^\Omega, k_z^\Omega\rangle = \langle \phi k_z^\Omega, k_z^\Omega\rangle = \int_\Omega \phi(w)|k_z^\Omega(w)|^2 dV(w).
$$

As a consequence of Theorem 3 and Dini's Theorem we have the following corollary.

Corollary 1. *Let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega_j \subset \Omega_{j+1}$ for all j and $\Omega = \cup_{j=1}^\infty \Omega_j$ be a non-trivial Bergman domain. Assume that $\phi \in L^q(\Omega)$ for some $0 < q < \infty$ so that T_ϕ is bounded on $A^2(\Omega)$. Then there exists a subsequence $\{j_k\}$ and functions $\phi_k \in L^\infty(\Omega_{j_k})$ such that $B_{\Omega_{j_k}}\phi_k \to B_\Omega\phi$ uniformly on compact subsets of $\Omega\backslash\mathcal{Z}$. If Ω is bounded, then $EB_{\Omega_{j_k}}\phi_k \to B_\Omega\phi$ in $L^p(\Omega)$ as $k \to \infty$ for all $0 < p < \infty$.*

We note that, as Proposition 2 below shows, ϕ_k in the corollary above might have to be different from $R_{\Omega_k}\phi$.

Remark 2. If the domain Ω is not bounded, then the Berezin transform $B_\Omega T_\phi$ of the Toeplitz operator of a bounded symbol ϕ does not have to be in $L^p(\Omega)$. For instance, let $\phi(z) = \mathrm{Re}(z)$ and $\Omega = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$. We note that $K^\Omega(z, z) \neq 0$ for any $z \in \Omega$ as $(z + 1)^{-1}$ is square integrable on Ω. Since ϕ is bounded and harmonic, we conclude that $B_\Omega T_\phi = \phi$ which is not in $L^p(\Omega)$ for any $0 < p < \infty$.

In the following proposition we compute the asymptotics of the Berezin transform of $\log|z|$ on annuli that converge to the punctured disc. Also it shows that the first conclusion in Theorem 3 is not true if we drop the assumption that the symbol is bounded. The function $\log|z| \in L^p(\mathbb{D} \setminus \{0\})$ for all $0 < p < \infty$ and, Lemma 6 implies that,

$$B_{\mathbb{D}\setminus\{0\}} \log|z| = B_{\mathbb{D}} \log|z| = \frac{1}{2}(|z|^2 - 1).$$

Proposition 1. *Let $A_r = \{z \in \mathbb{C} : r < |z| < 1\}$ and $\phi(z) = \log|z|$. Then*

$$B_{A_r}\phi(z) \to \frac{|z|^2}{4} - \frac{1}{4|z|^2}$$

uniformly on compact subsets of $\mathbb{D} \setminus \{0\}$ as $r \to 0^+$.

The following proposition shows that the last statement in Theorem 3 is not true in general for operators in the Toeplitz algebra. One can argue as follows. Let $\phi(z) = \log|z|$ be a symbol on $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. One can show that T_ϕ is compact on $A^2(\mathbb{D}^*)$ (as $A^2(\mathbb{D}^*) = A^2(\mathbb{D})$ and $\phi = 0$ on the unit circle). However, compact operators are in the Toeplitz algebra (see [7, Theorem 6]). Hence, T_ϕ is in the Toeplitz algebra; yet, by Proposition 2 below, $\{B_{A_r} T_\phi^{A_r}\}$ does not converge to $B_{\mathbb{D}^*} T_\phi$ in L^p.

Proposition 2. *Let $A_r = \{z \in \mathbb{C} : r < |z| < 1\}$, $\mathbb{D}^* = \mathbb{D}\setminus\{0\}$, and $\phi(z) = \log|z|$. Then T_ϕ is a compact operator on $A^2(\mathbb{D}^*)$ and*

$$\lim_{r\to 0^+} \|EB_{A_r} T_\phi^{A_r}\|_{L^p(\mathbb{D}^*)} = \infty,$$

while $\|B_{\mathbb{D}^*} T_\phi\|_{L^p(\mathbb{D}^*)} < \infty$ for all $1 \le p \le \infty$.

2. Proofs of Theorems 1, 2, 3 and Corollary 1

We start with a simple lemma.

Lemma 1. *Let Ω be a non-trivial Bergman domain in $\mathbb{C}^n$ and $U \subset \Omega$ be a subdomain. Then $R_U^* K_z^U = K_z^\Omega$ for $z \in U$.*

Proof. For $z \in U$ and $f \in A^2(\Omega)$ we have

$$f(z) = \langle R_U f, K_z^U \rangle_U = \langle f, R_U^* K_z^U \rangle_\Omega.$$

Because of the uniqueness of the Bergman kernel, we conclude that $R_U^* K_z^U = K_z^\Omega$. □

We will need the following results of Ramadanov and Skwarczyński (see [11, Theorem 12.1.23 and Theorem 12.1.24] and also [12,13,10,15]).

Theorem 4 *(Ramadanov). Let Ω_j be an increasing sequence of domains in $\mathbb{C}^n$ such that $\Omega = \cup_{j=1}^\infty \Omega_j$. Then, $K^{\Omega_j} \to K^\Omega$ as $j \to \infty$ locally uniformly on $\Omega \times \Omega$.*

Theorem 5 *(Skwarczyński). Let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$. Then, $K^{\Omega_j} \to K^{\Omega}$ as $j \to \infty$ locally uniformly on $\Omega \times \Omega$ if and only if $K^{\Omega_j}(w,w) \to K^{\Omega}(w,w)$ as $j \to \infty$ for all $w \in \Omega$.*

Let U be a subdomain of a domain Ω. Since

$$K^{\Omega}(z,z) = \sup\{|f(z)|^2 : f \in A^2(\Omega) \text{ and } \|f\| = 1\},$$

we have $0 \leq K^{\Omega}(z,z) \leq K^{U}(z,z)$ for every $z \in U$. Hence, if $K^{\Omega}(z,z) \neq 0$, then $K^{U}(z,z) \neq 0$.

