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B-Convexity, the Analytic Radon–Nikodym Property, and Individual Stability of C_0-Semigroups

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Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X, with generator A and growth bound ω. Assume that $x_0 \in X$ is such that the local resolvent $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension to the right half-plane $\{\operatorname{Re}\lambda > 0\}$. We prove the following results:

(i) If X has Fourier type $p \in (1, 2]$, then $\lim_{t\to\infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0$ for all $\beta > 1/p$ and $\lambda_0 > \omega$.

(ii) If X has the analytic RNP, then $\lim_{t\to\infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0$ for all $\beta > 1$ and $\lambda_0 > \omega$.

(iii) If X is arbitrary, then weak-$\lim_{t\to\infty} T(t)(\lambda_0 - A)^{-\beta}x_0 = 0$ for all $\beta > 1$ and $\lambda_0 > \omega$.

As an application we prove a Tauberian theorem for the Laplace transform of functions with values in a B-convex Banach space.

Key Words: C_0-semigroup; individual stability; resolvent estimates; B-convex; Fourier type; analytic Radon–Nikodym property; Tauberian theorems.

0. INTRODUCTION

In this paper we address the problem to find sufficient conditions on the local spectra of individual orbits of a C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ to ensure

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their strong convergence to zero. In certain work [1, 2, 9, 15] it has become increasingly clear that most of the "global" stability theory can be localized to individual orbits $T(\cdot)x$ by replacing the assumptions on the spectrum of the generator A by assumptions on the local spectrum of A at x.

For example, it is proven by Weis and Wrobel [22] that $\mathbf{T}$ is exponentially stable, i.e., there exist $M > 0$ and $\omega > 0$ such that $\|T(t)x\| \le Me^{-\omega t}\|x\|_{D(A)}$ for all $x \in D(A)$, if the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ exists and is uniformly bounded in the right half-plane $\{\operatorname{Re}\lambda > 0\}$. A little later and independently, in [15] the following local version of this result was proved: if $x_0 \in X$ is such that the map $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension to $\{\operatorname{Re}\lambda > 0\}$, then for each $\lambda_0 \in \varrho(A)$ there exists a constant $M > 0$ such that

$$\|T(t)R(\lambda_0, A)x_0\| \le M(1+t), \qquad t \ge 0.$$

By a standard resolvent expansion argument, the Weis–Wrobel result is an immediate consequence of this. In [9], for Hilbert spaces it was proved that actually,

$$\lim_{t\to\infty} \|T(t)R(\lambda_0, A)x_0\| = 0.$$

In this paper, we extend the result of [9] into various directions.

Let $p \in [1, 2]$. A Banach space X has *Fourier type* p if the Fourier transform extends to a bounded linear operator from $L^p(\mathbb{R}, X)$ into $L^q(\mathbb{R}, X)$, $1/p + 1/q = 1$. Trivially, every Banach space has Fourier type $p = 1$, but certain spaces have nontrivial Fourier type; see Section 1.

A Banach space X has the *analytic Radon–Nikodym property* if for every $f \in H^p(D, X)$, the Hardy space of all X-valued holomorphic functions on the unit disc D, the radial limits $\lim_{r\uparrow 1} f(re^{i\theta})$ exist for almost all $\theta \in [0, 2\pi]$. This property will be discussed in more detail in Section 2.

Our main results read as follows.

THEOREM 0.1. *Let X be a Banach space with Fourier type $p \in (1, 2]$ and let A be the generator of a C_0-semigroup $\mathbf{T}$ on X. If $x_0 \in X$ is such that the map $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension in the open right half-plane, then for all $\beta > 1/p$ and $\lambda_0 > \omega_0(\mathbf{T})$ we have*

$$\lim_{t\to\infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.$$

THEOREM 0.2. *Let X be a Banach space with the analytic Radon–Nikodym property and let A be the generator of a C_0-semigroup $\mathbf{T}$ on X. If $x_0 \in X$ is such that the map $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension in the open right half-plane, then for all $\beta > 1$ and $\lambda_0 > \omega_0(\mathbf{T})$ we have*

$$\lim_{t\to\infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.$$

THEOREM 0.3. *Let A be the generator of a C_0-semigroup $\mathbf{T}$ on an arbitrary Banach space X. If $x_0 \in X$ is such that the map $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension in the open right half-plane, then for all $\beta > 1$ and $\lambda_0 > \omega_0(\mathbf{T})$ we have*

$$\textit{weak-}\lim_{t\to\infty} T(t)(\lambda_0 - A)^{-\beta}x_0 = 0.$$

In these results, $\omega_0(\mathbf{T})$ denotes the growth bound of $\mathbf{T}$, i.e., the infimum of all $\omega \in \mathbb{R}$ such that $\|T(t)\| \le Me^{\omega t}$ for some $M > 0$ and all $t \ge 0$. The restriction to real λ_0 is not essential; by a standard rescaling argument the same results hold for $\lambda_0 \in \mathbb{C}$ with $\operatorname{Re}\lambda_0 > \omega_0(\mathbf{T})$.

We also present a simple example which shows that $\lim_{t\to\infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0$ may fail for all $\beta \ge 0$ if no restrictions on the Banach space X are imposed.

The paper is organized as follows. In Section 1 we prove Theorems 0.1 and 0.3 and we give a simple application to Tauberian theory for the Laplace transform of functions with values in a Banach space with nontrivial Fourier type. In Section 2 we present the proof of Theorem 0.2 and a second proof of Theorem 0.3.

1. STABILITY AND B-CONVEXITY

Let A be a closed, densely defined operator in a Banach space X such that $(0, \infty) \subset \varrho(A)$, the resolvent set of A, and assume that there is a constant $M > 0$ such that

$$\|R(\lambda, A)\| \le \frac{M}{1+\lambda}, \qquad \lambda > 0. \tag{1.1}$$

As is well known, fractional powers of $-A$ can be defined, and for $0 < \beta < 1$ we have the representation,

$$(-A)^{-\beta}x = \frac{\sin \pi\beta}{\pi}\int_0^\infty t^{-\beta}R(t, A)x\,dt, \qquad x \in X. \tag{1.2}$$

For the theory of fractional powers the reader is referred to [21].

If A is the generator of a C_0-semigroup $\mathbf{T}$, then for all $\lambda_0 > \omega_0(\mathbf{T})$ the operator $A - \lambda_0$ satisfies an estimate of the type (1.1), and the fractional powers of $\lambda_0 - A$ are well defined. We assume that the reader is familiar with the elementary theory of C_0-semigroups; we refer to [14, 17].

Let $p \in [1, 2]$. A Banach space Y has *Fourier type* p if the Y-valued Hausdorff–Young theorem holds; i.e., if the Fourier transform extends to a bounded linear operator from $L^p(\mathbb{R}, Y)$ into $L^q(\mathbb{R}, Y)$, $1/p + 1/q = 1$.

Here, as usual, for $f \in L^p(\mathbb{R}, Y) \cap L^1(\mathbb{R}, Y)$, the Fourier transform $\mathcal{F} f$ is defined by

$$\mathcal{F} f(s) := \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt, \qquad s \in \mathbb{R}.$$

Every Banach space has Fourier type 1 but only Banach spaces which are isomorphic to Hilbert spaces have Fourier type 2 [12]. The classical spaces $L^p(\mu)$ have Fourier type $\min\{p, q\}$, $1/p + 1/q = 1$ [18].

