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The existence and stability of conic shock waves

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Abstract

We study the existence of the nonsymmetrical conic shock wave produced by a supersonic flow past a distorted conic projectile. For the weak conic shock wave, we establish the existence and its linear stability using the mathematical model of an isentropic irrotational flow.
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1. Introduction

The mathematical model for shock waves in hydrodynamics is the quasi-linear hyperbolic system of Euler equations:

$$\begin{cases} \partial_t \rho + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0, \\ \partial_t(\rho v_i) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3, \\ \partial_t(\rho E) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j E + p v_j) = 0. \end{cases} \tag{1.1}$$

In (1.1), $(\rho, \mathbf{v})$ are the density and the velocity of the gas particles, $E = e + \frac{1}{2}|\mathbf{v}|^2$ is the total energy, and the pressure $p = p(\rho, E)$ is a known function.

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The shock front can be described by certain jump discontinuity in the solution of the system (1.1), see [5,8,19]. It is a typical free boundary problem because the position of the shock front is unknown beforehand and should be determined as part of the solution.

An important problem in the shock wave study is the supersonic flow around solid projectiles. Physical observation shows that as the solid projectile travels through the air, a steady shock front will be produced in front of the solid object. If the projectile has a blunt head, the shock front will be detached from the projectile. And if the projectile has a narrow and sharp pointed head, the shock front will be attached to the head of the projectile.

Two simplest special cases of attached shock waves were discussed in the book *Supersonic Flow and Shock Waves* [5] by R. Courant and K.O. Friedrichs.

One case is the two-dimensional oblique shock front produced by a sharp plane wedge [5, IV.C]. More generalized such shock waves are further studied in [11,18] where the wedge sides are general curves, and a 3-dimensional version was studied in [3].

Another special case is the three-dimensional symmetrical conical shock produced by a sharp symmetrical regular cone [5, VI.B]. The discussion in [5] employed the simplified irrotational isentropic mathematical model where the system of Eqs. (1.1) can be reduced to a second order scalar equation.

For the irrotational flow, we have $\nabla \times \mathbf{v} = 0$, i.e.,

$$\partial_{x_i} v_j = \partial_{x_j} v_i, \quad i, j = 1, 2, 3.$$

Consequently in the conservation of momentum equations in (1.1), we have

$$v_j \partial_{x_j} v_i = v_j \partial_{x_i} v_j = \frac{1}{2} \partial_{x_i} v_j^2.$$

Therefore, for an isentropic irrotational compressible flow, the conservation of mass and momentum in the system (1.1) can be rewritten as:

$$\begin{cases} \partial_t \rho + \sum_{j=1}^3 \partial_{x_j} (\rho v_j) = 0, \\ \partial_t v_i + \frac{1}{2} \sum_{j=1}^3 \partial_{x_j} (|\mathbf{v}|^2 + \delta_{ij} h) = 0, \quad i = 1, 2, 3. \end{cases} \tag{1.2}$$

Here $h(\rho)$ is the specific enthalpy determined within a constant by the thermodynamic equation of state and the formula

$$h'(\rho) = \frac{a^2(\rho)}{\rho} > 0 \tag{1.3}$$

where $a(\rho)$ is the sound speed with $a^2(\rho) = dp/d\rho > 0$. For an ideal gas,

$$h = \frac{\gamma p}{(\gamma - 1)\rho}.$$

In the case of polytropic gas $p = A\rho^\gamma$,

$$h = \frac{A\gamma\rho^{\gamma-1}}{(\gamma - 1)} = \frac{a^2}{(\gamma - 1)}.$$

From the irrotationality assumption, we can introduce a velocity potential ϕ such that $\nabla\phi = \mathbf{v}$. Then the momentum equations in (1.2) can be integrated to obtain the Bernoulli's relation

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + h(\rho) = C, \tag{1.4}$$

where C is a constant. Since $h'(\rho) > 0$, the density ρ can be determined through the inverse function of the enthalpy by $\partial_t\phi$ and $\nabla\phi$:

$$\rho = h^{-1}\left(C - \partial_t\phi - \frac{1}{2}|\nabla\phi|^2\right) \equiv H\left(C - \partial_t\phi - \frac{1}{2}|\nabla\phi|^2\right). \tag{1.5}$$

Substituting (1.5) into the equation of conservation of mass in (1.2), we obtain a second order quasi-linear wave equation for the velocity potential ϕ

$$\partial_t H + \sum_{i=1}^{3} \partial_{x_i}(\phi_{x_i} H) = 0.$$

In particular for the steady flow considered in this paper, all the time derivatives vanish and we obtain a second order quasi-linear equation for the velocity potential ϕ

$$\sum_{i=1}^{3} \partial_{x_i}(\phi_{x_i} H) = 0. \tag{1.6}$$

From relations (1.3) and (1.5), it is easy to obtain that

$$H/H' = a^2. \tag{1.7}$$

Therefore, Eq. (1.6) can be further rewritten into

$$\begin{aligned}&\left(\frac{v_1^2}{a^2} - 1\right)\phi_{x_1x_1} + \left(\frac{v_2^2}{a^2} - 1\right)\phi_{x_2x_2} + \left(\frac{v_3^2}{a^2} - 1\right)\phi_{x_3x_3}\\ &\quad + \frac{2v_1v_2}{a^2}\phi_{x_1x_2} + \frac{2v_1v_3}{a^2}\phi_{x_1x_3} + \frac{2v_2v_3}{a^2}\phi_{x_2x_3} = 0\end{aligned} \tag{1.8}$$

with $v_i = \phi_{x_i}$ $(i = 1, 2, 3)$.

