J. Math. Anal. Appl. 502 (2021) 125200

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Convergence for a planar elliptic problem with large exponent Neumann data

Habib Fourti [a,b,*]

[a] University of Monastir, Faculty of Sciences, 5019 Monastir, Tunisia
[b] Laboratory LR 13 ES 21, University of Sfax, Faculty of Sciences, 3000 Sfax, Tunisia

ARTICLE INFO

Article history:
Received 9 November 2019
Available online 2 April 2021
Submitted by M. Musso

Keywords:
Nonlinear boundary value problem
Large exponent
Asymptotic analysis
Concentration of solutions

ABSTRACT

We study positive solutions u_p of the nonlinear Neumann elliptic problem $\Delta u = u$ in Ω, $\partial u/\partial\nu = |u|^{p-1}u$ on $\partial\Omega$, where Ω is a bounded open smooth domain in $\mathbb{R}^2$. We investigate the asymptotic behavior of families of solutions u_p satisfying an energy bound condition when the exponent p is getting large. Inspired by the work of Davila-del Pino-Musso [8], we prove that u_p is developing m peaks $x_i \in \partial\Omega$, in the sense $u_p^p / \int_{\partial\Omega} u_p^p$ approaches the sum of m Dirac masses at the boundary and we determine the localization of these concentration points.

1. Introduction

The purpose of this paper is to study the following problem

$$\begin{cases} \Delta u = u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = |u|^{p-1}u & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $p > 1$, $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and $\partial/\partial\nu$ denotes the derivative with respect to the outward normal to $\partial\Omega$. Elliptic problem with nonlinear boundary condition has been widely studied in the past by many authors and it is still an area of intensive research, see for instance [2,5–8,16,18,24].
Problem (1.1) has a variational structure. Indeed, its solutions are in a one-to-one correspondence with the critical points of the functional:

$$E_p(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 + u^2 \, dx - \frac{1}{p+1}\int_{\partial\Omega} |u|^{p+1} \, d\sigma(x)$$

* Correspondence to: University of Monastir, Faculty of Sciences, 5019 Monastir, Tunisia.
E-mail addresses: habib.fourti@fsm.rnu.tn, habib40@hotmail.fr.

https://doi.org/10.1016/j.jmaa.2021.125200

defined on the Sobolev space $H^1(\Omega)$. Trace and Sobolev embeddings tell us that we have

$$H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$$

and that the embeddings are compact for every $p > 1$. Since in dimension 2, any exponent $p > 1$ is subcritical (with respect to the Sobolev embedding) it is well known, by standard variational methods, that (1.1) has at least one positive solution.
Our main result provides a description of the asymptotic behavior, as $p \to +\infty$, of positive solutions of (1.1) under a uniform bound of their energy, namely we consider any family (u_p) of positive solutions to (1.1) satisfying the condition

$$p \int_\Omega |\nabla u_p|^2 + u_p^2 \, dx \to \beta \in \mathbb{R}, \quad \text{as } p \to +\infty. \tag{1.2}$$

Our strategy goes along the method developed by Davila, del Pino and Musso in [8] when they analyzed a nonlinear exponential Neumann boundary condition. Indeed, Davila et al. were interested to the following problem

$$\begin{cases} \Delta u = u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where ε is a small parameter. They proved that any family of solutions u_ε for which $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon}$ is bounded develops, up to subsequences, a finite number m of peaks $\xi_i \in \partial\Omega$, $\varepsilon \int_{\partial\Omega} e^{u_\varepsilon} \to 2m\pi$, and reciprocally, they established that at least two such families exist for any given $m \geq 1$.

There is another source of motivation for problem we are considering here. Its analogous usual elliptic equation is

$$\begin{cases} \Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

known as Lane-Emden equation. Such equation has been investigated widely in the last decades, see for example [1,10–13,17]. Concerning general positive solutions (i.e. not necessarily with least energy) of the Lane-Emden Dirichlet problem, a first asymptotic analysis was carried out in [11] showing that, under the corresponding energy bound condition, all solutions (u_p) concentrate at a finite number of points in $\overline{\Omega}$. Later the same authors gave in [12] a description of the asymptotic behavior of u_p as $p \to \infty$. They completed this study in a recent work with Grossi [10]. More precisely, they showed quantization of the energy to multiples of $8\pi e$ and proved convergence to $\sqrt{e}$ of the L^∞- norm, thus confirming the conjecture made in [12]. A proof of this quantization conjecture was also independently done by Thizy [25].

Going back to (1.1), the asymptotic behavior of general positive solutions has not been studied yet. Before stating our theorem let us review some known facts. In [24], Takahashi studied (1.1) by analyzing the asymptotic behavior of least energy solutions (hence positive), as $p \to \infty$. He proved that the least energy solutions remain bounded uniformly with respect to p and develop one peak on the boundary. The location of this blow-up point is associated with a critical point of the Robin function $H(x, x)$ on the boundary, where H is the regular part of the Green function of the corresponding linear Neumann problem. More precisely, the Green function $G(x, y)$ is the solution of the problem

$$\begin{cases} \Delta_x G(x,y) = G(x,y) & \text{in } \Omega, \\ \dfrac{\partial G}{\partial \nu_x}(x,y) = \delta_y(x) & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

for all $y \in \partial\Omega$ and its regular part

$$H(x,y) = G(x,y) - \frac{1}{\pi}\log\frac{1}{|x-y|}. \tag{1.6}$$

Note that least energy solutions (u_p) of this 2-dimensional semi-linear Neumann problem satisfy the condition

$$p\int\limits_{\Omega} |\nabla u_p|^2 + u_p^2\, dx \to 2\pi e, \quad \text{as } p \to +\infty,$$

which is a particular case of (1.2). Later, following a similar argument firstly introduced in [1], Castro [6] identified a limit problem by showing that suitable scaling of the least energy solutions (u_p) converges in $C^1_{loc}(\overline{\mathbb{R}^2_+})$ to a regular solution U of the Liouville problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial U}{\partial \nu} = e^U & \text{on } \partial\mathbb{R}^2_+ \\ \int_{\partial\mathbb{R}^2_+} e^U < \infty \ (= 2\pi) \text{ and } \sup_{\overline{\mathbb{R}^2_+}} U < \infty. \end{cases} \tag{1.7}$$

He also proved that $\|u_p\|_\infty$ converges to $\sqrt{e}$ as $p \to +\infty$, as it had been previously conjectured in [24]. All these results are respectively similar to those contained in [20,21] and [1] which focus on the least energy solution of the Lane-Emden problem in the plane.

However, problem (1.1) may have positive solutions with an arbitrarily large number of boundary peaks, as shown by Castro in [6]. Indeed, he proved that given any integer $m \geq 1$, problem (1.1) has at least two families of positive solutions u_p, each of them satisfying

$$pu_p(x)^{p+1} \rightharpoonup 2\pi e \sum_{i=1}^{m} \delta_{\xi_i} \quad \text{weakly in the sense of measure in } \partial\Omega,$$

as $p \to +\infty$, and the peaks of these two solutions are located near points $\xi = (\xi_i, \dots, \xi_m) \in (\partial\Omega)^m$ corresponding to two distinct critical points of the following functional defined on $(\partial\Omega)^m$

$$\varphi_m(x_1, \dots, x_m) := -\big[\sum_{i=1}^{m} H(x_i, x_i) + \sum_{j\neq i} G(x_i, x_j)\big].$$

It is natural to ask whether these properties hold for all families of positive solutions (u_p) satisfying (1.2), as $p \to \infty$. To our knowledge, a complete answer to this conjecture has not been given so far, while partial results are available as we describe below. In fact we extend the concentration result in [8], concerning elliptic problem with exponential Neumann data, to a large exponent one. More precisely, we prove that $u_p^p / \int_{\partial\Omega} u_p^p$ approaches the sum of m Dirac masses at the boundary. The location of these possible points of concentration may be further characterized as solutions of a system of equations defined explicitly in terms of the gradients of the above Green function and its regular part.