A Banach space Y is called *B-convex* if Y does not contain the spaces l_1^n uniformly, or equivalently, if it has nontrivial type, i.e., if it has type p for some $p \in (1, 2]$. The spaces $L^p(\mu)$ are B-convex and more generally, every Lebesgue–Bochner space $L^p(\mu, Y)$ with Y B-convex is B-convex (cf. [13, p. 247]) and every uniformly convex Banach space is B-convex. For more details the reader should consult [19]. Every B-convex Banach space has nontrivial Fourier type; i.e., Fourier type p for some $p \in (1, 2]$ [4], and conversely it is easy to show that a space with nontrivial Fourier type is B-convex (cf. [3, p. 354]).

In most of the results of this section, we investigate the behaviour of the map $t \mapsto PT(t)(\lambda_0 - A)^{-\beta} x_0$, assuming certain growth conditions on $\lambda \mapsto PR(\lambda, A)x_0$; here, P is an arbitrary bounded linear operator from X into some B-convex Banach space Y. Although we are primarily interested in the case $Y = X$ and $P = I$, this slightly more general setting allows the following applications:

- Taking $Y = \mathbb{C}$ and $P = x^* \in X^*$ we obtain weak analogues of our results;
- We may consider the translation semigroup on $X = BUC(\mathbb{R}_+, Y)$ and the map $P\colon X \to Y$, $Pf := f(0)$. In this way the asymptotic behaviour of Y-valued BUC-functions can be studied via semigroup techniques;
- It may be possible to apply our results to matrix semigroups, taking for P a coordinate projection. Matrix semigroups arise, e.g., in the study of delay equations and higher order abstract Cauchy problems.

The first lemma imposes no restrictions on the Fourier type of Y.

LEMMA 1.1. *Let X and Y be Banach spaces and let $P\colon X \to Y$ be a bounded linear operator. Let A be the generator of a C_0-semigroup $\mathbf{T}$ on X and let $x_0 \in X$ be such that the map $\lambda \mapsto PR(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ in the open right half-plane. Suppose there exist $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, $M > 0$, and $\alpha \in [-1, \infty)$ such that*

$$\|F(\lambda)\| \le M(1 + |\lambda|)^{\alpha}, \qquad 0 < \operatorname{Re}\lambda < \omega_0.$$

Fix $\lambda_0 > \max\{0, \omega_0(\mathbf{T})\}$. For all $\beta \ge 0$ with $\beta > \alpha$ the function $\lambda \mapsto PR(\lambda, A)(\lambda_0 - A)^{-\beta} x_0$ $(\operatorname{Re}\lambda > \omega_0(\mathbf{T}))$ admits a holomorphic extension

$g(\lambda)$ in the open right half-plane, and for all $\omega_1 \in (0, \min\{\omega_0, \lambda_0\})$ there exists a constant $C > 0$ such that

$$\|g(\lambda)\| \le C(1+|\lambda|)^{\max\{\alpha-\beta,-1\}}, \qquad 0 < \operatorname{Re}\lambda < \omega_1. \tag{1.3}$$

Proof. Fix $\lambda_0 > \max\{0, \omega_0(\mathbf{T})\}$ and $0 < \omega_1 < \min\{\omega_0, \lambda_0\}$. Upon replacing ω_0 by some smaller number and upon replacing ω_1 by a larger number, we may assume that $\max\{0, \omega_0(\mathbf{T})\} < \omega_1 < \omega_0 < \lambda_0$.

Let $\beta = n + \delta$ with $n \in \mathbb{N}$ and $0 \le \delta < 1$ and put $y_0 := R(\lambda_0, A)^n x_0$. In view of the identity,

$$R(\lambda, A)y_0 = \frac{R(\lambda, A)x_0}{(\lambda_0-\lambda)^n} - \sum_{k=0}^{n-1} \frac{R(\lambda_0, A)^{k+1}x_0}{(\lambda_0-\lambda)^{n-k}},$$

the map $\lambda \mapsto PR(\lambda, A)y_0$ admits a holomorphic extension $F_1(\lambda)$ to $\{\operatorname{Re}\lambda > 0\}$ which satisfies

$$\|F_1(\lambda)\| \le M'(1+|\lambda|)^{\max\{\alpha-n,-1\}}, \qquad 0 < \operatorname{Re}\lambda < \omega_0, \tag{1.4}$$

for some constant $M' > 0$.

If $\delta = 0$ (so $\beta = n$), then $g = F_1$ and the proof is complete. Therefore, in the rest of the proof we will assume that $\delta \in (0, 1)$.

We have

$$\begin{aligned} g(\lambda) &= PR(\lambda, A)(\lambda_0 - A)^{-\beta}x_0 \\ &= PR(\lambda, A)(\lambda_0 - A)^{-\delta}y_0, \qquad \operatorname{Re}\lambda > \omega_0(\mathbf{T}). \end{aligned}$$

Hence by (1.2) and the resolvent identity, for $\operatorname{Re}\lambda > \omega_0(\mathbf{T})$ we have

$$\begin{aligned} g(\lambda) &= \frac{\sin\pi\delta}{\pi}\int_0^\infty t^{-\delta} PR(\lambda, A)R(\lambda_0+t, A)y_0\,dt \\ &= \frac{\sin\pi\delta}{\pi}\int_0^\infty t^{-\delta}\frac{PR(\lambda, A)y_0 - PR(\lambda_0+t, A)y_0}{t+\lambda_0-\lambda}\,dt. \end{aligned}$$

Passing to the holomorphic extension, we see that

$$g(\lambda) = \frac{\sin\pi\delta}{\pi}\int_0^\infty t^{-\delta}\frac{F_1(\lambda) - F_1(\lambda_0+t)}{t+\lambda_0-\lambda}\,dt; \tag{1.5}$$

by (1.4) and the fact that $\alpha < \beta = n + \delta$ this integral converges absolutely and defines a holomorphic extension of g in the strip $\{0 < \operatorname{Re}\lambda < \lambda_0\}$.

For $\omega > 0$ consider the functions $g_\omega\colon \mathbb{R} \to Y$ defined by

$$g_\omega(s) := g(\omega - is), \qquad s \in \mathbb{R}.$$

Then $g_\omega(s) = PR(\omega - is, A)(\lambda_0 - A)^{-\beta}x_0$ for $\omega > \omega_0(\mathbf{T})$. Noting that $\|R(\lambda, A)\| \le \text{const}\cdot(\operatorname{Re}\lambda - \omega_0)^{-1}$ for all $\operatorname{Re}\lambda > \lambda_0$, we see that $c :=$

$\sup_{\tau\geq\lambda_0}\tau\|F_1(\tau)\| < \infty$. Hence by (1.4) and (1.5), for all $0 < \omega < \omega_1$ and $s \in \mathbb{R}$ we have

$$\begin{aligned}\|g_\omega(s)\| &\leq \frac{\sin\pi\delta}{\pi}\int_0^\infty t^{-\delta}\frac{M'(1+(\omega^2+s^2)^{1/2})^{\max\{\alpha-n,\,-1\}}+c(\lambda_0+t)^{-1}}{((t+\lambda_0-\omega)^2+s^2)^{1/2}}\,dt\\ &\leq \text{const}\cdot(1+s^2)^{1/2\cdot\max\{\alpha-n-\delta,\,-1\}},\end{aligned}$$

where the constant is independent of $s \in \mathbb{R}$ and $\omega \in (0, \omega_1)$. ∎

We can now state and prove the first main result.

