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Fractional Order Continuity and Some Properties about Integrability and Differentiability of Real Functions*

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In this paper a certain function space $\mathbf{C}_\alpha$, $0 \leq \alpha \leq 1$, larger than the space of continuous functions, is introduced in order to study new properties and the extension of some already known results about the Riemann–Liouville fractional integral and derivative operators.

Sufficient conditions for the continuity of $I_a^{1-\alpha} f$ are given. Furthermore, necessary conditions are given for the pointwise existence of fractional derivatives. The existence of a derivative of order β, from the existence of a certain derivative of order α, $\beta < \alpha$, is also analyzed.

Key Words: fractional calculus; α-continuity; fractional derivative; fractional integral; Riemann–Liouville operator; index rule.

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1. INTRODUCTION

Some authors have studied the operators of Riemann–Liouville (R–L):

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{1.1}$$

$$D_a^\alpha f(x) = D I_a^{1-\alpha} f(x), \tag{1.2}$$

with $\alpha \in [0,1]$, $a \in \mathbb{R}$ and $x > a$, on different function spaces within which it is pointed out that Hardy–Littlewood [2], Riesz [6], Erdélyi [1], Oldham–Spanier [5], McBride [3], Srivastava–Buschman [8], Miller–Ross [4], and especially Samko–Kilbas–Marichev [7] had made a deeper study.

In this paper we introduce a certain function space $\mathbf{C}_\alpha$, $\alpha \in [0,1]$. The function of this space are called a α-continuous function, and it satisfies the following relation:

$$\mathbf{C}(\Omega) \equiv \mathbf{C}_1(\Omega) \subset \mathbf{C}_\alpha(\Omega) \subset \mathbf{L}(\Omega), \quad \forall \alpha \in [0,1],$$

where $\mathbf{C}(\Omega)$ is the space of the continuous functions and $L(\Omega)$ is the space of the Lebesgue integrable functions, on the same interval $\Omega \subset \mathbb{R}$.

Moreover, properties of the R–L operators I_a^α and D_a^α are studied on those spaces, and the conditions, under which some of those properties hold, are weakened.

We work out a basic theory, for the fractional calculus of R–L, similar to that of the classical differential calculus.

In the sequel we use the following notation:

$f \in \mathbf{F}(\Omega)$, where $\mathbf{F}(\Omega)$ is the set of the real functions of a single real variable with domain in $\Omega \subset \mathbb{R}$.

- $\mathbf{M}(\Omega) = \{f : f \text{ is Lebesgue mesurable on } \Omega\}$.
- $\mathbf{AC}(\Omega) = \{f : f \text{ is absolutely continuous on } \Omega\}$.
- $\mathbf{AC}^n(\Omega) = \{f : f, f^{(i)} \in \mathbf{C}(\Omega)\ \forall i = 1, \ldots, (n-1)$ and $f^{(n-1)}(x) \in \mathbf{AC}(\Omega)\}$.
- $\mathbf{H}^\lambda(\Omega) = \{f : f \text{ verify the Hölder conditions of order } \lambda \text{ on } \Omega\}$.
- $[\lambda]$ denotes the integral part of a number λ.
- $f(x) = O(|x-a|^{\nu-1})$ in $x = a$, with $\nu, a \in \mathbb{R}$, means

$$\lim_{x \to a} \frac{f(x)}{|x-a|^{\nu-1}} = k < \infty \text{ and } k \neq 0.$$

2. DEFINITIONS

Here we introduce the concept of α-continuity, $\alpha \in [0,1]$, together with the definitions of the R–L fractional integral and derivative operators.

2.1. α-*Continuity*

DEFINITION 2.1.1. Let $f \in \mathbf{M}(\Omega)$, $\alpha \in [0, 1)$, and $x_0 \in \Omega$. f is called α-continuous in x_0 if, there exists $\lambda \in [0, 1-\alpha)$ for which $g(x) = |x - x_0|^\lambda f(x)$ is a continuous function in x_0. Moreover, f is called 1-continuous in x_0 if it is continuous in x_0.

As usual, it is said that "f is a α-continuous function on Ω if it is α-continuous for every x in Ω," and it is denoted:

$$\mathbf{C}_\alpha(\Omega) = \{f \in F(\Omega) : f \text{ is } \alpha\text{-continuous in } \Omega\}$$

EXAMPLE 2.1.1. Next are α-continuous functions:

(1) $\forall \alpha \in (0, 1)$, any measurable and bounded function on $\mathbb{R}$.

(2) $\forall \alpha \in (0, 1)$, $f(x) = [x]$, $x > 0$.

(3) $\forall \alpha \in [0, 3/4)$, $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{if } x \in \mathbb{Q}^c \cap [0, 1] \\ \frac{\sin(x)}{(x^2-1)^{1/4}}, & \text{if } x > 1. \end{cases}$

But

(4) $\forall \alpha \in [0, 1)$, $f(x) = (x - a)^{\alpha-1}$ is not α-continuous in $x = a$.