It is easy to see that for a supersonic flow ($|\mathbf{v}| > a$), Eq. (1.8) is normally hyperbolic, i.e., the symmetric matrix $A_v = (\alpha_{ij})$ consisting of the coefficients of the principal part of (1.8) has one positive and two negative eigenvalues, and for a subsonic flow ($|\mathbf{v}| < a$), Eq. (1.8) is elliptic.

The special case of three-dimensional symmetrical conical shock waves produced by a sharp symmetrical regular cone was studied in [5, VI.B] using the simplified irrotational isentropic mathematical model where the system of Eqs. (1.1) can be reduced to the second order scalar equation (1.8).

Let us look at a sharp solid symmetrical regular cone moving at a steady supersonic speed through a compressible isentropic gas, with the axis of the cone parallel to the velocity. The flow in front of the shock front is uniform. From the geometrical symmetry, Eq. (1.8) governing the flow behind the shock front becomes a nonlinear ordinary differential equation, and the conic shock wave can be formulated as a boundary value problem for this ordinary differential equation. It was shown in [5] that one solution to the boundary value problem corresponds the weak conic shock front observed in aerodynamical experiments.

However, a rigorous stability discussion of such conic waves using the irrotational isentropic mathematical model (1.8) has not been available and the problem of general non-symmetrical conic shock waves remains open.

In this paper, we study the existence of distorted non-symmetrical conic shock waves generated by a projectile which is a small perturbation to a sharp symmetrical regular cone, using the same irrotational isentropic model as in [5]. In particular, Theorem 4.1 on the linearized problem provides the linear stability of the symmetrical conic shock wave solution obtained in [5] for the irrotational isentropic conic shock wave model. The justification of using the model in the study can also be found in [12,17].

In particular, for a uniform flow in the direction of x_3-coordinate, we have $v_1 = v_2 = 0$ and $v_3 = u_0$ in (1.8). The corresponding velocity potential function is $\phi_0 = u_0 x_3$. And it is readily checked that (1.8) is hyperbolic for $u_0 > a$ and elliptic for $u_0 < a$.

Throughout this paper, we will assume the uniform supersonic incoming flow be parallel to the x_3-coordinate. Let the surface of the solid distorted cone be given by $x_3 = m(x_1, x_2)$, and the corresponding shock front be given by $x_3 = \mu(x_1, x_2)$. Obviously, we have $m(x_1, x_2) > \mu(x_1, x_2)$. For the non-symmetric conic shock problem, the boundary conditions accompanying (1.8) on the solid surface and on the shock front are:

- On the solid surface $x_3 = m(x_1, x_2)$, the velocity of fluid is tangent to the surface:
$$m_{x_1}\phi_{x_1} + m_{x_2}\phi_{x_2} - \phi_{x_3} = 0. \tag{1.9}$$
- On the shock front $x_3 = \mu(x_1, x_2)$:
  - ϕ is continuous across the shock front:
$$\phi = \phi_0 = u_0 x_3; \tag{1.10}$$
  - the Rankine–Hugoniot condition
$$(\mu_{x_1}\phi_{x_1} + \mu_{x_2}\phi_{x_2} - \phi_{x_3})H = -u_0\rho_0. \tag{1.11}$$

It was shown in [5, VI.B] that given the supersonic incoming flow, there is a critical value m_c such that a symmetric solid cone surface $x_3 = m_1\sqrt{x_1^2 + x_2^2}$ with $m_1 > m_c$ will produce an attached symmetrical conic weak shock. We will call this symmetrical conic shock in [5] throughout this paper the *background shock wave* and the corresponding solution of the velocity potential ϕ to the boundary value problem (1.8), (1.9), (1.10), (1.11) the *background solution* Φ. For the background conic shock wave, the corresponding shock front is then given by

$$x_3 = \mu_1\sqrt{x_1^2 + x_2^2}$$

with constant $\mu_1 < m_1$.

The main result of this paper in Section 2 states that if

$$m(x_1, x_2) - m_1\sqrt{x_1^2 + x_2^2}$$

is of the order $O((x_1^2 + x_2^2)^N)$ with sufficiently large N, i.e., if the distorted non-symmetrical solid cone is a small perturbation of a symmetrical regular cone, then there exists a corresponding non-symmetrical distorted conic shock wave solution near the tip of the cone $x_3 = 0$.

It is worth mentioning that the conical shock wave studied in this paper is different from two-shock waves resulted from the shock interaction or an initial jump. The latter has

been studied, within context of the Euler system of equations in [2,10,15], and within the context of a second order wave equation in [16]. From the viewpoint of physics, the flow in the domain behind the shock front under study in this paper is supersonic, while it is subsonic in [2,10,15,16]. Geometrically, the conic domain has a higher order of singularity in contrast with the domain in [2,10,15,16] which is isomorphic to a quarter space.