In order to state our main result we introduce some notations. Let

$$v_p = \frac{u_p}{\int_{\partial\Omega} u_p^p \, d\sigma(x)},$$

where u_p is a positive solution of (1.1) satisfying (1.2). We define the blow-up set S of v_{p_n} to be the subset of $\partial\Omega$ such that $x \in S$ if there exist a subsequence, still denoted by v_{p_n}, and a sequence x_n in $\overline{\Omega}$ with

$$v_{p_n}(x_n) \to +\infty \quad \text{and} \quad x_n \to x.$$

Now, we are able to state the following result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then for any sequence v_{p_n} of v_p with $p_n \to \infty$, there exists a subsequence (still denoted by v_{p_n}) such that the following statements hold true.*

(1) *There exists a finite collection of distinct points $x_i \in \partial\Omega$, $\quad i = 1, \ldots, m$ such that $S = \{x_i,\ 1 \le i \le m\}$.*
(2)

$$f_n := \frac{u_{p_n}^{p_n}}{\int_{\partial\Omega} u_{p_n}^{p_n}\, d\sigma(x)} \rightharpoonup^* \sum_{i=1}^{m} a_i \delta_{x_i}$$

in the sense of Radon measures on $\partial\Omega$ where

$$a_i = \lim_{r\to 0} \lim_{n\to+\infty} \left(\frac{2\pi}{(p_n+1)(\int_{\partial\Omega} u_{p_n}^{p_n} d\sigma(x))^2} \int\limits_{\partial\Omega\cap B_r(x_i)} u_{p_n}^{p_n+1} d\sigma(x) \right)^{\frac{1}{2}}, \quad \forall 1 \le i \le m. \tag{1.8}$$

(3) $v_{p_n} \to \sum_{i=1}^{m} a_i G(., x_i)$ *in $C^1_{loc}(\overline{\Omega} \setminus S)$, $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \le t < \infty$, where G is the Green's function for Neumann problem (1.5).*
(4) *The concentration points x_i, $i = 1, \ldots, m$ satisfy*

$$a_i \nabla_{\tau(x_i)} H(x_i, x_i) + \sum_{\ell\neq i} a_\ell \nabla_{\tau(x_i)} G(x_i, x_\ell) = 0, \tag{1.9}$$

where $\tau(x_i)$ is a tangent vector to $\partial\Omega$ at x_i.

As we mentioned before to prove this result we will proceed as in [8]. But, to adopt their argument, a blow up technique is needed to get some useful estimates. Indeed in both [6] and [24] the authors established the facts that

$$c_1 \le \|u_p\|_{L^\infty(\overline{\Omega})} \le c_2 \quad \text{and} \quad c_3 \le p \int\limits_{\partial\Omega} |u_p|^p d\sigma(x) \le c_4 \tag{1.10}$$

for some positive constants c_1, c_2, c_3 and c_4 independent of p for the case of least energy solutions to pursue the analysis. In our case we prove that these estimates hold true for general solutions (not necessarily positive) of (1.1) satisfying the bound energy condition (1.2). We point out that this last condition is very crucial in our framework to analyze the asymptotic behavior of the families (u_p). In fact, by using a suitable rescaling of the solution, we proved that the rescaled function about the maximum point of $|u_p|$, which is located on the boundary of Ω, converges to the bubble not only for least energy solution (shown in [6]) but also for finite energy ones. This information allowed us to obtain (1.10). This will be the subject of Proposition 2.1 which is the analogous result of [12, Proposition 2.2] and [13, Theorem 2.1] concerning Lane-Emden equation.
Let us point out that, as in [8] we have boundary concentration phenomena due to the nonlinear condition. But the exponent nonlinearity brings us some difficulties in our analysis.

Remark 1.2. In contrast with the exponential nonlinearity studied in [8], the argument of Davila et al. does not give the value of the coefficients or weights a_i's nor the quantization of the energy result.

To determine the data a_i's we think that new ideas are needed. May be a detailed local analysis is required to overcome this difficulty. However arguing as in Brezis and Merle [3], we get $a_i \geq \pi/L_0$ where L_0 will be defined in (3.2).
We conjecture that the a_i's are equal and more precisely we have

$$a_i = l^{-1} 2\pi\sqrt{e}, \quad \forall 1 \leq i \leq m$$

where $l = \lim_{p\to+\infty} p \int_{\partial\Omega} u_p(x)^p$.
If we combine this conjecture with results of Theorem 1.1 we get, for any family of positive solutions (u_p) of (1.1) satisfying (1.2), the following results

(i) up to subsequence

$$pu_p(x) \to 2\pi\sqrt{e}\sum_{i=1}^{m} G(x, x_i) \text{ as } p \to +\infty, \text{ in } C^1_{loc}(\bar{\Omega} \setminus \mathcal{S}),$$

where G is the Green's function for Neumann problem (1.5);
(ii) $(x_1, \ldots, x_m)$ is a critical point of φ_m, that is the concentration points x_i, $i = 1, \ldots, m$ satisfy

$$\nabla_{\tau(x_i)} H(x_i, x_i) + \sum_{\ell\neq i} \nabla_{\tau(x_i)} G(x_i, x_\ell) = 0. \tag{1.11}$$

We also conjecture that

$$\|u_p\|_{L^\infty(\overline{\Omega})} \to \sqrt{e} \text{ and } p\int_{\Omega} |\nabla u_p(x)|^2 + u_p^2(x)\, dx \to m.2\pi e, \quad \text{as } p \to +\infty.$$

This complete picture or behavior needs more accurate analysis. Verification of these conjectures remains as the future work [9].

The remainder of this paper is organized as follows: Section 2 is devoted to the asymptotic behavior of a general family (u_p) of nontrivial solutions of (1.1) satisfying (1.2). In Section 3 we give the proof of our theorem.

2. General asymptotic analysis

It was first proved in [18] for more general nonlinearities, that there exists at least one solution which changes sign. If the nonlinearity is odd in u, as in our case, it is mentioned in [18] that there exist infinitely many sign-changing solutions by a standard argument (see the reference therein), so it makes sense to study the properties of both positive and sign-changing solutions.

This section is mostly devoted to the study of the asymptotic behavior of a general family $(u_p)_{p>1}$ of nontrivial solutions of (1.1) satisfying the uniform upper bound

$$p\int_{\Omega} |\nabla u_p|^2 + u_p^2\, dx \leq C, \quad \text{for some } C > 0 \text{ independent of } p. \tag{2.1}$$

Recall that in [24] it has been proved that for any family $(u_p)_{p>1}$ of nontrivial solutions of (1.1) the following lower bound holds

$$\liminf_{p\to+\infty} p \int_\Omega |\nabla u_p|^2 + u_p^2 \, dx \geq 2\pi e, \tag{2.2}$$

so the constant C in (2.1) is intended to satisfy $C \geq 2\pi e$. Moreover if u_p is sign-changing then we also know that (see again [24])

$$\liminf_{p\to+\infty} p \int_\Omega |\nabla u_p^\pm|^2 + (u_p^\pm)^2 dx \geq 2\pi e. \tag{2.3}$$

We recall that the energy functional associated to (1.1) satisfies

$$E_p(u) = \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\partial\Omega)}, \quad u \in H^1(\Omega).$$

Since for a solution u of (1.1)

$$E_p(u) = (\frac{1}{2} - \frac{1}{p+1})\|u\|^2_{H^1(\Omega)} = (\frac{1}{2} - \frac{1}{p+1})\|u\|^{p+1}_{L^{p+1}(\partial\Omega)}, \tag{2.4}$$

then (2.1), (2.2) and (2.3) are equivalent to uniform upper and lower bounds for the energy E_p or for the $L^{p+1}(\partial\Omega)$-norm, indeed

$$\limsup_{p\to+\infty} \; 2pE_p(u_p) = \limsup_{p\to+\infty} \; p\int_{\partial\Omega} |u_p|^{p+1} \, d\sigma(x) = \limsup_{p\to+\infty} \; p\int_\Omega |\nabla u_p|^2 + u_p^2 \, dx \leq C$$
$$\liminf_{p\to+\infty} \; 2pE_p(u_p) = \liminf_{p\to+\infty} \; p\int_{\partial\Omega} |u_p|^{p+1} \, d\sigma(x) = \liminf_{p\to+\infty} \; p\int_\Omega |\nabla u_p|^2 + u_p^2 \, dx \geq 2\pi e$$

and if u_p is sign-changing, also

$$\liminf_{p\to+\infty} \; 2pE_p(u_p^\pm) = \liminf_{p\to+\infty} \; p\int_{\partial\Omega} |u_p^\pm|^{p+1} \, d\sigma(x) = \liminf_{p\to+\infty} \; p\int_\Omega |\nabla u_p^\pm|^2 + (u_p^\pm)^2 \, dx \geq 2\pi e,$$

we will use all these equivalent formulations throughout the paper.