THEOREM 1.2. *Let P be a bounded linear operator from a Banach space X into a Banach space Y with Fourier type $p \in (1, 2]$. Let A be the generator of a C_0-semigroup $\mathbf{T}$ on X and let $x_0 \in X$ be such that the map $\lambda \mapsto PR(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ in the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, $M > 0$, and $\alpha \in [-1, \infty)$ such that*

$$\|F(\lambda)\| \leq M(1+|\lambda|)^\alpha, \qquad 0 < \operatorname{Re}\lambda < \omega_0,$$

then for all $\beta \geq 0$ with $\beta > \alpha + 1/p$ and for all $\lambda_0 > \omega_0(\mathbf{T})$ we have

$$PT(\cdot)(\lambda_0 - A)^{-\beta}x_0 \in L^q(\mathbb{R}_+, Y), \qquad \frac{1}{p}+\frac{1}{q} = 1.$$

Proof. Without loss of generality we may assume that $\omega_0(\mathbf{T}) \geq 0$. Fix $\lambda_0 > \omega_0(\mathbf{T})$. By taking a smaller value of ω_0, we may furthermore assume that $\omega_0(\mathbf{T}) < \omega_0 < \lambda_0$. Fix $\omega_1 \in (\omega_0(\mathbf{T}), \omega_0)$.

Let the functions g_ω be defined as in the proof of Lemma 1.1. In view of $\beta - \alpha > 1/p$ and $p > 1$ the estimate obtained there shows that $g_\omega \in L^p(\mathbb{R}, Y)$, uniformly for $\omega \in (0, \omega_1)$. Let $C := \sup_{0<\omega<\omega_1}\|g_\omega\|_p$.

Because Y has Fourier type p, the Fourier transform $G_\omega := (1/2\pi)\mathcal{F}g_\omega$ of g_ω defines an element of $L^q(\mathbb{R}, Y)$.

Let $\omega \in (0, \omega_1)$ be fixed. We claim that

$$G_\omega(t) = e^{-\omega t}PT(t)(\lambda_0 - A)^{-\beta}x_0, \quad \text{for a.a. } t > 0.$$

To see this we define, for each $r > 0$, $g_{\omega,r} := g_\omega \cdot \chi_{[-r,r]}$. Then $\lim_{r\to\infty} g_{\omega,r} = g_\omega$ in the norm of $L^p(\mathbb{R}, Y)$, so for the Fourier transforms $G_{\omega,r} = (1/2\pi)\mathcal{F}g_{\omega,r}$ we have $\lim_{r\to\infty} G_{\omega,r} = G_\omega$ in $L^q(\mathbb{R}, Y)$. Let Γ be the rectangle spanned by the points $\omega - ir$, $\omega + ir$, $\omega_0 + ir$, and $\omega_0 - ir$. By Cauchy's theorem, for all $t > 0$ we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\omega-ir}^{\omega+ir} e^{zt}g(z)\,dz\\ &\quad= \frac{1}{2\pi i}\int_{\omega_0-ir}^{\omega_0+ir} e^{zt}g(z)\,dz + R_r(t)\\ &\quad= \frac{1}{2\pi i}\int_{\omega_0-ir}^{\omega_0+ir} e^{zt}PR(z, A)(\lambda_0 - A)^{-\beta}x_0\,dz + R_r(t), \qquad (1.6)\end{aligned}$$

where $R_r(t)$ represents the integrals over the two horizontal parts of Γ. From (1.3) we see that $\lim_{r\to\infty}\|R_r(t)\| = 0$ for all $t > 0$. Also, by the complex inversion theorem for the Laplace transform, the Cesàro means of the integral on the right-hand side in (1.6) converge to $PT(t)(\lambda_0 - A)^{-\beta}x_0$ as $r \to \infty$; here we use that $\omega_0 > \omega_0(\mathbf{T})$. It follows that for all $t > 0$,

$$\lim_{m\to\infty}\frac{1}{m}\int_0^m \frac{1}{2\pi i}\int_{\omega-ir}^{\omega+ir} e^{zt}g(z)\,dz\,dr = PT(t)(\lambda_0 - A)^{-\beta}x_0. \tag{1.7}$$

On the other hand, for $t > 0$ we have

$$G_{\omega,r}(t) = \frac{1}{2\pi}\int_{-r}^{r} e^{-ist}g(\omega - is)\,ds = \frac{1}{2\pi i}e^{-\omega t}\int_{\omega-ir}^{\omega+ir} e^{zt}g(z)\,dz. \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$\begin{aligned}\lim_{m\to\infty}\left(\left(\frac{1}{m}\int_0^m G_{\omega,r}\,dr\right)(t)\right) &= \lim_{m\to\infty}\frac{1}{m}\int_0^m G_{\omega,r}(t)\,dr\\ &= e^{-\omega t}PT(t)(\lambda_0 - A)^{-\beta}x_0,\end{aligned}$$

for all $t > 0$. In the first identity we used the fact that the map $r \mapsto G_{\omega,r}$ is continuous as a map into $C_0(\mathbb{R}, Y)$ by the Riemann–Lebesgue lemma. Therefore the integrals with respect to r can be regarded as Bochner integrals in $C_0(\mathbb{R}, Y)$ and we may use the continuity of point evaluations.

We also have

$$\lim_{m\to\infty}\left(\frac{1}{m}\int_0^m G_{\omega,r}\,dr\right) = \lim_{r\to\infty} G_{\omega,r} = G_\omega,$$

in the norm of $L^q(\mathbb{R}, Y)$. Because norm convergent sequences have pointwise a.e. convergent subsequences, we see that $G_\omega(t) = e^{-\omega t}PT(t)(\lambda_0 - A)^{-\beta}x_0$ for almost all $t > 0$ and the claim is proved.

It follows that $t \mapsto e^{-\omega t}PT(t)(\lambda_0 - A)^{-\beta}x_0$ defines an element of $L^q(\mathbb{R}_+, Y)$ and

$$\|e^{-\omega(\cdot)}PT(\cdot)(\lambda_0 - A)^{-\beta}x_0\|_q \le \|G_\omega\|_q \le \frac{c_p}{2\pi}\|g_\omega\|_p \le \frac{c_pC}{2\pi}.$$

By the monotone convergence theorem, upon letting $\omega \downarrow 0$ we obtain

$$\|PT(\cdot)(\lambda_0 - A)^{-\beta}x_0\|_q \le \frac{c_pC}{2\pi}.$$

∎

Let $x_0 \in X$ and $x_0^* \in X^*$ be such that the map $\lambda \mapsto \langle x_0^*, R(\lambda, A)x_0\rangle$ admits a bounded holomorphic extension to $\{\operatorname{Re}\lambda > 0\}$. Taking $Y = \mathbb{C}$ and $P = x_0^*$, Theorem 1.2 shows that

$$\int_0^\infty |\langle x_0^*, T(t)(\lambda_0 - A)^{-\beta}x_0\rangle|^q < \infty,$$

for all $p \in (1, 2]$, $\beta > 1/p$; $1/p + 1/q = 1$. This is an individual version of [16, Theorem 5.1], and this observation can be used to show that for $\alpha = 0$ and $p = 2$, the bound $\beta > \alpha + 1/p$ $(= \frac{1}{2})$ in Theorem 1.2 is optimal in the sense that a counterexample exists for all $\beta \in [0, \frac{1}{2})$. Indeed, assume that the theorem holds for $\alpha = 0$, $p = 2$, and some $\beta \geq 0$. Suppose that $\mathbf{T}$ is a C_0-semigroup on a Banach space X whose resolvent $R(\lambda, A)$ is uniformly bounded in $\{\operatorname{Re}\lambda > 0\}$. Let $\lambda_0 > \max\{0, \omega_0(\mathbf{T})\}$. Then by the observation just made,

$$\int_0^\infty |\langle x^*, T(t)(\lambda_0 - A)^{-\beta}x\rangle|^2 < \infty, \qquad \forall x \in X,\ x^* \in X^*.$$