The special difficulties in the study of conic shock waves are the following. First of all, the high order singularity at the vertex of the conic shock wave implies a higher order singularity in the corresponding differential equations. Secondly, the free boundary of the shock front intersects with the fixed boundary of solid surface at the vertex of the conic projectile. The standard partial hodograph transformation which changes the free boundary of shock front into a fixed one in [13] would also at the same time change the fixed solid surface into a new free boundary. In this paper we use a generalized nonlinear partial hodograph transformation to overcome this dilemma. This generalized hodograph transform is a composition of the usual partial hodograph transformation and a nonlinear transformation of the unknown function. It will change the shock front and the solid surface into fixed boundaries at the same time.

This paper is arranged as follows.

In Section 2, we introduce a singular conical coordinates system to formulate the problem and state the main theorem of the paper. A generalized nonlinear partial hodograph transformation is employed in Section 3 to transform the free boundary problem into an equivalent boundary value problem with fixed boundaries. The main theorem is then restated in the new coordinates. For this new boundary value problem with fixed boundaries, we derive in Section 4 the energy estimate for the linearized problem which also provides the linearized stability of the conic shock wave for the isentropic irrotational flow model in [5]. Finally, a linear iteration process is used to prove the existence of distorted conic shock wave solution.

2. Conical coordinates, main theorem

Because of the conical geometrical feature of the problem, it is natural for us to introduce the conical coordinates (z, y, θ):

$$x_1 = yz\cos\theta, \qquad x_2 = yz\sin\theta, \qquad x_3 = z, \tag{2.1}$$

where y is the radial direction, θ is the angular direction in the cross section perpendicular to the cone axis, and z is the axial direction of the cone which coincides with the direction of the uniform incoming supersonic flow.

In the conical coordinates (z, y, θ), the symmetrical background conic shock wave solution has $y = b_1$ as its symmetrical conic solid surface and $y = s_1$ as its position for the shock front. And for the distorted non-symmetrical conical shock wave, the distorted solid surface is given by $y = b(z, \theta)$ and shock front is defined by $y = s(z, \theta)$.

We require that the distorted solid surface is sufficiently close to the symmetric cone, i.e.,

$$b(z, \theta) - b_1 = \mathrm{O}\big(z^N\big), \tag{2.2}$$

with N being a sufficiently large number. The main question of the non-symmetric conic shock wave problem is to determine the location of the distorted shock front $y = s(z,\theta)$ together with the velocity potential $\phi(y,z,\theta)$ in the domain $b(z,\theta) < y < s(z,\theta)$, $z > 0$. From the assumption (2.2), we would expect that the background conic shock front and the non-symmetric distorted conic shock front also satisfy a similar relation

$$s(z,\theta) - s_1 = \mathrm{O}(z^N) \quad \text{near } z = 0. \tag{2.3}$$

Let's denote v_y the velocity component in the (x_1,x_2)-plane in the radial direction, i.e., the y-direction in polar coordinates (y,θ), and v_θ the velocity component in the (x_1,x_2)-plane perpendicular to the radial direction, i.e., the θ-direction in polar coordinates (y,θ). We have the following relations for v_y and v_θ:

$$\begin{aligned} v_y &= v_1\cos\theta + v_2\sin\theta = \frac{\phi_y}{z}, \\ v_\theta &= v_2\cos\theta - v_1\sin\theta = \frac{\phi_\theta}{zy}. \end{aligned} \tag{2.4}$$

At the background solution Φ, we obviously have $v_\theta = 0$.

Let β be the angle such that

$$\tan\beta = \sqrt{x_1^2 + x_2^2}/x_3 = y.$$

Let v_t denote the velocity component of the flow tangential to the surface $\beta =$ constant and pointing to the positive direction of x_3-coordinate, and v_n the velocity component of the flow in the normal direction to the surface $\beta =$ constant, pointing to the upstream of the shock front (i.e., the negative direction of x_3-coordinate). Obviously, we have

$$\begin{aligned} v_t &= v_3\cos\beta + v_y\sin\beta, \\ v_n &= v_y\cos\beta - v_3\sin\beta. \end{aligned} \tag{2.5}$$

It is readily checked that we have the following relations

$$\begin{cases} v_1 = \phi_{x_1} = \dfrac{\cos\theta}{z}\phi_y - \dfrac{\sin\theta}{zy}\phi_\theta, \\ v_2 = \phi_{x_2} = \dfrac{\sin\theta}{z}\phi_y + \dfrac{\cos\theta}{zy}\phi_\theta, \\ v_3 = \phi_{x_3} = \phi_z - \dfrac{y}{z}\phi_y. \end{cases} \tag{2.6}$$

From the boundedness of the flow velocity vector (v_1,v_2,v_3) and the relation (2.4) we conclude that the solution ϕ and ϕ_z, ϕ_y/z, ϕ_θ/z are all bounded near $z = 0$.