Observe that by the assumption in (2.1) we have that

$$E_p(u_p) \to 0, \quad \|u_p\|_{H^1(\Omega)} \to 0, \quad \text{as } p \to +\infty$$
$$E_p(u_p^\pm) \to 0, \quad \|u_p^\pm\|_{H^1(\Omega)} \to 0, \quad \text{as } p \to +\infty \qquad (\text{if } u_p \text{ is sign-changing})$$

so in particular $u_p^\pm \to 0$ a.e. as $p \to +\infty$.
In this section, we will show that the solutions u_p do not vanish as $p \to +\infty$ (both $u_p^\pm$ do not vanish if u_p is sign-changing) and that moreover, differently with what happens in higher dimension, they do not blow-up. The last information is a consequence of the existence of the first bubble which is obtained by the rescaling respect to the maximum point. A uniform upper and lower bounds of the quantity $p\int_{\partial\Omega} u_p^p \, d\sigma(x)$ is also obtained (see Proposition 2.1 below). All these estimates are required to adopt the argument developed in [8]. Our key result is the following:

Proposition 2.1. *Let* (u_p) *be a family of solutions to* (1.1) *satisfying* (2.1). *Then* (*i*) *(No vanishing on the boundary).*

$$\|u_p\|^{p-1}_{L^\infty(\partial\Omega)} \geq \lambda_1,$$

where $\lambda_1 = \lambda_1(\Omega)(> 0)$ *is the first eigenvalue of the eigenvalue problem*

$$\begin{cases} \Delta u = u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

defined on $H^1(\Omega)$.
If u_p *is sign-changing then also* $\|u_p^\pm\|^{p-1}_{L^\infty(\partial\Omega)} \geq \lambda_1$.
Moreover

$$\liminf_{p\to+\infty} \|u_p\|_{L^\infty(\partial\Omega)} \geq 1 \text{ and } \liminf_{p\to+\infty} \|u_p\|_{L^\infty(\overline{\Omega})} \geq 1. \tag{2.6}$$

(*ii*) *(Existence of the first bubble). Let* $(x_p^+)_p \subset \overline{\Omega}$ *such that* $|u_p(x_p^+)| = \|u_p\|_{L^\infty(\overline{\Omega})}$. *Then*

$$x_p^+ \in \partial\Omega, \quad \text{for all } p > 1. \tag{2.7}$$

Let us set

$$\mu_p^+ := \left(p|u_p(x_p^+)|^{p-1}\right)^{-1} \tag{2.8}$$

and for $t \in \widetilde{\Omega}_p^+ := \{t \in \overline{\mathbb{R}^2_+} : y_p + \mu_p^+ t \in \Psi(\overline{\Omega} \cap B_R(x_p^+))\}$

$$z_p(t) := \frac{p}{u_p(x_p^+)}\left(u_p\left(\Psi^{-1}(y_p + \mu_p^+ t)\right) - u_p(x_p^+)\right), \tag{2.9}$$

where $y_p = \Psi(x_p^+)$ *and* Ψ *is the change of coordinates introduced in* (2.13).
Then $\mu_p^+ \to 0$ *as* $p \to +\infty$ *and*

$$z_p \longrightarrow U \text{ in } C^1_{loc}(\overline{\mathbb{R}^2_+}) \text{ as } p \to +\infty$$

where

$$U(t_1, t_2) = \log\left(\frac{4}{t_1^2 + (t_2 + 2)^2}\right) \tag{2.10}$$

is the solution of the Liouville problem (1.7) *satisfying* $U(0) = 0$.
(*iii*) *(No blow-up). There exists* $C > 0$ *such that*

$$\|u_p\|_{L^\infty(\overline{\Omega})} \leq C, \quad \text{for all } p > 1. \tag{2.11}$$

(*iv*) *There exist constants* $c, C > 0$, *such that for all* p *sufficiently large we have*

$$c \leq p \int_{\partial\Omega} |u_p|^p d\sigma(x) \leq C. \tag{2.12}$$

(*v*) $\sqrt{p}u_p \rightharpoonup 0$ *in* $H^1(\Omega)$ *as* $p \to +\infty$.

Proof. Point (i) has been first proved for positive solutions in [24], here we follow the proof in [17, Proposition 2.5]. If u_p is sign-changing, just observe that $u_p^\pm \in H^1(\Omega)$, where we know that

$$0 < 2\pi e - \varepsilon \overset{(2.2)/(2.3)}{\leq} p \int_\Omega |\nabla u_p^\pm|^2 + (u_p^\pm)^2 \, dx \overset{(2.1)}{\leq} C < +\infty.$$

Moreover, the trace inequality

$$\lambda_1(\Omega) \int_{\partial\Omega} u^2 \, d\sigma(x) \leq \int_\Omega |\nabla u|^2 + u^2 \, dx$$

holds for all $u \in H^1(\Omega)$, where $\lambda_1(\Omega) > 0$ is the least Steklov eigenvalue for (2.5). Thus we have

$$\begin{aligned}\int_\Omega |\nabla u_p^\pm|^2 + (u_p^\pm)^2 \, dx = \int_{\partial\Omega} |u_p^\pm|^{p+1} \, d\sigma(x) &\leq \|u_p^\pm\|^{p-1}_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u_p^\pm|^2 \, d\sigma(x) \\ &\leq \frac{\|u_p^\pm\|^{p-1}_{L^\infty(\partial\Omega)}}{\lambda_1(\Omega)} \int_\Omega |\nabla u_p^\pm|^2 + (u_p^\pm)^2 \, dx.\end{aligned}$$

Hence $\|u_p^\pm\|^{p-1}_{L^\infty(\partial\Omega)} \geq \lambda_1(\Omega)$ and $\|u_p^\pm\|^{p-1}_{L^\infty(\overline{\Omega})} \geq \|u_p^\pm\|^{p-1}_{L^\infty(\partial\Omega)} \geq \lambda_1(\Omega)$.
If u_p is not sign-changing just observe that either $u_p = u_p^+$ or $u_p = u_p^-$ and the same proof as before applies.

The proof of (ii) follows the same ideas in [6] where the same result has been proved for least energy (positive) solutions. In the sequel we will adopt the same method in [2] and [6] based on flattening the boundary near the maximum point then using a classical blow up argument introduced in [1].
Before doing so, we start by proving (2.7). We argue by contradiction. Suppose that there exists $p > 1$ such that $x_p^+ \in \Omega$. Recall that x_p^+ is a point where $|u_p|$ achieves its maximum. Without loss of generality, we can assume that:

$$u_p(x_p^+) = \max_{\overline{\Omega}} u_p > 0.$$

Hence x_p^+ is an interior local maximum and $u_p(x_p^+) > 0$. By continuity of u_p, there exists $r > 0$ such that $u_p(x) > 0$, for each $x \in B_r(x_p^+)$ and from (1.1) we get $\Delta u_p > 0$ in $B_r(x_p^+)$. Maximum principle implies that u_p is a constant function in $B_r(x_p^+)$. Therefore $u_p = \Delta u_p = 0$ in $B_r(x_p^+)$ which contradicts $u_p(x_p^+) > 0$.
If $u_p(x_p^+) < 0$ then x_p^+ is a minimum and a similar argument holds. Thus (2.7) is proved.
Next, we prove the remaining part of (ii). Recall that x_p^+ is a maximum point of $|u_p|$ in $\overline{\Omega}$. Without loss of generality, we may assume that

$$u_p(x_p^+) = \max_{\overline{\Omega}} u_p > 0.$$

By (i) we have that $pu_p(x_p^+)^{p-1} \to +\infty$ as $p \to +\infty$, so (2.6) holds and moreover $\mu_p^+ \to 0$, where μ_p^+ is defined in (2.8).
From (2.7), we have $x_p^+ \in \partial\Omega$ and $\|u_p\|_{L^\infty(\overline{\Omega})} = \|u_p\|_{L^\infty(\partial\Omega)}$. Up to a subsequence, x_p^+ converges to some $\overline{x} \in \partial\Omega$. Assume that $\overline{x}$ is located in the origin and the unit outward normal to $\partial\Omega$ at 0 is $(-e_2)$ where e_2 is the second element of a canonical basis in $\mathbb{R}^2$. It will be convenient to work in fixed half balls. For this reason, we need some change of coordinates. This program was done in many works (see for example [14] and [22]).
Since we will assume that $\partial\Omega$ is a C^2 surface, we know that there is an $R > 0$ and a $C^2(\mathbb{R})$ function ρ such that (after a possible renumbering and reorientation of coordinates)

$$\partial\Omega \cap B_R(0) = \{x \in B_R(0) : \quad x_2 = \rho(x_1)\}$$
$$\Omega' := \Omega \cap B_R(0) = \{x \in B_R(0) : \quad x_2 > \rho(x_1)\}$$

and moreover, the mapping

$$\Omega' \ni x \mapsto y = \Psi(x) \in \Omega'' \subseteq \mathbb{R}^2$$

defined by

$$\begin{cases} y_1 := x_1, \\ y_2 := x_2 - \rho(x_1), \end{cases} \tag{2.13}$$

is one-to-one. Define $\Phi := \Psi^{-1}$. Note that Ψ is a C^2 function that transforms the set Ω' (in what we refer to as x space) into a set Ω'' in the half-space $y_2 > 0$ (of y space). Note also that the point $\overline{x} = 0$ is mapped to the origin of y space.
Our task now is changing the partial differential equation (1.1) satisfied by u_p in Ω' into y coordinates. We define

$$\widetilde{u}_p(y) := u_p(\Phi(y)), \quad \text{for all } y \in \Omega''.$$

Let $\varphi \in \mathcal{D}(B_R(0) \cap \overline{\Omega})$. Multiplying (1.1) by φ, integrating by part over $B_R(0) \cap \Omega$ and using the change of variable $x = \Phi(y)$, we find

$$\begin{aligned} &\int_{\Omega''} \nabla u_p(\Phi(y)).\nabla\varphi(\Phi(y)) + u_p(\Phi(y))\varphi(\Phi(y))dy \\ &= \int_{\partial\Omega'' \cap \partial\mathbb{R}^2_+} |u_p(\Phi(y_1,0))|^{p-1} u_p(\Phi(y_1,0))\varphi(\Phi(y_1,0))dy_1. \end{aligned} \tag{2.14}$$