Lemma 2. *Let $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega_j \subset \Omega_{j+1}$ for all j and $\Omega = \cup_{j=1}^{\infty}\Omega_j$ be a non-trivial Bergman domain. Then for each compact set $K \subset \Omega \backslash \mathcal{Z}$, we have*

$$\lim_{j\to\infty} \sup_{z\in K} \|R^*_{\Omega_j} k_z^{\Omega_j} - k_z^{\Omega}\|_{L^2(\Omega)} = 0.$$

Proof. First we note that $0 \leq K^{\Omega}(z,z) \leq K^{\Omega_j}(z,z)$ for all j and $z \in K$. So since $K \subset \Omega \setminus \mathcal{Z}$ we have $K^{\Omega_j}(z,z) \neq 0$ for all j so that $K \subset \Omega_j$. Let j_0 be chosen such that $K \subset \Omega_{j_0}$. Lemma 1 implies that $R^*_{\Omega_j} k_z^{\Omega_j} = K_z^{\Omega}/\sqrt{K^{\Omega_j}(z,z)}$ for $j \geq j_0$. Then for $z \in K$ and $j \geq j_0$ we have

$$\begin{aligned}
\|R^*_{\Omega_j} k_z^{\Omega_j} - k_z^{\Omega}\|_{L^2(\Omega)} &= \left\| \frac{K_z^{\Omega}}{\sqrt{K^{\Omega_j}(z,z)}} - \frac{K_z^{\Omega}}{\sqrt{K^{\Omega}(z,z)}} \right\|_{L^2(\Omega)} \\
&= \left\| k_z^{\Omega} \left(1 - \sqrt{K^{\Omega}(z,z)}/\sqrt{K^{\Omega_j}(z,z)} \right) \right\|_{L^2(\Omega)} \\
&= \left| 1 - \sqrt{K^{\Omega}(z,z)}/\sqrt{K^{\Omega_j}(z,z)} \right|.
\end{aligned}$$

Ramadanov's Theorem (Theorem 4) implies that $K^{\Omega}(z,z)/K^{\Omega_j}(z,z) \to 1$ uniformly on K as $j \to \infty$. Therefore, $\sup_{z\in K} \|R^*_{\Omega_j} k_z^{\Omega_j} - k_z^{\Omega}\|_{L^2(\Omega)} \to 0$ as $j \to \infty$. $\square$

The following Lemma, which is used in the proof of Theorem 1, might be of interest on its own right.

Lemma 3. *Let Ω be a non-trivial Bergman domain in $\mathbb{C}^n$ and $U \subset \Omega$ be a subdomain. Let T be a bounded operator on $A^2(\Omega)$. Then*

$$\frac{B_{\Omega}T(z)}{B_U(R_U T R_U^*)(z)} = \frac{K^U(z,z)}{K^{\Omega}(z,z)}$$

for $z \in U \backslash \mathcal{Z}$.

Proof. For $z \in U\backslash\mathcal{Z}$, we use Lemma 1 to get

$$\begin{aligned}
B_U(R_U T R_U^*)(z) =& \langle T R_U^* k_z^U, R_U^* k_z^U \rangle_{\Omega} \\
=& \frac{\langle T K_z^{\Omega}, K_z^{\Omega} \rangle_{\Omega}}{K^U(z,z)} \\
=& \frac{K^{\Omega}(z,z)}{K^U(z,z)} B_{\Omega}T(z).
\end{aligned}$$

Hence the proof of Lemma 3 is complete. $\square$

Corollary 2. *Let Ω be a non-trivial Bergman domain in $\mathbb{C}^n, U \subset \Omega$ be a subdomain, and T be a bounded linear operator on $A^2(\Omega)$. Assume that $p \in \overline{U}$ and $1 \le \alpha < \infty$ such that $\frac{K^U(z,z)}{K^\Omega(z,z)} \to \alpha$ as $z \to p$, $z \in U \backslash \mathcal{Z}$. Then $B_\Omega T$ is continuous at p if and only if $B_U(R_U T R_U^*)$ is continuous at p.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The proof of locally uniform convergence is a result of Theorem 4 together with Lemma 3. Indeed, Theorem 4 implies that

$$K^{\Omega_j}(z,z)/K^\Omega(z,z) \to 1$$

locally uniformly on $\Omega \times \Omega$ as $j \to \infty$. Then Lemma 3 implies that

$$B_{\Omega_j} R_{\Omega_j} T R^*_{\Omega_j} \to B_\Omega T$$

locally uniformly on Ω as $j \to \infty$.

To prove the second part we assume that Ω is bounded and $0 < p < \infty$. From the first part of the proof, we know that $B_{\Omega_j} R_{\Omega_j} T R^*_{\Omega_j} \to B_\Omega T$ uniformly on compact sets as $j \to \infty$. Furthermore, $|B_\Omega T(z)| \le \|T\|$ and $|EB_{\Omega_j} R_{\Omega_j} T R^*_{\Omega_j}(z)| \le \|T\|$ for all $z \in \Omega$ and all j. Then, using the Lebesgue Dominated Convergence Theorem, we conclude that $EB_{\Omega_j} R_{\Omega_j} T R^*_{\Omega_j} \to B_\Omega T$ in $L^p(\Omega)$ as $j \to \infty$. $\square$

Lemma 4. *Let Ω be a non-trivial Bergman domain and $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ for all j. Assume that $K_{\Omega_j}(z,z) \to K_\Omega(z,z)$ as $j \to \infty$ for every $z \in \Omega$. Then for each compact set $K \subset \Omega \backslash \mathcal{Z}$, we have*

$$\lim_{j\to\infty} \sup_{z\in K} \|R^{\Omega_j}_\Omega k^{\Omega_j}_z - k^\Omega_z\|_{L^2(\Omega)} = 0.$$

Proof. If $K^\Omega(z,z) > 0$ for some $z \in \Omega$, then $K^{\Omega_j}(z,z) > 0$ for large j because $K^{\Omega_j}(z,z)$ increases to $K^\Omega(z,z)$ as $j \to \infty$. Furthermore, there exists an open neighborhood of z for which the normalized Bergman kernels k^{Ω_j} and k^Ω are well-defined for j large enough. Since $K \subset \Omega \backslash \mathcal{Z}$ is compact, all of the functions in the statement are well-defined for large j, and the limit makes sense.