For the second order derivatives of ϕ, we have

$$\begin{cases} \phi_{x_1x_1} = \dfrac{\cos^2\theta}{z^2}\phi_{yy} + \dfrac{\sin^2\theta}{(zy)^2}\phi_{\theta\theta} - \dfrac{\sin 2\theta}{z^2 y}\phi_{y\theta} + \dfrac{\sin^2\theta}{z^2 y}\phi_y + \dfrac{\sin 2\theta}{(zy)^2}\phi_\theta, \\ \phi_{x_2x_2} = \dfrac{\sin^2\theta}{z^2}\phi_{yy} + \dfrac{\cos^2\theta}{(zy)^2}\phi_{\theta\theta} + \dfrac{\sin 2\theta}{z^2 y}\phi_{y\theta} + \dfrac{\cos^2\theta}{z^2 y}\phi_y - \dfrac{\sin 2\theta}{(zy)^2}\phi_\theta, \\ \phi_{x_3x_3} = \phi_{zz} + \dfrac{y^2}{z^2}\phi_{yy} - \dfrac{2y}{z}\phi_{zy} + \dfrac{2y}{z^2}\phi_y, \\ \phi_{x_1x_2} = \dfrac{\sin 2\theta}{2z^2}\phi_{yy} - \dfrac{\sin 2\theta}{2(zy)^2}\phi_{\theta\theta} + \dfrac{\cos 2\theta}{z^2 y}\phi_{y\theta} - \dfrac{\sin 2\theta}{2z^2 y}\phi_y - \dfrac{\cos 2\theta}{(zy)^2}\phi_\theta, \\ \phi_{x_1x_3} = \dfrac{\cos\theta}{z}\phi_{zy} - \dfrac{y\cos\theta}{z^2}\phi_{yy} - \dfrac{\sin\theta}{zy}\phi_{z\theta} + \dfrac{\sin\theta}{z^2}\phi_{y\theta} - \dfrac{\cos\theta}{z^2}\phi_y, \\ \phi_{x_2x_3} = \dfrac{\sin\theta}{z}\phi_{zy} - \dfrac{y\sin\theta}{z^2}\phi_{yy} + \dfrac{\cos\theta}{zy}\phi_{z\theta} - \dfrac{\cos\theta}{z^2}\phi_{y\theta} - \dfrac{\sin\theta}{z^2}\phi_y. \end{cases} \tag{2.7}$$

Rewriting Eq. (1.8) in the new conic coordinates (z, y, θ) and then multiplying the two sides of the equation by z^2, we obtain

$$\begin{aligned} &z^2 a_{00}\phi_{zz} + a_{11}\phi_{\theta\theta} + a_{22}\phi_{yy} + 2z a_{01}\phi_{z\theta} + 2z a_{02}\phi_{zy} + 2a_{12}\phi_{y\theta} \\ &\quad + z a_0\phi_z + a_1\phi_\theta + a_2\phi_y = 0, \quad \text{in } b(z,\theta) < y < s(z,\theta). \end{aligned} \tag{2.8}$$

The coefficients a_{ij} and a_i in (2.8) are all bounded functions of ϕ_z, ϕ_θ/z, ϕ_y/z. At the background solution Φ, the coefficients a_{ij} can be written explicitly as:

$$\begin{cases} a_{00} = \left(\dfrac{v_3}{a}\right)^2 - 1, \\ a_{11} = \dfrac{1}{y^2}\left(\dfrac{v_\theta^2}{a^2} - 1\right) = -\dfrac{1}{y^2}, \\ a_{22} = \dfrac{(v_r - y v_3)^2}{a^2} - \left(1 + y^2\right) = \dfrac{1}{\cos^2\beta}\left(\dfrac{v_n^2}{a^2} - 1\right), \\ a_{01} = \dfrac{v_3 v_\theta}{a^2 y} = 0, \\ a_{02} = \dfrac{v_3 v_r}{a^2} - y\left(\dfrac{v_3^2}{a^2} - 1\right) = \dfrac{1}{\cos\beta}\left(\dfrac{v_3 v_n}{a^2} + \sin\beta\right), \\ a_{12} = \dfrac{v_r v_\theta}{a^2 y} - \dfrac{v_\theta v_3}{a^2} = 0. \end{cases} \tag{2.9}$$

As in [5] and [4], we will always assume in this paper that the background shock is a weak shock and the normal component of the velocity v_n behind the shock front is close to the one ahead of the shock, i.e.,

$$v_n/\sin\beta \sim u_{0n}/\sin\beta = u_0. \tag{2.10}$$

It is readily checked that $a_{00} > 0$, $a_{11} < 0$ and $a_{22} < 0$ in (2.8). Eq. (2.8) is normally hyperbolic in $z > 0$ and having z as time-like direction. Eq. (2.8), however, is degenerate at $z = 0$.

Next let's examine the boundary conditions in the conic coordinates (z, y, θ). On the solid surface $y = b(z, \theta)$ we have

$$(b + zb_z)\phi_z + \frac{b_\theta}{y^2}\left(\frac{\phi_\theta}{z}\right) - \big(1 + y(b + zb_z)\big)\left(\frac{\phi_y}{z}\right) = 0 \quad \text{on } y = b(z, \theta). \tag{2.11}$$

On the shock front $y = s(z, \theta)$, we have the velocity potential $\phi(y, z, \theta) = u_0 z$ with u_0 being the constant supersonic velocity in front of the shock wave. Since the directional derivative of ϕ normal to the shock front is never zero behind the shock front, we can use $\phi(y, z, \theta) = u_0 z$ to describe the shock front. Then the Rankine–Hugoniot condition (1.11) can be written as

$$\big(\phi_{x_1}^2 + \phi_{x_2}^2 + \phi_{x_3}(\phi_{x_3} - u_0)\big)H = (\phi_{x_3} - u_0)u_0\rho_0. \tag{2.12}$$