For each $x \in X$ and $x^* \in X^*$ put

$$f_{x,x^*}(t) := \langle x^*, T(t)(\lambda_0 - A)^{-\beta}x\rangle, \qquad t \geq 0.$$

Then $f_{x,x^*} \in L^2(\mathbb{R}_+)$ and by general considerations involving the closed graph theorem there exists a constant $C > 0$ such that $\|f_{x,x^*}\|_2 \leq C\|x\| \cdot \|x^*\|$ for all $x \in X$ and $x^* \in X^*$. By the Plancherel theorem, $s \mapsto \langle x^*, R(is, A)(\lambda_0 - A)^{-\beta}x\rangle \in L^2(\mathbb{R})$. Hence for all $\gamma > \frac{1}{2}$ and $\omega > 0$, by Hölder's inequality the function,

$$g_{\omega,x,x^*}(s) := (\omega + is)^{-\gamma}\langle x^*, R(-is, A)(\lambda_0 - A)^{-\beta}x\rangle$$

belongs to $L^1(\mathbb{R})$. In particular, the Fourier transforms $\mathcal{F}g_{\omega,x,x^*}$ are bounded.

Claim. $(1/2\pi)\mathcal{F}g_{\omega,x,x^*}(t) = \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta}x\rangle$ for all $t > 0$.

Indeed, for $t > 0$ we have, with $A_\omega := A - \omega$,

$$\begin{aligned}\frac{1}{2\pi}&\mathcal{F}g_{\omega,x,x^*}(t)\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ist}(\omega + is)^{-\gamma}\langle x^*, R(-is, A)(\lambda_0 - A)^{-\beta}x\rangle\, ds\\ &= \frac{1}{2\pi i}e^{\omega t}\int_{\operatorname{Re}\lambda = -\omega} e^{\lambda t}(-\lambda)^{-\gamma}\langle x^*, R(\lambda, A_\omega)(\lambda_0 - A)^{-\beta}x\rangle\, d\lambda.\end{aligned}$$

If $x \in D(A) = D(A_\omega)$, then by [16, Lemma 3.3] the right most hand equals

$$e^{\omega t}\langle x^*, T_\omega(t)(-A_\omega)^{-\gamma}(\lambda_0 - A)^{-\beta}x\rangle = \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta}x\rangle,$$

where $T_\omega(t) := e^{-\omega t}T(t)$. For general $x \in X$, we choose a sequence $x_n \to x$ with $x_n \in D(A)$ for all n. Then $f_{x_n,x^*} \to f_{x,x^*}$ in $L^2(\mathbb{R}_+)$ for all $x^* \in X^*$, hence $g_{\omega, x_n, x^*} \to g_{\omega, x, x^*}$ in $L^1(\mathbb{R})$, and so $\mathcal{F} g_{\omega, x_n, x^*} \to \mathcal{F} g_{\omega, x, x^*}$ in $C_0(\mathbb{R})$. Therefore, for all $t > 0$,

$$\begin{aligned}\frac{1}{2\pi}\mathcal{F} g_{\omega, x, x^*}(t) &= \lim_{n\to\infty} \frac{1}{2\pi}\mathcal{F} g_{\omega, x_n, x^*}(t) \\ &= \lim_{n\to\infty} \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta} x_n\rangle \\ &= \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta} x\rangle.\end{aligned}$$

This proves the claim.

It follows that $t \mapsto \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta} x\rangle$ is bounded, and because this is true for all $x \in X$, $x^* \in X^*$, and $\gamma > \frac{1}{2}$, the uniform boundedness theorem and standard arguments involving fractional powers show that

$$\sup_{t\geq 0} \|T(t)(\lambda_0 - A)^{-\beta-\gamma}\| < \infty,$$

for all $\gamma > \frac{1}{2}$. On the other hand, in [22] for each $\delta \in [0, 1)$ an example of a C_0-semigroup $\mathbf{T}$ is given which has uniformly bounded resolvent in the right half-plane and satisfies

$$\limsup_{t\to\infty} \|T(t)(\lambda_0 - A)^{-\delta}\| = \infty.$$

Thus, if Theorem 1.2 holds for $\alpha = 0$, $p = 2$, and some $\beta \geq 0$, we must have $\beta \geq \frac{1}{2}$.

For $Y = X$, $P = I$, and $p \in (1, 2]$, Theorem 1.2 has the following consequence:

COROLLARY 1.3. *Let X be a Banach space with Fourier type $p \in (1, 2]$, let A be the generator of a C_0-semigroup $\mathbf{T}$ on X. Let $x_0 \in X$ be such that the local resolvent $\lambda \mapsto R(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ in the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, $M > 0$ and $\alpha \in [-1, \infty)$ such that*

$$\|F(\lambda)\| \leq M(1 + |\lambda|)^{\alpha}, \qquad 0 < \operatorname{Re}\lambda < \omega_0,$$

then for all $\beta \geq 0$ with $\beta > \alpha + 1/p$ and all $\lambda_0 > \omega_0(\mathbf{T})$ we have

$$\lim_{t\to\infty} \|T(t)(\lambda_0 - A)^{-\beta} x_0\| = 0.$$

Proof. By Theorem 1.2 applied to the case $Y = X$ and $P = I$ we find that the function $f(t) := T(t)(\lambda_0 - A)^{-\beta} x_0$ defines an element of $L^q(\mathbb{R}_+, X)$, $1/p + 1/q = 1$. Hence a standard argument (cf. the proof of [17, Theorem 4.4.1]) shows that $\lim_{t\to\infty} \|f(t)\| = 0$. ∎

For $\alpha = 0$, this gives Theorem 0.1.

Recalling that a B-convex Banach space X has nontrivial Fourier type, we see from Corollary 1.3 that

$$\lim_{t\to\infty} \|T(t)R(\lambda, A)x_0\| = 0,$$

whenever $\mathbf{T}$ is a C_0-semigroup on a B-convex space X and $x_0 \in X$ is such that the local resolvent $R(\lambda, A)x_0$ admits a bounded holomorphic extension to the open right half-plane. This improves the result of [9] mentioned in the Introduction.