Let $0 < \varepsilon < 1$ be given. For each $z \in K$, we choose a compact $S_z \subset \Omega$ so that $\left\|k^\Omega_z\right\|_{L^2(\Omega \backslash S_z)} < \varepsilon$. Recall that the map $z \mapsto k^\Omega_z$ is continuous from $\Omega \backslash \mathcal{Z}$ to $L^2(\Omega)$ (see (1)). For any $z \in \Omega \setminus \mathcal{Z}$ we choose an open set $U_z \subset \Omega \backslash \mathcal{Z}$ so that $z \in U_z$ and $\left\|k^\Omega_z - k^\Omega_w\right\|_{L^2(\Omega)} < \varepsilon$ when $w \in U_z$. Then

$$\left\|k^\Omega_w\right\|_{L^2(\Omega \backslash S_z)} < \varepsilon + \left\|k^\Omega_z\right\|_{L^2(\Omega \backslash S_z)} < 2\varepsilon$$

for $w \in U_z$. Since K is compact, there exist $z_1, \cdots, z_m \in K$ so that $K \subset \cup_{j=1}^m U_{z_j}$. The set $S = \cup_{j=1}^m S_{z_j} \subset \Omega$ is compact as well and

$$\sup_{w\in K} \left\|k^\Omega_w\right\|_{L^2(\Omega \backslash S)} < 2\varepsilon.$$

Using Theorem 5, we have

$$\sup_{z\in K, w\in S} \left|k^{\Omega_j}_z(w) - k^\Omega_z(w)\right| < \frac{\varepsilon}{\sqrt{Vol(S)+1}} \tag{2}$$

and

$$\sup_{z\in K, w\in S}\left||k_z^{\Omega_j}(w)|^2-|k_z^{\Omega}(w)|^2\right|<\frac{\varepsilon^2}{Vol(S)+1}$$

for large enough j. Then by integrating the above inequality over S and using $\left\|k_z^{\Omega}\right\|_{L^2(\Omega\setminus S)}<2\varepsilon$ we get

$$\|k_z^{\Omega_j}\|^2_{L^2(S)}\geq \|k_z^{\Omega}\|^2_{L^2(S)}-\varepsilon^2>1-4\varepsilon^2-\varepsilon^2=1-5\varepsilon^2,$$

which implies $\|k_z^{\Omega_j}\|_{L^2(\Omega\setminus S)}<\sqrt{5}\varepsilon$ when j is large enough. Then using (2) we get

$$\begin{aligned}\|R_{\Omega}^{\Omega_j}k_z^{\Omega_j}-k_z^{\Omega}\|_{L^2(\Omega)}&\leq \left\|k_z^{\Omega_j}-k_z^{\Omega}\right\|_{L^2(S)}+\|k_z^{\Omega}\|_{L^2(\Omega\setminus S)}+\|k_z^{\Omega_j}\|_{L^2(\Omega\setminus S)}\\&<(3+\sqrt{5})\varepsilon\end{aligned}$$

for j large and $z\in K$. Hence,

$$\lim_{j\to\infty}\sup_{z\in K}\|R_{\Omega}^{\Omega_j}k_z^{\Omega_j}-k_z^{\Omega}\|_{L^2(\Omega)}=0.$$

The proof is finished. $\square$

Proof of Theorem 2. For $z\in\Omega\setminus\mathcal{Z}$, we define $f(z)=B_{\Omega}T(z)$ and

$$\begin{aligned}f_j(z)=&B_{\Omega_j}(R_{\Omega}^{\Omega_j})^*TR_{\Omega}^{\Omega_j}(z)\\g_j(z)=&\langle TR_{\Omega}^{\Omega_j}k_z^{\Omega_j},k_z^{\Omega}\rangle_{L^2(\Omega)}\end{aligned}$$

for each j. Then

$$f_j(z)=\langle (R_{\Omega}^{\Omega_j})^*TR_{\Omega}^{\Omega_j}k_z^{\Omega_j},k_z^{\Omega_j}\rangle_{L^2(\Omega_j)}=\langle TR_{\Omega}^{\Omega_j}k_z^{\Omega_j},R_{\Omega}^{\Omega_j}k_z^{\Omega_j}\rangle_{L^2(\Omega)}.$$

Let $K\subset\Omega$ be a compact set. By Cauchy-Schwarz inequality we have

$$\begin{aligned}\sup_{z\in K}|g_j(z)-f(z)|&=\sup_{z\in K}\left|\langle TR_{\Omega}^{\Omega_j}k_z^{\Omega_j}-Tk_z^{\Omega},k_z^{\Omega}\rangle\right|\\&\leq\sup_{z\in K}\left\|TR_{\Omega}^{\Omega_j}k_z^{\Omega_j}-Tk_z^{\Omega}\right\|_{L^2(\Omega)}\\&\leq\|T\|\sup_{z\in K}\left\|R_{\Omega}^{\Omega_j}k_z^{\Omega_j}-k_z^{\Omega}\right\|_{L^2(\Omega)}.\end{aligned}$$

The last term above converges to zero by Lemma 4. Therefore, the sequence $\{g_j\}$ converges to f uniformly on K.

Using Cauchy-Schwarz inequality again we have

$$|f_j(z)-g_j(z)|=\left|\langle TR_{\Omega}^{\Omega_j}k_z^{\Omega_j},R_{\Omega}^{\Omega_j}k_z^{\Omega_j}-k_z^{\Omega}\rangle\right|\leq\|T\|\|R_{\Omega}^{\Omega_j}k_z^{\Omega_j}-k_z^{\Omega}\|_{L^2(\Omega)}.$$

Lemma 4 implies that the last term above converges to zero uniformly on K. Hence, $|f_j-g_j|\to 0$ uniformly on K as $j\to\infty$. Therefore, $\{f_j\}$ converges to f uniformly on K.

As in the proof of Theorem 1 we prove the second part as follows. We assume that Ω is bounded. From the previous part of this proof we know that $\{f_j\}$ converges to f uniformly on compact subset of Ω. Furthermore, $\|f_j\|_{L^\infty(\Omega)}\leq\|T\|$ for all j. Then using the Lebesgue Dominated Convergence Theorem, we conclude that $\{f_j\}$ converges to f in $L^p(\Omega)$ as $j\to\infty$ for all $0<p<\infty$. $\square$

Now we are ready to prove Theorem 3.