In the (z, y, θ) coordinates Eq. (2.12) becomes

$$\begin{aligned}&\left[\left(\frac{\phi_y}{z}\right)^2 + \frac{1}{y^2}\left(\frac{\phi_\theta}{z}\right)^2 + \left(\phi_z - \frac{y\phi_y}{z} - u_0\right)^2\right]H \\ &\quad + \left(\phi_z - \frac{y\phi_y}{z} - u_0\right)u_0(H - \rho_0) = 0 \quad \text{on } y = s(z, \theta).\end{aligned} \tag{2.13}$$

In addition, it is easy to verify that in the new conic coordinates the absolute value of the flow velocity has the expression:

$$|\nabla\phi|^2 = \left|\left(\frac{1}{z}\cos\theta\phi_y - \frac{\sin\theta}{zy}\phi_\theta, \sin\theta\frac{\phi_y}{z} + \frac{\cos\theta}{zy}\phi_\theta, \phi_z - \frac{y}{z}\phi_y\right)\right|^2. \tag{2.14}$$

Eq. (2.8) for the velocity potential $\phi(y, z, \theta)$, together with the boundary value conditions on the solid cone-like surface (2.11) and on the shock front (2.13), constitute the mathematical formulation of the non-symmetrical conic shock waves. The following is the main theorem on the existence of conic shock waves of this paper.

Theorem 2.1. *Consider the free boundary value problem consisting of Eq.* (2.8) *in the domain* $b(z, \theta) < y < s(z, \theta)$, $0 < z$, $0 \leqslant \theta \leqslant 2\pi$, *and the boundary condition* (2.11) *on the fixed boundary* $y = b(z, \theta)$, *and the boundary conditions* (2.13) *and* $\phi(y, z, \theta) = u_0 z$ *on the unknown boundary* $y = s(z, \theta)$.

Assume that

- *the solid symmetric regular cone* $y = b_1$ *satisfy* $b_1 < m_c^{-1}$ *as in* [5, VI.B] *such that a background weak conic shock wave exists*;
- *the non-symmetrical solid conic surface* $y = b(z, \theta)$ *be a sufficiently smooth function and* (2.2) *be satisfied for sufficiently large* N.

Then the free boundary value problem (2.8), (2.11), (2.13) *has a unique classical solution* (ϕ, s) *in a neighborhood of* $z = 0$ *which corresponds to a weak perturbed conic shock wave.*

Remark 2.1. In Theorem 2.1, the condition (2.2) for large N means that the non-symmetrically distorted conic solid projectile is tangential up to the high order N to the

symmetric regular cone at the tip of the cone $z = 0$, i.e., the perturbation is sufficiently small. We will use this fact in the proof of the existence of the distorted conic shock wave in the process of linear iteration in Section 4. The background symmetric conic shock wave solution will be taken as the first approximation in the linear iteration.

Since ϕ is of the order $O(z)$ near $z = 0$, we can replace the unknown function ϕ by a new function ψ such that $\phi(z, y, \theta) = z\psi(z, y, \theta)$. To further simplify the notation and formally remove the degeneracy at $z = 0$, we set

$$t = \ln z, \quad z = e^t. \tag{2.15}$$

The transform (2.15) moves the point $z = 0$ to $t = -\infty$ and formally removes singularity at $z = 0$ in (2.8), (2.11) and (2.13). In addition, the transform (2.15) will also allow us in Section 4 to use only the usual hyperbolic η-weighted norm instead of a combination of η-weighted and z^k-weighted norms used in [1] and [15].

In the new (t, y, θ) coordinates and for the new function ψ, Eqs. (2.8), (2.11) and (2.13) become

$$\begin{aligned} &a_{00}(\psi + 2\psi_t + \psi_{tt}) + a_{11}\psi_{yy} + a_{22}\psi_{\theta\theta} + 2a_{01}\psi_{ty} + 2a_{02}\psi_{t\theta} + 2a_{12}\psi_{y\theta} \\ &\quad + (a_0 - a_{00})(\psi + \psi_t) + (a_1 + 2a_{01})\psi_y + (a_2 + 2a_{02})\psi_\theta \\ &\quad = 0 \quad \text{in } b(t, \theta) < y < s(t, \theta). \end{aligned} \tag{2.16}$$

$$(b + b_t)(\psi + \psi_t) + \frac{b_\theta}{y^2}\psi_\theta - \big(1 + y(b + b_t)\big)\psi_y = 0 \quad \text{on } y = b(t, \theta). \tag{2.17}$$

$$\begin{aligned} G(\psi_t, \psi_\theta, \psi_y, \psi) &\equiv \big[\psi_y^2 + y^{-2}\psi_\theta^2 + (\psi_t - y\psi_y)^2\big]H + (\psi_t - y\psi_y)u_0(H - \rho_0) \\ &= 0 \quad \text{on } y = s(t, \theta). \end{aligned} \tag{2.18}$$

In Eq. (2.18), we have used the fact that the shock front $y = s(t, \theta)$ can be implicitly defined by $\psi = u_0$. In contrast to Eq. (2.8), the coefficients in Eq. (2.16) are all smooth functions of their arguments and have no singularity at $t = -\infty$.

Therefore the proof of Theorem 2.1 is reduced to the proof of the existence of the solution ψ for (2.16)–(2.18) near $t = -\infty$.