Let $\varphi_1(y) := \varphi(\Phi(y)),$ for each $y \in \Omega''$. A simple computation shows that

$$\nabla u_p(\Phi(y)) = \nabla\widetilde{u}_p(y) - \left(\rho'(y_1)\frac{\partial\widetilde{u}_p}{\partial y_2}(y), 0\right). \tag{2.15}$$

The above relation holds also for φ and φ_1.
Using (2.14), (2.15) and Green's formula, we can prove that the functions $\widetilde{u}_p$ satisfy the following problem

$$\begin{cases} \Delta\widetilde{u}_p - \widetilde{u}_p - 2\rho'(y_1)\dfrac{\partial^2\widetilde{u}_p}{\partial y_1 \partial y_2} - \rho''(y_1)\dfrac{\partial\widetilde{u}_p}{\partial y_2} + (\rho'(y_1))^2\dfrac{\partial^2\widetilde{u}_p}{\partial y_2^2} = 0 \quad \text{in } \Omega'', \\ \dfrac{\partial\widetilde{u}_p}{\partial\nu} + \rho'(y_1)\dfrac{\partial\widetilde{u}_p}{\partial y_2} - (\rho'(y_1))^2\dfrac{\partial\widetilde{u}_p}{\partial y_2} = |\widetilde{u}_p|^{p-1}\widetilde{u}_p \quad \text{on } \partial\Omega'' \cap \partial\mathbb{R}^2_+. \end{cases} \tag{2.16}$$

Let $\overline{R}$ be such that $B_{\overline{R}}(0) \cap \{y|y_2 > 0\} \subset \Omega''$ and define $B^+_{\overline{R}}(0) := B_{\overline{R}}(0) \cap \mathbb{R}^2_+$ and $D_{\overline{R}}(0) := B_{\overline{R}}(0) \cap \partial\mathbb{R}^2_+$ (the flat boundary of $B^+_{\overline{R}}(0)$). In particular we can look at problem (2.16) as being defined only in the half-ball $B^+_{\overline{R}}(0)$, that is

$$\begin{cases} \Delta\widetilde{u}_p - \widetilde{u}_p - 2\rho'(y_1)\dfrac{\partial^2\widetilde{u}_p}{\partial y_1 \partial y_2} - \rho''(y_1)\dfrac{\partial\widetilde{u}_p}{\partial y_2} + (\rho'(y_1))^2\dfrac{\partial^2\widetilde{u}_p}{\partial y_2^2} = 0 \quad \text{in } B^+_{\overline{R}}(0), \\ \dfrac{\partial\widetilde{u}_p}{\partial\nu} + \rho'(y_1)\dfrac{\partial\widetilde{u}_p}{\partial y_2} - (\rho'(y_1))^2\dfrac{\partial\widetilde{u}_p}{\partial y_2} = |\widetilde{u}_p|^{p-1}\widetilde{u}_p \quad \text{on } D_{\overline{R}}(0). \end{cases} \tag{2.17}$$

Now we perform a classical blow up argument. Let p_0 be a sufficiently large integer such that $y_p = \Psi(x_p^+) \in B_{\overline{R}/4}(0)$ for all $p > p_0$. Then we consider

$$z_p(t) := \frac{p}{\widetilde{u}_p(y_p)}\big(\widetilde{u}_p(y_p + \mu_p^+ t) - \widetilde{u}_p(y_p)\big), \quad \forall t \in \overline{B^+}_{\overline{R}/(2\mu_p^+)}, \ \forall p > p_0$$

where $\widetilde{u}_p(y_p) = u_p(x_p^+)$. For simplicity we shall write x_p for x_p^+, μ_p for μ_p^+, $B_{\overline{R}}^+$ for $B_{\overline{R}}^+(0)$ and $D_{\overline{R}}$ for $D_{\overline{R}}(0)$. The function z_p satisfies the following system

$$\begin{cases} \Delta z_p - \mu_p^2 z_p - \mu_p^2 p - 2\rho'(\mu_p t_1 + y_{p,1})\dfrac{\partial^2 z_p}{\partial t_1 \partial t_2} \\ \quad -\mu_p \rho''(\mu_p t_1 + y_{p,1})\dfrac{\partial z_p}{\partial t_2} - (\rho'(\mu_p t_1 + y_{p,1}))^2 \dfrac{\partial^2 z_p}{\partial t_2^2} = 0 \quad \text{in } B^+_{\mu_p^{-1}\overline{R}/2}, \\ \dfrac{\partial z_p}{\partial \nu} + \rho'(\mu_p t_1 + y_{p,1})\dfrac{\partial z_p}{\partial t_2} \\ \quad -(\rho'(\mu_p t_1 + y_{p,1}))^2 \dfrac{\partial z_p}{\partial t_2} = |1 + \dfrac{z_p}{p}|^{p-1}(1 + \dfrac{z_p}{p}) \quad \text{on } D_{\mu_p^{-1}\overline{R}/2}, \\ |1 + \frac{z_p}{p}| \le 1, \quad z_p(0) = 0 \quad \text{and} \quad z_p \le 0, \end{cases} \tag{2.18}$$

where (t_1, t_2) is the coordinates of t and $y_{p,1}$ is the first component of y_p. We rewrite (2.18) as follows:

$$\begin{cases} -L_p z_p + \mu_p^2 z_p = -\mu_p^2 p \quad \text{in } B^+_{\mu_p^{-1}\overline{R}/2}, \\ N_p z_p = |1 + \frac{z_p}{p}|^{p-1}(1 + \frac{z_p}{p}) \quad \text{on } D_{\mu_p^{-1}\overline{R}/2}, \\ |1 + \frac{z_p}{p}| \le 1, \quad z_p(0) = 0 \quad \text{and} \quad z_p \le 0, \end{cases} \tag{2.19}$$

where $L_p := \Delta - 2\rho'(\mu_p t_1 + y_{p,1})\dfrac{\partial^2 .}{\partial t_1 \partial t_2} - \mu_p \rho''(\mu_p t_1 + y_{p,1})\dfrac{\partial .}{\partial t_2} - (\rho'(\mu_p t_1 + y_{p,1}))^2 \dfrac{\partial^2 .}{\partial t_2^2}$ and $N_p := \dfrac{\partial\ .}{\partial \nu} + \rho'(\mu_p t_1 + y_{p,1})\dfrac{\partial\ .}{\partial t_2} - (\rho'(\mu_p t_1 + y_{p,1}))^2 \dfrac{\partial\ .}{\partial t_2}$.

Remark 2.2. Observe that $\rho'(0) = 0$ and the continuity of ρ'' imply that

- $L_p \to_{p\to\infty} \Delta$,
- $N_p \to_{p\to\infty} \dfrac{\partial\ .}{\partial \nu}$.

For fixed $r > 0$ we consider $p_1 > p_0$ large enough so that $8\mu_p r < \overline{R}$ for all $p > p_1$, and consider the problem of finding w_p solution of

$$\begin{cases} -L_p w + \mu_p^2 w = -p\mu_p^2 & \text{in } B^+_{4r}, \\ N_p w = |1 + \frac{z_p}{p}|^{p-1}(1 + \frac{z_p}{p}) & \text{on } D_{4r}, \\ w = 0 & \text{on } S_{4r}, \end{cases} \tag{2.20}$$

where $S_{4r} = \partial B_{4r} \cap \mathbb{R}^2_+$ (the curved boundary of B^+_{4r}). Firstly, the existence of such $w_p \in H^1(B^+_{4r})$ is guaranteed by Lax-Milgram theorem and it satisfies

$$\|w_p\|_{H^1(B^+_{4r})} \le C\bigg(\|\mu_p^2 p\|_{L^2(B^+_{4r})} + \bigg\||1 + \frac{z_p}{p}|^p\bigg\|_{L^2(D_{4r})}\bigg).$$