The last example suggests the following:

DEFINITION 2.1.2. Let $f \in \mathbf{F}(\Omega)$ be. The function $f(x)$ is called a-singular of order α, $0 \le \alpha < 1$, if $f(x) = O(|x - a|^{\alpha-1})$ in $x = a$.

2.2. a-*Integrability of Order* α

DEFINITION 2.2.1. Let $\alpha \in \mathbb{R}^+$, $f \in \mathbf{F}(\Omega)$ and $[a, x] \subset \Omega$. Then

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\,dt$$

is called the Riemann–Liouville a-integral of order α of f.

DEFINITION 2.2.2. Let $\alpha \in \mathbb{R}^+$, $f \in \mathbf{F}(\Omega)$ and $[a, x_0] \subset \Omega$: it is said that f is a-integrable of order α in x_0 if $I_a^\alpha f(x_0)$ exists and it is finite.

Let $E \subset \Omega$ be, such that $a \le x$, $\forall x \in E$. It is said that "f is a-integrable of order α on E if f is a-integrable of order α for every $x \in E$," and it is denoted

$${}_a\mathbf{I}_\alpha(E) = \{f \in F(\Omega) \colon f \text{ is a-integrable of order } \alpha \text{ on } E\}.$$

2.3. *a-Differentiability of Order α*

DEFINITION 2.3.1. Let $\alpha \in \mathbb{R}^+$ be such that $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$. Let $f \in \mathbf{F}(\Omega)$ and $[a, x_0] \subset \Omega$ be. It is said that f is a-differentiable of order α in x_0 if there exists

$$D_a^\alpha f(x_0) = (D^n I^{n-\alpha} f(x))_{x=x_0},$$

and it is finite, D^n being the classical derivative operator of order n.

Let $E \subset \Omega$ be such that $a \leq x$, $\forall x \in E$. It is said that "f is a-differentiable of order α on E if f is a-differentiable of order α for every $x \in E$", and it is denoted

$${}_a\mathbf{D}_\alpha(E) = \{f \in F(\Omega)\colon f \textit{ is } \text{a-differentiable} \textit{ of order } \alpha \textit{ on } E\}.$$

DEFINITION 2.3.2. Let $f \in {}_a\mathbf{D}_\alpha(E)$, $x \in E$ and $a \leq x$. The function

$$D_a^\alpha f(x) = D^n I_a^{n-\alpha} f(x),$$

is called the a-derivative of order α function of f.

EXAMPLE 2.3.1. (a) $\forall a \in \mathbb{R} - \{1\}$, the function $f(x) = x - 1$ is not a-differentiable of order α, $\alpha \in (0,1)$, in $x = a$, but it is differentiable of order 1 in any $x \in \mathbb{R}$.

(b) $\forall \alpha, a \in (0,1)$, the function $f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2 - x, & \text{if } x > 1, \end{cases}$

is a-differentiable of order α in $x = 1$, and

$$(D_a^\alpha f(x))_{x=1} = \frac{a(1-a)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{(1-a)^{1-\alpha}}{\Gamma(2-\alpha)}.$$

However, f is not differentiable of order 1 in $x = 1$.

(c) Let $f(x) = \begin{cases} \frac{x^\alpha}{\Gamma(1+\alpha)}, & \text{if } 0 \leq x \leq 1 \\ \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{2(x-1)^\alpha}{\Gamma(1+\alpha)}, & \text{if } x > 1, \end{cases}$

f is continuous, but is not 0-differentiable of order α in $x = 1$, $\forall \alpha \in (0,1)$ and

$$(D_0^\alpha f(x))_{x=1^-} = 1; \quad (D_0^\alpha f(x))_{x=1^+} = -1.$$

3. PROPERTIES

Some properties of α-continuous functions are given now, which will be used later in order to establish some properties involving the a-integrability and the a-differentiability of order α of these functions.

3.1. *Properties Involving the α-Continuity*

Let $\alpha, \beta \in [0, 1]$ and set $[a, b] \in \Omega$ a real interval.

(1) $\mathbf{C}_\alpha(\Omega)$ is a linear space over $\mathbb{R}$.

(2) If $f \in \mathbf{C}_\alpha([a, b])$, then $\#\{x_0 \in [a, b] / \lim_{x \to x_0} f(x) = \pm\infty\} < \infty$, where $\#$ denotes the cardinal of a set.

(3) If $f \in \mathbf{C}_\alpha([a, b])$, then $\exists x_1, x_2, \ldots, x_n \in [a, b]$ and $\exists \lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1 - \alpha)$, such that the function $h(x) = (\prod_{i=1}^{n} |x - x_i|^{\lambda_i}) f(x)$ is bounded on $[a, b]$.

