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Matrix biorthogonal polynomials: Eigenvalue problems and non-Abelian discrete Painlevé equations
A Riemann–Hilbert problem perspective

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ABSTRACT

In this paper we use the Riemann–Hilbert problem, with jumps supported on appropriate curves in the complex plane, for matrix biorthogonal polynomials and apply it to find Sylvester systems of differential equations for the orthogonal polynomials and its second kind functions as well. For this aim, Sylvester type differential Pearson equations for the matrix of weights are shown to be instrumental. Several applications are given, in order of increasing complexity. First, a general discussion of non-Abelian Hermite biorthogonal polynomials on the real line, understood as those whose matrix of weights is a solution of a Sylvester type Pearson equation with coefficients first degree matrix polynomials, is given. All of these are applied to the discussion of possible scenarios leading to eigenvalue problems for second order linear differential operators with matrix eigenvalues. Nonlinear matrix difference equations are discussed next. Firstly, for the general Hermite situation a general non linear relation (non trivial because of the non commutativity features of the setting) for the recursion coefficients is gotten. In the next case of higher difficulty, degree two polynomials are allowed in the Pearson equation, but the discussion is simplified by considering only a left Pearson equation. In the case, the support of the measure is on an appropriate branch of a hyperbola. The recursion coefficients are shown to fulfill a non-Abelian extension of the alternate discrete Painlevé I equation. Finally, a discussion is given for the case of degree three polynomials as coefficients in the left Pearson equation characterizing the matrix of weights. However, for simplicity only odd polynomials

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are allowed. In this case, a new and more general matrix extension of the discrete Painlevé I equation is found.

Contents

1. Introduction

Matrix extensions of real orthogonal polynomials were first discussed back in 1949 by Krein [50,51] and thereafter were studied sporadically until the last decade of the XX century, being some relevant papers [7,12,41]. Then, in 1984, Aptekarev and Nikishin, for a kind of discrete Sturm–Liouville operators, solved the corresponding scattering problem in [7], and found that the polynomials that satisfy a relation of the form

$$xP_k(x) = A_kP_{k+1}(x) + B_kP_k(x) + A^*_{k-1}P_{k-1}(x), \qquad k = 0, 1, \dots,$$

are orthogonal with respect to a positive definite measure; i.e., they derived a matrix version of Favard's theorem.

In a period of 20 years, from 1990 to 2010, it was found that matrix orthogonal polynomials (MOP) satisfy, in some cases, properties as do the classical orthogonal polynomials. The first explicit (nontrivial) example of matrix-valued orthogonal polynomials satisfying a second-order differential equation was given by Grünbaum in [42] as a byproduct of [44–46]. Later, in a very different way, other examples were obtained in [33].

Let us mention, for example, that for matrix versions of Laguerre, Hermite and Jacobi polynomials, i.e., the scalar-type Rodrigues' formula [34,35] and a second order differential equation [13,31,33] has been discussed. It also has been proven [32] that operators of the form $D = \partial^2F_2(t) + \partial^1F_1(t) + \partial^0F_0$ have as eigenfunctions different infinite families of MOP's. A new family of MOP's satisfying second order differential equations, whose three term recurrence relation coefficients do not behave asymptotically as the identity matrix, was found in [13]; see also [15]. We have studied [4,5] matrix extensions of the generalized polynomials studied in [1,2]. Recently, in [6], the Christoffel transformation to matrix orthogonal polynomials on the real line (MOPRL) were extended to obtaining a new matrix Christoffel formula, and in [8,9] more general transformations —of Geronimus and Uvarov type— were also considered.

It was 26 years ago, in 1992, when Fokas, Its and Kitaev, in the context of 2D quantum gravity, discovered that certain Riemann–Hilbert problem was solved in terms of orthogonal polynomials on the real line (OPRL), [37]. Namely, it was found that the solution of a 2×2 Riemann–Hilbert problem can be expressed in terms of orthogonal polynomials on the real line and its Cauchy transforms. Later, Deift and Zhou combined these ideas with a non-linear steepest descent analysis in a series of papers [26,27,29,30] which was the seed for a large activity in the field. To mention just a few relevant results let us cite the study of strong asymptotic with applications in random matrix theory, [26,28], the analysis of determinantal point processes [23,24,52,53], orthogonal Laurent polynomials [56,57] and Painlevé equations [25,49].

The study of equations for the recursion coefficients for OPRL or orthogonal polynomials in the unit circle constitutes a subject of current interest. The question of how the form of the weight and its properties, for example to satisfy a Pearson type equation, translates to the recursion coefficients has been treated in several places, for a review see [63]. In 1976, Freud [38] studied weights in $\mathbb{R}$ of exponential variation $w(x) = |x|^\rho \exp(-|x|^m)$, $\rho > -1$ and $m > 0$. For $m = 2, 4, 6$ he constructed relations among them as well as determined its asymptotic behavior. However, Freud did not find the role of the discrete Painlevé I, that was discovered later by Magnus [55]. For the unit circle and a weight of the form $w(\theta) = \exp(k \cos\theta)$, $k \in \mathbb{R}$, Periwal and Shevitz [60,61], in the context of matrix models, found the discrete Painlevé II equation for the recursion relations of the corresponding orthogonal polynomials. This result was rediscovered later and connected with the Painlevé III equation [48]. In [10] the discrete Painlevé II was found using the Riemann–Hilbert problem given in [11], see also [62]. For a nice account of the relation of these discrete Painlevé equations and integrable systems see [22], and for a survey on the subject of differential and discrete Painlevé equations cf. [19]. We also mention the recent paper [20] where a discussion on the relationship between the recurrence coefficients of orthogonal polynomials with respect to a semiclassical Laguerre weight and classical solutions of the fourth Painlevé equation can be found. Also, in [21] the solution of the discrete alternate Painlevé equations is presented in terms of the Airy function.

In [16] the Riemann–Hilbert problem for this matrix situation and the appearance of non-Abelian discrete versions of Painlevé I were explored, showing singularity confinement [17]. The singularity analysis for a matrix discrete version of the Painlevé I equation was performed. It was found that the singularity confinement holds generically, i.e. in the whole space of parameters except possibly for algebraic subvarieties. The situation was considered in [18] for the matrix extension of the Szegő polynomials in the unit circle and corresponding non-Abelian versions discrete Painlevé II equations. For an alternative discussion of the use of Riemann–Hilbert problem for MOPRL see [47].

Let us mention that in [58,59] and [14] the MOP are expressed in terms of Schur complements that play the role of determinants in the standard scalar case. In [14] an study of matrix Szegő polynomials and the relation with a non Abelian Ablowitz–Ladik lattice is carried out, and in [3] the CMV ordering is applied to study orthogonal Laurent polynomials in the circle.

In this work we obtain Sylvester systems of differential equations for the orthogonal polynomials and its second kind functions, directly from a Riemann–Hilbert problem, with jumps supported on appropriate curves in the complex plane. The differential properties for the weight function are fundamental. In this case we consider a Sylvester type differential Pearson equation for the matrix of weights. We also study whenever the orthogonal polynomials and its second kind functions are solutions of a second order linear differential operator with matrix eigenvalues. This is done by stating an appropriate boundary value problem for the matrix of weights. In particular, special attention is paid to non-Abelian Hermite biorthogonal polynomials on the real line, understood as those whose matrix of weights is a solution of a Sylvester type Pearson equation with given first order matrix polynomial coefficients. In Theorem 5 we give conditions such that Hermite type matrix biorthogonal polynomials and corresponding second kind functions are eigenfunctions of second order differential operators.

Several applications are given, in order of increasing complexity, as well. First, we return to the non-Abelian Hermite biorthogonal polynomials on the real line, and give nonlinear matrix difference equations for the recurrent coefficients of the non-Abelian Hermite biorthogonal polynomials. Next, we consider the orthogonal polynomials and functions of second kind associated with matrix of weights, that satisfy a differential matrix Pearson equation with degree two polynomials as coefficients. To simplify the discussion, only a left Pearson equation is considered. In this case, the support of the measure belongs to an appropriate branch of a hyperbola, and the recursion coefficients are shown to fulfill a non-Abelian extension of the scalar alternate discrete Painlevé I equation. Finally, a discussion is given for the case of degree three polynomials as coefficients in the left Pearson equation characterizing the matrix of weights. However, for simplicity only odd polynomials are allowed. In this case, a new and more general matrix extension of the discrete Painlevé equation is found. To end this study we present a comparison with the results already obtained by several authors in the scalar and matrix cases.

The layout of the paper is as follows. In § 2 we introduce the basic objects and results fundamental to the rest of the work. Then, § 3 is devoted to study the interplay between fundamental matrices with constant jump and structure formulas. In § 4 and 5 we characterize sequences of orthogonal polynomials whose matrix weight satisfies a Pearson–Sylvester matrix differential equation by means of a Sylvester matrix differential system and a second order differential operator. Finally, in § 6 we show how to derive Painlevé equations for the matrix recurrence coefficients of orthogonal polynomial sequences associated with matrix weight functions of "exponential" type.

2. Riemann–Hilbert problem for matrix biorthogonal polynomials

2.1. Matrix biorthogonal polynomials

Let

$$W = \begin{bmatrix} W^{(1,1)} & \dots & W^{(1,N)} \\ \vdots & \ddots & \vdots \\ W^{(N,1)} & \dots & W^{(N,N)} \end{bmatrix} \in \mathbb{C}^{N\times N},$$

be a $N \times N$ matrix of weights with support on a smooth oriented non self-intersecting unbounded curve γ, without end point, in the complex plane $\mathbb{C}$, i.e. $W^{(j,k)}$ is, for each $j,k \in \{1,\dots,N\}$, a complex weight with support on γ. We define the *moment of order* n associated with W as

$$W_n = \frac{1}{2\pi\,\mathrm{i}} \int_\gamma z^n W(z)\,\mathrm{d}\,z, \qquad n \in \mathbb{Z}_+ := \{0,1,2,\dots\}.$$

We say that W is *regular* if the matrix moments, W_n, $n \in \mathbb{Z}_+$, exist and the matrix of moments,

$$\mathsf{U}_n = [W_{j+k}]_{j,k=0,\dots,n} = \begin{bmatrix} W_0 & \cdots & W_n \\ \vdots & \ddots & \vdots \\ W_n & \cdots & W_{2n} \end{bmatrix},$$

is such that

$$\det \mathsf{U}_n \neq 0, \qquad n \in \mathbb{Z}_+. \tag{1}$$

In this way, we define a *sequence of matrix monic polynomials*, $\{P_n^{\mathsf{L}}(z)\}_{n\in\mathbb{Z}_+}$, were $\deg P_n^{\mathsf{L}}(z)=n$, $n\in\mathbb{Z}_+$, *left orthogonal* and *right orthogonal*, $\{P_n^{\mathsf{R}}(z)\}_{n\in\mathbb{Z}_+}$, were $\deg P_n^{\mathsf{R}}(z)=n$, $n\in\mathbb{Z}_+$, with respect to a regular matrix measure W, by the conditions,

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma P_n^{\mathsf{L}}(z)W(z)z^k\,\mathrm{d}\,z=\delta_{n,k}C_n^{-1}, \tag{2}$$

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma z^kW(z)P_n^{\mathsf{R}}(z)\,\mathrm{d}\,z=\delta_{n,k}C_n^{-1}, \tag{3}$$

for $k=0,1,\dots,n$ and $n\in\mathbb{Z}_+$, where C_n is a nonsingular matrix.

We can see that sequence of monic polynomials $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{Z}_+}$ are defined by (2) with respect to a regular matrix weight, W. In fact, taking into account a representation for P_n^{L} as

$$P_n^{\mathsf{L}}(z)=p^0_{\mathsf{L},n}z^n+p^1_{\mathsf{L},n}z^{n-1}+\cdots+p^{n-1}_{\mathsf{L},n}z+p^n_{\mathsf{L},n}$$

such that for each $j=0,1,\dots,n-1$

$$\int_\gamma P_n^{\mathsf{L}}(z)W(z)z^j\,\mathrm{d}\,z=p^0_{\mathsf{L},n}W_{n+j}+p^1_{\mathsf{L},n}W_{n+j-1}+\cdots+p^{n-1}_{\mathsf{L},n}W_{j+1}+p^n_{\mathsf{L},n}W_j=0,$$

and with $j=n$

$$\int_\gamma P_n^{\mathsf{L}}(z)W(z)z^n\,\mathrm{d}\,z=p^0_{\mathsf{L},n}W_{2n}+p^1_{\mathsf{L},n}W_{2n-1}+\cdots+p^{n-1}_{\mathsf{L},n}W_{n+1}+p^n_{\mathsf{L},n}W_n=C_n^{-1}.$$

In matrix notation we have

$$\begin{bmatrix}p^n_{\mathsf{L},n} & p^{n-1}_{\mathsf{L},n} & \cdots & p^1_{\mathsf{L},n} & p^0_{\mathsf{L},n}\end{bmatrix}\mathbf{U}_n=\begin{bmatrix}0 & 0 & \cdots & 0 & C_n^{-1}\end{bmatrix}.$$

From (1) we know that the above linear system has an unique solution, i.e. there exists and are unique the matrices $p^n_{\mathsf{L},n},p^{n-1}_{\mathsf{L},n},\dots,p^1_{\mathsf{L},n},p^0_{\mathsf{L},n}$, and so the sequence $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{Z}_+}$ is uniquely defined up to a multiplicative nonsingular matrix defined by (2).

This last sentence is a direct consequence of the non-singularity of the last block of $\boldsymbol{U}_n^{-1}$, i.e. the one in the position $(n+1),(n+1)$, of the matrix $\boldsymbol{U}_n^{-1}$, as (see for instance [40])

$$\mathbf{U}_n^{-1}=\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]$$

with $D=\Big(W_{2n}-\begin{bmatrix}W_n & \cdots & W_{2n-1}\end{bmatrix}\mathbf{U}_{n-1}^{-1}\begin{bmatrix}W_n^{\mathsf{T}} & \cdots & W_{2n-1}^{\mathsf{T}}\end{bmatrix}^{\mathsf{T}}\Big)^{-1}$, and $\det D=\dfrac{\det\mathbf{U}_{n-1}}{\det\mathbf{U}_n}$. The same can be seen for $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{Z}_+}$.

Notice that neither the matrix of weights is requested to be Hermitian nor the curve γ to be on the real line, i.e., we are dealing, in principle with nonstandard orthogonality and, consequently, with biorthogonal matrix polynomials instead of orthogonal matrix polynomials.

The matrix of weights induces a *sesquilinear like form* in the set of matrix polynomials $\mathbb{C}^{N\times N}[z]$ given by

$$\langle P, Q\rangle_W := \int_\gamma P(z)W(z)Q(z)\,\mathrm{d}\,z,$$

in the sense that, for all $P, P_1, P_2, Q, Q_1, Q_2 \in \mathbb{C}^{N\times N}[z]$ and $A, B \in \mathbb{C}^{N\times N}$ we have

$$\begin{aligned}\langle P_1+P_2, Q_1+Q_2\rangle_W &= \langle P_1, Q_1\rangle_W + \langle P_2, Q_1\rangle_W + \langle P_1, Q_2\rangle_W + \langle P_2, Q_2\rangle_W,\\ \langle A\,P, Q\,B\rangle_W &= A\,\langle P, Q\rangle_W\,B.\end{aligned}$$

Moreover, we say that $\{P_n^{\mathsf{L}}(z)\}_{n\in\mathbb{Z}_+}$ and $\{P_n^{\mathsf{R}}(z)\}_{n\in\mathbb{Z}_+}$ are biorthogonal with respect to a matrix weight functions W, as from (2) and (3)

$$\frac{1}{2\pi\,\mathrm{i}}\langle P_n^{\mathsf{L}}, P_m^{\mathsf{R}}\rangle_W = \delta_{n,m}C_n^{-1}, \qquad n, m \in \mathbb{Z}_+. \tag{4}$$

As the polynomials are chosen to be monic, we can write

$$\begin{aligned}P_n^{\mathsf{L}}(z) &= Iz^n + p^1_{\mathsf{L},n}z^{n-1} + p^2_{\mathsf{L},n}z^{n-2} + \cdots + p^n_{\mathsf{L},n},\\ P_n^{\mathsf{R}}(z) &= Iz^n + p^1_{\mathsf{R},n}z^{n-1} + p^2_{\mathsf{R},n}z^{n-2} + \cdots + p^n_{\mathsf{R},n},\end{aligned}$$

with matrix coefficients $p^k_{\mathsf{L},n}, p^k_{\mathsf{R},n} \in \mathbb{C}^{N\times N}$, $k = 0, \ldots, n$ and $n \in \mathbb{Z}_+$ (imposing that $p^0_{\mathsf{L},n} = p^0_{\mathsf{R},n} = I$, $n \in \mathbb{Z}_+$). Here $I \in \mathbb{C}^{N\times N}$ denotes the identity matrix.

2.2. Three term relations

From (2) we deduce that the Fourier coefficients of the expansion

$$zP_n^{\mathsf{L}}(z) = \sum_{k=0}^{n+1} \ell^n_{\mathsf{L},k} P_k^{\mathsf{L}}(z),$$

are given by $\ell^n_{\mathsf{L},k} = 0_N$, $k = 0, 1, \ldots, n-2$ (here we denote the zero matrix by 0_N), $\ell^n_{\mathsf{L},n-1} = C_n^{-1}C_{n-1}$ (is a direct consequence of orthogonality conditions), $\ell^n_{\mathsf{L},n+1} = I$ (as $P_n^{\mathsf{L}}(z)$ are monic polynomials) and $\ell^n_{\mathsf{L},n} = p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1} =: \beta_n^{\mathsf{L}}$ (by comparison of the coefficients, assuming $C_0 = I$).

Hence, assuming the orthogonality relations (2), we conclude that the sequence of monic polynomials $\{P_n^{\mathsf{L}}(z)\}_{n\in\mathbb{Z}_+}$ is defined by the three term recurrence relations

$$zP_n^{\mathsf{L}}(z) = P_{n+1}^{\mathsf{L}}(z) + \beta_n^{\mathsf{L}}P_n^{\mathsf{L}}(z) + \gamma_n^{\mathsf{L}}P_{n-1}^{\mathsf{L}}(z), \qquad n \in \mathbb{Z}_+, \tag{5}$$

with recursion coefficients

$$\beta_n^{\mathsf{L}} := p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1}, \qquad \gamma_n^{\mathsf{L}} := C_n^{-1}C_{n-1},$$

with initial conditions, $P_{-1}^{\mathsf{L}} = 0_N$ and $P_0^{\mathsf{L}} = I$.