Proof of Theorem 3. It is enough to prove the result for finite product of Toeplitz operators as it is easy to conclude the theorem for the finite sums of such operators. So let $T = T_{\phi_m} \cdots T_{\phi_1}$ where $\phi_1, \dots, \phi_m \in L^\infty(\Omega)$. One can easily show that $B_\Omega T \in L^\infty(\Omega)$ and $B_{\Omega_j} T^{\Omega_j} \in L^\infty(\Omega_j)$ for all j. Furthermore, one can show that

$$\max\{\|B_{\Omega_j} T^{\Omega_j}\|_{L^\infty(\Omega_j)}, \|B_\Omega T\|_{L^\infty(\Omega)}\} \le \|\phi_1\|_{L^\infty(\Omega)} \cdots \|\phi_m\|_{L^\infty(\Omega)}.$$

Let $f_j(z) = |B_\Omega T(z) - EB_{\Omega_j} T^{\Omega_j}(z)|$ for $z \in \Omega$. Then

$$\|f_j\|_{L^\infty(\Omega)} \le 2\|\phi_1\|_{L^\infty(\Omega)} \cdots \|\phi_m\|_{L^\infty(\Omega)} \tag{3}$$

for all j.

We will use induction to prove that

$$\sup\{|T^{\Omega_j} k_z^{\Omega_j}(w) - T k_z^\Omega(w)| : z, w \in K\} \to 0$$

as $j \to \infty$. So first let us assume that $T = T_{\phi_1}$ is a Toeplitz operator. Let K be a compact set in $\Omega \setminus \mathcal{Z}$. As in the proof of Lemma 4 for a given $\varepsilon > 0$, there exists a compact set $S \subset \Omega$ and $j_0 \in \mathbb{N}$ such that $K \Subset \Omega_j$, $\|k_z^\Omega\|_{L^2(\Omega\setminus S)} < \varepsilon$ for all $z \in K$, and $\|k_z^{\Omega_j}\|_{L^2(\Omega_j\setminus S)} < \varepsilon$ for all $z \in K$ and $j \ge j_0$. Let us consider the following equalities.

$$\begin{aligned} T_{\phi_1} k_z^\Omega(w) - T_{\phi_1}^{\Omega_j} k_z^{\Omega_j}(w) =& \langle \phi_1 k_z^\Omega, K^\Omega(.,w)\rangle_\Omega - \langle \phi_1 k_z^{\Omega_j}, K^{\Omega_j}(.,w)\rangle_{\Omega_j} \\ =& \langle \phi_1 k_z^\Omega, K^\Omega(.,w)\rangle_S - \langle \phi_1 k_z^{\Omega_j}, K^{\Omega_j}(.,w)\rangle_S \\ &+ \langle \phi_1 k_z^\Omega, K^\Omega(.,w)\rangle_{\Omega\setminus S} - \langle \phi_1 k_z^{\Omega_j}, K^{\Omega_j}(.,w)\rangle_{\Omega_j\setminus S}. \end{aligned}$$

There exists $C_K > 1$ such that $1/C_K \le K^{\Omega_j}(w,w) \le C_K$ for all $w \in K$ and all $j \ge j_0$ since by Theorem 4, the continuous functions $\{K^{\Omega_j}(w,w)\}$ converges to $K^\Omega(w,w)$ uniformly on K.

Without loss of generality we can assume that

$$\begin{aligned} \|k_z^\Omega\|_{L^2(\Omega\setminus S)} &< \frac{\varepsilon}{\sqrt{K^\Omega(w,w)}}, \\ \|k_z^{\Omega_j}\|_{L^2(\Omega_j\setminus S)} &< \frac{\varepsilon}{\sqrt{K^{\Omega_j}(w,w)}} \end{aligned}$$

for $j \ge j_0$ and all $z, w \in K$. Then

$$\left|\langle \phi_1 k_z^\Omega, K^\Omega(.,w)\rangle_{\Omega\setminus S}\right| + \left|\langle \phi_1 k_z^{\Omega_j}, K^{\Omega_j}(.,w)\rangle_{\Omega_j\setminus S}\right| \le 2\varepsilon\|\phi_1\|_{L^\infty(\Omega)}$$

for all $z, w \in K$. Also

$$\sup\left\{\left|\langle \phi_1 k_z^\Omega, K^\Omega(.,w)\rangle_S - \langle \phi_1 k_z^{\Omega_j}, K^{\Omega_j}(.,w)\rangle_S\right| : z, w \in K\right\} \to 0$$

as $j \to \infty$ (a consequence of Theorem 4). Then

$$\limsup_{j\to\infty} \sup\left\{\left|T_{\phi_1} k_z^\Omega(w) - T_{\phi_1}^{\Omega_j} k_z^{\Omega_j}(w)\right| : z, w \in K\right\} \le 2\varepsilon\|\phi_1\|_{L^\infty(\Omega)}.$$

Since ε is arbitrary, we conclude that

$$\sup\left\{\left|T^{\Omega}_{\phi_1}k^{\Omega}_z(w)-T^{\Omega_j}_{\phi_1}k^{\Omega_j}_z(w)\right|:z,w\in K\right\}\to 0$$

as $j\to\infty$. We note that for $z\in\Omega_j$ we have

$$\begin{aligned}\left|B_{\Omega_j}T^{\Omega_j}_{\phi_1}(z)-B_{\Omega}T_{\phi_1}(z)\right|=&\frac{\left|\sqrt{\frac{K^{\Omega}(z,z)}{K^{\Omega_j}(z,z)}}\left\langle T^{\Omega_j}_{\phi_1}k^{\Omega_j}_z,K^{\Omega_j}_z\right\rangle_{\Omega_j}-\left\langle T_{\phi_1}k^{\Omega}_z,K^{\Omega}_z\right\rangle_{\Omega}\right|}{\sqrt{K^{\Omega}(z,z)}}\\ =&\frac{1}{\sqrt{K^{\Omega}(z,z)}}\left|\sqrt{\frac{K^{\Omega}(z,z)}{K^{\Omega_j}(z,z)}}T^{\Omega_j}_{\phi_1}k^{\Omega_j}_z(z)-T_{\phi_1}k^{\Omega}_z(z)\right|.\end{aligned}\tag{4}$$

Hence $B_{\Omega_j}T^{\Omega_j}_{\phi_1}\to B_{\Omega}T_{\phi_1}$ uniformly on compact subsets of $\Omega\backslash\mathcal{Z}$ as $j\to\infty$.