Besides, the absolute value of the flow velocity $|\nabla\phi|$ in the new coordinates (t, y, θ) can be expressed by the new function ψ:

$$|\nabla\phi|^2 = \left|\left(\cos\theta\psi_y - \frac{\sin\theta}{y}\psi_\theta, \sin\theta\psi_y + \frac{\cos\theta}{y}\psi_\theta, \psi + \psi_t - y\psi_y\right)\right|^2. \tag{2.19}$$

3. Generalized nonlinear hodograph transformation

In the boundary value problem (2.16)–(2.18), the boundary $y = b(t, \theta)$ is the fixed boundary with the given function $b(t, \theta)$. However, the shock front boundary $y = s(t, \theta)$ is unknown and should be determined along with the unknown function ψ. To study such free boundary problems, the typical approach is to perform a transformation to fix the unknown free boundary. Naturally the transform would involve the function describing the

free boundary. The resulting problem would be a boundary value problem in a domain with fixed boundaries.

One useful method to fix a free boundary is the partial hodograph transformation. The partial hodograph transform takes the unknown function $\psi(t, y, \theta)$ as one of the new coordinates, and has been successfully used in the study of the Stefan problems in heat transfer [9,14] and the shock wave problems in [13]. In order that the transformed new problem be equivalent to the original one, it is crucial that the vector $\nabla\psi$ is not perpendicular to the normal direction of the free boundary. In the symmetric conic wave problem, it is easily derived that $\forall y \in [b_1, s_1]$, we have $\psi_y > 0$ for the background solution, see [5, VI.B]. Hence the partial hodograph transformation should be valid throughout the domain between distorted solid cone $y = b(t, \theta)$ and the distorted conic shock front $y = s(t, \theta)$.

However in the problem (2.16)–(2.18), while such a transformation would fix the free boundary $y = s(t, \theta)$, it will at the same time also change the fixed solid boundary $y = b(t, \theta)$ into a new unknown free boundary.

To overcome this difficulty, we introduce the generalized nonlinear hodograph transformation which will fix the free boundary $y = s(t, \theta)$ while at the same time change the fixed boundary $y = b(t, \theta)$ into a new fixed boundary. This generalized hodograph transformation has the form

$$y_0 = t, \qquad y_1 = \theta, \qquad y_2 = \frac{y - b}{y - b + u_0 - \psi}. \tag{3.1}$$

The transformation (3.1) can be decomposed into two parts. The first step is to introduce a new unknown function $\overline{\psi}$, i.e., we perform a nonlinear transform of unknown functions:

$$\overline{\psi}(t, \theta, y) = \frac{y - b}{y - b + u_0 - \psi}. \tag{3.2}$$

Then we take the unknown function $\overline{\psi}$ as an independent variable in the new coordinates, i.e., we perform a standard partial hodograph transformation:

$$y_0 = t, \qquad y_1 = \theta, \qquad y_2 = \overline{\psi}(t, \theta, y). \tag{3.3}$$

Then the transformation (3.1) is the composition of the standard partial hodograph transformation (3.3) and the nonlinear transform of unknown functions (3.2). Under the transformation (3.1), it is obvious that the fixed solid boundary $y = b(t, \theta)$ becomes a new fixed boundary $y_2 = 0$ and the free shock front boundary $y = s(t, \theta)$ becomes a fixed boundary $y_2 = 1$.

In order that the transformation (3.1) would change the problem (2.16)–(2.18) into an equivalent problem, we must show that (3.1) is uniformly nonsingular and invertible in the domain $b(t, \theta) \leqslant y \leqslant s(t, \theta)$. For this we have the following

Lemma 3.1. *For the nonlinear generalized hodograph transformation* (3.1)*, there is a number* $\varepsilon_0 > 0$ *such that in the neighborhood of the background shock wave solution, we have*

$$y - b + u_0 - \psi \geqslant \varepsilon_0 > 0 \quad \textit{and} \quad \partial_y y_2 \geqslant \varepsilon_0 > 0 \quad \textit{in } b(t, \theta) \leqslant y \leqslant s(t, \theta). \tag{3.4}$$

Hence the transformation (3.1) *is uniformly nonsingular and invertible in the domain* $b(t,\theta) \leqslant y \leqslant s(t,\theta)$.

Proof of Lemma 3.1. Since we consider the domain $b \leqslant y \leqslant s$, we always have $y - b \geqslant 0$. From (2.4) and $\phi = z\psi$, we have $\psi_y = v_y$ which is the flow velocity in the radial direction. For the background solution, we have $v_y > 0$ in the domain $b < y < s$, hence $\psi_y > 0$. On the other hand, $\psi = u_0$ at $y = s(t,\theta)$. Therefore we conclude

$$y - b + u_0 - \psi \geqslant 0 \quad \text{in } b(t,\theta) \leqslant y \leqslant s(t,\theta). \tag{3.5}$$

To show that $y - b + u_0 - \psi$ is uniformly bounded away from zero in the domain, we can consider the domain in two separate regions. In the region outside a neighborhood of the free boundary $y = s(t,\theta)$, we have $u_0 - \psi \geqslant \varepsilon_0 > 0$ because $\psi = u_0$ at $y = s(t,\theta)$ and $\psi_y > 0$. In the region outside a neighborhood of the solid cone surface $y = b(t,\theta)$, we obviously have $y - b \geqslant \varepsilon_0 > 0$. Therefore, we have

$$y - b + u_0 - \psi \geqslant \varepsilon_0 > 0 \quad \text{in } b(t,\theta) \leqslant y \leqslant s(t,\theta). \tag{3.6}$$