Any sequence of monic matrix polynomials, $\{P_n^{\mathsf{R}}(z)\}_{n\in\mathbb{Z}_+}$, with $\deg P_n^{\mathsf{R}} = n$, biorthogonal with respect to $\{P_n^{\mathsf{L}}(z)\}_{n\in\mathbb{Z}_+}$ and $W(z)$, i.e. (4) is fulfilled, also satisfies a three term relation. To prove this we compute the Fourier coefficients of $zP_m^{\mathsf{R}}(z)$ in the expansion

$$zP_n^{\mathsf{R}}(z) = \sum_{k=0}^{n+1} P_k^{\mathsf{R}}(z)\ell^n_{\mathsf{R},k}, \qquad \ell^n_{\mathsf{R},k} = \frac{1}{2\pi\,\mathrm{i}}\int_\gamma zP_k^{\mathsf{L}}(z)W(z)P_n^{\mathsf{R}}(z)\,\mathrm{d}\,z.$$

From (2) we have $\ell^n_{\mathsf{R},n+1}=I$, $\ell^n_{\mathsf{R},n}=C_n\beta^{\mathsf{L}}_nC_n^{-1}$, $\ell^n_{\mathsf{R},n-1}=C_{n-1}C_n^{-1}$, and $\ell^n_{\mathsf{R},k}=0_N$, $k=0,\dots,n-2$, i.e. the sequence of monic polynomials $\big\{P^{\mathsf{R}}_n(z)\big\}_{n\in\mathbb{Z}_+}$ satisfies

$$P^{\mathsf{R}}_{-1}=0_N, \qquad P^{\mathsf{R}}_0=I, \qquad zP^{\mathsf{R}}_n(z)=P^{\mathsf{R}}_{n+1}(z)+P^{\mathsf{R}}_n(z)\beta^{\mathsf{R}}_n+P^{\mathsf{R}}_{n-1}(z)\gamma^{\mathsf{R}}_n, \qquad n\in\mathbb{Z}_+, \tag{6}$$

where

$$\beta^{\mathsf{R}}_n:=C_n\beta^{\mathsf{L}}_nC_n^{-1}, \qquad \gamma^{\mathsf{R}}_n:=C_n\gamma^{\mathsf{L}}_nC_n^{-1}=C_{n-1}C_n^{-1},$$

and the orthogonality conditions (3) are satisfied.

2.3. Second kind functions

We define the *sequence of second kind matrix functions* by

$$Q^{\mathsf{L}}_n(z):=\frac{1}{2\pi\,\mathrm{i}}\int_\gamma\frac{P^{\mathsf{L}}_n(z')}{z'-z}W(z')\,\mathrm{d}\,z', \tag{7}$$

$$Q^{\mathsf{R}}_n(z):=\frac{1}{2\pi\,\mathrm{i}}\int_\gamma W(z')\frac{P^{\mathsf{R}}_n(z')}{z'-z}\,\mathrm{d}\,z', \tag{8}$$

for $n\in\mathbb{Z}_+$. From the orthogonality conditions (2) and (3) we have, for all $n\in\mathbb{Z}_+$, the following asymptotic expansion near infinity for the sequence of functions of the second kind

$$Q^{\mathsf{L}}_n(z)=-C_n^{-1}\big(Iz^{-n-1}+q^1_{\mathsf{L},n}z^{-n-2}+\cdots\big), \tag{9}$$

$$Q^{\mathsf{R}}_n(z)=-\big(Iz^{-n-1}+q^1_{\mathsf{R},n}z^{-n-2}+\cdots\big)C_n^{-1}. \tag{10}$$

From now on we assume that the measures $W^{(j,k)}$, $j,k\in\{1,\dots,N\}$ are Hölder continuous. Hence using the Plemelj's formula, cf. [39], applied to (7) and (8), the following fundamental jump identities hold

$$\big(Q^{\mathsf{L}}_n(z)\big)_+-\big(Q_n(z)^{\mathsf{L}}\big)_-=P^{\mathsf{L}}_n(z)W(z), \tag{11}$$

$$\big(Q^{\mathsf{R}}_n(z)\big)_+-\big(Q^{\mathsf{R}}_n(z)\big)_-=W(z)P^{\mathsf{R}}_n(z), \tag{12}$$

$z\in\gamma$, where, $\big(f(z)\big)_\pm=\lim\limits_{\epsilon\to0^\pm}f(z+i\epsilon)$; here $\pm$ indicates the positive/negative region according to the orientation of the curve γ.

Now, multiplying equation (5) on the right by W and integrating we get, using the definition (7) of $\big\{Q^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{Z}_+}$, that

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma\frac{z'P^{\mathsf{L}}_n(z')}{z'-z}W(z')\,\mathrm{d}\,z'=Q^{\mathsf{L}}_{n+1}(z)+\beta^{\mathsf{L}}_nQ^{\mathsf{L}}_n(z)+C_n^{-1}C_{n-1}Q^{\mathsf{L}}_{n-1}(z).$$

As $\frac{z'}{z'-z}=1+\frac{z}{z'-z}$, from the orthogonality conditions (2) we conclude that

$$zQ^{\mathsf{L}}_n(z)=Q^{\mathsf{L}}_{n+1}(z)+\beta^{\mathsf{L}}_nQ^{\mathsf{L}}_n(z)+C_n^{-1}C_{n-1}Q^{\mathsf{L}}_{n-1}(z), \qquad n\in\mathbb{Z}_+,$$

with initial conditions

$$Q^{\mathsf{L}}_{-1}(z)=Q^{\mathsf{R}}_{-1}(z)=-C_{-1}^{-1}\quad\text{and}\quad Q^{\mathsf{L}}_{0}(z)=Q^{\mathsf{R}}_{0}(z)=S_W(z):=\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\gamma}\frac{W(z')}{z'-z}\,\mathrm{d}\,z',$$

where $S_W(z)$ is the Stieltjes–Markov like transformation of the matrix of weights W.

Sometimes in the literature some authors distinguish between Markov transforms and Stieltjes transform when we are dealing with measure defined on a bounded or an unbounded interval, respectively, of the real line. Here we unified the notion as the scalar Markov convergence theorem (stated for the bounded case) is still valid for the unbounded case when the moment problem is determined.

It can be seen that

$$P^{\mathsf{L}}_n(z)Q_0(z)=-\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\gamma}\frac{P^{\mathsf{L}}_n(z')-P^{\mathsf{L}}_n(z)}{z'-z}W(z')\,\mathrm{d}\,z'+\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\gamma}\frac{P^{\mathsf{L}}_n(z')}{z'-z}W(z')\,\mathrm{d}\,z',$$

i.e. we have the Hermite–Padé like formula for the left orthogonal polynomials,

$$P^{\mathsf{L}}_n(z)S_W(z)+P^{\mathsf{L},(1)}_{n-1}(z)=Q^{\mathsf{L}}_n(z),\qquad n\in\mathbb{Z}_+,$$

where

$$P^{\mathsf{L},(1)}_{n-1}(z)=\frac{1}{2\pi\,\mathrm{i}}\int\frac{P^{\mathsf{L}}_n(z')-P^{\mathsf{L}}_n(z)}{z'-z}W(z')\,\mathrm{d}\,w,\qquad n\in\mathbb{Z}_+,$$

is a polynomial of degree at most $n-1$ said to be the *first kind associated polynomial with respect to* $\{P^{\mathsf{L}}_n(z)\}_{n\in\mathbb{Z}_+}$ *and* $W(z)$. Similarly, for the right situation we have the associated

$$P^{\mathsf{R},(1)}_{n}(z)=\frac{1}{2\pi i}\int\limits_{\gamma}W(z')\frac{P^{\mathsf{R}}_{n+1}(z')-P^{\mathsf{R}}_{n+1}(z)}{z'-z}\,\mathrm{d}\,w,\qquad n\in\mathbb{Z}_+,$$

and the corresponding Hermite–Padé like formula for the right orthogonal polynomials,

$$S_W(z)P^{\mathsf{R}}_n(z)+P^{\mathsf{R},(1)}_{n-1}(z)=Q^{\mathsf{R}}_n(z)\qquad n\in\mathbb{Z}_+.$$

2.4. Reductions: from biorthogonality to orthogonality

We consider two possible reductions:

i) When the matrix of weights $W(z)$ with support on γ is symmetric, i.e. $(W(z))^\top=W(z)$, $z\in\gamma$, then

$$P^{\mathsf{R}}_n(z)=\big(P^{\mathsf{L}}_n(z)\big)^\top,\qquad Q^{\mathsf{R}}_n(z)=\big(Q^{\mathsf{L}}_n(z)\big)^\top,\qquad z\in\mathbb{C}.$$

Moreover,

$$\langle P^{\mathsf{L}}_n,(P^{\mathsf{L}}_n)^\top\rangle_W=\int\limits_{\gamma}P^{\mathsf{L}}_n(x)W(x)(P^{\mathsf{L}}_n(x))^\top\,\mathrm{d}\,x.$$

ii) When the matrix of weights $W(z)$ is Hermitian positive definite with support on $\gamma\subset\mathbb{R}$, i.e. $(W(x))^\dagger=W(x)$, $x\in\mathbb{R}$, then

$$P^{\mathsf{R}}_n(z)=\big(P^{\mathsf{L}}_n(\bar z)\big)^\dagger,\qquad Q^{\mathsf{R}}_n(z)=\big(Q^{\mathsf{L}}_n(\bar z)\big)^\dagger,\qquad z\in\mathbb{C}.$$

In this case we have

$$\langle P^{\mathsf{L}}_n, (P^{\mathsf{L}}_n)^\dagger\rangle_W = \int_{\mathbb{R}} P^{\mathsf{L}}_n(x)W(x)(P^{\mathsf{L}}_n(x))^\dagger \,\mathrm{d}\, x.$$

2.5. Fundamental and transfer matrices vs. Riemann–Hilbert problems

We can summarize the left three term relation as follows

$$\begin{bmatrix} P^{\mathsf{L}}_{n+1}(z) & Q^{\mathsf{L}}_{n+1}(z) \\ -C_n P^{\mathsf{L}}_n(z) & -C_n Q^{\mathsf{L}}_n(z)\end{bmatrix} = \begin{bmatrix} zI-\beta^{\mathsf{L}}_n & C_n^{-1} \\ -C_n & 0_N\end{bmatrix}\begin{bmatrix} P^{\mathsf{L}}_n(z) & Q^{\mathsf{L}}_n(z) \\ -C_{n-1} P^{\mathsf{L}}_{n-1}(z) & -C_{n-1} Q^{\mathsf{L}}_{n-1}(z)\end{bmatrix};$$

and

$$\begin{bmatrix} P^{\mathsf{L},(1)}_n(z) \\ -C_n P^{\mathsf{L},(1)}_{n-1}(z)\end{bmatrix} = \begin{bmatrix} zI-\beta^{\mathsf{L}}_n & C_n^{-1} \\ -C_n & 0_N\end{bmatrix}\begin{bmatrix} P^{\mathsf{L},(1)}_{n-1}(z) \\ -C_{n-1} P^{\mathsf{L},(1)}_{n-2}(z)\end{bmatrix}.$$

In terms of the left *fundamental matrix* $Y^{\mathsf{L}}_n(z)$ and the left *transfer matrix* $T^{\mathsf{L}}_n(z)$,

$$Y^{\mathsf{L}}_n(z) := \begin{bmatrix} P^{\mathsf{L}}_n(z) & Q^{\mathsf{L}}_n(z) \\ -C_{n-1} P^{\mathsf{L}}_{n-1}(z) & -C_{n-1} Q^{\mathsf{L}}_{n-1}(z)\end{bmatrix}, \qquad T^{\mathsf{L}}_n(z) := \begin{bmatrix} zI-\beta^{\mathsf{L}}_n & C_n^{-1} \\ -C_n & 0_N\end{bmatrix},$$

we rewrite the above identities as follows

$$Y^{\mathsf{L}}_{n+1}(z) = T^{\mathsf{L}}_n(z) Y^{\mathsf{L}}_n(z), \qquad n\in\mathbb{Z}_+.$$

From these we see that $\det Y^{\mathsf{L}}_n(z) = \det Y^{\mathsf{L}}_0(z) = 1$, as $\det T^{\mathsf{L}}_n = 1$ on $\mathbb{C}\setminus\gamma$ for $n\in\mathbb{Z}_+$.

For the right orthogonality, we similarly obtain from (6) that

$$\begin{bmatrix} P^{\mathsf{R}}_{n+1}(z) & -P^{\mathsf{R}}_n(z)C_n \\ Q^{\mathsf{R}}_{n+1}(z) & -Q^{\mathsf{R}}_n(z)C_n\end{bmatrix} = \begin{bmatrix} P^{\mathsf{R}}_n(z) & -P^{\mathsf{R}}_{n-1}(z)C_{n-1} \\ Q^{\mathsf{R}}_n(z) & -Q^{\mathsf{R}}_{n-1}(z)C_{n-1}\end{bmatrix}\begin{bmatrix} zI-\beta^{\mathsf{R}}_n & -C_n \\ C_n^{-1} & 0_N\end{bmatrix}$$

and also

$$\begin{bmatrix} P^{\mathsf{R},(1)}_n(z) & -P^{\mathsf{R},(1)}_{n-1}(z)C_n\end{bmatrix} = \begin{bmatrix} P^{\mathsf{R},(1)}_{n-1}(z) & -P^{\mathsf{R},(1)}_{n-2}(z)C_n\end{bmatrix}\begin{bmatrix} zI-\beta^{\mathsf{R}}_n & -C_n \\ C_n^{-1} & 0_N\end{bmatrix}$$

as we have the Hermite–Padé like formula for the right orthogonal polynomials,

$$Q^{\mathsf{R}}_0(z)\, P^{\mathsf{R}}_m(z) + P^{\mathsf{R},(1)}_{m-1}(z) = Q^{\mathsf{R}}_m(z)\,.$$

Taking the right versions of *fundamental matrix* $Y^{\mathsf{R}}_n(z)$ and *transfer matrix* $T^{\mathsf{R}}_n(z)$,

$$Y^{\mathsf{R}}_n(z) := \begin{bmatrix} P^{\mathsf{R}}_n(z) & -P^{\mathsf{R}}_{n-1}(z)C_{n-1} \\ Q^{\mathsf{R}}_n(z) & -Q^{\mathsf{R}}_{n-1}(z)C_{n-1}\end{bmatrix}, \qquad T^{\mathsf{R}}_m(z) := \begin{bmatrix} zI-\beta^{\mathsf{R}}_n & -C_n \\ C_n^{-1} & 0_N\end{bmatrix},$$

we see that $\det Y^{\mathsf{R}}_n(z) = \det\, Y^{\mathsf{R}}_0(z) = 1$, because $\det T^{\mathsf{R}}_n = 1$ on $\mathbb{C}\setminus\gamma$ for $n\in\mathbb{Z}_+$.

Note that,

$$T^{\mathsf{R}}_n(z) = \begin{bmatrix} C_n & 0_N \\ 0_N & -C_n^{-1}\end{bmatrix} T^{\mathsf{L}}_n(z)\begin{bmatrix} C_n & 0_N \\ 0_N & -C_n^{-1}\end{bmatrix}^{-1}, \qquad n\in\mathbb{Z}_+.$$

Now we can state the following left Riemann–Hilbert problem.

Theorem 1. *The matrix function $Y_n^{\mathsf{L}}(z)$ is, for each $n \in \mathbb{Z}_+$, the unique solution of the Riemann–Hilbert problem; which consists in the determination of a $2N \times 2N$ complex matrix function such that:*

(RH1): $Y_n^{\mathsf{L}}(z)$ *is holomorphic in* $\mathbb{C} \setminus \gamma$*;*
(RH2): *has the following asymptotic behavior near infinity,*

$$Y_n^{\mathsf{L}}(z) = \big(I + \mathrm{O}(z^{-1})\big) \begin{bmatrix} Iz^n & 0_N \\ 0_N & Iz^{-n} \end{bmatrix};$$

(RH3): *satisfies the jump condition*

$$\big(Y_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} I & W(z) \\ 0_N & I \end{bmatrix}, \qquad z \in \gamma.$$

As well as its right version.

Theorem 2. *The matrix function $Y_n^{\mathsf{R}}(z)$ is, for each $n \in \mathbb{Z}_+$, the unique solution of the Riemann–Hilbert problem; which consists in the determination of a $2N \times 2N$ complex matrix function such that:*

(RH1): $Y_n^{\mathsf{R}}(z)$ *is holomorphic in* $\mathbb{C} \setminus \gamma$*;*
(RH2): *has the following asymptotic behavior near infinity,*

$$Y_n^{\mathsf{R}}(z) = \begin{bmatrix} Iz^n & 0_N \\ 0_N & Iz^{-n} \end{bmatrix} \big(I + \mathrm{O}(z^{-1})\big);$$

(RH3): *satisfies the jump condition*

$$\big(Y_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} I & 0_N \\ W(z) & I \end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)_-, \qquad z \in \gamma.$$

Remark 1. Conditions (RH2) and (RH3) are direct consequences of the representation of the second kind functions (9), (10) and the inverse formulas (11), (12), respectively.