Next we show the induction step. Let $\widetilde{T}=T_{\phi_{m-1}}\cdots T_{\phi_1}$ and $\widetilde{T}^{\Omega_j}=T_{\phi_{m-1}|_{\Omega_j}}\cdots T_{\phi_1|_{\Omega_j}}$. As the induction hypothesis we assume that $\widetilde{T}^{\Omega_j}k^{\Omega_j}_z\to\widetilde{T}k^{\Omega}_z$ uniformly on compact subsets as $j\to\infty$. Then

$$\begin{aligned}Tk^{\Omega}_z(w)-T^{\Omega_j}k^{\Omega_j}_z(w)=&\langle\phi_m\widetilde{T}k^{\Omega}_z,K^{\Omega}(.,w)\rangle_{\Omega}-\langle\phi_m\widetilde{T}^{\Omega_j}k^{\Omega_j}_z,K^{\Omega_j}(.,w)\rangle_{\Omega_j}\\ =&\langle\phi_m\widetilde{T}k^{\Omega}_z,K^{\Omega}(.,w)\rangle_{S}-\langle\phi_m\widetilde{T}^{\Omega_j}k^{\Omega_j}_z,K^{\Omega_j}(.,w)\rangle_{S}\\ &+\langle\phi_m\widetilde{T}k^{\Omega}_z,K^{\Omega}(.,w)\rangle_{\Omega\backslash S}\\ &-\langle\phi_m\widetilde{T}^{\Omega_j}k^{\Omega_j}_z,K^{\Omega_j}(.,w)\rangle_{\Omega_j\backslash S}\end{aligned}$$

As in the previous case, we have

$$\begin{aligned}\left|\langle\phi_m\widetilde{T}k^{\Omega}_z,K^{\Omega}(.,w)\rangle_{\Omega\backslash S}\right|\le&\|\phi_m\|_{L^{\infty}(\Omega)}\|\widetilde{T}\|\|k^{\Omega}_z\|_{L^2(\Omega\backslash S)}\sqrt{K^{\Omega}(w,w)}\\ \le&\varepsilon\|\phi_m\|_{L^{\infty}(\Omega)}\cdots\|\phi_1\|_{L^{\infty}(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\left|\langle\phi_m\widetilde{T}^{\Omega_j}k^{\Omega_j}_z,K^{\Omega_j}(.,w)\rangle_{\Omega_j\backslash S}\right|\le&\|\phi_m\|_{L^{\infty}(\Omega)}\|\widetilde{T}^{\Omega_j}\|\|k^{\Omega_j}_z\|_{L^2(\Omega_j\backslash S)}\sqrt{K^{\Omega_j}(w,w)}\\ \le&\varepsilon\|\phi_m\|_{L^{\infty}(\Omega)}\cdots\|\phi_1\|_{L^{\infty}(\Omega)}.\end{aligned}$$

Then

$$\begin{aligned}&\left|\langle\phi_m\widetilde{T}k^{\Omega}_z,K^{\Omega}(.,w)\rangle_{\Omega\backslash S}\right|+\left|\langle\phi_m\widetilde{T}^{\Omega_j}k^{\Omega_j}_z,K^{\Omega_j}(.,w)\rangle_{\Omega_j\backslash S}\right|\\ &\qquad\le 2\varepsilon\|\phi_m\|_{L^{\infty}(\Omega)}\cdots\|\phi_1\|_{L^{\infty}(\Omega)}\end{aligned}$$

for all $z,w\in K$. Furthermore, by induction hypothesis, we have

$$\sup\{|\widetilde{T}^{\Omega_j}k^{\Omega_j}_z(w)-\widetilde{T}k^{\Omega}_z(w)|:z,w\in K\}\to 0$$

as $j\to\infty$. Then

$$\sup\left\{\langle\phi_m\widetilde{T}k^{\Omega}_z,K^{\Omega}(.,w)\rangle_S-\langle\phi_m\widetilde{T}^{\Omega_j}k^{\Omega_j}_z,K^{\Omega_j}(.,w)\rangle_S:z,w\in K\right\}\to 0$$

as $j\to\infty$. Hence,

$$\sup\{|T^{\Omega_j}k^{\Omega_j}_z(w)-Tk^{\Omega}_z(w)|:z,w\in K\}\to 0$$

as $j \to \infty$. Similar to (4) one can show that

$$\left|B_{\Omega_j} T^{\Omega_j}(z) - B_\Omega T(z)\right| = \frac{1}{\sqrt{K^\Omega(z,z)}} \left| \sqrt{\frac{K^\Omega(z,z)}{K^{\Omega_j}(z,z)}} T^{\Omega_j} k_z^{\Omega_j}(z) - T k_z^\Omega(z) \right|.$$

Therefore, $f_j \to 0$ uniformly on K as $j \to \infty$.

To prove the second part we assume that Ω is bounded. Then the Lebesgue Dominated Convergence Theorem together with (3) implies that $\int_\Omega |f_j(z)|^p dV(z) \to 0$ as $j \to \infty$. Hence, $EB_{\Omega_j} T^{\Omega_j} \to B_\Omega T$ in $L^p(\Omega)$ as $j \to \infty$. $\square$

Using very similar arguments as in the proof of Theorem 3 one can prove the following corollary.

Corollary 3. *Let Ω be a non-trivial Bergman domain and $\{\Omega_j\}$ be a sequence of domains in $\mathbb{C}^n$ such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ for all j. Assume $K^{\Omega_j}(z,z) \to K^\Omega(z,z)$ as $j \to \infty$ for every $z \in \Omega$. Let $T = \sum_{m=1}^{l} T_{\phi_{m,1}} \cdots T_{\phi_{m,k_m}}$ be a finite sum of finite products of Toeplitz operators with bounded symbols on Ω_1 and $T^{\Omega_j} = \sum_{m=1}^{l} T_{\phi_{m,1}|_{\Omega_j}} \cdots T_{\phi_{m,k_m}|_{\Omega_j}}$ for each j. Then $B_{\Omega_j} T^{\Omega_j} \to B_\Omega T$ uniformly on compact subsets of $\Omega \setminus \mathcal{Z}$ as $j \to \infty$. Furthermore, if Ω_1 is bounded, then $EB_{\Omega_j} T^{\Omega_j} \to B_\Omega T$ in $L^p(\Omega)$ as $j \to \infty$ for all $0 < p < \infty$.*

We finish this section with the proof of Corollary 1.