We next discuss the analogue of Corollary 1.3 for general operators P. Although the proof of Corollary 1.3 breaks down, for slightly larger values of β we can prove:

THEOREM 1.4. *Let P be a bounded operator from a Banach space X into a B-convex Banach space Y. Let A be the generator of a C_0-semigroup* $\mathbf{T}$ *on X and let $x_0 \in X$ be such that the map $\lambda \mapsto PR(\lambda, A)x_0$ extends to a holomorphic function $F(\lambda)$ in the open right half-plane. If there exist $\omega_0 >$* $\max\{0, \omega_0(\mathbf{T})\}$, $M > 0$, *and* $\alpha \in [-1, \infty)$ *such that*

$$\|F(\lambda)\| \le M(1 + |\lambda|)^\alpha, \qquad 0 < \operatorname{Re}\lambda < \omega_0,$$

then for all $\beta > \alpha + 1$ and $\lambda_0 >$ $\max\{0, \omega_0(\mathbf{T})\}$ *we have*

$$\lim_{t\to\infty} \|PT(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.$$

Proof. Without loss of generality we may assume that $\omega_0(\mathbf{T}) \ge 0$. Fix $\lambda_0 > \omega_0(\mathbf{T})$. Let $p \in (1, 2]$ be the Fourier type of Y. Then Y also has Fourier type p' for all $p' \in (1, p]$. Hence, because $\beta > 0$ by assumption, upon replacing p by a smaller value we may assume that $\beta > 1/q$, $1/p + 1/q = 1$. This enables us to choose $\delta \ge 0$ such that $\delta > \alpha + 1/p$ in such a way that $1/q < \gamma := \beta - \delta < 1$. Consider the functions,

$$f(t) := PT(t)(\lambda_0 - A)^{-\delta}x_0, \qquad g(t) := PT(t)(\lambda_0 - A)^{-\beta}x_0; \qquad t \ge 0.$$

By Theorem 1.2, $f \in L^q(\mathbb{R}_+, Y)$. For $t \ge 0$ we have

$$\begin{aligned} g(t) &= PT(t)(\lambda_0 - A)^{-\delta-\gamma}x_0 \\ &= PT(t)(\lambda_0 - A)^{-\delta}\left(\frac{\sin \pi\gamma}{\pi}\int_0^\infty s^{-\gamma}R(\lambda_0 + s, A)x_0\, ds\right) \\ &= \frac{\sin \pi\gamma}{\pi}P(\lambda_0 - A)^{-\delta}\int_0^\infty s^{-\gamma}\int_0^\infty e^{-(\lambda_0+s)r}T(t + r)x_0\, dr\, ds \\ &= \frac{\sin \pi\gamma}{\pi}\int_0^\infty s^{-\gamma}\int_0^\infty e^{-(\lambda_0+s)r}f(t + r)x_0\, dr\, ds. \end{aligned} \tag{1.9}$$

Now,

$$\begin{aligned}
&\left\| \int_0^\infty e^{-(\lambda_0+s)r} f(t+r)x_0 \, dr \right\| \\
&\qquad \le \left(\int_0^\infty e^{-(\lambda_0+s)rp} \, dr \right)^{1/p} \cdot \left(\int_0^\infty \|f(t+r)\|^q \, dr \right)^{1/q} \\
&\qquad = \frac{1}{(p(\lambda_0+s))^{1/p}} \left(\int_t^\infty \|f(r)\|^q \, dr \right)^{1/q}.
\end{aligned}$$

Combining this estimate with (1.9) yields

$$\|g(t)\| \le \frac{\sin \pi\gamma}{\pi p^{1/p}} \int_0^\infty s^{-\gamma}(\lambda_0+s)^{-(1/p)} \, ds \cdot \left(\int_t^\infty \|f(r)\|^q \, dr \right)^{1/q}.$$

Because $1/q < \gamma < 1$, the first integral in the previous expression is absolutely convergent, and the second integral tends to 0 as $t \to \infty$. This proves that $\lim_{t\to\infty} \|g(t)\| = 0$. ∎

Theorem 0.3 is a special case of Theorem 1.4 by taking $\alpha = 0$, $Y = \mathbb{C}$, and $P = x^*$. Of course, Theorem 0.3 can be proved without reference to B-convexity: Take $Y = X$ and $P = x^*$ in the proofs of Theorems 1.2 and 1.5 and use the Hausdorff–Young theorem instead of the Fourier type. A similar remark applies to Corollary 2.3.

For $\alpha = 0$, Theorem 1.4 fails for every $0 \le \beta < 1$ (the case $\beta = 1$ remains open). Indeed, consider the case that the resolvent $R(\lambda, A)$ itself is uniformly bounded in $\{\operatorname{Re} \lambda > 0\}$. Then the assumptions of Theorem 1.4 are satisfied for $\alpha = 0$, all $x_0 \in X$, and all functionals $P = x^* \in X^*$. Hence if the theorem holds for some $\beta \ge 0$, then from the uniform boundedness principle we conclude

$$\sup_{t\ge 0} \|T(t)(\lambda_0 - A)^{-\beta}\| < \infty.$$

For $0 \le \beta < 1$, this contradicts the example in [22] cited in the discussion after Theorem 1.2.

We next turn to a version of Theorem 1.4 which holds for $\beta > \alpha + 1/p$ rather than $\beta > \alpha + 1$. The price for this is the a priori assumption that $PT(\cdot)x_0$ is *bounded*.

THEOREM 1.5. *Let P be a bounded linear operator from a Banach space X into a Banach space Y with Fourier type $p \in (1, 2]$. Let A be the generator of a C_0-semigroup $\mathbf{T}$ on X and let $x_0 \in X$ be such that the orbit $t \mapsto PT(t)x_0$ is bounded and $\lambda \mapsto PR(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ to the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, $M > 0$ and $\alpha \in [-1, \infty)$ such that*

$$\|F(\lambda)\| \le M(1 + |\lambda|)^\alpha, \qquad 0 < \operatorname{Re} \lambda < \omega_0,$$

then for all $\lambda_0 > \omega_0(\mathbf{T})$ *and* $\beta \geq 1$ *with* $\beta > \alpha + 1/p$ *we have*

$$\lim_{t\to\infty} \|PT(t)(\lambda_0 - A)^{-\beta} x_0\| = 0.$$

Proof. Without loss of generality we may assume that $\omega_0(\mathbf{T}) \geq 0$. Fix $\lambda_0 > \omega_0(\mathbf{T})$ and $\beta \geq 1$ with $\beta > \alpha + 1/p$. For each $\delta \geq 0$ consider the function,

$$f_\delta(t) := PT(t)(\lambda_0 - A)^{-\delta} x_0, \qquad t \geq 0.$$

We have to show that $\lim_{t\to\infty} \|f_\beta(t)\| = 0$. Theorem 1.2 shows that $f_\beta \in L^q(\mathbb{R}, Y)$, $1/p + 1/q = 1$.