Remark 2. For the symmetric and Hermitian reductions these two Riemann–Hilbert problems are equivalent and for the fundamental matrices we have

$$\begin{aligned} Y_n^{\mathsf{R}}(z) &= \big(Y_n^{\mathsf{L}}(z)\big)^{\top}, && \text{symmetric case,} \\ Y_n^{\mathsf{R}}(z) &= \big(Y_n^{\mathsf{L}}(\bar{z})\big)^{\dagger}, && \text{Hermitian case.} \end{aligned}$$

In both cases, we will use the notation

$$Y_n(z) := Y_n^{\mathsf{L}}(z).$$

We define the family of *normalized left fundamental matrices* $\big\{S_n^{\mathsf{L}}(z)\big\}_{n \in \mathbb{Z}_+}$ associated with $\big\{Y_n^{\mathsf{L}}(z)\big\}_{n \in \mathbb{Z}_+}$ by means of

$$S_n^{\mathsf{L}}(z) := Y_n^{\mathsf{L}}(z) \begin{bmatrix} Iz^{-n} & 0_N \\ 0_N & Iz^n \end{bmatrix}, \qquad n \in \mathbb{Z}_+.$$

Taking into account the representation of $\big\{P_n^{\mathsf{L}}(z)\big\}_{n \in \mathbb{Z}_+}$ and $\big\{Q_n^{\mathsf{L}}(z)\big\}_{n \in \mathbb{Z}_+}$ in (5), we arrive to the asymptotic representation for the normalized fundamental matrices

$$S_n^{\mathsf{L}}(z)=I+\begin{bmatrix} p^1_{\mathsf{L},n} & -C_n^{-1}\\ -C_{n-1} & q^1_{\mathsf{L},n-1}\end{bmatrix} z^{-1}+\begin{bmatrix} p^2_{\mathsf{L},n} & -C_n^{-1}q^1_{\mathsf{L},n}\\ -C_{n-1}p^1_{\mathsf{L},n-1} & q^2_{\mathsf{L},n-1}\end{bmatrix} z^{-2}+O(z^{-3}),$$

for $z\to\infty$, where

$$\begin{aligned} p^1_{\mathsf{L},n}-p^1_{\mathsf{L},n+1}&=\beta_n^{\mathsf{L}},\\ p^2_{\mathsf{L},n}-p^2_{\mathsf{L},n+1}&=\beta_n^{\mathsf{L}}p^1_{\mathsf{L},n}+C_n^{-1}C_{n-1},\\ p^3_{\mathsf{L},n}-p^3_{\mathsf{L},n+1}&=\beta_n^{\mathsf{L}}p^2_{\mathsf{L},n}+C_n^{-1}C_{n-1}p^1_{\mathsf{L},n-1},\end{aligned}$$

and

$$\begin{aligned} q^1_{\mathsf{L},n}-q^1_{\mathsf{L},n-1}&=\beta_n^{\mathsf{R}},\\ q^2_{\mathsf{L},n}-q^2_{\mathsf{L},n-1}&=\beta_n^{\mathsf{R}}q^1_{\mathsf{L},n}+C_nC_{n+1}^{-1}.\end{aligned}$$

Observe that we will also have the following asymptotics for $z\to\infty$,

$$\begin{aligned}\big(S_n^{\mathsf{L}}(z)\big)^{-1}=I-\begin{bmatrix} p^1_{\mathsf{L},n} & -C_n^{-1}\\ -C_{n-1} & q^1_{\mathsf{L},n-1}\end{bmatrix} z^{-1}&\\ +\left(\begin{bmatrix} p^1_{\mathsf{L},n} & -C_n^{-1}\\ -C_{n-1} & q^1_{\mathsf{L},n-1}\end{bmatrix}^2-\begin{bmatrix} p^2_{\mathsf{L},n} & -C_n^{-1}q^1_{\mathsf{L},n}\\ -C_{n-1}p^1_{\mathsf{L},n-1} & q^2_{\mathsf{L},n-1}\end{bmatrix}\right)&z^{-2}+O(z^{-3}).\end{aligned}$$

For the right version we have *normalized right fundamental matrices* $\big\{S_n^{\mathsf{R}}(z)\big\}_{n\in\mathbb{Z}_+}$ associated with $\big\{Y_n^{\mathsf{R}}(z)\big\}_{n\in\mathbb{Z}_+}$

$$S_n^{\mathsf{R}}(z)=\begin{bmatrix} I\,z^{-n} & 0_N\\ 0_N & I\,z^{n}\end{bmatrix} Y_m^{\mathsf{R}}(z),$$

with asymptotic behavior at infinity given by

$$S_n^{\mathsf{R}}(z)=I+\begin{bmatrix} p^1_{\mathsf{R},n} & -C_{n-1}\\ -C_n^{-1} & q^1_{\mathsf{R},n-1}\end{bmatrix} z^{-1}+\begin{bmatrix} p^2_{\mathsf{R},n} & -p^1_{\mathsf{R},n-1}C_{n-1}\\ -q^1_{\mathsf{R},n}C_n^{-1} & q^2_{\mathsf{R},n-1}\end{bmatrix} z^{-2}+O(z^{-3}),$$

for $z\to\infty$, and the asymptotics for the inverse matrix is

$$\begin{aligned}\big(S_n^{\mathsf{R}}(z)\big)^{-1}=I-\begin{bmatrix} p^1_{\mathsf{R},n} & -C_{n-1}\\ -C_n^{-1} & q^1_{\mathsf{R},n-1}\end{bmatrix} z^{-1}&\\ +\left(\begin{bmatrix} p^1_{\mathsf{R},n} & -C_{n-1}\\ -C_n^{-1} & q^1_{\mathsf{R},n-1}\end{bmatrix}^2-\begin{bmatrix} p^2_{\mathsf{R},n} & -p^1_{\mathsf{R},n-1}C_{n-1}\\ -q^1_{\mathsf{R},n}C_n^{-1} & q^2_{\mathsf{R},n-1}\end{bmatrix}\right)&z^{-2}+O(z^{-3}).\end{aligned}$$

Here

$$\begin{aligned} p^1_{\mathsf{R},n}-p^1_{\mathsf{L},n+1}&=\beta_n^{\mathsf{R}},\\ p^2_{\mathsf{R},n}-p^2_{\mathsf{L},n+1}&=p^1_{\mathsf{L},n}\beta_n^{\mathsf{R}}+C_{n-1}C_n^{-1},\\ p^3_{\mathsf{R},n}-p^3_{\mathsf{R},n+1}&=p^2_{\mathsf{L},n}\beta_n^{\mathsf{R}}+p^1_{\mathsf{L},n-1}C_{n-1}C_n^{-1},\end{aligned}$$

and

$$q^1_{\mathsf{R},n} - q^1_{\mathsf{L},n-1} = \beta^{\mathsf{L}}_n,$$
$$q^2_{\mathsf{R},n} - q^2_{\mathsf{L},n-1} = q^1_{\mathsf{L},n}\beta^{\mathsf{L}}_n + C^{-1}_{n+1}C_n.$$

Theorem 3. *Let Y^{L}_n and Y^{R}_n be, for each $n \in \mathbb{Z}_+$, the unique solutions of the Riemann–Hilbert problems in Theorems 1 and 2, respectively; then*

$$(Y^{\mathsf{L}}_n(z))^{-1} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} Y^{\mathsf{R}}_n(z) \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \qquad n \in \mathbb{Z}_+. \tag{13}$$

Proof. Let us remember that $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$ satisfies (5), i.e.

$$zP^{\mathsf{L}}_n(z) = P^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n P^{\mathsf{L}}_n(z) + C^{-1}_n C_{n-1} P^{\mathsf{L}}_{n-1}(z), \qquad n \in \mathbb{Z}_+,$$

with initial conditions $P^{\mathsf{L}}_{-1} = 0_N$ and $P^{\mathsf{L}}_0 = I$; and $\{P^{\mathsf{R}}_n\}_{n\in\mathbb{Z}_+}$ satisfies (6), i.e.

$$tP^{\mathsf{R}}_n(t) = P^{\mathsf{R}}_{n+1}(t) + P^{\mathsf{R}}_n(t) C_n \beta^{\mathsf{L}}_n C^{-1}_n + P^{\mathsf{R}}_{n-1}(t) C_{n-1} C^{-1}_n, \qquad n \in \mathbb{Z}_+,$$

with initial conditions $P^{\mathsf{R}}_{-1} = 0_N$ and $P^{\mathsf{R}}_0 = I$. Multiplying the first equation on the left by $P^{\mathsf{R}}_n(t)C_n$ and the second one on the right by $C_n P^{\mathsf{L}}_n(z)$ and summing up, we arrive after applying telescoping rule

$$(z-t)\sum_{k=0}^{n} P^{\mathsf{R}}_k(t) C_k P^{\mathsf{L}}_k(z) = P^{\mathsf{R}}_n(t) C_n P^{\mathsf{L}}_{n+1}(z) - P^{\mathsf{R}}_{n+1}(t) C_n P^{\mathsf{L}}_n(z), \qquad n \in \mathbb{Z}_+; \tag{14}$$

hence for $t = z$,

$$P^{\mathsf{R}}_n(z) C_n P^{\mathsf{L}}_{n+1}(z) = P^{\mathsf{R}}_{n+1}(z) C_n P^{\mathsf{L}}_n(z), \qquad n \in \mathbb{Z}_+; \tag{15}$$

As $\{Q^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$ (respectively, $\{Q^{\mathsf{R}}_n\}_{n\in\mathbb{Z}_+}$) satisfies (5) (respectively, (6)), with initial conditions $Q^{\mathsf{L}}_{-1} = Q^{\mathsf{R}}_{-1} = -C^{-1}_{-1}$, $Q^{\mathsf{L}}_0 = Q^{\mathsf{R}}_0 = S_W(z)$, proceeding in the same way with $\{Q^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$ and $\{Q^{\mathsf{R}}_n\}_{n\in\mathbb{Z}_+}$ in place of $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$ and $\{P^{\mathsf{R}}_n\}_{n\in\mathbb{Z}_+}$, respectively, we arrive, for all $n \in \mathbb{Z}_+$, to

$$(z-t)\sum_{k=0}^{n} Q^{\mathsf{R}}_k(t) C_k Q^{\mathsf{L}}_k(z) = Q^{\mathsf{R}}_n(t) C_n Q^{\mathsf{L}}_{n+1}(z) - Q^{\mathsf{R}}_{n+1}(t) C_n Q^{\mathsf{L}}_n(z) + S_W(z) - S_W(t); \tag{16}$$

hence for $t = z$,

$$Q^{\mathsf{R}}_n(z) C_n Q^{\mathsf{L}}_{n+1}(z) = Q^{\mathsf{R}}_{n+1}(z) C_n Q^{\mathsf{L}}_n(z), \qquad n \in \mathbb{Z}_+. \tag{17}$$

Applying the same procedure mixing the P's and the Q's we get, for all $n \in \mathbb{Z}_+$,

$$(z-t)\sum_{k=0}^{n} Q^{\mathsf{R}}_k(t) C_k P^{\mathsf{L}}_k(z) = Q^{\mathsf{R}}_n(t) C_n P^{\mathsf{L}}_{n+1}(z) - Q^{\mathsf{R}}_{n+1}(t) C_n P^{\mathsf{L}}_n(z) + I, \tag{18}$$

$$(z-t)\sum_{k=0}^{n} P^{\mathsf{R}}_k(t) C_k Q^{\mathsf{L}}_k(z) = P^{\mathsf{R}}_n(t) C_n Q^{\mathsf{L}}_{n+1}(z) - P^{\mathsf{R}}_{n+1}(t) C_n Q^{\mathsf{L}}_n(z) - I, \tag{19}$$

and when $t = z$ we arrive to, for all $n \in \mathbb{Z}_+$,

$$Q^{\mathsf{R}}_{n+1}(z)C_nP^{\mathsf{L}}_n(z) - Q^{\mathsf{R}}_n(z)C_nP^{\mathsf{L}}_{n+1}(z) = I, \tag{20}$$

$$P^{\mathsf{R}}_n(z)C_nQ^{\mathsf{L}}_{n+1}(z) - P^{\mathsf{R}}_{n+1}(z)C_nQ^{\mathsf{L}}_n(z) = I. \tag{21}$$

Equations (14), (16), (18) and (19) are known in the literature as Christoffel-Darboux formulas. Now, from (15), (17), (20) and (21) we conclude that

$$\begin{bmatrix} -Q^{\mathsf{R}}_{n-1}(z)C_{n-1} & -Q^{\mathsf{R}}_n(z) \\ P^{\mathsf{R}}_{n-1}(z)C_{n-1} & P^{\mathsf{R}}_n(z) \end{bmatrix} Y^{\mathsf{L}}_n(z) = I, \qquad n \in \mathbb{Z}_+,$$

and as

$$\begin{bmatrix} -Q^{\mathsf{R}}_{n-1}(z)C_{n-1} & -Q^{\mathsf{R}}_n(z) \\ P^{\mathsf{R}}_{n-1}(z)C_{n-1} & P^{\mathsf{R}}_n(z) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} Y^{\mathsf{R}}_n(z) \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \qquad n \in \mathbb{Z}_+,$$

we get the desired result. □

Corollary 1. *In the conditions of Theorem 3 we have that for all $n \in \mathbb{Z}_+$,*

$$Q^{\mathsf{L}}_n(z)P^{\mathsf{R}}_{n-1}(z) - P^{\mathsf{L}}_n(z)Q^{\mathsf{R}}_{n-1}(z) = C^{-1}_{n-1}, \tag{22}$$

$$P^{\mathsf{L}}_{n-1}(z)Q^{\mathsf{R}}_n(z) - Q^{\mathsf{L}}_{n-1}(z)P^{\mathsf{R}}_n(z) = C^{-1}_{n-1}, \tag{23}$$

$$Q^{\mathsf{L}}_n(z)P^{\mathsf{R}}_n(z) - P^{\mathsf{L}}_n(z)Q^{\mathsf{R}}_n(z) = 0. \tag{24}$$

Proof. As we already proved that the matrix

$$\begin{bmatrix} -Q^{\mathsf{R}}_{n-1}(z)C_{n-1} & -Q^{\mathsf{R}}_n(z) \\ P^{\mathsf{R}}_{n-1}(z)C_{n-1} & P^{\mathsf{R}}_n(z) \end{bmatrix},$$

is the inverse of $Y^{\mathsf{L}}_n(z)$, i.e.

$$Y^{\mathsf{L}}_n(z) \begin{bmatrix} -Q^{\mathsf{R}}_{n-1}(z)C_{n-1} & -Q^{\mathsf{R}}_n(z) \\ P^{\mathsf{R}}_{n-1}(z)C_{n-1} & P^{\mathsf{R}}_n(z) \end{bmatrix} = I;$$

and multiplying the two matrices we get the result. □

Corollary 2. *In the conditions of Theorem 3 we have that for all $n \in \mathbb{Z}_+$,*

$$(S^{\mathsf{L}}_n(z))^{-1} = I + \begin{bmatrix} q^1_{\mathsf{R},n-1} & C^{-1}_n \\ C_{n-1} & p^1_{\mathsf{R},n} \end{bmatrix} z^{-1} + \begin{bmatrix} q^2_{\mathsf{R},n-1} & q^1_{\mathsf{R},n}C^{-1}_n \\ p^2_{\mathsf{R},n-1}C_{n-1} & p^2_{\mathsf{R},n} \end{bmatrix} z^{-2} + \cdots,$$

$$(S^{\mathsf{R}}_n(z))^{-1} = I + \begin{bmatrix} q^1_{\mathsf{L},n-1} & C_{n-1} \\ C^{-1}_n & p^1_{\mathsf{L},n} \end{bmatrix} z^{-1} + \begin{bmatrix} q^2_{\mathsf{L},n-1} & C_{n-1}p^1_{\mathsf{L},n-1} \\ C^{-1}_n p^2_{\mathsf{L},n-1} & p^2_{\mathsf{L},n} \end{bmatrix} z^{-2} + \cdots.$$

3. Constant jump on the support, structure matrices and zero curvature

So far we discussed the connection between biorthogonal families of matrix polynomials for a given matrix of weights W and a specific Riemann–Hilbert problem. Now, to derive difference and/or differential equations satisfied by these families of matrix polynomials we will move to a simpler setting and we will assume that the following hold

i) The matrix of weights factors out as $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$, $z \in \gamma$.

ii) The factors W^{L} and W^{R} are the restriction to the curve γ of matrices of entire functions $W^{\mathsf{L}}(z)$ and $W^{\mathsf{R}}(z)$, $z \in \mathbb{C}$.

iii) The right logarithmic derivative $h^{\mathsf{L}}(z) := \big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1}$ and the left logarithmic derivative $h^{\mathsf{R}}(z) := \big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)'$ exist and are entire functions.

We underline that for a given matrix of weights $W(z)$ we will have many possible factorization $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$. Indeed, if we define an equivalence relation $(W^{\mathsf{L}}, W^{\mathsf{R}}) \sim (\widetilde{W}^{\mathsf{L}}, \widetilde{W}^{\mathsf{R}})$ if and only if $W^{\mathsf{L}}W^{\mathsf{R}} = \widetilde{W}^{\mathsf{L}}\widetilde{W}^{\mathsf{R}}$, then each matrix of weights W can be though as a class of equivalence, and can be described by the orbit

$$\Big\{(W^{\mathsf{L}}\phi, \phi^{-1}W^{\mathsf{R}}),\ \phi(z) \text{ is a nonsingular matrix of entire functions}\Big\}.$$

3.1. Constant jump on the support

Given assumptions i) and ii), for each factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$, we introduce the *constant jump fundamental matrices* which will be instrumental in what follows

$$Z^{\mathsf{L}}_n(z) := Y^{\mathsf{L}}_n(z)\begin{bmatrix} W^{\mathsf{L}}(z) & 0_N \\ 0_N & (W^{\mathsf{R}}(z))^{-1}\end{bmatrix}, \tag{25}$$

$$Z^{\mathsf{R}}_n(z) := \begin{bmatrix} W^{\mathsf{R}}(z) & 0_N \\ 0_N & (W^{\mathsf{L}}(z))^{-1}\end{bmatrix} Y^{\mathsf{R}}_n(z), \qquad n \in \mathbb{Z}_+. \tag{26}$$

Taking inverse on (25) and applying (13) we see that Z^{R}_n given in (26) admits the representation

$$Z^{\mathsf{R}}_n(z) = \begin{bmatrix} 0 & -I \\ I & 0\end{bmatrix}(Z^{\mathsf{L}}_n(z))^{-1}\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}, \qquad n \in \mathbb{Z}_+. \tag{27}$$

Proposition 1. *For each factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$, the constant jump fundamental matrices $Z^{\mathsf{L}}_n(z)$ and $Z^{\mathsf{R}}_n(z)$ are, for each $n \in \mathbb{Z}_+$, characterized by the following properties:*

i) *They are holomorphic on $\mathbb{C} \setminus \gamma$.*

ii) *We have the following asymptotic behaviors*

$$Z^{\mathsf{L}}_n(z) = \big(I + \mathrm{O}(z^{-1})\big)\begin{bmatrix} z^n W^{\mathsf{L}}(z) & 0_N \\ 0_N & Iz^{-n}(W^{\mathsf{R}}(z))^{-1}\end{bmatrix},$$
$$Z^{\mathsf{R}}_n(z) = \begin{bmatrix} z^n W^{\mathsf{R}}(z) & 0_N \\ 0_N & (W^{\mathsf{L}}(z))^{-1}z^{-n}\end{bmatrix}\big(I + \mathrm{O}(z^{-1})\big),$$

for $z \to \infty$.

iii) *They present the following* constant jump condition *on γ*

$$\big(Z^{\mathsf{L}}_n(z)\big)_+ = \big(Z^{\mathsf{L}}_n(z)\big)_-\begin{bmatrix} I & I \\ 0_N & I\end{bmatrix}, \qquad \big(Z^{\mathsf{R}}_n(z)\big)_+ = \begin{bmatrix} I & 0_N \\ I & I\end{bmatrix}\big(Z^{\mathsf{R}}_n(z)\big)_-,$$

for all $z \in \gamma$ in the support on the matrix of weights.