Proof of Corollary 1. Let $\phi \in L^q(\Omega)$ and let $K \subset \Omega \backslash \mathcal{Z}$ be compact. First assume that ϕ is real valued and $\phi \geq 0$ on Ω. For each $k \geq 1$ we define $\phi_k = \min\{\phi, k\}$. Hence, $\phi_k \in L^\infty(\Omega)$ and $B_\Omega \phi_k(z)$ increases to $B_\Omega \phi(z)$ for each $z \in \Omega$. By Dini's Theorem, $B_\Omega \phi_k$ converges uniformly to $B_\Omega \phi$ on K. By Theorem 3, for each $k \geq 1$ there exists j_k so that

$$\sup_{z \in K} |EB_{\Omega_{j_k}} \phi_k(z) - B_\Omega \phi_k(z)| \leq \frac{1}{k}.$$

This means that $EB_{\Omega_{j_k}} \phi_k$ converges uniformly to $B_\Omega \phi$ on K. If Ω is bounded and $p > 0$, then by the last statement of Theorem 3, we can find j_k so that $\|EB_{\Omega_{j_k}} \phi_k - B_\Omega \phi_k\|_{L^p(\Omega)} \leq 1/k$. By Monotone Convergence Theorem, we conclude that $\|B_\Omega \phi_k - B_\Omega \phi\|_{L^p(\Omega)} \to 0$ as $k \to \infty$. Therefore, $\|EB_{\Omega_{j_k}} \phi_k - B_\Omega \phi\|_{L^p(\Omega)} \to 0$ as $k \to \infty$. Now let $\phi \in L^q(\Omega)$ be real valued. Then we write $\phi = \phi^+ - \phi^-$ where $\phi^+, \phi^- \geq 0$ on Ω. Since $B_\Omega \phi = B_\Omega \phi^+ - B_\Omega \phi^-$, we can apply the first part of the proof to each term. Finally, if ϕ is complex valued then we can apply the previous part of the proof to the real and imaginary parts of ϕ. $\square$

3. Proofs of Propositions 1 and 2

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane. The Poisson kernel (see, for instance, [14, Definition 1.2.3]) on the unit disk is defined as

$$P(z,\zeta) = \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1 - |z|^2}{|\zeta - z|^2},$$

where $z \in \mathbb{D}$, $|\zeta| = 1$.

Lemma 5. *Let $0 < s < 1$ and $z \in \mathbb{D}$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} (P(sz, e^{it}))^2 dt = \frac{1 + s^2|z|^2}{1 - s^2|z|^2}.$$

Proof. Let us fix $z = \rho e^{i\theta}$. In (5), we use the property that

$$P(s\rho e^{i\theta}, e^{it}) = P(s\rho e^{it}, e^{i\theta});$$

and in (6) we use the facts that P, the Poisson kernel, is the kernel of the integral operator that solves the Dirichlet problem and $P(., e^{it})$ is harmonic on $\mathbb{D}$ (see [14]).

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left(P(sz, e^{it})\right)^2 dt =& \frac{1}{2\pi}\int_0^{2\pi} P(sz, e^{it})P(sz, e^{it})dt \\
=& \frac{1}{2\pi}\int_0^{2\pi} P(s\rho e^{i\theta}, e^{it})P(sz, e^{it})dt \\
=& \frac{1}{2\pi}\int_0^{2\pi} P(s\rho e^{it}, e^{i\theta})P(sz, e^{it})dt \qquad (5)\\
=& P(s^2\rho z, e^{i\theta}) \qquad (6)\\
=& \frac{1 - s^4|z|^4}{(1 - s^2|z|^2)^2} \\
=& \frac{1 + s^2|z|^2}{1 - s^2|z|^2}.
\end{aligned}$$

Hence, the proof of Lemma 5 is complete. □

A function $u(z, w)$ in $\mathbb{D}^2$ is said to be separately subharmonic if when one of the variables is fixed in $\mathbb{D}$, u is subharmonic in the other variable.

Lemma 6. *Let $G_a(z) = \log\left|\frac{a - z}{1 - \overline{a}z}\right|$ be the Green's function for $\mathbb{D}$ with pole at $a \in \mathbb{D}$. Then*

$$B_{\mathbb{D}} G_a(z) = \frac{1}{2}\left(\left|\frac{a - z}{1 - \overline{a}z}\right|^2 - 1\right)$$

and the function $u(z, a) = B_{\mathbb{D}} G_a(z)$, defined for $(z, a) \in \mathbb{D}^2$, is separately subharmonic on $\mathbb{D}^2$.

Proof. First suppose that $a = 0$. Using Lemma 5 in the fourth equality below we get

$$\begin{aligned}
B_{\mathbb{D}} G_0(z) =& \frac{(1 - |z|^2)^2}{\pi}\int_{\mathbb{D}} \frac{\log|w|}{|1 - \overline{w}z|^4} dV(w) \\
=& \frac{(1 - |z|^2)^2}{\pi}\int_0^1 s\log s \int_0^{2\pi} \frac{1}{|1 - se^{-it}z|^4} dtds \\
=& 2(1 - |z|^2)^2 \int_0^1 \frac{s\log s}{(1 - s^2|z|^2)^2} \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - s^2|z|^2)}{|e^{it} - sz|^2}\frac{(1 - s^2|z|^2)}{|e^{it} - sz|^2} dtds \\
=& 2(1 - |z|^2)^2 \int_0^1 \frac{s\log s}{(1 - s^2|z|^2)^2} \frac{1 + s^2|z|^2}{(1 - s^2|z|^2)} ds
\end{aligned}$$

$$=2(1-|z|^2)^2\int_0^1 \frac{s(1+s^2|z|^2)\log s}{(1-s^2|z|^2)^3}ds.$$

One can show that

$$\int \frac{x(1+|z|^2x^2)\log x}{(1-|z|^2x^2)^3}dx = \frac{x^2\log x}{2(|z|^2x^2-1)^2}+\frac{1}{4|z|^2(|z|^2x^2-1)}+C.$$