Moreover, observe that for each $q \ge 2$, and all $p > p_1$

$$\int_{B_{4r}^{+}} |\mu_p^2 p|^q dt \le C$$

since we have $r \le C\mu_p^{-1}$ and (i) holds. Also

$$\begin{aligned}\int_{D_{4r}} |1+\frac{z_p}{p}|^{pq} d\sigma(t) &\le \int_{D_{\mu_p^{-1}R/2}} |1+\frac{z_p}{p}|^{pq} d\sigma(t) \\ &= \mu_p^{-1} \int_{D_{R/2}(y_p)} \frac{|\widetilde{u}_p(y)|^{pq}}{\widetilde{u}_p(y_p)^{pq}} d\sigma(y) \\ &\le \mu_p^{-1} \frac{1}{u_p(x_p)^{pq}} \int_{\partial\Omega} |u_p(x)|^{pq} d\sigma(x) \\ &\le \frac{p}{u_p(x_p)^2} \int_{\partial\Omega} |u_p(x)|^{p+1} d\sigma(x) \\ &\le C,\end{aligned}$$

where the last inequality holds from Proposition 2.1 (i) and (2.1). Hence using a result from [23] (see also [6] page 270) we conclude that when $q > 4$, w_p must be in $W^{\frac{1}{2}+t,q}(B_{4r}^{+})$ for $0 < t < 2/q$ with

$$\|w_p\|_{W^{\frac{1}{2}+t,q}(B_{4r}^{+})} \le C\left(\|\mu_p^2 p\|_{L^q(B_{4r}^{+})} + \left\| |1+\frac{z_p}{p}|^p \right\|_{L^q(D_{4r})}\right) \le C, \tag{2.21}$$

where the constant C is independent of p since the coefficients of the operator (L_p, N_p) were uniformly bounded. Furthermore, (2.21) implies that w_p is L^∞ bounded.
Consider now the function $\varphi_p := w_p - z_p + \|w_p\|_{L^\infty(B_{4r}^{+})}$ which solves

$$\begin{cases} -L_p\varphi + \mu_p^2\varphi = \mu_p^2 \|w_p\|_{L^\infty(B_{4r}^{+})} & \text{in } B_{4r}^{+}, \\ N_p\varphi = 0 & \text{on } D_{4r}, \\ \varphi \ge 0 & \text{in } B_{4r}^{+}. \end{cases}$$

Note that, for p large, we have $N_p\varphi = 0$ is equivalent to $\frac{\partial\varphi}{\partial\nu} = 0$ since the function $(t_1, t_2) \mapsto \rho'(\mu_p t_1 + y_{p,1})$ converges uniformly to 0. Hence the function φ_p satisfies

$$\begin{cases} -L_p\varphi + \mu_p^2\varphi = \mu_p^2 \|w_p\|_{L^\infty(B_{4r}^{+})} & \text{in } B_{4r}^{+}, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } D_{4r}, \\ \varphi \ge 0 & \text{in } B_{4r}^{+}. \end{cases}$$

For $t = (t_1, t_2) \in B_{4r}$, we define the function

$$\hat{\varphi}_p = \begin{cases} \varphi_p(t) & \text{if } \ t_2 \ge 0, \\ \varphi_p(t_1, -t_2) & \text{if } \ t_2 < 0. \end{cases}$$

Clearly $\hat{\varphi}_p$ is a non-negative solution of $-L_p\varphi + \mu_p^2\varphi = \mu_p^2 \|w_p\|_{L^\infty(B_{4r}^{+})}$ in B_{4r}. Applying Harnack inequality ([15, Theorem 4.17]), we obtain for every $a \ge 1$

$$\left(\frac{1}{|B_{3r}|}\int\limits_{B_{3r}}\hat{\varphi}_p^a\right)^{\frac{1}{a}} \leq C\left\{\inf_{B_{3r}}\hat{\varphi}_p + \left\|\mu_p^2\|w_p\|_{L^\infty(B_{4r}^+)}\right\|_{L^2(B_{4r})}\right\}$$
$$\leq C\left\{\varphi_p(0) + \left\|\mu_p^2\|w_p\|_{L^\infty(B_{4r}^+)}\right\|_{L^2(B_{4r})}\right\}$$
$$\leq C$$

where we have used the facts that $z_p(0) = 0$ and w_p is uniformly bounded in B_{4r}^+. By interior elliptic regularity (see for instance [15, Theorem 9.13]) now we obtain that

$$\|\hat{\varphi}_p\|_{W^{2,q}(B_{2r})} \leq C\left(\left\|\mu_p^2\|w_p\|_{L^\infty(B_{4r}^+)}\right\|_{L^q(B_{3r})} + \|\hat{\varphi}_p\|_{L^q(B_{3r})}\right) \leq C.$$

Hence, we get that

$$\varphi_p \text{ is uniformly bounded in } W^{2,q}(B_{2r}^+) \text{ for } q > 1. \tag{2.22}$$

It follows using (2.21) and (2.22) that

$$\|z_p\|_{W^{\frac{1}{2}+t,q}(B_{2r}^+)} \leq C \tag{2.23}$$

for $q > 4$, $0 < t < 2/q$ and any $p > p_1$. Finally, Shauder regularity will tell us that z_p is bounded in $C^{1,\alpha}(B_r^+)$ for some $0 < \alpha < 1$, independently of $p > 1$ large. Thus by Arzela-Ascoli Theorem and a diagonal process on $r \to \infty$, after passing to a subsequence

$$z_p \to U \text{ in } C_{loc}^1(\overline{\mathbb{R}_+^2}) \text{ as } p \to +\infty. \tag{2.24}$$

Since $\rho'(0) = 0$ and $\mu_p \to 0$ and by using Remark 2.2, we conclude that U satisfies the following problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}_+^2 \\ \frac{\partial U}{\partial \nu} = e^U & \text{on } \partial\mathbb{R}_+^2. \end{cases} \tag{2.25}$$

Moreover, we have $U(0) = 0$ and $U \leq 0$.
Next we show that, if U satisfies (2.24) and (2.25), then we have

$$\int\limits_{\partial\mathbb{R}_+^2} e^U < \infty. \tag{2.26}$$

In order to prove (2.26), let us observe that for any $R > 0$ and each $|t_1| < R$, we have

$$(p+1)[\log|1 + \frac{z_p(t_1,0)}{p}| - \frac{z_p(t_1,0)}{p+1}] \longrightarrow_{p\to+\infty} 0.$$

So we can use Fatou's Lemma to write

$$\int\limits_{-R}^{R} e^{U(t_1,0)}dt_1 \overset{(2.24)+\text{Fatou}}{\leq} \int\limits_{-R}^{R} e^{z_p(t_1,0)-(p+1)[\log|1+\frac{z_p(t_1,0)}{p}|-\frac{z_p(t_1,0)}{p+1}]}dt_1 + o_p(1)$$
$$\leq \int\limits_{D_R(0)} |1 + \frac{z_p(t)}{p}|^{p+1}d\sigma(t) + o_p(1)$$

$$
\begin{aligned}
&\leq \int\limits_{D_R(0)} \frac{|\widetilde{u}_p(y_p+\mu_p t)|^{p+1}}{\widetilde{u}_p(y_p)^{p+1}} d\sigma(t) + o_p(1) \\
&\leq \mu_p^{-1} \int\limits_{D_{R\mu_p}(y_p)} \frac{|\widetilde{u}_p(y)|^{p+1}}{\widetilde{u}_p(y_p)^{p+1}} d\sigma(y) + o_p(1) \\
&\leq \frac{p}{\|u_p\|_\infty^2} \int\limits_{\partial\Omega} |u_p(x)|^{p+1} d\sigma(x) + o_p(1) \\
&\overset{(2.6)}{\leq} \frac{p}{(1-\varepsilon)^2} \int\limits_{\partial\Omega} |u_p(x)|^{p+1} d\sigma(x) + o_p(1) \overset{(2.1)}{\leq} C < +\infty,
\end{aligned}
$$

so that $e^U \in L^1(\partial\mathbb{R}^2_+)$ and (2.26) is proved.
Recall that U is a non positive solution of (2.25). Using (2.26), U satisfies the Liouville problem (1.7). By virtue of the classification due to P. Liu [19] (see also [16, Theorem 1.3]), the solution U must be of the form

$$U(t_1,t_2) = \log \frac{2\mu_2}{(t_1-\mu_1)^2 + (t_2+\mu_2)^2},$$

for some $\mu_2 > 0$ and $\mu_1 \in \mathbb{R}$. Since $U(0) = 0$ and $U \leq 0$, arguing as in [6] (see page 265) we obtain (2.10). Last an easy computation shows that $\int_{\partial\mathbb{R}^2_+} e^U = 2\pi$.

Point (iii) has been first proved in [24] in the case of least energy solutions using same ideas contained in [20], here we write a simpler proof which follows directly from (ii) by applying Fatou's lemma. An analogous argument can be found in [6] arguing as in [1, Lemma 3.1]. Indeed, for each $p > p_0$ we have

$$
\begin{aligned}
\|u_p\|_\infty^2 2\pi = \|u_p\|_\infty^2 \int\limits_{\partial\mathbb{R}^2_+} e^{U(t)} d\sigma(t) &\overset{(ii)\text{-Fatou}}{\leq} \|u_p\|_\infty^2 \int\limits_{D_{\mu_p^{-1}R}} \left|1 + \frac{z_p(t)}{p}\right|^{p+1} d\sigma(t) \\
&\leq \|u_p\|_\infty^2 \int\limits_{D_{\mu_p^{-1}R}} \frac{|\widetilde{u}_p(y_p+\mu_p t)|^{p+1}}{\widetilde{u}_p(y_p)^{p+1}} d\sigma(t) \\
&\leq \|u_p\|_\infty^2 \mu_p^{-1} \int\limits_{D_R(y_p)} \frac{|\widetilde{u}_p(y)|^{p+1}}{\widetilde{u}_p(y_p)^{p+1}} d\sigma(y) \\
&\overset{(2.8)}{\leq} p \int\limits_{\partial\Omega} |u_p(x)|^{p+1} d\sigma(x) \overset{(2.1)}{\leq} C < +\infty,
\end{aligned}
$$

where R is chosen such that $R \leq \frac{\overline{R}}{2}$.