Proof. We only give the proofs for the left case because their right ones follow from (27).

i) As the $W^{\mathsf{L}}(z)$ and $W^{\mathsf{R}}(z)$ are matrices of entire functions the holomorphicity properties of Z^{L}_n are inherited from that of the fundamental matrices Y^{L}_n.

ii) It follows from the asymptotic of the fundamental matrices.
iii) From the definition of $Z^{\mathsf{L}}_n(z)$ we have

$$\big(Z^{\mathsf{L}}_n(z)\big)_+ = \big(Y^{\mathsf{L}}_n(z)\big)_+ \begin{bmatrix} W^{\mathsf{L}}(z) & 0_N \\ 0_N & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix},$$

and taking into account Theorem 1 we arrive to

$$\big(Z^{\mathsf{L}}_n(z)\big)_+ = \big(Y^{\mathsf{L}}_n(z)\big)_- \begin{bmatrix} I & W^{\mathsf{L}}(z)W^{\mathsf{R}}(z) \\ 0_N & I \end{bmatrix} \begin{bmatrix} W^{\mathsf{L}}(z) & 0_N \\ 0_N & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix};$$

now, as

$$\begin{bmatrix} I & W^{\mathsf{L}}(z)W^{\mathsf{R}}(z) \\ 0_N & I \end{bmatrix} \begin{bmatrix} W^{\mathsf{L}}(z) & 0_N \\ 0_N & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix} = \begin{bmatrix} W^{\mathsf{L}}(z) & 0_N \\ 0_N & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix} \begin{bmatrix} I & I \\ 0_N & I \end{bmatrix},$$

we get the desired *constant* jump condition for $Z^{\mathsf{L}}_n(z)$. $\square$

Remark 3. For the symmetric and Hermitian reductions we assume

$$\begin{aligned} &W^{\mathsf{L}}(z) = \rho(z), && W^{\mathsf{R}}(z) = (\rho(z))^{\top}, && W(z) = \rho(z)\big(\rho(z)\big)^{\top}, && Z^{\mathsf{R}}(z) = \big(Z^{\mathsf{L}}(z)\big)^{\top}, && \text{symmetric},\\ &W^{\mathsf{L}}(z) = \rho(z), && W^{\mathsf{R}}(z) = (\rho(\bar z))^{\dagger}, && W = \rho(z)\big(\rho(\bar z)\big)^{\dagger}, && Z^{\mathsf{R}}(z) = \big(Z^{\mathsf{L}}(\bar z)\big)^{\dagger}, && \text{Hermitian}. \end{aligned}$$

In both cases, we will use the notation

$$Z_n(z) := Z^{\mathsf{L}}_n(z).$$

3.2. Structure matrices

In parallel to the matrices $Z^{\mathsf{L}}_n(z)$ and $Z^{\mathsf{R}}_n(z)$, for each factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$, we introduce what we call *structure matrices* given in terms of the *right derivative* and *left derivative* (logarithmic derivatives), respectively,

$$M^{\mathsf{L}}_n(z) := \big(Z^{\mathsf{L}}_n(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1}, \qquad M^{\mathsf{R}}_n(z) := \big(Z^{\mathsf{R}}_n(z)\big)^{-1}\big(Z^{\mathsf{R}}_n(z)\big)'.$$

It is not difficult to prove that

$$M^{\mathsf{R}}_n(z) = -\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} M^{\mathsf{L}}_n(z) \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \qquad n \in \mathbb{Z}_+.$$

Proposition 2. *The following properties hold:*

i) *The structure matrices $M^{\mathsf{L}}_n(z)$ and $M^{\mathsf{R}}_n(z)$, defined on subsection 3.2, are, for each $n \in \mathbb{Z}_+$, matrices of entire functions in the complex plane.*
ii) *The transfer matrix satisfies*

$$T^{\mathsf{L}}_n(z)Z^{\mathsf{L}}_n(z) = Z^{\mathsf{L}}_{n+1}(z), \qquad Z^{\mathsf{R}}_n(z)T^{\mathsf{R}}_n(z) = Z^{\mathsf{R}}_{n+1}(z), \qquad n \in \mathbb{Z}_+.$$

iii) *The zero curvature formulas*

$$\begin{bmatrix} I & 0_N \\ 0_N & 0_N \end{bmatrix} = M^{\mathsf{L}}_{n+1}(z)T^{\mathsf{L}}_n(z) - T^{\mathsf{L}}_n(z)M^{\mathsf{L}}_n(z), \tag{28}$$

$$\begin{bmatrix} I & 0_N \\ 0_N & 0_N \end{bmatrix} = T^{\mathsf{R}}_n(z)\, M^{\mathsf{R}}_{n+1}(z) - M^{\mathsf{R}}_n(z)T^{\mathsf{R}}_n(z), \tag{29}$$

$n \in \mathbb{Z}_+$, *are fulfilled.*

iv) *The* second order *zero curvature formulas*

$$\begin{bmatrix} I & 0_N \\ 0_N & 0_N \end{bmatrix} M^{\mathsf{L}}_n(z) + M^{\mathsf{L}}_{n+1}(z) \begin{bmatrix} I & 0_N \\ 0_N & 0_N \end{bmatrix} = \big(M^{\mathsf{L}}_{n+1}(z)\big)^2 T^{\mathsf{L}}_n(z) - T^{\mathsf{L}}_n(z)\big(M^{\mathsf{L}}_n(z)\big)^2, \tag{30}$$

$$\begin{bmatrix} I & 0_N \\ 0_N & 0_N \end{bmatrix} M^{\mathsf{R}}_{n+1}(z) + M^{\mathsf{R}}_n(z) \begin{bmatrix} I & 0_N \\ 0_N & 0_N \end{bmatrix} = T^{\mathsf{R}}_n(z)\big(M^{\mathsf{R}}_{n+1}(z)\big)^2 - \big(M^{\mathsf{R}}_n(z)\big)^2 T^{\mathsf{R}}_n(z), \tag{31}$$

$n \in \mathbb{Z}_+$, *are satisfied.*

Proof. Again we only give the proofs for the left case. We begin to prove that the sequence of matrix functions $\big\{M^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{Z}_+}$ is a sequence of matrices with coefficients given by entire functions. In fact, $\big(M^{\mathsf{L}}_n\big)_+ = \Big(\big(Z^{\mathsf{L}}_n\big)'\Big)_+ \Big(\big(Z^{\mathsf{L}}_n\big)^{-1}\Big)_+$, and applying the *constant* jump condition we get

$$\big(M^{\mathsf{L}}_n(z)\big)_+ = \Big(\big(Z^{\mathsf{L}}_n\big)'\Big)_- \begin{bmatrix} I & I \\ 0_N & I \end{bmatrix}^{-1} \begin{bmatrix} I & I \\ 0_N & I \end{bmatrix} \Big(\big(Z^{\mathsf{L}}_n\big)^{-1}\Big)_- = \big(M^{\mathsf{L}}_n(z)\big)_-.$$

It follows from the definition of Z^{L}_n that

$$T^{\mathsf{L}}_n(z) = Y^{\mathsf{L}}_{n+1}(z)\big(Y^{\mathsf{L}}_n(z)\big)^{-1} = Z^{\mathsf{L}}_{n+1}(z)\big(Z^{\mathsf{L}}_n(z)\big)^{-1}.$$

Taking derivatives on $T_n(z)$ we get

$$\big(T^{\mathsf{L}}_n(z)\big)' = \big(Z^{\mathsf{L}}_{n+1}(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1} - Z^{\mathsf{L}}_{n+1}(z)\big(Z^{\mathsf{L}}_n(z)\big)^{-1}\big(Z^{\mathsf{L}}_n(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1}, \qquad n \in \mathbb{Z}_+,$$

and so, taking into account that

$$\big(Z^{\mathsf{L}}_{n+1}(z)\big)'\big(Z^{\mathsf{L}}_n(z)\big)^{-1} = \big(Z^{\mathsf{L}}_{n+1}(z)\big)'\big(Z^{\mathsf{L}}_{n+1}(z)\big)^{-1} Z^{\mathsf{L}}_{n+1}(z)\big(Z^{\mathsf{L}}_n(z)\big)^{-1} = M^{\mathsf{L}}_{n+1}T^{\mathsf{L}}_n\,,$$

we get (28). Using the same ideas we derive (29).

Now, multiplying (28) on the left by M^{L}_{n+1} we get

$$M^{\mathsf{L}}_{n+1} \begin{bmatrix} I & 0_N \\ 0_N & 0_N \end{bmatrix} = \big(M^{\mathsf{L}}_{n+1}(z)\big)^2 T^{\mathsf{L}}_n(z) - \big(M^{\mathsf{L}}_{n+1}T^{\mathsf{L}}_n(z)\big)M^{\mathsf{L}}_n(z),$$

and again by (28) applied to the term $M^{\mathsf{L}}_{n+1}T^{\mathsf{L}}_n(z)$ we get (30). □

Higher order transfer matrices

$$T^{\mathsf{L}}_{n,\ell}(z) := T^{\mathsf{L}}_{n+\ell}(z)\cdots T^{\mathsf{L}}_n(z), \qquad T^{\mathsf{R}}_{n,\ell}(z) := T^{\mathsf{R}}_n(z)\cdots T^{\mathsf{L}}_{n+\ell}(z),$$

satisfy

$$Y^{\mathsf{L}}_{n+\ell}(z) = T^{\mathsf{L}}_{n,\ell}(z)Y^{\mathsf{L}}_n(z), \qquad Y^{\mathsf{R}}_{n+\ell}(z) = Y^{\mathsf{R}}_n(z)T^{\mathsf{R}}_{n,\ell}(z).$$

Proposition 3. *The following zero-curvature conditions hold, for all $n, \ell \in \mathbb{Z}_+$,*

$$\big(T^{\mathsf{L}}_{n,\ell}(z)\big)' = M^{\mathsf{L}}_{n+\ell+1}(z)T^{\mathsf{L}}_{n}(z) - T^{\mathsf{L}}_{n}(z)M^{\mathsf{L}}_{n}(z), \qquad \big(T^{\mathsf{R}}_{n,\ell}(z)\big)' = T^{\mathsf{R}}_{n}(z)\, M^{\mathsf{R}}_{n+\ell+1}(z) - M^{\mathsf{R}}_{n}(z)T^{\mathsf{R}}_{n}(z).$$

Proof. As before we only give a discussion for the left situation. It is done by induction: First of all recall that $\ell = 0$ is just the already proven zero-curvature condition. Now, assuming that it holds for ℓ we prove it for $\ell + 1$:

$$\begin{aligned}
\big(T^{\mathsf{L}}_{n,\ell+1}(z)\big)' &= \big(T^{\mathsf{L}}_{n+\ell+1}(z)T^{\mathsf{L}}_{n,\ell}(z)\big)' = \big(T^{\mathsf{L}}_{n+\ell+1}(z)\big)'T^{\mathsf{L}}_{n,\ell}(z) + T^{\mathsf{L}}_{n+\ell+1}(z)\big(T^{\mathsf{L}}_{n,\ell}(z)\big)' \\
&= \big(M^{\mathsf{L}}_{n+\ell+2}(z)T^{\mathsf{L}}_{n+\ell+1}(z) - T^{\mathsf{L}}_{n+\ell+1}(z)M^{\mathsf{L}}_{n+\ell+1}(z)\big)T^{\mathsf{L}}_{n,\ell}(z) \\
&\quad + T^{\mathsf{L}}_{n+\ell+1}(z)\big(M^{\mathsf{L}}_{n+\ell+1}(z)T^{\mathsf{L}}_{n,\ell}(z) - T^{\mathsf{L}}_{n,\ell}(z)M^{\mathsf{L}}_{n}(z)\big), \\
&= M^{\mathsf{L}}_{n+\ell+2}(z)T^{\mathsf{L}}_{n+\ell+1}(z)T^{\mathsf{L}}_{n,\ell}(z) - T^{\mathsf{L}}_{n+\ell+1}(z)T^{\mathsf{L}}_{n,\ell}(z)M^{\mathsf{L}}_{n}(z),
\end{aligned}$$

and the result is proven. $\square$

Proposition 4 *(Computing the structure matrices). If the subindex $+$ indicates that only the positive powers of the asymptotic expansion about infinity are kept, for each factorization $W = W^{\mathsf{L}}W^{\mathsf{R}}$, we have for all $n \in \mathbb{Z}_+$, the following power expansions for the structure matrices, defined on subsection 3.2,*

$$M^{\mathsf{L}}_{n}(z) = \left(S^{\mathsf{L}}_{n}(z)\begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1} & 0_N \\ 0_N & -\big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)' \end{bmatrix}\big(S^{\mathsf{L}}_{n}(z)\big)^{-1}\right)_{+}, \tag{32}$$

$$M^{\mathsf{R}}_{n}(z) = \left(\big(S^{\mathsf{R}}_{n}(z)\big)^{-1}\begin{bmatrix} \big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)' & 0_N \\ 0_N & -\big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1} \end{bmatrix}S^{\mathsf{R}}_{n}(z)\right)_{+}. \tag{33}$$

Proof. Using assumption i) in Proposition 2, we find the expressions for the left structure matrix, $M^{\mathsf{L}}_{n}(z)$, in terms of $S^{\mathsf{L}}_{n}(z)$ and $W(z) = W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)$. For doing so we require the use of the definition of $S^{\mathsf{L}}_{n}(z)$, i.e.

$$Z^{\mathsf{L}}_{n}(z) = S^{\mathsf{L}}_{n}(z)\begin{bmatrix} z^n W^{\mathsf{L}}(z) & 0_N \\ 0_N & z^{-n}\big(W^{\mathsf{R}}(z)\big)^{-1} \end{bmatrix},$$

and consequently, we find

$$\begin{aligned}
M^{\mathsf{L}}_{n}(z) &= \big(S^{\mathsf{L}}_{n}(z)\big)'\big(S^{\mathsf{L}}_{n}(z)\big)^{-1} \\
&\quad + S^{\mathsf{L}}_{n}(z)\begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1} + nz^{-1} & 0_N \\ 0_N & -\big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)' - nz^{-1} \end{bmatrix}\big(S^{\mathsf{L}}_{n}(z)\big)^{-1}.
\end{aligned}$$

Given assumption iii) in the beginning of this section, on the entire character of the right derivative, $\big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1}$, and of the left derivative, $\big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)'$, and since $\big(S^{\mathsf{L}}_{n}(z)\big)'\big(S^{\mathsf{L}}_{n}(z)\big)^{-1}$ has only negative powers of z in its Laurent expansion, and given that the structure matrix $M^{\mathsf{L}}(z)$ has entire coefficients, the asymptotic expansion of $M^{\mathsf{L}}_{n}(z)$ about ∞ must be a power expansion.

A similar approach holds for the right context, and we can determine $M^{\mathsf{R}}_{n}(z)$ in terms of $S^{\mathsf{R}}_{n}(z)$ and $W(z)$. Indeed, from

$$Z^{\mathsf{R}}_{n}(z) = \begin{bmatrix} W^{\mathsf{R}}(z)z^n & 0_N \\ 0_N & (W^{\mathsf{L}}(z))^{-1}z^{-n} \end{bmatrix} S^{\mathsf{R}}_{n}(z),$$

we get

$$M_n^{\mathsf{R}}(z)=\big(S_n^{\mathsf{R}}(z)\big)^{-1}\big(S_n^{\mathsf{R}}(z)\big)' \\ +\big(S_n^{\mathsf{R}}(z)\big)^{-1}\begin{bmatrix}\big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)'+nz^{-1} & 0_N\\ 0_N & -\big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1}-nz^{-1}\end{bmatrix}S_n^{\mathsf{R}}(z),$$

and reasoning as for the left case we derive the desired result. □

Notice that given the matrices of entire functions $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$ the structure matrices, using (32), can be explicitly determined in terms of the coefficients in $S_n^{\mathsf{L}}(z)$ and $S_n^{\mathsf{R}}(z)$. Moreover, when $h^{\mathsf{L}}(z),h^{\mathsf{R}}(z)\in\mathbb{C}^{N\times N}[z]$ are matrix polynomials, only the first elements, as much as the degree of the corresponding polynomial, in the asymptotic expansions of $S_n^{\mathsf{L}}(z)$ and $S_n^{\mathsf{R}}(z)$ are involved, and we will have that $M_n^{\mathsf{L}}(z),M_n^{\mathsf{R}}(z)\in\mathbb{C}^{2N\times 2N}[z]$ are also polynomials with degree $\deg M_n^{\mathsf{L}}(z),\deg M_n^{\mathsf{L}}(z)=\max(h_n^{\mathsf{L}}(z),h_n^{\mathsf{R}}(z))$.

Remark 4. For the reductions we have

$$\begin{aligned}M_n^{\mathsf{R}}(z)&=\big(M_n^{\mathsf{L}}(z)\big)^{\top}, && \text{symmetric,}\\ M_n^{\mathsf{R}}(z)&=\big(M_n^{\mathsf{L}}(\bar z)\big)^{\dagger}, && \text{Hermitian.}\end{aligned}$$

In both cases, we will use the notation

$$M_n(z):=M_n^{\mathsf{L}}(z).$$

4. Matrix Pearson equations and differential equations

4.1. Matrix Pearson equations

As we have seen, the left and right logarithmic derivatives, $h^{\mathsf{L}}(z)=\big(W^{\mathsf{L}}(z)\big)'\big(W^{\mathsf{L}}(z)\big)^{-1}$ and $h^{\mathsf{R}}(z)=\big(W^{\mathsf{R}}(z)\big)^{-1}\big(W^{\mathsf{R}}(z)\big)'$, play an important role in the discussion of the structure matrices. This motivates us to adopt the following strategy: assume that instead of a given matrix of weights we are provided with two matrices, say $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$, of entire functions such that the following two matrix Pearson equations are satisfied

$$\frac{\mathrm{d}\,W^{\mathsf{L}}}{\mathrm{d}\,z}=h^{\mathsf{L}}(z)W^{\mathsf{L}}(z), \tag{34}$$

$$\frac{\mathrm{d}\,W^{\mathsf{R}}}{\mathrm{d}\,z}=W^{\mathsf{R}}(z)h^{\mathsf{R}}(z); \tag{35}$$

and given solutions to them we construct the corresponding matrix of weights $W=W^{\mathsf{L}}W^{\mathsf{R}}$. Moreover, this matrix of weights is also characterized by a Pearson equation.