(4) If $f \in \mathbf{C}_\alpha([a, b])$, then $f \in L([a, b])$.

(5) If $\beta < \alpha$, then $\mathbf{C}_\alpha(\Omega) \subset \mathbf{C}_\beta(\Omega)$.

The above properties are easily proved from definition 2.1.1.

Now we give some properties of the operator I_a^α, which complement the already known properties of this operator. See, for example, Samko–Kilbas–Marichev [7].

3.2. *Properties about a-Integrability of Order α*

Let $\alpha, \beta \in \mathbb{R}^+$ and set $[a, b]$ a real interval.

(1) ${}_a\mathbf{I}_\alpha([a, b])$ is a linear space over $\mathbb{R}$ and I_a^α is a linear operator.

(2) If $\alpha \leq \beta$, then ${}_a\mathbf{I}_\alpha([a, b]) \subset {}_a\mathbf{I}_\beta([a, b])$ and ${}_a\mathbf{I}_\alpha((a, b]) \subset {}_a\mathbf{I}_\beta((a, b])$.

(3) If $c \in [a, b]$, then ${}_a\mathbf{I}_\alpha([a, b]) \subset {}_c\mathbf{I}_\alpha([c, b])$.

(4) If $f \in \mathbf{L}((a, b))$ and it is bounded on $[a, b]$ then $f \in {}_a\mathbf{I}_\alpha([a, b])$.

(5) Let $\alpha \in [0, 1]$ and $\beta \geq 1 - \alpha$ be. Then $\mathbf{C}_\alpha([a, b]) \subset {}_a\mathbf{I}_\beta([a, b])$.

(6) (a) If $f \in \mathbf{C}([a, b])$, then $I_a^\alpha f \in \mathbf{C}([a, b])$.

(b) If $f \in \mathbf{AC}([a, b])$, then $I_a^\alpha f \in \mathbf{AC}([a, b])$.

(7) Let $\alpha \in [0, 1]$ and $f \in \mathbf{C}_\alpha([a, b])$ be. Then $I_a^{1-\alpha} f$ is continuous on $[a, b]$ and

$$I_a^{1-\alpha} f(a) = \lim_{x \to a^+} I_a^{1-\alpha} f(x) = 0, \quad \text{for } 0 \leq \alpha < 1.$$

Also, if f is a-singular of order α and $f \in \mathbf{C}_\alpha((a, b])$, the property holds, but $I_a^{1-\alpha} f(a) = cte < \infty$, for $0 \leq \alpha < 1$.

Proof. The properties (1), (2), (3) and (4) are easily proved from Definition 2.2.2. The properties (5) and (7) follow from 3.1.2 and 3.1.3. The property (6) is in Samko–Kilbas–Marichev [7].

On the other hand, in [7] the index rule of the Riemann–Liouville fractional integral operators is given, in the next cases:

(a) If $f \in \mathbf{L}(a, b)$, the index rule is true almost everywhere on $[a, b]$.

(b) If $1 \le \alpha + \beta$ and $f \in \mathbf{L}(a, b)$, the index rule holds everywhere in (a, b).

Conditions under which this index rule holds are extended through the spaces $\mathbf{C}_\alpha$ as follows:

(8) If $\alpha + \beta < 1$ and $f \in \mathbf{C}_\gamma([a, b])$, with $1 - (\alpha + \beta) \le \gamma \le 1$ then

$$I_a^\alpha (I_a^\beta f(x)) = I_a^{\alpha+\beta} f(x), \quad \forall x \in [a, b].$$

Moreover, this holds also if f is a-singular of order γ and $f \in \mathbf{C}_\gamma((a, b])$.

Proof. It follows from the Fubini and Tonelli–Hobson theorems.

(9) Let $n \in \mathbb{N}$ and $\alpha \in [0, 1]$ be. If $f \in \mathbf{C}_{1-\alpha}([a, b])$ then:

$$D_a^n I_a^{n+\alpha} f(x) = I_a^\alpha f(x), \quad \forall x \in [a, b]$$

Proof. It follows from above properties 3.2.6 and 3.2.7.

3.3. *Properties Involving the α-Differentiability*

(1) ${}_a\mathbf{D}_\alpha([a, b])$ is a linear space over $\mathbb{R}$ and the operator D_a^α is linear.

In Samko–Kilbas–Marichev [7], sufficient conditions are given for the existence, almost everywhere, of the a-differentiability of order α of a Lebesgue integrable function. In the next property, we give necessary conditions for the pointwise a-differentiability of order α.

(2) Let $\alpha \in (0, 1]$, $x_0 \in (a, b)$ and $f \in \mathbf{C}_\alpha([a, b])$ be. If f is a-differentiable of order α in x_0 and it has only a finite number of discontinuities in some neighborhood of x_0, then f is continuous in x_0.