Let $\delta = n + \gamma$ with $n \in \mathbb{N}$ and $\gamma \in [0, 1)$. If $\gamma \in (0, 1)$, then

$$\begin{aligned}
\|PT(\tau)(\lambda_0 - A)^{-\gamma} x_0\| &= \frac{\sin \pi\gamma}{\gamma} \left\| \int_0^\infty r^{-\gamma} PT(\tau) R(\lambda_0 + r, A) x_0 \, dr \right\| \\
&= \frac{\sin \pi\gamma}{\gamma} \left\| \int_0^\infty r^{-\gamma} \int_0^\infty e^{-(\lambda_0 + r)s} PT(\tau + s) x_0 \, ds \, dr \right\| \\
&\leq \frac{\sin \pi\gamma}{\gamma} \int_0^\infty C r^{-\gamma} (\lambda_0 + r)^{-1} \, dr,
\end{aligned}$$

where $C := \sup_{t\geq 0} \|PT(t)x_0\|$. If $\gamma = 0$, then $\|PT(\tau)x_0\| \leq C$. In either case, we see that $C_\gamma := \sup_{\tau\geq 0} \|PT(\tau)(\lambda_0 - A)^{-\gamma} x_0\| < \infty$. Using this, we obtain

$$\begin{aligned}
\|f_\delta(t)\| &= \left\| \int_0^\infty \cdots \int_0^\infty e^{-\lambda_0(s_1 + \cdots + s_n)} \right. \\
&\qquad \left. \times PT(t + s_1 + \cdots + s_n)(\lambda_0 - A)^{-\gamma} x_0 \, ds_n \cdots ds_1 \right\| \\
&\leq C_\gamma \lambda_0^{-n},
\end{aligned}$$

for all $t \geq 0$, so f_δ is bounded. In particular, such an estimate holds for f_β. Also, f_β is differentiable and

$$f'_\beta(t) = PT(t)A(\lambda_0 - A)^{-\beta} x_0 = -f_{\beta-1}(t) + \lambda_0 f_\beta(t).$$

Therefore, also $f'_\beta(\cdot)$ is bounded (here we use that $\beta \geq 1$) and hence the bounded function $f_\beta(\cdot)$ is uniformly continuous. Then also $\|f_\beta(\cdot)\|^q = \|PT(\cdot)(\lambda_0 - A)^{-\beta} x_0\|^q$ is bounded and uniformly continuous, and it is an immediate consequence of Theorem 1.2 that $\|f_\beta(t)\| \to 0$ as $t \to \infty$. ∎

Assuming boundedness *and uniform continuity* of $PT(\cdot)x_0$, we obtain a stronger result. Let us say that a function F is *polynomially bounded* in the strip $\{0 < \operatorname{Re} \lambda < \omega_0\}$ if there exist $M > 0$ and $n \in \mathbb{N}$ such that

$$\|F(\lambda)\| \leq M(1 + |\lambda|)^n, \qquad 0 < \operatorname{Re} \lambda < \omega_0. \tag{1.10}$$

COROLLARY 1.6. *Let P be a bounded linear operator from X into a B-convex space Y. Let A be the generator of a C_0-semigroup $\mathbf{T}$ on X and let $x_0 \in X$ be such that the orbit $t \mapsto PT(t)x_0$ is bounded and uniformly continuous. If the map $\lambda \mapsto PR(\lambda, A)x_0$ extends to a holomorphic function in the open right half-plane which is polynomially bounded in $\{0 < \operatorname{Re}\lambda < \omega_0\}$ for some $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, then $\lim_{t\to\infty} \|PT(t)x_0\| = 0$.*

Proof. Fix $\lambda > \omega_0(\mathbf{T})$. Let $\mathbf{S}$ denote the left translation semigroup on the space $Z := BUC(\mathbb{R}_+, Y)$ defined by $(S(t)f)(s) = f(t+s)$; $s, t \geq 0$. The function $f(t) := PT(t)x_0$ defines an element of Z. From the identity,

$$\begin{aligned} &PT(t)R(\lambda, A)^{n+1}x_0 \\ &\quad = \int_0^\infty \cdots \int_0^\infty e^{-\lambda(s_1+\cdots+s_{n+1})} PT(s_1 + \cdots + s_{n+1} + t)x_0 \, ds_{n+1} \cdots ds_1, \end{aligned}$$

also it is easy to see that $f_\lambda(t) := PT(t)R(\lambda, A)^{n+1}x_0$ defines an element of Z; here $n \in \mathbb{N}$ is chosen such that (1.10) holds.

By Theorem 1.5,

$$\lim_{t\to\infty} \|S(t)f_\lambda\|_Z = \lim_{t\to\infty} \Big(\sup_{s\geq 0} \|PT(t+s)R(\lambda, A)^{n+1}x_0\| \Big) = 0.$$

Therefore, $f_\lambda \in Z_0 := \{f \in Z\colon \lim_{t\to\infty} \|S(t)f\|_Z = 0\}$. For $\operatorname{Re}\lambda > \omega_0(\mathbf{T})$ and $s \geq 0$ we have, denoting by B the generator of $\mathbf{S}$,

$$\begin{aligned} &(R(\lambda, B)^{n+1}f)(s) \\ &\quad = \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1+\cdots+t_{n+1})} (S(t_1 + \cdots + t_{n+1})f)(s) \, dt_{n+1} \cdots dt_1 \\ &\quad = \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1+\cdots+t_{n+1})} PT(t_1 + \cdots + t_{n+1} + s)x_0 \, dt_{n+1} \cdots dt_1 \\ &\quad = PT(s)R(\lambda, A)^{n+1}x_0 = f_\lambda(s). \end{aligned}$$

Hence $f = \lim_{\lambda\to\infty} \lambda^{n+1}R(\lambda, B)^{n+1}f = \lim_{\lambda\to\infty} \lambda^{n+1}f_\lambda \in Z_0$ by the closedness of Z_0. Hence $\lim_{t\to\infty} \|S(t)f\| = 0$, and thus $\lim_{t\to\infty} \|PT(t)x_0\| = \lim_{t\to\infty} \|(S(t)f)(0)\| = 0$. ∎

The technique of this proof goes back to Kantorovitz [10]; see [2] for another application.

The following example shows that our results break down if no restrictions on the Banach space X are imposed.

EXAMPLE 1.7. Let $X = C_0(\mathbb{R})$ and consider the left translation group $\mathbf{S}$ on X. Let B be its generator. Let $f \in X$ be any nonzero function with support in $[0, 1]$. Then for all $\operatorname{Re}\lambda > 0$ and $s \in \mathbb{R}$ we have

$$|(R(\lambda, B)f)(s)| = \left| \int_0^\infty e^{-\lambda t} f(s+t) \, dt \right| \leq \|f\|_\infty.$$

Consequently,

$$\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, B)f\|_\infty \le \|f\|_\infty,$$

but because **S** is isometric and $(\lambda_0 - B)^{-\beta}$ is injective we see that

$$\lim_{t\to\infty} \|S(t)(\lambda_0 - B)^{-\beta} f\|_\infty = \|(\lambda_0 - B)^{-\beta} f\|_\infty \ne 0; \qquad \forall \beta \ge 0,\ \lambda_0 > 0.$$

As an application of Corollary 1.6 we derive a Tauberian theorem for the Laplace transform of functions in $L^\infty(\mathbb{R}_+, Y)$, where Y is a B-convex Banach space. This serves merely as an illustration of what can be done with the preceding theory; by considering bounded, uniformly continuous orbits much of the sharpness of the preceding results is lost and it may well be that more direct methods lead to a sharper Tauberian theorem (cf. the remarks at the end of the paper).

LEMMA 1.8. *Let Y be a B-convex Banach space and assume that the Laplace transform $\hat g$ of a function $g \in BUC(\mathbb{R}_+, Y)$ is polynomially bounded in some strip* $\{0 < \operatorname{Re}\lambda < \omega_0\}$. *Then* $\lim_{t\to\infty} \|g(t)\| = 0$.