(iv) follows directly from (iii). Indeed on the one hand

$$0 < C \overset{(2.2)-(2.4)}{\leq} p \int\limits_{\partial\Omega} |u_p|^{p+1}\, d\sigma(x) \leq \|u_p\|_\infty p \int\limits_{\partial\Omega} |u_p|^p\, d\sigma(x) \overset{(iii)}{\leq} Cp \int\limits_{\partial\Omega} |u_p|^p\, d\sigma(x)$$

On the other hand by Hölder inequality

$$p \int\limits_{\partial\Omega} |u_p|^p\, d\sigma(x) \leq |\partial\Omega|^{\frac{1}{p+1}} p \left(\int\limits_{\partial\Omega} |u_p|^{p+1}\, d\sigma(x) \right)^{\frac{p}{p+1}} \overset{(2.1)}{\leq} C.$$

To prove (v) we need (iv). Indeed let us note that, since (2.1) holds, there exists $w \in H^1(\Omega)$ such that, up to a subsequence, $\sqrt{p}u_p \rightharpoonup w$ in $H^1(\Omega)$. We want to show that $w = 0$ a.e. in Ω.

Using the equation (1.1), for any test function $\varphi \in C^\infty(\overline{\Omega})$, we have

$$\Big|\int_\Omega \nabla(\sqrt{p}u_p)\nabla\varphi + \sqrt{p}u_p\varphi\, dx\Big| = \sqrt{p}\Big|\int_{\partial\Omega} |u_p|^{p-1}u_p\varphi\, d\sigma(x)\Big| \leq \frac{\|\varphi\|_\infty}{\sqrt{p}} p\int_{\partial\Omega} |u_p|^p\, d\sigma(x) \overset{(iv)}{\leq} \frac{\|\varphi\|_\infty}{\sqrt{p}} C$$

for p large. Hence

$$\int_\Omega \nabla w\nabla\varphi + w\varphi\, dx = 0 \quad \forall\varphi \in C^\infty(\overline{\Omega}),$$

which implies that $w = 0$ a.e. in Ω. $\square$

3. Proof of Theorem 1.1

We start with the following interesting result contained in [8], which is a variant of an estimate of Brezis and Merle [3].

Lemma 3.1. *Consider the linear equation*

$$\begin{cases} \Delta u = u & \text{in } \Omega \\ \frac{\partial u}{\partial\nu} = h & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

with $h \in L^1(\partial\Omega)$.

For any $0 < k < \pi$ there exists a constant C depending on k and Ω such that for any $h \in L^1(\partial\Omega)$ and u the solution of (3.1) we have

$$\int_{\partial\Omega} \exp\left[\frac{k|u(x)|}{\|h\|_{L^1(\partial\Omega)}}\right] d\sigma(x) \leq C.$$

Let u_p be a family of positive solutions to (1.1) satisfying (1.2). We recall that $v_p = u_p/\int_{\partial\Omega} u_p^p\, d\sigma(x)$ and $f_p = u_p^p/\int_{\partial\Omega} u_p^p\, d\sigma(x)$. Hence v_p satisfies

$$\begin{cases} \Delta v_p = v_p & \text{in } \Omega \\ \frac{\partial v_p}{\partial\nu} = f_p & \text{on } \partial\Omega. \end{cases}$$

We now define the quantity:

$$L_0 = \limsup_{p\to+\infty} \frac{p\,\gamma_p}{e} \tag{3.2}$$

where

$$\gamma_p = \int_{\partial\Omega} u_p^p\, d\sigma(x).$$

Note that the quantity L_0 is a positive real number by (2.12).

In the sequel, we denote any sequence u_{p_n} of u_p by u_n and γ_{p_n} of γ_p by γ_n.

Since u_n has the property

$$\int_{\partial\Omega} f_n \, d\sigma(x) = \int_{\partial\Omega} \frac{u_n^{p_n}}{\int_{\partial\Omega} u_n^{p_n} \, d\sigma(x)} \, d\sigma(x) = 1$$

we can subtract a subsequence of u_n, still denoted by u_n, such that there exists a positive bounded measure μ in $M(\partial\Omega)$, the set of all real bounded Borel measures on $\partial\Omega$, such that $\mu(\partial\Omega) \leq 1$ and

$$\int_{\partial\Omega} f_n \varphi \longrightarrow \int_{\partial\Omega} \varphi d\mu$$

for all $\varphi \in C(\partial\Omega)$ where

$$v_n = u_n/\gamma_n \quad \text{and} \quad f_n = \gamma_n^{p_n-1} v_n^{p_n}.$$

To analyze the measure μ, we introduce some notations. For any $\delta > 0$, we call x_0 a δ-regular point if there exists a function φ in $C(\partial\Omega)$, $0 \leq \varphi \leq 1$, with $\varphi = 1$ in a neighborhood of x_0 such that

$$\int_{\partial\Omega} \varphi \, d\mu < \frac{\pi}{L_0 + 2\delta}. \tag{3.3}$$

We define

$$\Sigma(\delta) = \{x_o \in \partial\Omega : \quad x_o \text{ is not a } \delta\text{-regular point}\}.$$

Our next lemma plays a central role in the proof of Theorem 1.1. It says that smallness of μ at a point x_0 implies boundedness of v_n near x_0.

Lemma 3.2. *Let $x_0 \in \partial\Omega$ be a δ-regular point for some $\delta > 0$. Then v_n is bounded in $L^\infty(B_{R_0}(x_0) \cap \Omega)$ for some $R_0 > 0$.*

Proof. Let x_0 be a regular point. From the definition of regular points, there exists $R > 0$ such that

$$\int_{\partial\Omega \cap B_R(x_0)} f_n \, d\sigma(x) < \frac{\pi}{L_0 + \delta}$$

holds for all n large. Put $a_n = \chi_{B_R(x_0)} f_n$ and $b_n = (1 - \chi_{B_R(x_0)}) f_n$ where $\chi_{B_R(x_0)}$ denotes the characteristic function of $B_R(x_0)$. Split $v_n = v_{1n} + v_{2n}$, where v_{1n} and v_{2n} are solutions to

$$\begin{cases} \Delta v_{1n} = v_{1n} & \text{in } \Omega \\ \frac{\partial v_{1n}}{\partial \nu} = a_n \text{ on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta v_{2n} = v_{2n} & \text{in } \Omega \\ \frac{\partial v_{2n}}{\partial \nu} = b_n \text{ on } \partial\Omega \end{cases}$$

respectively.
By the maximum principle, we have $v_{1n}, \ v_{2n} > 0$. Since $b_n = 0$ on $B_R(x_0)$, elliptic estimates imply that

$$\|v_{2n}\|_{L^\infty(B_{R/2}(x_0) \cap \Omega)} \leq C \|v_{2n}\|_{L^1(B_R(x_0) \cap \Omega)} \leq C, \tag{3.4}$$

where we used the fact $\|v_{2n}\|_{L^1(\Omega)} = \|\Delta v_{2n}\|_{L^1(\Omega)} = \|b_n\|_{L^1(\partial\Omega)} \leq C$ for the last inequality. Thus we have to consider v_{1n} only.
Claim: There exists some $q > 1$ such that

$$\int_{B_{R/2}(x_0)\cap\partial\Omega} f_n^q \, d\sigma(x) \leq C.$$

Indeed, let t be such that $t' = L_0 + \delta/2$ where t' is the Hölder conjugate of t. From

$$\int_{\partial\Omega\cap B_R(x_0)} f_n \, d\sigma(x) < \frac{\pi}{L_0 + \delta}$$

using Lemma 3.1, we have

$$\int_{\partial\Omega\cap B_R(x_0)} \exp\Big[(L_0 + \delta/2)|v_{1n}|\Big] \, d\sigma(x) \leq C. \tag{3.5}$$

Now observe that $\log(x) \leq x/e$ for $x > 0$. As in [20] (see page 759), we have

$$p_n \log \frac{u_n}{\gamma_n^{1/p_n}} \leq t' \frac{u_n}{\gamma_n}$$

for n large enough because $\lim_{n\to\infty} \gamma_n^{1/p_n} = 1$ which follows from (2.12). Hence

$$f_n \leq e^{t' v_n}, \quad (f_n)^t e^{-tv_{1n}} \leq e^{(t'+t)v_{2n} + t' v_{1n}}.$$

Therefore since v_{2n} is uniformly bounded on $B_{R/2}(x_0) \cap \Omega$, we have

$$(f_n)^t e^{-tv_{1n}} \leq C e^{t' v_{1n}} \quad \text{on } B_{R/2}(x_0) \cap \partial\Omega. \tag{3.6}$$