On the other hand, we have

$$\partial_y y_2 = \frac{\psi_y(y-b) + (u_0 - \psi)}{[(y-b) + (u_0 - \psi)]^2}. \tag{3.7}$$

Using the same argument, we have

$$\partial_y y_2 \geqslant \varepsilon_0 > 0 \quad \text{in } b(t,\theta) \leqslant y \leqslant s(t,\theta). \tag{3.8}$$

Combining (3.6) and (3.8), we obtain (3.4) and this completes the proof of Lemma 3.1. □

By Lemma 3.1, the transformation (3.1) is uniformly nonsingular and invertible, and hence the free boundary problem (2.16)–(2.18) will be changed into an equivalent boundary value problem with fixed boundaries $y_2 = 0$ and $y_2 = 1$ under the transformation (3.1).

Let us denote the inverse transformation to (3.1) by

$$t = y_0, \qquad \theta = y_1, \qquad y = u(y_0, y_1, y_2). \tag{3.9}$$

Then we have the identity

$$y = u\big(y_0, y_1, y_2(t,\theta,y)\big) = u\big(t,\theta,y_2(t,\theta,y)\big).$$

The function $u(y_0, y_1, y_2)$ can be treated as the new unknown function in the new coordinates (y_0, y_1, y_2), and the boundary value problem (2.16)–(2.18) for ψ will be transformed into an equivalent problem for the unknown function $u(y_0, y_1, y_2)$. Once we find the function $u(y_0, y_1, y_2)$, then by the invertibility of the transformation, we can determine uniquely the original unknown function $\psi(t, y, \theta)$ from (3.1).

However, if we simply take $u(y_0, y_1, y_2)$ as the new unknown function, the equation for $u(y_0, y_1, y_2)$ obtained from (2.16) will have a singularity at $y_2 = 0$. This formal singularity comes from the special form of the transformation (3.1). Let us examine the nature of this singularity.

By (3.2), we have

$$\frac{\partial\psi}{\partial\overline{\psi}} = \frac{(y-b+u_0-\psi)^2}{y-b} = \frac{y-b}{\overline{\psi}^2}. \tag{3.10}$$

Since $d\psi = (\partial\psi/\partial\overline{\psi})d\overline{\psi}$, the principal part of the equation for $\overline{\psi}$ obtained from (2.16) will have the form

$$\left(\frac{y-b}{\overline{\psi}^2}\right)\left[a_{00}\overline{\psi}_{tt} + a_{11}\overline{\psi}_{yy} + a_{22}\overline{\psi}_{\theta\theta} + 2a_{01}\overline{\psi}_{ty} + 2a_{02}\overline{\psi}_{t\theta} + 2a_{12}\overline{\psi}_{y\theta}\right]. \tag{3.11}$$

The coefficients of the principal part in (3.11) obviously have singularity $(y-b)^{-1}$ because $\overline{\psi} = \mathrm{O}(y-b)$. But for the first order derivative terms of $\overline{\psi}$, some coefficients may have the singularity $(y-b)^{-2}$. This singularity comes from the terms such as

$$\psi_{ty} = \partial_y\left[\left(\frac{y-b}{\overline{\psi}^2}\right)\overline{\psi}_t\right] = \left(\frac{y-b}{\overline{\psi}^2}\right)\overline{\psi}_{ty} + \partial_y\left(\frac{y-b}{\overline{\psi}^2}\right)\overline{\psi}_t.$$

These singularities can not be canceled out by multiplying on both sides of the equation a factor $(y-b)$.

Since $y-b = u-b = \mathrm{O}(\overline{\psi}) = \mathrm{O}(y_2)$ near $y_2 = 0$, the equation for the unknown function $u(y_0, y_1, y_2)$ obtained under the partial hodograph transformation (3.3) will have a singularity at $y_2 = 0$.

To remove this formal singularity, we introduce a modified new unknown function $\omega(y_0, y_1, y_2)$ to replace the function $u(y_0, y_1, y_2)$. The function $\omega(y_0, y_1, y_2)$ is defined as follows:

$$\omega(y_0, y_1, y_2) = \frac{u(y_0, y_1, y_2) - b(y_0, y_1)}{y_2}.$$

The definition of the function $\omega(y_0, y_1, y_2)$ has taken into account the fact that $u-b = \mathrm{O}(y_2)$ near $y_2 = 0$ and hence removed the formal singularity in the coefficients. We will see later in (3.18) and (3.19) that the new equation for $\omega(y_0, y_1, y_2)$ has no singularity. We denote this modified new transformation of (3.9) as

$$t = y_0, \qquad \theta = y_1, \qquad \omega(y_0, y_1, y_2) = \frac{u(y_0, y_1, y_2) - b(y_0, y_1)}{y_2}. \tag{3.12}$$

We now begin to derive the boundary value problem satisfied by the unknown function $\omega(y_0, y_1, y_2)$ in the new coordinates (y_0, y_1, y_2) from the boundary value problem (2.16)–(2.18). For simplicity of notations, we denote briefly

$$\xi(t, \theta, y) = y - b + u_0 - \psi. \tag{3.13}$$

By (3.6) we have $\xi > 0$. Obviously the boundary value problem (2.16)–(2.18) for ψ is transformed under (3.13) into an equivalent boundary value problem for ξ with the same principal part, except for a negative sign.