Proof. Consider the left translation semigroup **S** in $BUC(\mathbb{R}_+, Y)$ with generator B. Let P be the bounded operator from $BUC(\mathbb{R}_+, Y)$ into Y defined by $Ph = h(0)$. Then $PS(t)g = g(t) \otimes \mathbf{1}$ and $PR(\lambda, B)g = \hat g(\lambda) \otimes \mathbf{1}$ for all $t \ge 0$ and $\operatorname{Re}\lambda > 0$. Because Y is B-convex, we can apply Corollary 1.6 to **S** and we can deduce that $\lim_{t\to\infty} \|g(t)\| = \lim_{t\to\infty} \|PS(t)g\| = 0$. ∎

THEOREM 1.9. *Let Y be a B-convex Banach space and let $f \in L^\infty(\mathbb{R}_+, Y)$. If the Laplace transform $\hat f$ is polynomially bounded in some strip* $\{0 < \operatorname{Re}\lambda < \omega_0\}$ *and can be holomorphically extended to a neighbourhood of* 0, *then*

$$\lim_{t\to\infty} \left\| \int_0^t f(s)\,ds - \hat f(0) \right\| = 0.$$

Proof. The proof is inspired by [2, Theorem 4.3].

Upon replacing $f(t)$ by $f(t) - e^{-t}\hat f(0)$ we may assume that $\hat f(0) = 0$. By a special case of Ingham's Tauberian theorem the function $g(t) := \int_0^t f(s)\,ds$ is bounded (see [11] for an elegant and elementary proof). Moreover, g is uniformly continuous and in view of $\hat f(0) = 0$, 0 is a removable singularity of $\hat g(\lambda) = \lambda^{-1}\hat f(\lambda)$. It follows that $\hat g$ is polynomially bounded in $\{0 < \operatorname{Re}\lambda < \omega_0\}$. Therefore by Lemma 1.8,

$$\lim_{t\to\infty} \left\| \int_0^t f(s)\,ds \right\| = \lim_{t\to\infty} \|g(t)\| = 0.$$

∎

2. STABILITY AND THE ANALYTIC RADON–NIKODYM PROPERTY

In this section we prove some analogues of the previous results for the case $p = 1$. As it turns out, this is possible if one assumes Y has the analytic Radon–Nikodym property.

We start by recalling some facts concerning vector-valued Hardy spaces over the disc $D = \{z \in \mathbb{C}: |z| < 1\}$.

For $p \in [1, \infty]$ we let $H^p(D, Y)$ denote the set of all holomorphic functions $f: D \to Y$ for which

$$\|f\|_p := \sup_{0<r<1} \left(\int_0^{2\pi} \|f(re^{i\theta})\|^p \, d\theta \right)^{1/p} < \infty.$$

In case $p = \infty$ we interpret the foregoing integral in terms of the supremum norm in the obvious way. It is not difficult to see that $H^p(D, Y)$ is a Banach space with respect to the norm $\|\cdot\|_p$. We let $H_0^p(D, Y)$ denote the closed subspace of $H^p(D, Y)$ consisting of all functions f for which the radial limits $\tilde{f}(e^{i\theta}) := \lim_{r\uparrow 1} f(re^{i\theta})$ exist for almost all θ. By Fatou's lemma,

$$\int_0^{2\pi} \|\tilde{f}(e^{i\theta})\|^p \, d\theta \le \liminf_{r\uparrow 1} \int_0^{2\pi} \|f(re^{i\theta})\|^p \, d\theta,$$

which shows that the boundary function $\tilde{f}$, if it exists a.e., belongs to $L^p(\Gamma)$, where $\Gamma = \{z \in \mathbb{C}: |z| = 1\}$. In this case, f can be recovered from $\tilde{f}$ by the Poisson integral,

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(e^{i\eta}) \frac{1 - r^2}{1 - 2r\cos(\theta - \eta) + r^2} \, d\eta.$$

Defining $f_r(e^{i\theta}) := f(re^{i\theta})$, as in the scalar case it follows from this representation that

$$\lim_{r\uparrow 1} \|\tilde{f} - f_r\|_{L^p(\Gamma)} = 0.$$

A Banach space Y is said to have the *analytic Radon–Nikodym property* if $H_0^p(D, Y) = H^p(D, Y)$. Equivalently, Y has the analytic Radon–Nikodym property if for all $f \in H^p(D, Y)$ the radial limits $\tilde{f}(e^{i\theta}) := \lim_{r\uparrow 1} f(re^{i\theta})$ exist for almost all θ, and in this case we actually have $f_r \to \tilde{f}$ in the L^p-norm.

The role of the exponent p needs some clarification: it can be shown that if $H_0^p(D, Y) = H^p(D, Y)$ holds for some $p \in [1, \infty]$, then it holds for all $p \in [1, \infty]$.

The following facts are well known:

(i) If Y has the Radon–Nikodym property, then Y has the analytic Radon–Nikodym property;

(ii) If Y has the analytic Radon–Nikodym property, then Y contains no closed subspace isomorphic to c_0;

(iii) A Banach lattice Y has the analytic Radon–Nikodym property if and only if Y contains no closed subspace isomorphic to c_0.

It follows from (i) that every reflexive Banach space and every separable dual Banach space has the analytic Radon–Nikodym property. By (iii), the spaces $L^1(\mu)$ have the analytic Radon–Nikoym property. The proofs can be found in [5, 6].

By mapping a rectangle conformally onto the unit disc it is not difficult to prove the following result; cf. [7].

PROPOSITION 2.1. *Let Δ and Δ_r, $0 < r < 1$, be the rectangles in $\mathbb{C}$ spanned by the points $\pm a \pm ib$ and $\pm ra \pm irb$, respectively. Let f be a holomorphic Y-valued function in the interior of Δ. Assume that Y has the analytic Radon–Nikodym property and that*

$$\sup_{0<r<1} \int_{\Delta_r} \|f(z)\| \, |dz| < \infty.$$

Then, the strong limits $\lim_{r\uparrow 1} f(rz)$ exist for almost all $z \in \Delta$ and define a function $\tilde{f} \in L^1(\Delta)$. Moreover,

$$\lim_{r\uparrow 1} \int_{\Delta} \|\tilde{f}(z) - f(rz)\| \, |dz| = 0.$$

∎

THEOREM 2.2. *Let P be a bounded operator from a Banach space X into a Banach space Y with the analytic Radon–Nikodym property. Let A be the generator of a C_0-semigroup $\mathbf{T}$ on X. Assume that for some $x_0 \in X$, the map $\lambda \mapsto PR(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ to the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, $M > 0$ and $\alpha \in [-1, \infty)$ such that*

$$\|F(\lambda)\| \le M(1 + |\lambda|)^{\alpha}, \qquad 0 < \operatorname{Re} \lambda < \omega_0,$$

then for all $\lambda_0 > \max\{0, \omega_0(\mathbf{T})\}$ and $\beta > \alpha + 1$ we have

$$\lim_{t\to\infty} \|PT(t)(\lambda_0 - A)^{-\beta} x_0\| = 0.$$

Proof. Without loss of generality we may assume that $\omega_0(\mathbf{T}) \geq 0$. Fix $\lambda_0 > \omega_0(\mathbf{T})$. By taking a smaller value of ω_0 we may assume that $\omega_0(\mathbf{T}) < \omega_0 < \lambda_0$.

Fix $\gamma \in (\alpha + 1, \beta)$ and let $\delta := \beta - \gamma$.