Proposition 5 *(Pearson Sylvester differential equation). Given two matrices of entire functions $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$, any solution of the Sylvester type matrix differential equation, which we call Pearson equation for the weight,*

$$\frac{\mathrm{d}\,W}{\mathrm{d}\,z}=h^{\mathsf{L}}(z)W(z)+W(z)h^{\mathsf{R}}(z) \tag{36}$$

is of the form $W=W^{\mathsf{L}}W^{\mathsf{R}}$ where the matrix factors W^{L} and W^{R} are solutions of (34) and (35), respectively.

Proof. Given solutions W^{L} and W^{R} of (34) and (35), respectively, it follows intermediately, just using the Leibniz law for derivatives, that $W=W^{\mathsf{L}}W^{\mathsf{R}}$ fulfills (36). Moreover, given a solution W of (36) we pick a solution W^{L} of (34), then it is easy to see that $(W^{\mathsf{L}})^{-1}W$ satisfies (35). $\square$

Remark 5. The matrix of weights W does not uniquely determine the left and the right matrix factors; indeed if $W=W^{\mathsf{L}}W^{\mathsf{R}}$, with factors solving (34) and (35), respectively, then $\widetilde{W}^{\mathsf{L}}=W^{\mathsf{L}}C$ and $\widetilde{W}^{\mathsf{R}}=C^{-1}W^{\mathsf{R}}$ for C being a nonsingular matrix, gives also another possible factorization $W=\widetilde{W}^{\mathsf{L}}\widetilde{W}^{\mathsf{R}}$, with factors solving the partial Pearson equations (34) and (35). This indeterminacy disappears when one considers the right and left derivatives of the factors.

Remark 6. Given two matrices of entire functions $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$ and a matrix of weights W characterized by the matrix Pearson equation (36) we have the left and right fundamental matrices $Y^{\mathsf{L}}_n(z)$ and $Y^{\mathsf{R}}_n(z)$ satisfying corresponding Riemann–Hilbert problems. The associated structure matrices are from (32) and (33) given by,

$$M^{\mathsf{L}}_n(z)=\left(S^{\mathsf{L}}_n(z)\begin{bmatrix}h^{\mathsf{L}}(z) & 0_N\\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix}\left(S^{\mathsf{L}}_n(z)\right)^{-1}\right)_+, \tag{37}$$

$$M^{\mathsf{R}}_n(z)=\left(\left(S^{\mathsf{R}}_n(z)\right)^{-1}\begin{bmatrix}h^{\mathsf{R}}(z) & 0_N\\ 0_N & -h^{\mathsf{L}}(z)\end{bmatrix}S^{\mathsf{R}}_n(z)\right)_+. \tag{38}$$

Remark 7. For the symmetric and Hermitian reductions, we have

$$\begin{aligned} h^{\mathsf{R}}(z)&=\left(h^{\mathsf{L}}(z)\right)^{\top}, && \text{symmetric,}\\ h^{\mathsf{R}}(z)&=\left(h^{\mathsf{L}}(\bar z)\right)^{\dagger}, && \text{Hermitian,}\end{aligned}$$

and (34) and (35) collapse into a single equation

$$\frac{\mathrm{d}\,\rho}{\mathrm{d}\,z}=h(z)\rho(z),$$

where $h(z):=h^{\mathsf{L}}(z)$, and the Pearson equation (36) reads

$$\begin{aligned} \frac{\mathrm{d}\,W}{\mathrm{d}\,z}&=h(z)W(z)+W(z)(h(z))^{\top}, && \text{symmetric,}\\ \frac{\mathrm{d}\,W}{\mathrm{d}\,z}&=h(z)W(z)+W(z)(h(\bar z))^{\dagger}, && \text{Hermitian.}\end{aligned} \tag{39}$$

4.2. Sylvester differential equations for the fundamental matrices

The differential structure determined by the Pearson equation for the matrix of weights induces a corresponding Sylvester differential equations for the fundamental matrices as follows.

Proposition 6 *(Sylvester differential linear equations). In the conditions of Proposition 5, the left fundamental matrix $Y^{\mathsf{L}}_n(z)$ and the right fundamental matrix $Y^{\mathsf{R}}_n(z)$ satisfy, for each $n\in\mathbb{Z}_+$, the following Sylvester matrix differential equations,*

$$\left(Y^{\mathsf{L}}_n(z)\right)'=M^{\mathsf{L}}_n(z)Y^{\mathsf{L}}_n(z)-Y^{\mathsf{L}}_n(z)\begin{bmatrix}h^{\mathsf{L}}(z) & 0_N\\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix}, \tag{40}$$

$$\big(Y_n^{\mathsf{R}}(z)\big)' = Y_n^{\mathsf{R}}(z)M_n^{\mathsf{R}}(z) - \begin{bmatrix} h^{\mathsf{R}}(z) & 0_N \\ 0_N & -h^{\mathsf{L}}(z)\end{bmatrix} Y_n^{\mathsf{R}}(z), \tag{41}$$

respectively.

Proof. As $M_n^{\mathsf{L}}(z) = \big(Z_n^{\mathsf{L}}(z)\big)'\big(Z_n^{\mathsf{L}}(z)\big)^{-1}$ is the right derivative of the constant jump structure matrix from (25) we get (40); (41) is proven analogously. □

We write

$$M_n^{\mathsf{L}}(z) = \begin{bmatrix} M_{1,1,n}^{\mathsf{L}}(z) & M_{1,2,n}^{\mathsf{L}}(z) \\ M_{2,1,n}^{\mathsf{L}}(z) & M_{2,2,n}^{\mathsf{L}}(z)\end{bmatrix}, \qquad M_n^{\mathsf{R}}(z) = \begin{bmatrix} M_{1,1,n}^{\mathsf{R}}(z) & M_{1,2,n}^{\mathsf{R}}(z) \\ M_{2,1,n}^{\mathsf{R}}(z) & M_{2,2,n}^{\mathsf{R}}(z)\end{bmatrix},$$

to express the previous results in the following manner.

Corollary 3. *The Sylvester matrix differential equations* (40) *and* (41) *split in the following Sylvester differential systems*

$$\begin{cases} \big(P_n^{\mathsf{L}}(z)\big)' + P_n^{\mathsf{L}}(z)h^{\mathsf{L}}(z) = M_{1,1,n}^{\mathsf{L}}(z)P_n^{\mathsf{L}}(z) - M_{1,2,n}^{\mathsf{L}}(z)C_{n-1}P_{n-1}^{\mathsf{L}}(z), \\ \big(P_{n-1}^{\mathsf{L}}(z)\big)' + P_{n-1}^{\mathsf{L}}(z)h^{\mathsf{L}}(z) = -C_{n-1}^{-1}M_{2,1,n}^{\mathsf{L}}(z)P_n^{\mathsf{L}}(z) + C_{n-1}^{-1}M_{2,2,n}^{\mathsf{L}}(z)C_{n-1}P_{n-1}^{\mathsf{L}}(z), \end{cases} \tag{42}$$

$$\begin{cases} \big(Q_n^{\mathsf{L}}(z)\big)' + Q_n^{\mathsf{L}}(z)h^{\mathsf{R}}(z) = M_{1,1,n}^{\mathsf{L}}Q_n^{\mathsf{L}}(z) - M_{1,2,n}^{\mathsf{L}}(z)C_{n-1}Q_{n-1}^{\mathsf{L}}(z), \\ \big(Q_{n-1}^{\mathsf{L}}(z)\big)' + Q_{n-1}^{\mathsf{L}}(z)h^{\mathsf{R}}(z) = -C_{n-1}^{-1}M_{2,1,n}^{\mathsf{L}}(z)Q_n^{\mathsf{L}}(z) + C_{n-1}^{-1}M_{2,2,n}^{\mathsf{L}}(z)C_{n-1}Q_{n-1}^{\mathsf{L}}(z), \end{cases} \tag{43}$$

$$\begin{cases} \big(P_n^{\mathsf{R}}(z)\big)' + h^{\mathsf{R}}(z)P_n^{\mathsf{R}}(z) = P_n^{\mathsf{R}}(z)M_{1,1,n}^{\mathsf{R}}(z) - P_{n-1}^{\mathsf{R}}(z)C_{n-1}M_{2,1,n}^{\mathsf{R}}(z), \\ \big(P_{n-1}^{\mathsf{R}}(z)\big)' + h^{\mathsf{R}}(z)P_{n-1}^{\mathsf{R}}(z) = -P_n^{\mathsf{R}}(z)M_{1,2,n}^{\mathsf{R}}(z)C_{n-1}^{-1} + P_{n-1}^{\mathsf{R}}(z)C_{n-1}M_{2,2,n}^{\mathsf{R}}(z)C_{n-1}^{-1}, \end{cases} \tag{44}$$

$$\begin{cases} \big(Q_n^{\mathsf{R}}(z)\big)' + h^{\mathsf{L}}(z)Q_n^{\mathsf{R}}(z) = Q_n^{\mathsf{R}}(z)M_{1,1,n}^{\mathsf{R}}(z) - Q_{n-1}^{\mathsf{R}}(z)C_{n-1}M_{2,1,n}^{\mathsf{R}}(z), \\ \big(Q_{n-1}^{\mathsf{R}}(z)\big)' + h^{\mathsf{L}}(z)Q_{n-1}^{\mathsf{R}}(z) = -Q_n^{\mathsf{R}}(z)M_{1,2,n}^{\mathsf{R}}(z)C_{n-1}^{-1} + Q_{n-1}^{\mathsf{R}}(z)C_{n-1}M_{2,2,n}^{\mathsf{R}}(z)C_{n-1}^{-1}. \end{cases} \tag{45}$$

We first observe from the linear differential systems (42) and (44) satisfied by the left and right matrix orthogonal polynomials, respectively, we will be able to extract in some scenarios, see next section on applications, a matrix eigenvalue problem for a second order matrix differential operator, with matrix eigenvalues. The differential systems (43) and (45) for the left and right second kind functions also provide interesting information, and we will use them to discover nonlinear equations satisfied by the recursion coefficients.

Remark 8. For the reductions we have

$$\big(Y_n(z)\big)' = M_n(z)Y_n(z) - Y_n(z)\begin{bmatrix} h(z) & 0_N \\ 0_N & -(h(z))^\top\end{bmatrix}, \qquad \text{symmetric,}$$

$$\big(Y_n(z)\big)' = M_n(z)Y_n(z) - Y_n(z)\begin{bmatrix} h(z) & 0_N \\ 0_N & -(h(\bar z))^\dagger.\end{bmatrix}, \qquad \text{Hermitian.}$$

5. Second order differential operators

We firstly derive, as a consequence of the Sylvester differential linear systems, second order differential equations fulfilled by the fundamental matrices, and therefore by the matrix biorthogonal polynomials and also by the corresponding second kind functions.

Following the standard use in Soliton Theory, given a matrix of holomorphic functions $A(z)$ we define its Miura transform by

$$\mathcal{M}(A) = A'(z) + (A(z))^2.$$

Observe that when A is a right (left) logarithmic derivative $A = w'w^{-1}$ ($A = w^{-1}w'$) we have $\mathcal{M}(A) = w''w^{-1}$ ($\mathcal{M}(A) = w^{-1}w''$).

Proposition 7 *(Second order linear differential equations). In the conditions of Proposition 5, the sequence of fundamental matrices, $\{Y_n^{\mathsf{L}}\}_{n\in\mathbb{Z}_+}$ and $\{Y_n^{\mathsf{R}}\}_{n\in\mathbb{Z}_+}$, satisfy*

$$\begin{aligned}
&\big(Y_n^{\mathsf{L}}(z)\big)'' + 2\big(Y_n^{\mathsf{L}}(z)\big)' \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix} + Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathcal{M}\big(h^{\mathsf{L}}(z)\big) & 0_N \\ 0_N & \mathcal{M}\big(-h^{\mathsf{R}}(z)\big)\end{bmatrix} \\
&\quad = \mathcal{M}\big(M_n^{\mathsf{L}}(z)\big) Y_n^{\mathsf{L}}(z), \qquad (46)\\
&\big(Y_n^{\mathsf{R}}(z)\big)'' + 2 \begin{bmatrix} h^{\mathsf{R}}(z) & 0_N \\ 0_N & -h^{\mathsf{L}}(z)\end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)' + \begin{bmatrix} \mathcal{M}\big(h^{\mathsf{R}}(z)\big) & 0_N \\ 0_N & \mathcal{M}\big(-h^{\mathsf{L}}(z)\big)\end{bmatrix} Y_n^{\mathsf{L}}(z) \\
&\quad = Y_n^{\mathsf{R}}(z) \mathcal{M}\big(M_n^{\mathsf{R}}(z)\big). \qquad (47)
\end{aligned}$$

Proof. We prove (46). First, let us take a derivative of (40) to get

$$\big(Y_n^{\mathsf{L}}(z)\big)'' + \big(Y_n^{\mathsf{L}}(z)\big)' \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix} + Y_n^{\mathsf{L}}(z) \begin{bmatrix} \big(h^{\mathsf{L}}(z)\big)' & 0_N \\ 0_N & -\big(h^{\mathsf{R}}(z)\big)'\end{bmatrix} = \big(M_n^{\mathsf{L}}(z)\big)' Y_n^{\mathsf{L}}(z) + M_n^{\mathsf{L}}(z)\big(Y_n^{\mathsf{L}}(z)\big)'$$

but again by (40)

$$M_n^{\mathsf{L}}(z)\big(Y_n^{\mathsf{L}}(z)\big)' = \big(M_n^{\mathsf{L}}(z)\big)^2 Y_n^{\mathsf{L}}(z) - M_n^{\mathsf{L}}(z) Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix}$$

and if we substitute

$$M_n^{\mathsf{L}}(z) Y_n^{\mathsf{L}}(z) = \big(Y_n^{\mathsf{L}}(z)\big)' + Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix}$$

we finally get

$$M_n^{\mathsf{L}}(z)\big(Y_n^{\mathsf{L}}(z)\big)' = \big(M_n^{\mathsf{L}}(z)\big)^2 Y_n^{\mathsf{L}}(z) - \big(Y_n^{\mathsf{L}}(z)\big)' \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix} - Y_n^{\mathsf{L}}(z) \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix}^2,$$

and the result follows. □

Definition 1. For the next corollary we need to introduce the following $\mathbb{C}^{2N\times 2N}$ valued functions in terms of the difference of two Miura maps

$$\mathsf{H}_n^{\mathsf{L}}(z) = \begin{bmatrix} \mathsf{H}_{1,1,n}^{\mathsf{L}}(z) & \mathsf{H}_{1,2,n}^{\mathsf{L}}(z) \\ \mathsf{H}_{2,1,n}^{\mathsf{L}}(z) & \mathsf{H}_{2,2,n}^{\mathsf{L}}(z)\end{bmatrix} = \mathcal{M}(M_n^{\mathsf{L}}(z)) - \mathcal{M}\left(\begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z)\end{bmatrix}\right), \qquad (48)$$

$$\mathsf{H}_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathsf{H}_{1,1,n}^{\mathsf{R}}(z) & \mathsf{H}_{1,2,n}^{\mathsf{R}}(z) \\ \mathsf{H}_{2,1,n}^{\mathsf{R}}(z) & \mathsf{H}_{2,2,n}^{\mathsf{R}}(z)\end{bmatrix} = \mathcal{M}(M_n^{\mathsf{R}}(z)) - \mathcal{M}\left(\begin{bmatrix} h^{\mathsf{R}}(z) & 0_N \\ 0_N & -h^{\mathsf{L}}(z)\end{bmatrix}\right). \qquad (49)$$

Corollary 4. *The second order matrix differential equations (46) and (47) split in the following differential relations*

$$
\begin{aligned}
&\big(P_n^{\mathsf{L}}\big)''(z)+2\big(P_n^{\mathsf{L}}\big)'(z)h^{\mathsf{L}}(z)+P_n^{\mathsf{L}}(z)\mathcal{M}(h^{\mathsf{L}}(z))\\
&\quad=\big(\mathcal{M}(h^{\mathsf{L}}(z))+\mathsf{H}^{\mathsf{L}}_{1,1,n}(z)\big)P_n^{\mathsf{L}}(z)-\mathsf{H}^{\mathsf{L}}_{1,2,n}(z)C_{n-1}P^{\mathsf{L}}_{n-1}(z)\,, &(50)\\
&\big(Q_n^{\mathsf{L}}\big)''(z)-2\big(Q_n^{\mathsf{L}}\big)'(z)h^{\mathsf{R}}(z)+Q_n^{\mathsf{L}}(z)\mathcal{M}(-h^{\mathsf{R}}(z))\\
&\quad=\big(\mathcal{M}(h^{\mathsf{L}}(z))+\mathsf{H}^{\mathsf{L}}_{1,1,n}(z)\big)Q_n^{\mathsf{L}}(z)-\mathsf{H}^{\mathsf{L}}_{1,2,n}(z)C_{n-1}Q^{\mathsf{L}}_{n-1}(z)\,, &(51)\\
&\big(P_n^{\mathsf{R}}\big)''(z)+2h^{\mathsf{R}}(z)\big(P_n^{\mathsf{R}}(z)\big)'(z)+\mathcal{M}(h^{\mathsf{R}}(z))P_n^{\mathsf{R}}(z)\\
&\quad=P_n^{\mathsf{R}}(z)\big(\mathcal{M}(h^{\mathsf{R}}(z))+\mathsf{H}^{\mathsf{R}}_{1,1,n}(z)\big)-P^{\mathsf{R}}_{n-1}(z)C_{n-1}\mathsf{H}^{\mathsf{R}}_{2,1,n}(z)\,, &(52)\\
&\big(Q_n^{\mathsf{R}}\big)''(z)-2h^{\mathsf{L}}(z)\big(Q_n^{\mathsf{R}}\big)'(z)+\mathcal{M}(-h^{\mathsf{L}}(z))Q_n^{\mathsf{R}}(z)\\
&\quad=Q_n^{\mathsf{R}}(z)\big(\mathcal{M}(h^{\mathsf{R}}(z))+\mathsf{H}^{\mathsf{R}}_{1,1,n}(z)\big)-Q^{\mathsf{R}}_{n-1}(z)C_{n-1}\mathsf{H}^{\mathsf{R}}_{2,1,n}(z). &(53)
\end{aligned}
$$

Proof. Is a direct consequence of Proposition 7. □

5.1. Adjoint operators

We now elaborate around the idea of adjoint operators in this matrix scenario.