Combining (3.5) and (3.6), we get that $f_n e^{-v_{1n}}$ is bounded in $L^t(B_{R/2}(x_0) \cap \partial\Omega)$.
Fix $\eta > 0$ small enough such that $\pi - \eta > \frac{\pi}{L_0+\delta}(t' + \eta)$. By Lemma 3.1 we have

$$\int_{\partial\Omega\cap B_R(x_0)} \exp[(t' + \eta)|v_{1n}|] \, d\sigma(x) \leq C.$$

Therefore $e^{v_{1n}}$ is bounded in $L^{t'+\eta}(\partial\Omega \cap B_R(x_0))$ and so $f_n = f_n e^{-v_{1n}}.e^{v_{1n}}$ is bounded in $L^q(\partial\Omega \cap B_{R/2}(x_0))$ for some $q > 1$. Hence we get the claim.
This fact and elliptic estimates imply that v_{1n} is uniformly bounded in $L^\infty(\Omega \cap B_{R/4}(x_0))$. Taking account of (3.4) and choosing $R_0 = R/4$ the desired result follows. □

Let's go back to the proof of Theorem 1.1. Taking account of (2.12) and Lemma 3.2, by the same argument of Ren and Wei (see [20] page 759), we have $S = \Sigma(\delta)$ for any $\delta > 0$. We get $S = \{x_o \in \partial\Omega : x_o$ is not a δ-regular point for any $\delta > 0\}$. Then

$$\mu(\{x_0\}) \geq \frac{\pi}{L_0 + 2\delta} \tag{3.7}$$

for all $x_0 \in S$ and for any $\delta > 0$.
Hence S is a finite nonempty set (since $\mu(\Omega) \leq 1$ and $\|v_n\|_{L^\infty(\partial\Omega)} \to +\infty$) and from Lemma 3.2 for every

$x \in \partial\Omega \setminus S$ we have that v_n is bounded in a neighborhood of x. Then v_n is bounded in compact subsets of $\partial\Omega \setminus S$ and so $f_n \to 0$ uniformly on compact subsets of $\partial\Omega \setminus S$ using (2.12). This shows that the support of μ is contained in S and therefore we can write

$$\mu = \sum_{i=1}^{m} a_i \delta_{x_i} \tag{3.8}$$

where $a_i > 0$ and $x_i \in \partial\Omega$. Hence we get parts (1) and (2) and we will come back later to the proof of (1.8). We point out that (3.7) and (3.8) imply that $a_i \geq \pi/L_0$.
Now, we need the following elliptic L^1 estimate by Brezis and Strauss [4] for weak solutions with the L^1 Neumann data.

Lemma 3.3. *Let u be a weak solution of*

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. Then we have $u \in W^{1,q}(\Omega)$ for all $1 \leq q < \frac{N}{N-1}$ and

$$\|u\|_{W^{1,q}(\Omega)} \leq C_q(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)})$$

holds.

Using Lemma 3.3, we have v_n is uniformly bounded in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$. Thus, by choosing a subsequence, we have a function v^* such that $v_n \rightharpoonup v^*$ weakly in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$, $v_n \to v^*$ strongly in $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \leq t < \infty$. The last convergence follows by the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^t(\Omega)$ for any $1 \leq t < q/(2-q)$. Thus by taking the limit in the equation

$$\int_{\Omega} (-\Delta\varphi + \varphi) v_n \, dx = \int_{\partial\Omega} f_n \varphi \, d\sigma(x) - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} v_n \, d\sigma(x)$$

for any $\varphi \in C^1(\overline{\Omega})$, we obtain

$$\int_{\Omega} (-\Delta\varphi + \varphi) v^* \, dx + \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} v^* \, d\sigma(x) = \sum_{i=1}^{m} a_i \varphi(x_i)$$

which implies v^* is the solution of the following problem

$$\begin{cases} \Delta v^* = v^* & \text{in } \Omega \\ \dfrac{\partial v^*}{\partial \nu} = \displaystyle\sum_{i=1}^{m} a_i \delta_{x_i} & \text{on } \partial\Omega \end{cases}$$

From this it follows that

$$v^*(x) = \sum_{i=1}^{m} a_i G(x, x_i).$$

In the sequel, we will prove that $v_n \to v^*$ in $C^1_{loc}(\overline{\Omega} \setminus S)$. We start by using Green representation for v_n:

$$v_n(x) = \int_{\partial\Omega} G(x,y) f_n(y)\, d\sigma(y), \tag{3.9}$$

where $G(x,y)$ is Green's function for Neumann problem (1.5). Suppose $x \in \Omega$ and $d = dist(x, \partial\Omega)$. Then, for $z \in B_{d/2}(x)$, we have $dist(z, \partial\Omega) \geq \frac{1}{2}d$, and

$$\begin{aligned} |u_n(z)| &\leq \int_{\partial\Omega} |\frac{1}{\pi} \log \frac{1}{|z-y|} + H(z,y)| f_n(y)\, d\sigma(y) \\ &\leq \int_{\partial\Omega} \left(\frac{1}{\pi} |\log \frac{2}{d}| + |H(z,y)| \right) f_n(y)\, d\sigma(y) \\ &\leq C(|\log d| + 1) \int_{\partial\Omega} f_n(y) d\sigma(y) \leq C, \quad \forall z \in B_{d/2}(x). \end{aligned} \tag{3.10}$$

Let K be a compact set in $\overline{\Omega} \setminus S$. From Lemma 3.2 and (3.10), we have that

$$v_n \leq C \text{ on } K.$$

Since v_n are bounded on K and satisfy $\Delta v_n - v_n = 0$ in $\mathring{K}$, we have by the elliptic regularity theory a subsequence of v_n, still denoted by v_n that approaches the same function v^* in $C^1(K)$.
We proved part (3).

Finally, we prove simultaneously (1.8) and Statement (4) of Theorem 1.1, that are the choice of the weights a_i's and the localization of the concentration points. We borrow the idea of [8] and derive Pohozaev-type identities in balls around the peak point. Let us concentrate on x_1. Without loss of generality, We may assume $x_1 = 0$. In the sequel, we use a particular straightening of the boundary introduced in [8]. That is a conformal diffeomorphism $\Phi_c : \Pi \cap B_{R_1} \longrightarrow \Omega \cap B_{r_1}$ which flattens the boundary $\partial\Omega$, where $\Pi = \{(y_1, y_2) \mid y_2 > 0\}$ denotes the upper half space and $R_1 > 0$ is a radius sufficiently small such that $(\partial\Omega \cap B_{r_1}) \cap S = \{0\}$. We may choose Φ_c is at least C^3, up to $\partial\Pi \cap B_{R_1}$, $\Phi_c(0) = 0$ and $D\Phi_c(0) = Id$. Set $\widetilde{u}_n(y) = u_n(\Phi_c(y))$ for $y = (y_1, y_2) \in \Pi \cap B_{R_1}$. Then by the conformality of Φ_c, $\widetilde{u}_n$ satisfies

$$\begin{cases} -\Delta \widetilde{u}_n + b(y)\widetilde{u}_n = 0 & \text{in} \quad \Pi \cap B_{R_1}, \\ \frac{\partial \widetilde{u}_n}{\partial \nu} = h(y)\widetilde{u}_n^{p_n} & \text{on} \quad \partial\Pi \cap B_{R_1}, \end{cases} \tag{3.11}$$

where $\widetilde{\nu}$ is the unit outer normal vector to $\partial(\Pi \cap B_{R_1})$, b and h are defined as

$$b(y) = |detD\Phi_c(y)|, \quad h(y) = |D\Phi_c(y)e|$$

with $e = (1,0)$. Note that $\widetilde{\nu}(y) = \nu(\Phi_c(y))$ for $y \in \partial\Pi \cap B_{R_1}$. Note also that, by using a clever idea of [8], we can modify Φ_c to prescribe the number

$$\alpha = \frac{\left(\frac{\partial h}{\partial y_1}\right)}{h(y)^2}\Bigg|_{y=0} = \left(\frac{\partial h}{\partial y_1}\right)(0).$$

Let R be such that $0 < R < R_1$. Applying now the Pohozaev identity to problem (3.11) we get

$$\int_{\Pi\cap B_R} b(y)\widetilde{u}_n^2(y)dy + \frac{1}{2} \int_{\Pi\cap B_R} (y - y_0, \nabla b(y))\widetilde{u}_n^2(y)dy$$

$$= \frac{1}{2} \int\limits_{\partial(\Pi \cap B_R)} (y - y_0, \widetilde{\nu}) b(y) \widetilde{u}_n^2(y) d\sigma(y) - \int\limits_{\partial(\Pi \cap B_R)} (y - y_0, \nabla \widetilde{u}_n(y)) \frac{\partial \widetilde{u}_n}{\partial \nu} d\sigma(y)$$
$$+ \frac{1}{2} \int\limits_{\partial(\Pi \cap B_R)} (y - y_0, \widetilde{\nu}) |\nabla \widetilde{u}_n|^2 d\sigma(y) \text{ for any } y_0 \in \mathbb{R}^2,$$

where and from now on, $\widetilde{\nu}$ will be used again to denote the unit normal to $\partial(H \cap B_R)$. The proof of the Pohozaev identity is standard and it is omitted here. Differentiating with respect to y_0, we have, in turn,

$$\int\limits_{\partial(\Pi \cap B_R)} \nabla \widetilde{u}_n(y) \frac{\partial \widetilde{u}_n}{\partial \widetilde{\nu}} \, d\sigma(y)$$
$$= \frac{1}{2} \int\limits_{\partial(\Pi \cap B_R)} \left(|\nabla \widetilde{u}_n|^2 + b(y) \widetilde{u}_n^2\right) \widetilde{\nu} d\sigma(y) - \frac{1}{2} \int\limits_{\Pi \cap B_R} \nabla b(y) \widetilde{u}_n^2 \, dy.$$