Therefore,

$$2(1-|z|^2)^2\int_0^1 \frac{s(1+s^2|z|^2)\log s}{(1-s^2|z|^2)^3}ds=\frac{1}{2}(|z|^2-1).$$

Let $a\in\mathbb{D}\backslash\{0\}$. Let $\psi_a(w)=\dfrac{a-w}{1-\overline{a}w}$ be the Möbius transform on the disk. Then, using [9, Chapter 2] (see also [17, Section 6.3]) we have

$$\begin{aligned} B_{\mathbb{D}}G_a(z) &= \int_{\mathbb{D}} G_a(\psi_z(w))dV(w)\\ &= \int_{\mathbb{D}} G_0(\psi_a\circ\psi_z(w))dV(w)\\ &= \int_{\mathbb{D}} G_0(\psi_{\psi_a(z)}(w))dV(w)\\ &= B_{\mathbb{D}}G_0(\psi_a(z))=\frac{1}{2}\left(\left|\frac{a-z}{1-\overline{a}z}\right|^2-1\right). \end{aligned}$$

Hence, the proof of Lemma 6 is complete. □

Proof of Proposition 1. The Bergman kernel of the annulus A_r is (see [11, Example 12.1.7 (c)])

$$K^{A_r}(z,w)=-\frac{1}{2\pi z\overline{w}\log r}+\frac{1}{\pi z\overline{w}}\sum_{k\neq 0}\frac{kz^k\overline{w}^k}{1-r^{2k}}.$$

Let K be a compact subset of $\mathbb{D}\setminus\{0\}$. Then for small enough $r>0$ the set K is a compact subset of A_r. Let us fix $z_0\in K\Subset A_r$ and let us break down the function $K^{A_r}(z_0,w)$ into four pieces as

$$K^{A_r}(z_0,w)=\psi^0_{r,z_0}(w)+\psi^1_{r,z_0}(w)+\psi^2_{r,z_0}(w)+\psi^3_{r,z_0}(w)$$

where

$$\begin{aligned} \psi^0_{r,z_0}(w) &= -\frac{1}{2\pi z_0\overline{w}\log r},\\ \psi^1_{r,z_0}(w) &= \frac{r^2}{(1-r^2)\pi z_0^2\overline{w}^2},\\ \psi^2_{r,z_0}(w) &= \frac{1}{\pi z_0^2}\sum_{k=2}^{\infty}\frac{k}{1-r^{2k}}\left(\frac{r}{z_0}\right)^{k-1}\left(\frac{r}{\overline{w}}\right)^{k+1}, \end{aligned}$$

$$\psi^3_{r,z_0}(w) = \frac{1}{\pi z_0 \overline{w}} \sum_{k=1}^{\infty} \frac{k z_0^k \overline{w}^k}{1-r^{2k}}.$$

One can check that the $\sup\{|\psi^1_{r,z_0}(w)| : z_0 \in K, w \in A_r\}$ and $\sup\{|\psi^3_{r,z_0}(w)| : z_0 \in K, w \in A_r\}$ stay bounded as $r \to 0^+$. Furthermore, $\sup\{|\psi^2_{r,z_0}(w)| : z_0 \in K, w \in A_r\}$ converges to zero as $r \to 0^+$.

Now we will estimate the Berezin transform of $\phi(w) = \log|w|$ on A_r at z_0. First we can write $|K^{A_r}(z_0,w)|^2$ as

$$|K^{A_r}(z_0,w)|^2 = \left|\psi^0_{r,z_0}(w)\right|^2 + \left|\psi^3_{r,z_0}(w)\right|^2 + \Psi_{r,z_0}(w)$$

where

$$\begin{aligned}\Psi_{r,z_0}(w) =& 2\mathrm{Re}\left(\psi^0_{r,z_0}(w)\sum_{j=1}^{3}\overline{\psi^j_{r,z_0}(w)} + \psi^1_{r,z_0}(w)\sum_{j=2}^{3}\overline{\psi^j_{r,z_0}(w)}\right)\\ &+ 2\mathrm{Re}\left(\psi^2_{r,z_0}(w)\overline{\psi^3_{r,z_0}(w)}\right) + \left|\psi^1_{r,z_0}(w)\right|^2 + \left|\psi^2_{r,z_0}(w)\right|^2.\end{aligned}$$

Now we will show that $\sup\left\{\left|\int_{A_r}\phi(w)\Psi_{r,z_0}(w)dV(w)\right| : z_0 \in K\right\} \to 0$ as $r \to 0^+$. Using polar coordinates we compute

$$\begin{aligned}\int_{A_r} |\phi(w)|\left|\psi^0_{r,z_0}(w)\right| dV(w) =& \frac{1}{|z_0|\log r}\int_r^1 \log\rho\, d\rho\\ =& \frac{r - r\log r - 1}{|z_0|\log r} \to 0\end{aligned}$$

uniformly on K as $r \to 0^+$. Hence using the fact that $\psi^1_{r,z_0}, \psi^2_{r,z_0}, \psi^3_{r,z_0}$ stay bounded uniformly on A_r for all $z_0 \in K$ we conclude that

$$\int_{A_r} \phi(w)\psi^0_{r,z_0}(w)\sum_{j=1}^{3}\overline{\psi^j_{r,z_0}(w)}dV(w) \to 0$$

uniformly on K as $r \to 0^+$. Similarly, we conclude that

$$\int_{A_r} \phi(w)\left|\psi^1_{r,z_0}(w)\right|^2 dV(w) \to 0$$

and

$$\int_{A_r} \phi(w)\psi^1_{r,z_0}(w)\sum_{j=2}^{3}\overline{\psi^j_{r,z_0}(w)}dV(w) \to 0$$

uniformly on K as $r \to 0^+$ because $\psi^1_{r,z_0}, \psi^2_{r,z_0}, \psi^3_{r,z_0}$ stay bounded uniformly on A_r for all $z_0 \in K$ and