Proof. It follows by using the contradiction method and the properties 3.1.2 and 3.1.3.

(3) Let $0 < \beta \le \alpha < 1$. If $f \in \mathbf{C}_\alpha([a, b])$ and $f \in {}_a\mathbf{D}_\alpha([a, b])$, then $f \in {}_a\mathbf{D}_\beta([a, b])$.

Proof. It follows from Definitions 2.1.1 and 2.3.1 and properties 3.2.7, 3.2.8, and 3.3.3.

The relations of inclusion between the spaces introduced above could be collected in the following way, for $0 \le \beta < \alpha < 1$, $[a, b] \subset \mathbb{R}$ and $\lambda \in \mathbb{R}$:

(i) $\mathbf{H}^\lambda([a, b]) \subset \mathbf{C}([a, b]) \subset \mathbf{C}_\alpha([a, b]) \subset \mathbf{C}_\beta([a, b])$.

(ii) $\mathbf{AC}([a, b]) \subset \mathbf{C}([a, b]) \subset \mathbf{C}_\alpha([a, b]) \subset \mathbf{C}_\beta([a, b])$.

(iii) ${}_a\mathbf{I}_\beta([a,b]) \subset {}_a\mathbf{I}_\alpha([a,b]) \subset {}_a\mathbf{I}_1([a,b])$.

(iv) $\{f \in \mathbf{F}([a,b]) : f(a)=0\} \cap \mathbf{D}([a,b]) \subset \mathbf{C}_\alpha([a,b]) \cap {}_a\mathbf{D}_\alpha([a,b]) \subset \mathbf{C}_\beta([a,b]) \cap {}_a\mathbf{D}_\beta([a,b])$.

3.4. *Index Rule for the Operators I_a^α and D_a^α*

It is well known that, in general, $D_a^\alpha D_a^\beta f \neq D_a^{\alpha+\beta} f$ and $I_a^\alpha D_a^\alpha f \neq f$. In [7] there are conditions under which the corresponding equalities hold. Certain relations in this way are now given.

Let $\alpha, \beta \in \mathbb{R}^+$ be, such that $n-1 < \alpha \leq n$, $m-1 < \beta \leq m$ with $n, m \in \mathbb{N}$:

1. Let $f \in {}_a\mathbf{D}_\beta([a,b])$ be, such that $D_a^\beta f \in {}_a\mathbf{D}_\alpha([a,b])$. Then

$$D_a^\alpha D_a^\beta f(x) = D_a^{\alpha+\beta} f(x) - \sum_{k=1}^{m} \frac{A_k (x-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)}, \quad \forall x \in (a,b],$$

where

$$A_k = \lim_{x \to a^+} D_a^{m-k} I_a^{m-\beta} f(x), \quad k = 1, 2, \ldots, m,$$

provided that

(i) $n+m-\alpha-\beta \geq 1$, or

(ii) $n+m-\alpha-\beta < 1$ and $f \in C\gamma([a,b])$, with $1-n-m+\alpha+\beta \leq \gamma \leq 1$.

2. Let $f \in {}_a\mathbf{D}_\beta([a,b])$ be, such that $D_a^\beta f \in {}_a\mathbf{I}_\alpha([a,b])$ and $I_a^\alpha D_a^\beta f \in \mathbf{L}(([a,b])$ is integrable. Then

$$I_a^\alpha D_a^\beta f(x) = \begin{cases} I_a^{\alpha-\beta} f(x) - \sum_{k=1}^{m} A_k \dfrac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} & \text{if } \alpha \geq \beta \\ D_a^{\beta-\alpha} f(x) - \sum_{k=1}^{m} A_k \dfrac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} & \text{if } \beta > \alpha \end{cases}$$

almost everywhere on $[a,b]$, with

$$A_k = \lim_{x \to a^+} D_a^{m-k} I_a^{m-\beta} f(x), \quad k = 1, 2, \ldots, m.$$

3. Let $f \in {}_a\mathbf{I}_\beta([a,b])$ be such that $I_a^\beta f \in {}_a\mathbf{D}_\alpha([a,b])$ and $f \in \mathbf{C}_\gamma([a,b])$ with $1-(n-\alpha-\beta) \leq \gamma \leq 1$, when $n-\alpha-\beta < 1$. Then

(i) If $\alpha \geq \beta$

$$D_a^\alpha I_a^\beta f = D_a^{\alpha-\beta} f, \quad \forall x \in [a,b].$$

(ii) If $\alpha < \beta$

$$D_a^\alpha I_a^\beta f = I_a^{\beta-\alpha} f, \quad \forall x \in [a,b].$$

If in (i) and (ii) we remove the condition $f \in \mathbf{C}_\gamma([a, b])$, then the results are true almost everywhere in $[a, b]$.

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