Since $\widetilde{\nu} = (\widetilde{\nu}_1, \widetilde{\nu}_2) = (0, -1)$ on $\partial\Pi \cap B_R$, the first component of the above vector equation reads

$$\int\limits_{\partial\Pi \cap B_R} (\widetilde{u}_n)_{y_1} h(y) \widetilde{u}_n^{p_n} d\sigma(y) + \int\limits_{\Pi \cap \partial B_R} (\widetilde{u}_n)_{y_1} \frac{\partial \widetilde{u}_n}{\partial \widetilde{\nu}} \, d\sigma(y)$$
$$= \frac{1}{2} \int\limits_{\Pi \cap \partial B_R} \left(|\nabla \widetilde{u}_n|^2 + b(y) \widetilde{u}_n^2\right) \widetilde{\nu}_1 d\sigma(y) - \frac{1}{2} \int\limits_{\Pi \cap B_R} b_{y_1}(y) \widetilde{u}_n^2 dy, \tag{3.12}$$

where $()_{y_1}$ denotes the derivative with respect to y_1.
Recall that $\gamma_n = \int_{\partial\Omega} u_n^{p_n} d\sigma(x)$. From the fact that $\widetilde{f}_n(y) = \frac{\widetilde{u}_n^{p_n}}{\gamma_n} \rightharpoonup a_1 \delta_0$ in the sense of Radon measures on $\partial\Pi \cap B_R$, (2.11) and (2.12), we see

$$\widetilde{g}_n(y) := \frac{1}{\gamma_n^2} \frac{\widetilde{u}_n^{p_n+1}(y)}{p_n + 1} = \frac{1}{(p_n + 1)\gamma_n} \widetilde{f}_n(y) \widetilde{u}_n(y)$$

satisfies that $supp(\widetilde{g}_n) \to \{0\}$ and $\int_{\partial\Pi \cap B_R} \widetilde{g}_n d\sigma(y) = O(1)$ as $n \to +\infty$. Thus, by choosing a subsequence, we have the convergence

$$\widetilde{g}_n(y) = \frac{1}{\gamma_n^2} \frac{\widetilde{u}_n^{p_n+1}(y)}{p_n + 1} \rightharpoonup C_1 \delta_0 \tag{3.13}$$

in the sense of Radon measures on $\partial\Pi \cap B_R$, where $C_1 = lim_{n \to +\infty} \int_{\partial\Pi \cap B_R} \widetilde{g}_n d\sigma(y)$ (up to a subsequence). By using this fact, we have

$$\frac{1}{\gamma_n^2} \int\limits_{\partial\Pi \cap B_R} (\widetilde{u}_n)_{y_1} h(y) \widetilde{u}_n^{p_n} d\sigma(y)$$
$$= \left[\frac{h(y)}{\gamma_n^2} \frac{\widetilde{u}_n^{p_n+1}(y)}{p_n + 1} \right]_{y_1 = -R}^{y_1 = R} - \int\limits_{\partial\Pi \cap B_R} h_{y_1}(y) \frac{\widetilde{u}_n^{p_n+1}(y)}{(p_n + 1)\gamma_n^2} d\sigma(y)$$
$$\to 0 - C_1 h_{y_1}(0) = -C_1 \alpha$$

as $n \to +\infty$.
Let $\widetilde{v}^*(y) = v^*(\Phi_c(y))$ denote the limit function in the y coordinates, and observe that

$$\widetilde{v}_{p_n} \to \widetilde{v}^* \text{ in } C^1_{loc}(\overline{\Pi} \cap B_R \backslash \{0\}).$$

Thus after dividing (3.12) by γ_n^2 and then letting $n \to \infty$, we obtain

$$\begin{aligned} -C_1\alpha + \int\limits_{\Pi\cap\partial B_R} \widetilde{v}^*_{y_1} \frac{\partial \widetilde{v}^*}{\partial \widetilde{\nu}} \, d\sigma(y) \\ = \frac{1}{2} \int\limits_{\Pi\cap\partial B_R} \big(|\nabla \widetilde{v}^*|^2 + b(y)(\widetilde{v}^*)^2\big)\widetilde{\nu}_1 \, d\sigma(y) - \frac{1}{2} \int\limits_{\Pi\cap B_R} b_{y_1}(y)(\widetilde{v}^*)^2 \, dy. \end{aligned} \tag{3.14}$$

At this point, we have the same formula as the equation (117) in [8], thus we obtain the result. Indeed decompose $v^*(x) = s(x) + w(x)$ where

$$s(x) = \frac{a_1}{\pi} \log \frac{1}{|x|}, \quad w(x) = a_1 H(x,0) + \sum_{j=2}^{m} a_j G(x, x_j).$$

We define then the corresponding functions in the new coordinates

$$\widetilde{s}(y) = s(\Phi_c(y)), \quad \widetilde{w}(y) = w(\Phi_c(y)), \quad y \in \Pi \cap B_{R_1}.$$

Using this decomposition and the fact that $\widetilde{w}$ satisfies

$$-\Delta \widetilde{w} + b(y)\widetilde{w} = -b(y)\widetilde{s}(y) \quad \text{in } \Pi \cap B_{R_1},$$

we derive from (3.14)

$$\begin{aligned} -C_1\alpha + \int\limits_{\Pi\cap\partial B_R} \widetilde{s}_{\widetilde{\nu}}\widetilde{s}_{y_1} + \widetilde{s}_{\widetilde{\nu}}\widetilde{w}_{y_1} + \widetilde{s}_{y_1}\widetilde{w}_{\widetilde{\nu}} \, d\sigma(y) \\ = \int\limits_{\Pi\cap\partial B_R} \big(\frac{1}{2}|\nabla \widetilde{s}|^2 + \nabla \widetilde{s}\nabla \widetilde{w} + \frac{1}{2} b(y)\widetilde{s}^2 + b(y)\widetilde{s}\widetilde{w}\big)\widetilde{\nu}_1 d\sigma(y) \\ - \int\limits_{\Pi\cap B_R} b_{y_1}(y)(\frac{1}{2}\widetilde{s}^2 + \widetilde{s}\widetilde{w}) \, dy + \int\limits_{\partial\Pi\cap B_R} \widetilde{w}_{\widetilde{\nu}}\widetilde{w}_{y_1} \, d\sigma(y) \\ - \int\limits_{\Pi\cap B_R} b(y)\widetilde{s}\widetilde{w}_{y_1} \, dy, \end{aligned} \tag{3.15}$$

where $\widetilde{s}_{\widetilde{\nu}}$ and $\widetilde{w}_{\widetilde{\nu}}$ are the partial derivatives with respect to $\widetilde{\nu}$ of the functions $\widetilde{s}$ and $\widetilde{w}$, respectively. Letting $R \to 0$ and using [8, Lemma 9.3] together with (3.15) we obtain

$$-\alpha C_1 + \frac{3\alpha}{4\pi} a_1^2 - a_1 \widetilde{w}_{y_1}(0) = \frac{\alpha}{4\pi} a_1^2 - \frac{a_1}{2}\widetilde{w}_{y_1}(0)$$

that is

$$\alpha\big(\frac{a_1^2}{2\pi} - C_1\big) = \frac{1}{2} a_1 \widetilde{w}_{y_1}(0).$$

Since $\alpha \in \mathbb{R}$ can be chosen arbitrarily, we conclude that

$$\widetilde{w}_{y_1}(0) = 0 \quad \text{and} \quad a_1^2 = 2\pi C_1 = 2\pi \lim_{R\to 0} \lim_{n\to+\infty} \int\limits_{\partial\Pi\cap B_R} \widetilde{g}_n d\sigma(y).$$

Consequently the desired conclusion of Theorem 1.1 (4) follows and by using a change of variables we get (1.8). □

Acknowledgment

The author wishes to express his sincere gratitude to Professor Filomena Pacella for her kind hospitality and support during his visits to the Department of Mathematics of the University of Roma “La Sapienza”. This work was partially supported by the “Istituto Nazionale di Alta Matematica”. He also thanks Professors Francesca De Marchis and Isabella Ianni for valuable discussions.

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