Let $g(\lambda)$ denote the holomorphic extension in the open right half-plane of the function $\lambda \mapsto PR(\lambda, A)(\lambda_0 - A)^{-\gamma} x_0$. Fix $\omega_1 \in (\omega_0(\mathbf{T}), \omega_0)$. On the strip $\{0 < \operatorname{Re} \lambda < \omega_1\}$ we define $h(\lambda) := (\omega_0 - \lambda)^{-\delta} g(\lambda)$. By Lemma 1.1, for each $\zeta \in \mathbb{C}$ with $0 < \operatorname{Re} \zeta < \omega_1$ the function,

$$s \mapsto h_\zeta(s) := h(\zeta - is) = (\omega_0 - \zeta + is)^{-\delta} g(\zeta - is)$$

belongs to $L^1(\mathbb{R}, Y)$, and the map $\zeta \mapsto h_\zeta$ is a bounded $L^1(\mathbb{R}, Y)$-valued holomorphic function on $\{0 < \operatorname{Re} \zeta < \omega_1\}$.

Arguing as in the proof of the Claim following Theorem 1.2 we see that for $\omega \in (\omega_0(\mathbf{T}), \omega_1)$ the Fourier transform of h_ω is given by

$$\frac{1}{2\pi} \mathcal{F} h_\omega(t) = e^{-\omega t} PT(t)(\omega_0 - A)^{-\delta}(\lambda_0 - A)^{-\gamma} x_0. \tag{2.1}$$

Hence by uniqueness of analytic continuation,

$$\frac{1}{2\pi} \mathcal{F} h_\zeta(t) = e^{-\zeta t} PT(t)(\omega_0 - A)^{-\delta}(\lambda_0 - A)^{-\gamma} x_0, \qquad 0 < \operatorname{Re} \zeta < \omega_1,$$

and we conclude that (2.1) holds for all $\omega \in (0, \omega_1)$.

Because Y has the analytic Radon–Nikodym property, we may apply Proposition 2.1 and we may conclude that the boundary function $\tilde{h}$ of h exists a.e. on $i\mathbb{R}$, defines an element in $L^1_{\mathrm{loc}}(i\mathbb{R}, Y)$, and that

$$\lim_{\omega \downarrow 0} \int_{-r}^{r} \|\tilde{h}(is) - h(\omega + is)\| \, ds = 0,$$

for all $r > 0$. But then (1.3) and the definition of h easily implies that we actually have $\tilde{h} \in L^1(i\mathbb{R}, Y)$ and

$$\lim_{\omega \downarrow 0} \int_{-\infty}^{\infty} \|\tilde{h}(is) - h(\omega + is)\| \, ds = 0.$$

Hence by passing to the limit $\omega \downarrow 0$ in (2.1), we obtain

$$\begin{aligned}
PT(\cdot)&(\omega_0 - A)^{-\delta}(\lambda_0 - A)^{-\gamma} x_0 \\
&= \lim_{\omega \downarrow 0} e^{-\omega t} PT(\cdot)(\omega_0 - A)^{-\delta}(\lambda_0 - A)^{-\beta} x_0 \\
&= \frac{1}{2\pi} \lim_{\omega \downarrow 0} \mathcal{F} h(\omega - i(\cdot))(t) = \frac{1}{2\pi} \mathcal{F} \tilde{h}(-i(\cdot))(t).
\end{aligned}$$

Therefore, $PT(\cdot)(\omega_0 - A)^{-\delta}(\lambda_0 - A)^{-\gamma} x_0 \in C_0(\mathbb{R}_+, Y)$ by the Riemann–Lebesgue lemma. Recalling that $\delta + \gamma = \beta$, by standard arguments involving fractional powers this will give the desired result. ∎

Theorem 0.2 is a special case of this.

Taking $Y = \mathbb{C}$ and $P := x_0^* \in X^*$, we obtain the following result, which contains Theorem 0.3 as a special case.

COROLLARY 2.3. *Let A be the generator of a C_0-semigroup $\mathbf{T}$ on a Banach space X. Assume that for some $x_0 \in X$ and $x_0^* \in X^*$, the map $\lambda \mapsto \langle x_0^*, R(\lambda, A)x_0\rangle$ admits a holomorphic extension $F(\lambda)$ to the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, $M > 0$ and $\alpha \in [-1, \infty)$ such that*

$$|F(\lambda)| \le M(1 + |\lambda|)^\alpha, \qquad 0 < \operatorname{Re}\lambda < \omega_0,$$

then for all $\lambda_0 > \max\{0, \omega_0(\mathbf{T})\}$ and $\beta \ge 0$ with $\beta > \alpha + 1$ we have

$$\lim_{t\to\infty} \langle x_0^*, T(t)(\lambda_0 - A)^{-\beta} x_0\rangle = 0.$$

Theorem 2.2 can be used to show that Corollary 1.6, and therefore also Theorem 1.9, remains valid if B-convexity is replaced by the analytic Radon–Nikodym property. It is possible, however, to modify the proof of [11] to prove in a more direct way the stronger result: if Y has the analytic Radon–Nikodym property and $f \in L^\infty(\mathbb{R}_+, Y)$ is such that for all $r > 0$ we have

$$\limsup_{\omega\downarrow 0} \int_{-r}^{r} \left\| \frac{\hat{f}(\omega + is) - \hat{f}(0)}{\omega + is} \right\| ds < \infty,$$

then $\lim_{t\to\infty} \| \int_0^t f(s)\,ds - \hat{f}(0)\| = 0$. This was shown by Chill [7] and suggests that it may be possible to prove a similar result assuming B-convexity. It is important in this context to point out that B-convexity and the analytic Radon-Nikodym property are unrelated concepts in the sense that none implies the other. In fact, $L^1[0, 1]$ has the analytic Radon–Nikodym property (by observation (iii) at the beginning of this section) but no nontrivial type, so it is not B-convex. The following example shows that there exist B-convex spaces without the analytic Radon–Nikodym property:

EXAMPLE 2.4. By the function space analogue of a result in [20] (the details are given in [24]), the operator of integration $I\colon L^1[0, 1] \to C[0, 1]$,

$$I(f)(t) := \int_0^t f(s)\,ds,$$

factors through a space with type 2. Denoting $f_0(t) := t$ and defining $T\colon C[0, 1] \to C[0, 1]$ by $T(f) := f - f(1)f_0$, also $J := T \circ I$ factors through a space with type 2. Identifying $[0, 1)$ with the unit circle Γ in the complex plane and letting $e_n(\theta) := \exp(2\pi i n\theta)$, $\theta \in \Gamma$, $n \in \mathbb{Z}$, we can represent J as an operator from $L^1(\Gamma)$ into $C(\Gamma)$ by

$$J(e_n) = \frac{e_n}{2\pi i n}, \qquad n \in \mathbb{Z}\backslash\{0\}, \qquad J(e_0) = 0.$$

Recalling that type passes to quotients, it follows that the quotient operator J_0: $L^1(\Gamma)/H_0^1 \to C(\Gamma)/A_0$ induced by J factors through a space with type 2; here H_0^1 and A_0 denote the closed linear span in $L^1(\Gamma)$ and $C(\Gamma)$, respectively, of $\{\theta \mapsto \exp(2\pi in\theta)$: $n = -1, -2, \ldots\}$. On the other hand, by a result of Pisier [8, Proposition V.5], J_0 cannot be factored through a space with the analytic Radon–Nikodym property.

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Note added in proof. V. Wrobel [23] shows that the bound $\beta > 1/p$ in Theorem 0.1 is the best possible, in the sense that a counterexample can be constructed for every $\beta \in [0, 1/p)$. Whether or not the theorem holds for $\beta = 1/p$ remains an open problem. In the same paper, an extension of Theorem 0.1 into a different direction is obtained.

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