Given a matrix valued differential operator L defined on certain domain $\mathcal{D}$, that belongs to the $N \times N$ complex matrix functions, we may consider the notion of adjoint operator L^* with respect to the sesquilinear like form

$$\langle f, g\rangle_W := \int_\gamma f(z)W(z)g(z)\,\mathrm{d}\,z.$$

The adjoint operator L^* of the differential operator L defined on the domain $\mathcal{D}$ is such that

$$\langle L(f), g\rangle_W = \langle f, L^*(g)\rangle_W, \qquad f(z), g(z) \in \mathcal{D}.$$

The existence of such adjoint is a delicate matter indeed. For a discussion of this subject see [43].

In our case and in what follows we will give explicit examples of such constructions.

From now on, and to be consistent with the definition of sesquilinear like form, $\langle\cdot,\cdot\rangle_W$, we restrict ourselves to the case when h^{L} and h^{R} are matrix polynomials of a specific degree.

Care must be taken at this point because in this definition of adjoint of a matrix differential operator we are not taking the transpose or the Hermitian conjugate of the matrix coefficients as was done in [31].

Definition 2. Motivated by (50) and (52) we introduce two linear operators $\boldsymbol{\ell}^{\mathsf{L}}$ and $\boldsymbol{\ell}^{\mathsf{R}}$, acting on the linear space of polynomials $\mathbb{C}^{N\times N}[z]$ as follows

$$\boldsymbol{\ell}^{\mathsf{L}}(P) := P'' + 2P'h^{\mathsf{L}} + P\mathcal{M}(h^{\mathsf{L}}), \qquad \boldsymbol{\ell}^{\mathsf{R}}(P) := P'' + 2h^{\mathsf{R}}P' + \mathcal{M}(h^{\mathsf{R}})P.$$

Lemma 1. *Let us assume that the matrix of weights $W(z)$ do satisfy the following boundary conditions*

$$W|_{\partial\gamma} = 0_N, \qquad \big(W' - 2h^{\mathsf{L}}W\big)\big|_{\partial\gamma} = 0_N, \qquad \big(W' - 2Wh^{\mathsf{R}}\big)\big|_{\partial\gamma} = 0_N, \tag{54}$$

where $\partial\gamma$ is the boundary of the curve γ, i.e. its endpoints. Then, $W(z)$ satisfies a Pearson Sylvester differential equation (36) if, and only if, $W(z)$ satisfies the following second order matrix differential equations

$$W'' - 2\big(h^{\mathsf{L}}W\big)' + \mathcal{M}(h^{\mathsf{L}})W = W\mathcal{M}(h^{\mathsf{R}})\,, \tag{55}$$

$$W'' - 2\big(Wh^{\mathsf{R}}\big)' + W\mathcal{M}(h^{\mathsf{R}}) = \mathcal{M}(h^{\mathsf{L}})W\,. \tag{56}$$

Proof. Taking derivative on (36), we get

$$W'' = \mathcal{M}(h^{\mathsf{L}})W + W\mathcal{M}(h^{\mathsf{R}}) + 2h^{\mathsf{L}}Wh^{\mathsf{R}}.$$

But, it is easy to see that

$$\left(h^{\mathsf{L}}W\right)' = \mathcal{M}(h^{\mathsf{L}})W + h^{\mathsf{L}}Wh^{\mathsf{R}}, \qquad \left(Wh^{\mathsf{R}}\right)' = W\mathcal{M}(h^{\mathsf{R}}) + h^{\mathsf{L}}Wh^{\mathsf{R}},$$

and so we arrive to (55) and (56).

The reciprocal result is a consequence of adding the equations (55), (56) and the boundary conditions (54). □

Now, we will see that the operators ℓ^{L} and ℓ^{R} are adjoint to each other with respect to the sesquilinear like form induced by the weight functions W.

Proposition 8. *Whenever $W(z)$ satisfies (36) and the boundary conditions (54), we have that*

$$\boldsymbol{\ell}^{\mathsf{R}} = \left(\boldsymbol{\ell}^{\mathsf{L}}\right)^*, \tag{57}$$

or, equivalently,

$$\langle \boldsymbol{\ell}^{\mathsf{L}}(P), \tilde{P}\rangle_W = \langle P, \boldsymbol{\ell}^{\mathsf{R}}(\tilde{P})\rangle_W, \qquad P(z), \tilde{P}(z) \in \mathbb{C}^{N\times N}[z].$$

Proof. By using the linearity of these operators it is sufficient to prove

$$\langle \boldsymbol{\ell}^{\mathsf{L}}(P^{\mathsf{L}}_n)\,,\, P^{\mathsf{R}}_k\rangle_W = \langle P^{\mathsf{L}}_n\,,\, \boldsymbol{\ell}^{\mathsf{R}}(P^{\mathsf{R}}_k)\rangle_W\,, \quad n,k \in \mathbb{Z}_+\,.$$

For the sake of simplicity, we omit, the z dependence on the integrands in the integrals. This way, the orthogonality reads,

$$\langle \boldsymbol{\ell}^{\mathsf{L}}(P^{\mathsf{L}}_n)\,,\, P^{\mathsf{R}}_k\rangle_W = \int_\gamma (P^{\mathsf{L}}_n)''\, W\, P^{\mathsf{R}}_k \,\mathrm{d}\,z + 2\int_\gamma (P^{\mathsf{L}}_n)'\,(h^{\mathsf{L}}\,W)\,P^{\mathsf{R}}_k\,\mathrm{d}\,z + \int_\gamma P^{\mathsf{L}}_n\,\mathcal{M}(h^{\mathsf{L}})\,W\,P^{\mathsf{R}}_k\,\mathrm{d}\,z\,,$$

and, using integration by parts, we find

$$\begin{aligned}
\langle \boldsymbol{\ell}^{\mathsf{L}}(P^{\mathsf{L}}_n), P^{\mathsf{R}}_k\rangle_W &= \left((P^{\mathsf{L}}_n)'WP^{\mathsf{R}}_k\right)\big|_{\partial\gamma} - \int_\gamma (P^{\mathsf{L}}_n)'\Big((WP^{\mathsf{R}}_k)' - 2h^{\mathsf{L}}W\Big)P^{\mathsf{R}}_k\,\mathrm{d}\,z + \int_\gamma P^{\mathsf{L}}_n\mathcal{M}(h^{\mathsf{L}})WP^{\mathsf{R}}_k\,\mathrm{d}\,z\\
&= \left((P^{\mathsf{L}}_n)'WP^{\mathsf{R}}_k\right)\big|_{\partial\gamma} - \Big(P^{\mathsf{L}}_n\Big((WP^{\mathsf{R}}_k)' - 2h^{\mathsf{L}}W\Big)P^{\mathsf{R}}_k\Big)\Big|_{\partial\gamma}\\
&\quad + \int_\gamma P^{\mathsf{L}}_n\,\left((W\,P^{\mathsf{R}}_k)'' - 2\,(h^{\mathsf{L}}\,W\,P^{\mathsf{R}}_k)' + \mathcal{M}(h^{\mathsf{L}})\,W\,P^{\mathsf{R}}_k\right)\,\mathrm{d}\,z\,.
\end{aligned}$$

Now, considering the boundary conditions (54) and taking into account that

$$(W\,P^{\mathsf{R}}_k)'' = W''\,P^{\mathsf{R}}_k + 2\,W'\,(P^{\mathsf{R}}_k)' + W\,(P^{\mathsf{R}}_k)''\,, \quad (h^{\mathsf{L}}\,W\,P^{\mathsf{R}}_k)' = (h^{\mathsf{L}}\,W)'\,P^{\mathsf{R}}_k + (h^{\mathsf{L}}\,W)\,(P^{\mathsf{R}}_k)'\,,$$

we arrive to

$$\langle \boldsymbol{\ell}^{\mathsf{L}}(P_n^{\mathsf{L}})\,,\,P_k^{\mathsf{R}}\rangle_W = \int_\gamma P_n^{\mathsf{L}}\big(W'' - 2(h^{\mathsf{L}}\,W)' + \mathcal{M}(h^{\mathsf{L}})W\big)P_k^{\mathsf{R}}\,\mathrm{d}\,z$$
$$+ 2\int_\gamma P_n^{\mathsf{L}}\big(W' - h^{\mathsf{L}}W\big)(P_k^{\mathsf{R}})'\,\mathrm{d}\,z + \int_\gamma P_n^{\mathsf{L}}W(P_k^{\mathsf{R}})''\,\mathrm{d}\,z;$$

and so

$$\langle \boldsymbol{\ell}^{\mathsf{L}}(P_n^{\mathsf{L}})\,,\,P_k^{\mathsf{R}}\rangle_W = \int_\gamma P_n^{\mathsf{L}}W\big((P_k^{\mathsf{R}})'' + 2\,h^{\mathsf{R}}(P_k^{\mathsf{R}})' + \mathcal{M}(h^{\mathsf{R}})P_k^{\mathsf{R}}\big)\,\mathrm{d}\,z, \qquad n,k \in \{0,1,2,\dots\}$$

or, equivalently,

$$\langle \boldsymbol{\ell}^{\mathsf{L}}(P_n^{\mathsf{L}})\,,\,P_k^{\mathsf{R}}\rangle_W = \langle P_n^{\mathsf{L}}\,,\,\boldsymbol{\ell}^{\mathsf{R}}(P_k^{\mathsf{R}})\rangle_W\,,$$

which completes the proof. □

Remark 9. For a symmetric or Hermitian reductions we find that

$$\begin{aligned} \boldsymbol{\ell}^{\mathsf{R}}(P) &= \big(\boldsymbol{\ell}^{\mathsf{L}}(P^\top)\big)^\top, && \text{symmetric},\\ \boldsymbol{\ell}^{\mathsf{R}}(P) &= \big(\boldsymbol{\ell}^{\mathsf{L}}(P^\dagger)\big)^\dagger, && \text{Hermitian}, \end{aligned}$$

where in the last case we take $x \in \mathbb{R}$. Relation (57) reads in this case as follows

$$\begin{aligned} \boldsymbol{\ell}^*(P) &= (\boldsymbol{\ell}(P^\top))^\top, && \text{symmetric},\\ \boldsymbol{\ell}^*(P) &= (\boldsymbol{\ell}(P^\dagger))^\dagger, && \text{Hermitian}; \end{aligned}$$

for P any matrix polynomial and $\boldsymbol{\ell} := \boldsymbol{\ell}^{\mathsf{L}}$.

Definition 3. Let α^{L} and α^{R} be two $N \times N$ matrices and define the following linear operators acting on the space of matrix polynomials $\mathbb{C}^{N\times N}[z]$ as follows

$$\mathcal{L}^{\mathsf{L}}(P) := P'' + 2P'h^{\mathsf{L}} + P\alpha^{\mathsf{L}}, \qquad \mathcal{L}^{\mathsf{R}}(P) := P'' + 2h^{\mathsf{R}}P' + \alpha^{\mathsf{R}}P.$$

Observe that

$$\mathcal{L}^{\mathsf{L}}(P) = \boldsymbol{\ell}^{\mathsf{L}}(P) - P\,\mathcal{M}(h^{\mathsf{L}}) + P\alpha^{\mathsf{L}}, \qquad \mathcal{L}^{\mathsf{R}}(P) = \boldsymbol{\ell}^{\mathsf{R}}(P - \mathcal{M}(h^{\mathsf{R}})P + \alpha^{\mathsf{R}}P.$$

We have the following characterization.

Theorem 4. *The following conditions are equivalent:*

i) $\mathcal{L}^{\mathsf{R}} = \big(\mathcal{L}^{\mathsf{L}}\big)^*$ *with respect to the matrix of weights* $W(z)$.
ii) *The matrix of weights* $W(z)$ *satisfies the matrix Pearson equation* (36) *with the boundary conditions* (54) *as well as fulfills the constraint*

$$\big(\alpha^{\mathsf{L}} - \mathcal{M}(h^{\mathsf{L}})\big)W = W\big(\alpha^{\mathsf{R}} - \mathcal{M}(h^{\mathsf{R}})\big). \tag{58}$$

iii) *The matrix of weights $W(z)$ satisfies the boundary conditions (54) as well as*

$$W'' - 2\big(h^{\mathsf{L}}W\big)' + \alpha^{\mathsf{L}}W = W\alpha^{\mathsf{R}}, \tag{59}$$

$$W'' - 2\big(Wh^{\mathsf{R}}\big)' + W\alpha^{\mathsf{R}} = \alpha^{\mathsf{L}}W. \tag{60}$$

Proof. Following the ideas in the proof of Proposition 8

$$\langle \mathcal{L}^{\mathsf{L}}(P), \tilde{P}\rangle_W = \langle P, \mathcal{L}^{\mathsf{R}}(\tilde{P})\rangle_W$$

if and only if

$$\langle -P\,\mathcal{M}(h^{\mathsf{L}}) + P\alpha^{\mathsf{L}}, \tilde{P}\rangle_W = \langle P, -\mathcal{M}(h^{\mathsf{R}})\tilde{P} + \alpha^{\mathsf{R}}\tilde{P}\rangle_W$$

that is (58) takes place, and so i) is equivalent to ii).

To prove that i) is equivalent to iii) observe that, adding (59) and (60), the following holds

$$W'' = \big(h^{\mathsf{L}}W\big)' + \big(Wh^{\mathsf{R}}\big)',$$

which transforms (36) if we integrate requesting boundary conditions (54). Moreover, if we subtract (59) and (60) we arrive directly to (58). □

Remark 10. For the symmetric or Hermitian reductions we find that

$$\mathcal{L}^{\mathsf{R}}(P) = \big(\mathcal{L}^{\mathsf{L}}(P^\top)\big)^\top, \qquad \text{symmetric},$$
$$\mathcal{L}^{\mathsf{R}}(P) = \big(\mathcal{L}^{\mathsf{L}}(P^\dagger)\big)^\dagger, \qquad \text{Hermitian},$$

where in the last case we take $x \in \mathbb{R}$.
Moreover, the following are equivalent conditions

i) Equations

$$\begin{aligned} \mathcal{L}^*(P) &= (\mathcal{L}(P^\top))^\top, && \text{symmetric},\\ \mathcal{L}^*(P) &= (\mathcal{L}(P^\dagger))^\dagger, && \text{Hermitian}; \end{aligned} \tag{61}$$

are satisfied by any matrix polynomial P, where $\mathcal{L} := \mathcal{L}^{\mathsf{L}}$.

ii) The matrix of weights $W(z)$ satisfies the matrix Pearson equation (39) with the boundary conditions

$$W|_{\partial\gamma} = 0_N, \qquad \big(W' - 2hW\big)\big|_{\partial\gamma} = 0_N, \tag{62}$$

as well as fulfills the constraint

$$\big(\alpha - \mathcal{M}(h)\big)W = W\big(\alpha^\top - \mathcal{M}(h^\top)\big), \qquad \text{symmetric},$$
$$\big(\alpha - \mathcal{M}(h)\big)W = W\big(\alpha^\dagger - \mathcal{M}((h(\bar{z}))^\dagger)\big), \qquad \text{Hermitian},$$

iii) The matrix of weights $W(z)$ satisfies the boundary conditions (62) as well as

$$\begin{aligned} W'' - 2\big(hW\big)' + \alpha W &= W\alpha^\top, \text{ symmetric},\\ W'' - 2\big(hW\big)' + \alpha W &= W\alpha^\dagger, \text{ Hermitian}. \end{aligned} \tag{63}$$

5.2. Eigenvalue problems

Now we discuss a result that links our results based on the Riemann–Hilbert problem with previous seminal results by Grünbaum and Durán [31,33,35,36]. The next theorem shows when the polynomials and associated functions of second kind are eigenfunctions of a second order operator.

Theorem 5 *(Eigenvalue problems for Hermite matrix case). Let $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$ be degree one matrix polynomials, i.e.*

$$h^{\mathsf{L}}(z)=A^{\mathsf{L}}z+B^{\mathsf{L}}, \qquad h^{\mathsf{R}}(z)=A^{\mathsf{R}}z+B^{\mathsf{R}}, \qquad A^{\mathsf{L}},A^{\mathsf{R}},B^{\mathsf{L}},B^{\mathsf{R}}\in\mathbb{C}^{N\times N},$$

with $A^{\mathsf{L}},A^{\mathsf{R}}$ negative definite, and $W(z)$ a matrix of weights that solves (59), (60) subject to the boundary conditions (54). Then, the following conditions are equivalent:

i) *The operators $\mathcal{L}^{\mathsf{L}}$ and $\mathcal{L}^{\mathsf{R}}$ are adjoint operators with respect to the matrix of weights $W(z)$, i.e. $\mathcal{L}^{\mathsf{R}}=\big(\mathcal{L}^{\mathsf{L}}\big)^*$.*

ii) *The biorthogonal polynomial sequences with respect to $W(z)$, say $\big\{P^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{Z}_+}$, $\big\{P^{\mathsf{R}}_n(z)\big\}_{n\in\mathbb{Z}_+}$, are eigenfunctions of $\mathcal{L}^{\mathsf{L}}$ and $\mathcal{L}^{\mathsf{R}}$, i.e. there exist $N\times N$ matrices, λ^{L}_n, λ^{R}_n such that*

$$\mathcal{L}^{\mathsf{L}}(P^{\mathsf{L}}_n)=\lambda^{\mathsf{L}}_nP^{\mathsf{L}}_n, \qquad \mathcal{L}^{\mathsf{R}}(P^{\mathsf{R}}_n)=P^{\mathsf{R}}_n\lambda^{\mathsf{R}}_n, \tag{64}$$

with $\lambda^{\mathsf{L}}_nC_n^{-1}=C_n^{-1}\lambda^{\mathsf{R}}_n$, $n\in\mathbb{Z}_+$.

iii) *The functions of second kind, $\big\{Q^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{Z}_+}$ and $\big\{Q^{\mathsf{R}}_n(z)\big\}_{n\in\mathbb{Z}_+}$, associated with the biorthogonal polynomials, $\big\{P^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{Z}_+}$ and $\big\{P^{\mathsf{R}}_n(z)\big\}_{n\in\mathbb{Z}_+}$, fulfill the second order differential equations,*

$$\big(Q^{\mathsf{L}}_n\big)''(z)-2\big(Q^{\mathsf{L}}_n\big)'(z)\,h^{\mathsf{R}}(z)+Q^{\mathsf{L}}_n(z)\,(\alpha^{\mathsf{R}}-2A^{\mathsf{R}})=\lambda^{\mathsf{L}}_n\,Q^{\mathsf{L}}_n(z), \tag{65}$$

$$\big(Q^{\mathsf{R}}_n\big)''(z)-2h^{\mathsf{L}}(z)\big(Q^{\mathsf{R}}_n\big)'(z)+(\alpha^{\mathsf{L}}-2A^{\mathsf{L}})\,Q^{\mathsf{R}}_n(z)=Q^{\mathsf{R}}_n\,\lambda^{\mathsf{R}}_n. \tag{66}$$