$$\begin{aligned}\int_{A_r} |\phi(w)|\left|\psi^1_{r,z_0}(w)\right| dV(w) =& -\frac{2r^2}{(1-r^2)|z_0|^2}\int_r^1 \frac{\log\rho}{\rho}d\rho\\ =& \frac{r^2(\log r)^2}{(1-r^2)|z_0|^2} \to 0\end{aligned}$$

uniformly on K as $r \to 0^+$. Finally, since ψ^3_{r,z_0} stays bounded uniformly on A_r while $\sup\{|\psi^2_{r,z_0}(w)| : z_0 \in K, w \in A_r\} \to 0$ as $r \to 0^+$ we get

$$\int\limits_{A_r} \phi(w) \left|\psi^2_{r,z_0}(w)\right|^2 dV(w) \to 0$$

and

$$\int\limits_{A_r} \phi(w)\psi^2_{r,z_0}(w)\overline{\psi^3_{r,z_0}(w)}dV(w) \to 0$$

uniformly on K as $r \to 0^+$. Therefore, we showed that

$$\sup\left\{\left|\int\limits_{A_r} \phi(w)\Psi_{r,z_0}(w)dV(w)\right| : z_0 \in K\right\} \to 0 \text{ as } r \to 0^+.$$

Now we turn to $\int_{A_r} \phi(w) \left|\psi^0_{r,z_0}(w)\right|^2 dV(w)$.

$$\int\limits_{A_r} \phi(w) \left|\psi^0_{r,z_0}(w)\right|^2 dV(w) = \frac{1}{2\pi|z_0|^2(\log r)^2}\int\limits_r^1 \frac{\log\rho}{\rho}d\rho = -\frac{1}{4\pi|z_0|^2}.$$

Finally,

$$K^{A_r}(z_0, z_0) \to K^{\mathbb{D}}(z_0, z_0) = \frac{1}{\pi(1 - |z_0|^2)^2}$$

uniformly for all $z_0 \in K$ as $r \to 0^+$ and

$$\sup\left\{\left|\psi^3_{r,z_0}(w)\right|^2 - \left|K^{\mathbb{D}}(w, z_0)\right|^2 : z_0 \in K, w \in \mathbb{D}\right\} \to 0$$

as $r \to 0^+$. Therefore, we have

$$\begin{aligned} B_{A_r}\phi(z_0) &= \int\limits_{A_r} \phi(w)\frac{|K^{A_r}(w, z_0)|^2}{K^{A_r}(z_0, z_0)}dV(w) \\ &= \int\limits_{A_r} \phi(w)\frac{|\psi^0_{r,z_0}(w)|^2}{K^{A_r}(z_0, z_0)}dV(w) + \int\limits_{A_r} \phi(w)\frac{|\psi^3_{r,z_0}(w)|^2}{K^{A_r}(z_0, z_0)}dV(w) \\ &\quad + \int\limits_{A_r} \phi(w)\frac{\Psi_{r,z_0}(w)}{K^{A_r}(z_0, z_0)}dV(w) \end{aligned}$$

and

$$B_{A_r}\phi(z_0) \to -\frac{(|z_0|^2 - 1)^2}{4|z_0|^2} + B_{\mathbb{D}}\phi(z_0) = \frac{|z_0|^2}{4} - \frac{1}{4|z_0|^2}$$

uniformly on K as $r \to 0^+$ because Lemma 6 implies that $B_{\mathbb{D}}\phi(z_0) = \frac{1}{2}(|z_0|^2 - 1)$. Therefore, we showed that

$$B_{A_r}\phi(z) \to \frac{|z|^2}{4} - \frac{1}{4|z|^2}$$

uniformly on compact subsets of $\mathbb{D} \setminus \{0\}$ as $r \to 0^+$. $\square$

Proof of Proposition 2. The functions $\{e_n : n = 0, 1, 2, \dots\}$ form an orthonormal basis for $A^2(\mathbb{D}^*)$ where $e_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$. Using integration by parts, we compute

$$T_\phi e_n(z) = \left(2(n+1) \int_0^1 r^{2n+1} \log r dr \right) z^n = -\frac{z^n}{2n+2} = -\frac{\sqrt{\pi}}{2(n+1)^{3/2}} e_n(z).$$

Hence, T_ϕ is a compact diagonal operator on $A^2(\mathbb{D}^*)$ and by [7, Theorem 6] it is in the Toeplitz algebra.

Let $f(z) = \frac{|z|^2}{4} - \frac{1}{4|z|^2}$. Proposition 1 implies that for any $\varepsilon > 0$ and any compact set $K \Subset \mathbb{D} \setminus \{0\}$ we can choose $r_0 > 0$ sufficiently small so that $K \Subset A_r$ and

$$\begin{aligned} \|EB_{A_r} T_\phi^{A_r}\|^p_{L^p(\mathbb{D}^*)} &= \int_{A_r} |B_{A_r}\phi(z)|^p dV(z) \geq \int_K |B_{A_r}\phi(z)|^p dV(z) \\ &\geq \int_K |f(z)|^p dV(z) - \varepsilon \end{aligned}$$

for all $0 < r \leq r_0$. Then

$$\liminf_{r \to 0^+} \|EB_{A_r} T_\phi^{A_r}\|^p_{L^p(\mathbb{D}^*)} \geq \|f\|^p_{L^p(K)} - \varepsilon.$$

Since K and ε are arbitrary, we conclude that

$$\liminf_{r \to 0^+} \|EB_{A_r} T_\phi^{A_r}\|^p_{L^p(\mathbb{D}^*)} \geq \|f\|^p_{L^p(\mathbb{D}^*)}.$$

Furthermore, one can show that $\|f\|_{L^p(\mathbb{D}^*)} = \infty$ if and only if $p \geq 1$. Therefore,

$$\lim_{r \to 0^+} \|EB_{A_r} T_\phi^{A_r}\|_{L^p(\mathbb{D}^*)} = \infty.$$

Finally, $\|B_{\mathbb{D}^*} T_\phi\|_{L^p(\mathbb{D}^*)} < \infty$ for all $1 \leq p \leq \infty$ because Lemma 6 implies that $B_{\mathbb{D}^*} T_\phi = (|z|^2 - 1)/2$. $\square$

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