Proof. ii) implies i). If $n\neq m$

$$\langle\mathcal{L}^{\mathsf{L}}(P^{\mathsf{L}}_n(z))\,,\,P^{\mathsf{R}}_m(z)\rangle_W=\lambda^{\mathsf{L}}_n\langle P^{\mathsf{L}}_n(z)\,,\,P^{\mathsf{R}}_m(z)\rangle_W=0_N\,,$$
$$\langle P^{\mathsf{L}}_n(z)\,,\,\mathcal{L}^{\mathsf{R}}(P^{\mathsf{R}}_m(z))\rangle_W=\langle P^{\mathsf{L}}_n(z)\,,\,P^{\mathsf{R}}_m(z)\rangle_W\lambda^{\mathsf{R}}_m=0_N\,;$$

and for $n=m$

$$\frac{1}{2\pi\,\mathrm{i}}\langle\mathcal{L}^{\mathsf{L}}(P^{\mathsf{L}}_n(z))\,,\,P^{\mathsf{R}}_n(z)\rangle_W=\lambda^{\mathsf{L}}_nC_n^{-1}\,, \qquad \frac{1}{2\pi\,\mathrm{i}}\langle P^{\mathsf{L}}_n(z)\,,\,\mathcal{L}^{\mathsf{R}}(P^{\mathsf{R}}_n(z))\rangle_W=C_n^{-1}\lambda^{\mathsf{R}}_n\,, \qquad n\in\mathbb{Z}_+,$$

which implies that $\langle\mathcal{L}^{\mathsf{L}}(P^{\mathsf{L}}_n(z))\,,\,P^{\mathsf{R}}_m(z)\rangle_W=\langle P^{\mathsf{L}}_n(z)\,,\,\mathcal{L}^{\mathsf{R}}(P^{\mathsf{R}}_m(z))\rangle_W$, $n,m\in\mathbb{Z}_+$.

i) implies ii). Let us note that the space of matrix polynomials of a given degree is invariant under the action of the operators $\mathcal{L}^{\mathsf{L}}$ and $\mathcal{L}^{\mathsf{R}}$; hence

$$\mathcal{L}^{\mathsf{L}}(P^{\mathsf{L}}_n)=\sum_{k=0}^{n}\lambda^{\mathsf{L}}_{n,k}P^{\mathsf{L}}_k.$$

Now, taking into account the biorthogonality of the sequences P^{L}_n and P^{R}_n with respect to W and using that the operators $\mathcal{L}^{\mathsf{L}}$ and $\mathcal{L}^{\mathsf{R}}$ are adjoint operators we have

$$\lambda^{\mathsf{L}}_{n,k} C_k^{-1} = \frac{1}{2\pi\,\mathrm{i}} \langle \mathcal{L}^{\mathsf{L}}(P^{\mathsf{L}}_n), P^{\mathsf{R}}_k \rangle_W = \frac{1}{2\pi\,\mathrm{i}} \langle P^{\mathsf{L}}_n, \mathcal{L}^{\mathsf{R}}(P^{\mathsf{R}}_k) \rangle_W = C_n^{-1} \lambda^{\mathsf{R}}_{n,k} \delta_{n,k}, \qquad n, m \in \mathbb{Z}_+,$$

so it holds that $\mathcal{L}^{\mathsf{L}}(P^{\mathsf{L}}_n) = \lambda^{\mathsf{L}}_n P^{\mathsf{L}}_k$ and also $\mathcal{L}^{\mathsf{R}}(P^{\mathsf{R}}_n) = \lambda^{\mathsf{R}}_n P^{\mathsf{R}}_k$ where $\lambda^{\mathsf{L}}_n C_n^{-1} = C_n^{-1} \lambda^{\mathsf{R}}_n$.

ii) implies iii) We return back to equations (50) and (64) and see that

$$\big[\mathcal{M}(h^{\mathsf{L}}(z')), P^{\mathsf{L}}_n(z')\big] + \mathsf{H}^{\mathsf{L}}_{1,1,n}(z') P^{\mathsf{L}}_n(z') - \mathsf{H}^{\mathsf{L}}_{1,2,n}(z') P^{\mathsf{L}}_{n-1}(z') = -P^{\mathsf{L}}_n(z')\,\alpha^{\mathsf{L}} + \lambda^{\mathsf{L}}_n\, P^{\mathsf{L}}_n(z')\,.$$

Now, multiplying this equation on the right by $W(z')/(z - z')$ and integrating along γ, taking into account the boundary conditions, we get

$$\mathcal{M}(h^{\mathsf{L}}(z))\, Q^{\mathsf{L}}_n(z) - Q^{\mathsf{L}}_n(z)\, \mathcal{M}(-h^{\mathsf{R}}(z)) + \mathsf{H}^{\mathsf{L}}_{1,1,n}(z) Q^{\mathsf{L}}_n(z) - \mathsf{H}^{\mathsf{L}}_{1,2,n}(z) Q^{\mathsf{L}}_{n-1}(z) = Q^{\mathsf{L}}_n(z)\,(2A^{\mathsf{R}} - \alpha^{\mathsf{R}}) + \lambda^{\mathsf{L}}_n\, Q^{\mathsf{L}}_n(z)\,.$$

Now, from (51) we get (65). We have proved that if $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$ satisfies a second order linear differential equation the associated functions of second kind also does.

We have that

$$\int_\gamma \frac{\mathcal{M}(h^{\mathsf{L}})(z')}{z' - z} P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' = \int_\gamma \frac{(A^{\mathsf{L}})^2 (z')^2 + \{A^{\mathsf{L}}, B^{\mathsf{L}}\} z' + A^{\mathsf{L}} + (B^{\mathsf{L}})^2}{z' - z} P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z',$$

with the anticommutator notation $\{A, B\} = AB + BA$. Now, as

$$\begin{aligned}\int_\gamma \frac{(z')^2}{z' - z} P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' &= \int_\gamma \frac{(z')^2 - z^2}{z' - z} P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' + z^2 Q^{\mathsf{L}}_n(z)\\ &= \int_\gamma (z' + z) P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' + z^2 Q^{\mathsf{L}}_n(z)\,,\end{aligned}$$

and, in the same way,

$$\begin{aligned}\int_\gamma \frac{z'}{z' - z} P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' &= \int_\gamma \frac{z' - z}{z' - z} P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' + z Q^{\mathsf{L}}_n(z)\\ &= \int_\gamma P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' + z Q^{\mathsf{L}}_n(z)\,,\end{aligned}$$

we finally obtain

$$\int_\gamma \frac{\mathcal{M}(h^{\mathsf{L}})(z')}{z' - z} P^{\mathsf{L}}_n(z') W(z')\,\mathrm{d}\,z' = \mathcal{M}(h^{\mathsf{L}})\, Q^{\mathsf{L}}_n(z), \qquad n \geq 2,$$

where we have used the orthogonality conditions for $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$. We also have

$$\begin{aligned}\int_\gamma P^{\mathsf{L}}_n(z') \frac{\mathcal{M}(h^{\mathsf{L}})(z') - \alpha^{\mathsf{L}}}{z' - z} W(z')\,\mathrm{d}\,z' &= \int_\gamma P^{\mathsf{L}}_n(z') W(z') \frac{\mathcal{M}(h^{\mathsf{R}})(z') - \alpha^{\mathsf{R}}}{z' - z}\,\mathrm{d}\,z'\\ &= Q^{\mathsf{L}}_n(z)\,(\mathcal{M}(h^{\mathsf{R}})(z) - \alpha^{\mathsf{R}}), \qquad n \geq 2.\end{aligned}$$

Using the same ideas we prove that

$$\int_\gamma \frac{\mathsf{H}^{\mathsf{L}}_{1,j,n}(z')}{z'-z} P^{\mathsf{L}}_{n-j+1}(z')W(z')\,\mathrm{d}\,z' = \mathsf{H}^{\mathsf{L}}_{1,j,n}(z)Q^{\mathsf{L}}_{n-j+1}(z), \qquad n\geq 1,\ j=1,2\,. \tag{67}$$

In fact, by definition (48) we know that the matrix polynomials $\mathsf{H}^{\mathsf{L}}_{1,j,n}(z')$ are of degree at most one, i.e.

$$\mathsf{H}^{\mathsf{L}}_{1,j,n}(z') = \mathsf{H}^{\mathsf{L},0}_{1,j,n}z' + \mathsf{H}^{\mathsf{L},1}_{1,j,n}\,, \qquad \mathsf{H}^{\mathsf{L},0}_{1,j,n}, \mathsf{H}^{\mathsf{L},1}_{1,j,n} \in \mathbb{C}^{N\times N}\,.$$

Summing and subtracting in (67) $\mathsf{H}^{\mathsf{L}}_{1,j,n}(z)$ we get in the left hand side

$$\int_\gamma \frac{\mathsf{H}^{\mathsf{L}}_{1,j,n}(z')}{z'-z} P^{\mathsf{L}}_{n-j+1}(z')W(z')\,\mathrm{d}\,z' = \int_\gamma \frac{\mathsf{H}^{\mathsf{L}}_{1,j,n}(z')-\mathsf{H}^{\mathsf{L}}_{1,j,n}(z)}{z'-z} P^{\mathsf{L}}_{n-j+1}(z')W(z')\,\mathrm{d}\,z' + \mathsf{H}^{\mathsf{L}}_{1,j,n}(z)Q^{\mathsf{L}}_{n-j+1}(z)\,;$$

hence, as

$$\frac{\mathsf{H}^{\mathsf{L}}_{1,j,n}(z')-\mathsf{H}^{\mathsf{L}}_{1,j,n}(z)}{z'-z} = \mathsf{H}^{\mathsf{L},0}_{1,j,n}\,,$$

we arrive to

$$\int_\gamma \frac{\mathsf{H}^{\mathsf{L}}_{1,j,n}(z')}{z'-z} P^{\mathsf{L}}_{n-j+1}(z')W(z')\,\mathrm{d}\,z' = \mathsf{H}^{\mathsf{L},0}_{1,j,n}\int_\gamma P^{\mathsf{L}}_{n-j+1}(z')W(z')\,\mathrm{d}\,z' + \mathsf{H}^{\mathsf{L}}_{1,j,n}(z)Q^{\mathsf{L}}_{n-j+1}(z)\,,$$

and by the orthogonality of $\{P^{\mathsf{L}}_{n-j+1}(z)\}_{n\in\mathbb{Z}_+}$ with respect to $W(z)$ we get for $j=1,2$, and for all $n=1,2,\ldots$, that (67) holds true.

From (52) and taking into account that $\mathcal{L}^{\mathsf{R}}(P^{\mathsf{R}}_n) = P^{\mathsf{R}}_n\,\lambda^{\mathsf{R}}_n$ we get

$$\big[P^{\mathsf{R}}_n(z'), \mathcal{M}(h^{\mathsf{R}})(z')\big] + P^{\mathsf{R}}_n(z')\mathsf{H}^{\mathsf{R}}_{1,1,n}(z') - P^{\mathsf{R}}_{n-1}(z')\mathsf{H}^{\mathsf{R}}_{2,1,n}(z') = -\alpha^{\mathsf{R}}\,P^{\mathsf{R}}_n(z') + P^{\mathsf{R}}_n(z')\,\lambda^{\mathsf{R}}_n\,.$$

Now, multiplying this equation on the left by $W(z')/(z-z')$ and integrate (using the boundary conditions) over γ, we get

$$Q^{\mathsf{R}}_n(z)\,\mathcal{M}(h^{\mathsf{R}})(z) - \mathcal{M}(-h^{\mathsf{L}})(z)Q^{\mathsf{R}}_n(z) + Q^{\mathsf{R}}_n(z)\mathsf{H}^{\mathsf{R}}_{1,1,n}(z) - Q^{\mathsf{R}}_{n-1}(z)\mathsf{H}^{\mathsf{R}}_{2,1,n}(z) = (2A^{\mathsf{L}}-\alpha^{\mathsf{L}})\,Q^{\mathsf{R}}_n + Q^{\mathsf{R}}_n\,\lambda^{\mathsf{R}}_n\,,$$

and so, from (53) we arrive to (66).

iii) implies ii). Taking derivatives with respect to z we get, after integration by parts and using the boundary conditions

$$(Q^{\mathsf{L}}_n)'(z) = \int_\gamma \frac{P^{\mathsf{L}}_n(z')W(z')}{(z'-z)^2}\,\mathrm{d}\,z'\,,$$

$$(Q^{\mathsf{L}}_n)''(z) = 2\int_\gamma \frac{P^{\mathsf{L}}_n(z')W(z')}{(z'-z)^3}\,\mathrm{d}\,z' = \int_\gamma \frac{(P^{\mathsf{L}}_n(z')W(z'))''}{z'-z}\,\mathrm{d}\,z'.$$

Moreover,

$$\begin{aligned}-2(Q^{\mathsf{L}}_n)'(z)h^{\mathsf{R}}(z) &= 2\int_\gamma P^{\mathsf{L}}_n(z')W(z')\frac{h^{\mathsf{R}}(z')-h^{\mathsf{R}}(z)}{(z'-z)^2}\,\mathrm{d}\,z' - 2\int_\gamma P^{\mathsf{L}}_n(z')W(z')\frac{h^{\mathsf{R}}(z')}{(z'-z)^2}\,\mathrm{d}\,z' \\ &= 2Q^{\mathsf{L}}_n(z)A^{\mathsf{R}} - 2\int_\gamma \frac{(P^{\mathsf{L}}_n(z')W(z')h^{\mathsf{R}}(z'))'}{z'-z}\,\mathrm{d}\,z'.\end{aligned}$$

Now, we plug all this information into (65) and deduce that

$$\int_{\gamma} \frac{(P^{\mathsf{L}}_n)''W + 2(P^{\mathsf{L}}_n)'(W' - Wh^{\mathsf{R}}) + P^{\mathsf{L}}_n(W'' - 2(Wh^{\mathsf{R}})') + W\alpha^{\mathsf{R}}}{z' - z}\,\mathrm{d}\,z' = \lambda^{\mathsf{L}}_n \int_{\gamma} \frac{P^{\mathsf{L}}_n W}{z' - z}\,\mathrm{d}\,z'\,;$$

by the hypothesis over W we get

$$\int_{\gamma} \frac{(P^{\mathsf{L}}_n)''(z') + 2(P^{\mathsf{L}}_n)'(z')h^{\mathsf{L}}(z') + P^{\mathsf{L}}_n\alpha^{\mathsf{L}} - \lambda^{\mathsf{L}}_n P^{\mathsf{L}}_n}{z' - z} W(z')\,\mathrm{d}\,z' = 0_N.$$

Hence, we get that $\{P^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$ satisfies (64). Using analogous arguments it can be proven that the equation (66) for $\{Q^{\mathsf{R}}_n\}_{n\in\mathbb{Z}_+}$ implies that $\{P^{\mathsf{R}}_n\}_{n\in\mathbb{Z}_+}$ satisfies (64). □

Let us emphasize that the results in iii) in the previous Theorem regarding the second kind functions, $\{Q^{\mathsf{L}}_n\}_{n\in\mathbb{Z}_+}$ and $\{Q^{\mathsf{R}}_n\}_{n\in\mathbb{Z}_+}$ are, to the best of our knowledge, completely new. Moreover, from Theorem 4, we see that W in Theorem 5 can be taken as a solution of a Pearson–Sylvester differential equation given by (36) and that satisfies (58).

Remark 11. For the symmetric or Hermitian reductions we take $h(z) = Az + B$, with A definite negative, and $W(z)$ a matrix of weights a solution of (63) subject to the boundary conditions (62). Then, the following conditions are equivalent:

i) Equation (61) is satisfied.
ii) The matrix orthogonal polynomials with respect to $W(z)$ are eigenfunctions of $\mathcal{L}$.
iii) The functions of second kind, $\{Q_n(z)\}_{n\in\mathbb{Z}_+}$, associated with the matrix orthogonal polynomials, $\{P_n(z)\}_{n\in\mathbb{Z}_+}$ fulfill the second order differential equations,

$$\begin{aligned} \big(Q_n\big)''(z) - 2\big(Q_n\big)'(z)\,(h(z))^{\top} + Q_n(z)\,(\alpha^{\top} - 2A^{\top}) &= \lambda_n\,Q_n(z), && \text{symmetric},\\ \big(Q_n\big)''(z) - 2\big(Q_n\big)'(z)\,(h(\bar z))^{\dagger} + Q_n(z)\,(\alpha^{\dagger} - 2A^{\dagger}) &= \lambda_n\,Q_n(z), && \text{Hermitian}. \end{aligned}$$

The equivalences, described in the previous remark, excluding the one for the second kind functions (which is new), coincide with those of [33]. Therefore, these results could be understood (in the sense the biorthogonality includes Hermitian and non Hermitian orthogonality) as an extension of those by Durán and Grünbaum to the non Hermitian orthogonality scenario.

6. Nonlinear difference equations for the recursion coefficients

Using the Riemann–Hilbert approach we will derive in this section nonlinear matrix difference equations fulfilled by the recursion coefficients. We will consider three different possibilities for the Pearson equations satisfied by the matrix of weights.

6.1. Nonlinear difference equations for Hermite matrix polynomials

We now explore the simplest case when $\max(\deg h^{\mathsf{L}}_n(z), \deg h^{\mathsf{R}}_n(z)) = 1$ in full generality. We take

$$h^{\mathsf{L}}(z) = A^{\mathsf{L}}z + B^{\mathsf{L}}, \qquad h^{\mathsf{R}}(z) = A^{\mathsf{R}}z + B^{\mathsf{R}},$$

for arbitrary matrices $A^{\mathsf{L}}, B^{\mathsf{L}}, A^{\mathsf{R}}, B^{\mathsf{R}} \in \mathbb{C}^{N\times N}$, with $A^{\mathsf{L}}, A^{\mathsf{R}}$ definite negative matrices. Thus, the matrix of weights $W(z)$ is a solution of the following Pearson equation (a Sylvester linear differential equation)

$$W'(z) = (A^{\mathsf{L}}z + B^{\mathsf{L}})W(z) + W(z)(A^{\mathsf{R}}z + B^{\mathsf{R}}).$$

For simplicity we take $\gamma = \mathbb{R}$. Hence, the structure matrices have, cf. (37) and (38), the following form

$$M^{\mathsf{L}}_n(z) = \mathcal{A}^{\mathsf{L}}z + \mathcal{K}^{\mathsf{L}}_n, \qquad \mathcal{A}^{\mathsf{L}} = \begin{bmatrix} A^{\mathsf{L}} & 0_N \\ 0_N & -A^{\mathsf{R}} \end{bmatrix}, \qquad \mathcal{K}^{\mathsf{L}}_n = \begin{bmatrix} B^{\mathsf{L}} + [p^1_{\mathsf{L},n}, A^{\mathsf{L}}] & C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1} \\ -C_{n-1}A^{\mathsf{L}} - A^{\mathsf{R}}C_{n-1} & -B^{\mathsf{R}} - [q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}] \end{bmatrix}. \tag{68}$$

The Sylvester differential system (40) for the left fundamental matrix is

$$\big(Y^{\mathsf{L}}_n(z)\big)' + \Big[Y^{\mathsf{L}}_n(z), \begin{bmatrix} A^{\mathsf{L}}z + B^{\mathsf{L}} & 0_N \\ 0_N & -A^{\mathsf{R}}z - B^{\mathsf{R}} \end{bmatrix}\Big] = \begin{bmatrix} [p^1_{\mathsf{L},n}, A^{\mathsf{L}}] & C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1} \\ -C_{n-1}A^{\mathsf{L}} - A^{\mathsf{R}}C_{n-1} & -[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}] \end{bmatrix} Y_n(z), \qquad n \in \mathbb{Z}_+,$$

that is, for all $n \in \mathbb{Z}_+$,

$$(P^{\mathsf{L}}_n)' + \Big[P^{\mathsf{L}}_n, A^{\mathsf{L}}z + B^{\mathsf{L}}\Big] = \big[p^1_{\mathsf{L},n}, A^{\mathsf{L}}\big]P^{\mathsf{L}}_n - \big(C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1}\big)C_{n-1}P^{\mathsf{L}}_{n-1}, \tag{69}$$

$$C_{n-1}(Q^{\mathsf{L}}_{n-1})' - \Big[C_{n-1}Q^{\mathsf{L}}_{n-1}, A^{\mathsf{R}}z + B^{\mathsf{R}}\Big] = \big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)Q^{\mathsf{L}}_n - \big[q^1_{n-1}, A^{\mathsf{R}}\big]C_{n-1}Q^{\mathsf{L}}_{n-1}, \tag{70}$$

$$\begin{aligned} C_{n-1}(P^{\mathsf{L}}_{n-1})' + C_{n-1}P_{n-1}\big(A^{\mathsf{L}}z + B^{\mathsf{L}}\big) + \big(A^{\mathsf{R}}z + B^{\mathsf{R}}\big)\,C_{n-1}P^{\mathsf{L}}_{n-1} \\ = \big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)P^{\mathsf{L}}_n - \big[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}\big]C_{n-1}P^{\mathsf{L}}_{n-1}, \end{aligned} \tag{71}$$

$$(Q^{\mathsf{L}}_n)' - Q^{\mathsf{L}}_n\big(A^{\mathsf{R}}z + B^{\mathsf{R}}\big) - \big(A^{\mathsf{L}}z + B^{\mathsf{L}}\big)Q^{\mathsf{L}}_n = \big[p^1_n, A^{\mathsf{L}}\big]Q^{\mathsf{L}}_n - \big(C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1}\big)C_{n-1}Q^{\mathsf{L}}_{n-1}. \tag{72}$$

Taking the $(n-1)$-th z power of the (69), the $-n$-th of (70), the $-(n-1)$-th of (70) and the $-(n+1)$-th of (72) we get, for all $n \in \mathbb{Z}_+$,

$$\begin{aligned} nI_N + \big[p^1_{\mathsf{L},n}, B^{\mathsf{L}}\big] + \big[p^2_{\mathsf{L},n}, A^{\mathsf{L}}\big] &= \big[p^1_{\mathsf{L},n}, A^{\mathsf{L}}\big]\,p^1_n - \big(C_n^{-1}A^{\mathsf{R}} + A^{\mathsf{L}}C_n^{-1}\big)C_{n-1}, \\ nI_N + \big[q^1_{n-1}, B^{\mathsf{R}}\big] + \big[q^2_{\mathsf{L},n-1}, A^{\mathsf{R}}\big] &= -\big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)C_n^{-1} + \big[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}\big]q^1_{\mathsf{L},n-1}, \\ C_{n-1}B^{\mathsf{L}} + B^{\mathsf{R}}C_{n-1} + C_{n-1}\big[p^1_{\mathsf{L},n-1}, A^{\mathsf{L}}\big] &= -\big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)\beta^{\mathsf{L}}_{n-1} - \big[q^1_{\mathsf{L},n-1}, A^{\mathsf{R}}\big]C_{n-1}, \\ B^{\mathsf{R}}C_n + C_nB^{\mathsf{L}} + \big[q^1_{\mathsf{L},n}, A^{\mathsf{R}}\big]C_n &= -C_n\big[p^1_{\mathsf{L},n}, A^{\mathsf{L}}\big] - \big(A^{\mathsf{R}}C_n + C_nA^{\mathsf{L}}\big)\beta^{\mathsf{L}}_n. \end{aligned}$$

After some cleaning we reckon that the system is, for all $n \in \mathbb{Z}_+$, equivalent to

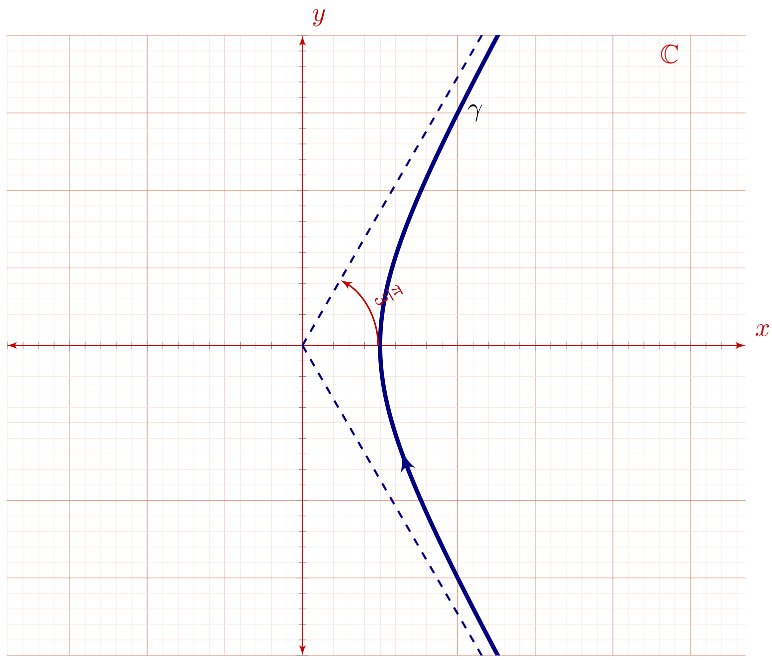
$$\begin{cases} I - \Big[\beta^{\mathsf{L}}_n, B^{\mathsf{L}} - \Big[\displaystyle\sum_{k=0}^{n-1}\beta^{\mathsf{L}}_k, A^{\mathsf{L}}\Big] + A^{\mathsf{L}}\beta^{\mathsf{L}}_n\Big] \\ \qquad\qquad = C_n^{-1}C_{n-1}A^{\mathsf{L}} - C_{n+1}^{-1}A^{\mathsf{R}}C_n - A^{\mathsf{L}}C_{n+1}^{-1}C_n + C_n^{-1}A^{\mathsf{R}}\,C_{n-1}, \\ C_{n-1}B^{\mathsf{L}} + B^{\mathsf{R}}C_{n-1} - C_{n-1}\Big[\displaystyle\sum_{k=0}^{n-2}\beta^{\mathsf{L}}_k, A^{\mathsf{L}}\Big] \\ \qquad\qquad = -\big(C_{n-1}A^{\mathsf{L}} + A^{\mathsf{R}}C_{n-1}\big)\beta^{\mathsf{L}}_{n-1} - \Big[\displaystyle\sum_{k=0}^{n-1}C_k\beta^{\mathsf{L}}_k(C_k)^{-1}, A^{\mathsf{R}}\Big]C_{n-1}. \end{cases}$$

6.2. A matrix extension of the alt-dPI

We now discuss the case $\max(h^{\mathsf{L}}_n(z), h^{\mathsf{R}}_n(z)) = 2$, but we perform a strong simplification as we take $h^{\mathsf{R}} = 0_N$ and $h^{\mathsf{L}} = \lambda + \mu z + \nu z^2$, with $\lambda, \mu, \nu \in \mathbb{C}^{N\times N}$ arbitrary matrices but for ν being negative definite nonsingular matrix. Thus, the Pearson equation will be

$$W'(z) = (\lambda + \mu z + \nu z^2)W(z). \tag{73}$$

We obviously drop off the notation that distinguish left and right polynomials and only describe the results for the left case. The integrals are taken along γ, a smooth curve for which we have a *simple* Riemann–Hilbert problem as depicted in the following diagram:



Branch of the hyperbola $3x^2 - y^2 = 3$

The structure matrix, cf. (37), is a second order polynomial $M_n(z) = M_n^0 z^2 + M_n^1 z + M_n^2$ with

$$M_n^0 = \begin{bmatrix} \nu & 0_N \\ 0_N & 0_N \end{bmatrix}, \qquad M_n^1 = \begin{bmatrix} \mu - [\nu, p_n^1] & \nu C_n^{-1} \\ -C_{n-1}\nu & \underline{0} \end{bmatrix},$$

$$M_n^2 = \begin{bmatrix} \lambda - [\beta, p_n^1] - [\nu, p_n^2] + \nu (p_n^1)^2 - p_n^1 \nu p_n^1 + \nu C_n^{-1} C_{n-1} & (\mu - [\nu, p_n^1] + \gamma\beta_n) C_n^{-1} \\ -C_{n-1}\left(\mu + p_{n-1}^1 \nu - \nu p_n^1\right) & -C_{n-1}\nu\, C_n^{-1} \end{bmatrix}.$$

Proposition 9 *(Matrix alt-dPI system). The recursion coefficients β_n, γ_n of the matrix orthogonal polynomials with matrix of weights a solution of the Pearson equation (73) are subject to the following system of equations, for all $n \in \mathbb{Z}_+$,*

$$\Big(\mu + \Big[\nu, \sum_{k=0}^{n-1} \beta_k\Big] + \gamma(\beta_n + \beta_{n+1})\Big)\gamma_{n+1} = -(n+1)I, \tag{74}$$

$$\begin{aligned} &\lambda + \gamma(\gamma_n + \gamma_{n+1} + \beta_n^2) - \mu\beta_n + \Big[\mu, \sum_{k=0}^{n-1} \beta_k\Big](I_N + \beta_n) \\ &+ \Big[\nu, \sum_{m=1}^{n-1} \gamma_m - \sum_{0 \le k < m \le n-1} \beta_m \beta_k\Big] + \Big[\nu, \sum_{k=0}^{n-1} \beta_k\Big] \sum_{k=0}^{n-1} \beta_k = 0_N. \end{aligned} \tag{75}$$

Proof. Given the asymptotics about ∞,

$$-C_n Q_n(z) = I_N z^{-n-1} + q_n^1 z^{-n-2} + \cdots,$$

we read the coefficient of z^{-n-1} coming from

$$C_{n-1}Q'_{n-1}(z) = -M^n_{2,1}(z)Q_n(z) + M^n_{2,2}(z)\,C_{n-1}Q_{n-1}(z)\,,$$

with $M^n_{2,1} = -C_{n-1}\nu z - C_{n-1}\big(\mu + p^1_{n-1}\nu - \nu p^1_n\big)\,,\quad M^n_{2,2} = -C_{n-1}\nu C_n^{-1}$, we get (74); and from

$$Q'_n(z) = M^n_{1,1}\,Q_n(z) - M^n_{1,2}(z)\,C_{n-1}\,Q_{n-1}(z)\,,$$

with

$$\begin{gathered}M^n_{1,1} = \nu z^2 + \big(\mu - \big[\nu, p^1_n\big]\big)z + \big(\lambda - \big[\mu, p^1_n\big] - \big[\nu, p^2_n\big] + \nu\big(p^1_n\big)^2 + \nu C_n^{-1}C_{n-1} - p^1_n\nu\, p^1_n\big)\\ M^n_{1,2} = \nu C_n^{-1}z + \big(\mu - \big[\nu, p^1_n\big] + \nu\beta_n\big)C_n^{-1}\,;\end{gathered}$$

we deduce (75) from the z^{-n-1}-coefficient. □

Another form of writing this result is

Proposition 10 *(Matrix alt-dPI system). Given matrix orthogonal polynomials with matrix of weights $W(z)$ supported on γ, a solution of the Pearson equation (73), the recursion coefficients γ_n can be expressed directly in terms of the recursion coefficients β_n, for all $n \in \mathbb{Z}_+$,*

$$\gamma_{n+1} = -(n+1)\Big(\beta + \Big[\gamma, \sum_{k=0}^{n-1}\beta_k\Big] + \gamma(\beta_n + \beta_{n+1})\Big)^{-1}.$$

The coefficients β_n fulfill, for all $n \in \mathbb{Z}_+$, the following non-Abelian alt-dPI,

$$\begin{aligned}\lambda + \nu\big(\gamma_n + \gamma_{n+1} + \beta_n^2\big) - \mu\beta_n + \Big[\beta, \sum_{k=0}^{n-1}\beta_k\Big]\big(I_N + \beta_n\big)&\\ + \Big[\nu, \sum_{m=1}^{n-1}\gamma_m - \sum_{0\leq k<m\leq n-1}\beta_m\beta_k\Big] + \Big[\nu, \sum_{k=0}^{n-1}\beta_k\Big]\sum_{k=0}^{n-1}\beta_k &= 0_N.\end{aligned}$$

Proof. From (74) we get the γ_n in terms of β_n, plugged this relation into the second one gives the following nonlinear equation for the matrices β_n. □

If we assume that $\nu = -I$ as expected strong simplifications occur. In the first place we find that

$$\gamma_{n+1} = -(n+1)(\mu - \beta_n - \beta_{n+1})^{-1},$$

and, secondly, we derive the following simplified version of a non-Abelian alt-dPI equation

$$\lambda - \beta_n^2 + n(\beta - \beta_{n-1} + \beta_n)^{-1} + (n+1)(\mu - \beta_n - \beta_{n+1})^{-1} - \mu\beta_n = -\Big[\mu, \sum_{k=0}^{n-1}\beta_k\Big]\big(I_N + \beta_n\big).$$

Moreover, when we choose $\nu = -I$ and $\mu = 0_N$ the non local terms disappear and the equation simplifies further to

$$-n(\beta_{n-1} + \beta_n)^{-1} - (n+1)(\beta_n + \beta_{n+1})^{-1} + \beta_n^2 = \lambda.$$

Let us remind the reader how the alt-dPI equation appeared for the first time. Going back to the scalar context, in Magnus' work [54], associated with the weight functions solution of the Pearson equation $W'(z) = \big(z^2+t\big)W(z)$, we can find the following scalar alternate discrete Painlevé I system

$$\gamma_n + \gamma_{n+1} + \beta_n^2 + t = 0,$$
$$n + \gamma_n\big(\beta_n + \beta_{n-1}\big) = 0,$$

which can be written as

$$-\frac{n}{\beta_n+\beta_{n-1}} - \frac{n+1}{\beta_n+\beta_{n+1}} + \beta_n^2 + t = 0.$$

6.3. The matrix dPI system

We now increase further the degree of the polynomials appearing in the Pearson equations. We consider the case with $\max(h_n^{\mathsf{L}}(z), h_n^{\mathsf{R}}(z)) = 3$, but we perform a strong simplification we take $h^{\mathsf{R}} = 0_N$ and $h^{\mathsf{L}} = \mu z + \nu z^3$, with $\mu, \nu \in \mathbb{C}^{N\times N}$ arbitrary matrices but for ν being negative definite nonsingular matrix. Now we take $\gamma = \mathbb{R}$. Observe that we have non taken the more general possible polynomial of degree three, but an odd one, with well defined parity on z, this simplifies widely the computations.

The associated Pearson type equation for a matrix of weights of Freud type:

$$W'(z) = (\mu z + \nu z^3)W(z) \tag{76}$$

The structure matrix, cf. (37), is a third order polynomial, that we write as follows

$$M_n(z) = M_n^0 z^3 + M_n^1 z^2 + M_n^2 z + M_n^3$$

with

$$M_n^0 = \begin{bmatrix} \nu & 0_N \\ 0_N & 0_N \end{bmatrix}, \qquad M_n^1 = \begin{bmatrix} 0_N & \mu C_n^{-1} \\ -C_{n-1}\mu & 0_N \end{bmatrix},$$
$$M_n^2 = \begin{bmatrix} \nu + [p_n^2, \nu] + \mu C_n^{-1} C_{n-1} & 0_N \\ 0_N & -C_{n-1}\nu C_n^{-1} \end{bmatrix}, \qquad M_n^3 = \begin{bmatrix} 0_N & \xi_n C_n^{-1} \\ -C_{n-1}\xi_{n-1} & 0_N \end{bmatrix},$$

where $\xi_n = \mu + [p_n^2, \nu] + \nu(C_n^{-1}C_{n-1} + C_{n+1}^{-1}C_n)$, $n \in \mathbb{Z}_+$.

With this at hand we find.

Proposition 11 *(Matrix dPI equation). The recursion coefficients γ_n of the matrix orthogonal polynomials with matrix of weights satisfying the Pearson equation (76) fulfill the following non-Abelian dPI equation*

$$\Big(\mu + \nu(\gamma_{n+2} + \gamma_{n+1} + \gamma_n) + \big[\nu, \sum_{k=1}^{n-1}\gamma_k\big]\Big)\gamma_{n+1} = -(n+1)I, \qquad n \in \mathbb{Z}_+.$$

Proof. Compare the coefficients of z^{-n-1} in the ODE for the second kind functions we get directly (without additional computations) the MdPI equations for the three term relation coefficients of $\big\{P_n(z)\big\}_{n\in\mathbb{Z}_+}$. □

Notice the appearance again of non local terms, that disappear if we take $\nu = -I$ and the matrix dPI reads

$$\gamma_{n+1} = n\gamma_n^{-1} - \gamma_n - \gamma_{n-1} - \mu, \qquad n \in \mathbb{Z}_+,$$

which was derived in the matrix context for the first time in [16] and the confinement of singularities for this relation was proven in [17,16], see also [47]. In 1995, Alphonse P. Magnus [54] for the Freud weight satisfying the Pearson equation $W'(z) = -\big(z^3 + 2tz\big)\,W(z)$ presented the following scalar discrete Painlevé I equation

$$\gamma_n\big(\gamma_{n-1} + \gamma_n + \gamma_{n+1}\big) + 2t\gamma_n = n.$$

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