Using the notation ξ, the generalized nonlinear hodograph transformation (3.1) can be written as

$$y_0 = t, \qquad y_1 = \theta, \qquad y_2 = \frac{y-b}{\xi}. \tag{3.14}$$

In particular, we have the following identity

$$y_2 = \frac{u-b}{\xi(y_0, y_1, u)}.$$

Combining (3.9), (3.12), (3.13) and (3.14) we obtain

$$\omega(y_0, y_1, y_2) = \xi\big(y_0, y_1, u(y_0, y_1, y_2)\big). \tag{3.15}$$

From (3.14) we compute the following differential relations

$$\begin{cases} dy_0 = dt, \qquad dy_1 = d\theta, \\ dy_2 = \dfrac{-\xi b_t - \xi_t(y-b)}{\xi^2}\,dt + \dfrac{-\xi b_\theta - \xi_\theta(y-b)}{\xi^2}\,d\theta + \dfrac{\xi - \xi_y(y-b)}{\xi^2}\,dy. \end{cases} \tag{3.16}$$

To solve (3.16) with respect of dy, we obtain

$$dy = \frac{\xi b_t + \xi_t(y-b)}{\xi - \xi_y(y-b)}\,dy_0 + \frac{\xi b_\theta + \xi_\theta(y-b)}{\xi - \xi_y(y-b)}\,dy_1 + \frac{\xi^2}{\xi - \xi_y(y-b)}\,dy_2. \tag{3.17}$$

Therefore we obtain the following relation between $(\omega_{y_0}, \omega_{y_1}, \omega_{y_2})$ and $(\xi_t, \xi_\theta, \xi_y)$ which decides the relation between the principal parts of Eqs. (2.16) and the new equation for $\omega(y_0, y_1, y_2)$:

$$\begin{pmatrix} \omega_{y_0} \\ \omega_{y_1} \\ \omega_{y_2} \end{pmatrix} = P(\xi_t, \xi_\theta, \xi_y) = \begin{pmatrix} \xi_t + \dfrac{\partial y}{\partial y_0}\xi_y \\ \xi_\theta + \dfrac{\partial y}{\partial y_1}\xi_y \\ \dfrac{\partial y}{\partial y_2}\xi_y \end{pmatrix} = \begin{pmatrix} \xi_t + \dfrac{\xi b_t + \xi_t(y-b)}{\xi - \xi_y(y-b)}\xi_y \\ \xi_\theta + \dfrac{\xi b_\theta + \xi_\theta(y-b)}{\xi - \xi_y(y-b)}\xi_y \\ \dfrac{\xi^2}{\xi - \xi_y(y-b)}\xi_y \end{pmatrix}. \tag{3.18}$$

The Jacobi matrix $D\omega'/D\xi'$ of the transformation (3.18) is:

$$\begin{aligned} J &\equiv \frac{\partial(\omega_{y_0}, \omega_{y_1}, \omega_{y_2})}{\partial(\xi_t, \xi_\theta, \xi_y)} \\ &= \begin{pmatrix} \dfrac{\xi}{\xi - \xi_y(y-b)} & 0 & \dfrac{[\xi b_t + \xi_t(y-b)]\xi}{[\xi - \xi_y(y-b)]^2} \\ 0 & \dfrac{\xi}{\xi - \xi_y(y-b)} & \dfrac{[\xi b_\theta + \xi_\theta(y-b)]\xi}{[\xi - \xi_y(y-b)]^2} \\ 0 & 0 & \dfrac{\xi^3}{[\xi - \xi_y(y-b)]^2} \end{pmatrix}. \end{aligned} \tag{3.19}$$

From (3.6) and (3.13) we have

$$\begin{aligned} \xi - \xi_y(y-b) &= (y-b) + (u_0 - \psi) - (1-\psi_y)(y-b) \\ &= (u_0 - \psi) + \psi_y(y-b) > 0. \end{aligned} \tag{3.20}$$

This implies that the matrix (3.19) is nonsingular everywhere in the region $b(t,\theta) < y < s(t,\theta)$. Consequently the transformation (3.18) is invertible in the considered region. Therefore the free boundary problem (2.16)–(2.18) for the unknown function ψ is

transformed under the transformation (3.12) into an equivalent problem for the new unknown function $\omega(y_0, y_1, y_2)$. In particular, the new problem is defined in the domain $0 < y_2 < 1$ with fixed boundaries. We denote this new second-order quasi-linear partial differential equation for the unknown function $\omega(y_0, y_1, y_2)$ in the fixed domain $0 < y_2 < 1$ as:

$$\sum_{i,j=0,1,2} \alpha_{ij}(\omega, \nabla\omega)\omega_{y_i y_j} + I(\omega, \nabla\omega) = 0$$
$$\text{in } -\infty < y_0 < -T,\ 0 \leqslant y_1 < 2\pi,\ 0 < y_2 < 1. \tag{3.21}$$

By the relations (3.18) and (3.19), it is readily checked that the coefficient matrix (α_{ij}) of the principal part in Eq. (3.21) is congruent with the matrix (a_{ij}) in Eq. (2.16) by the following relation:

$$(a_{ij}) = J^t(\alpha_{ij})J, \tag{3.22}$$

where the matrix J is the non-singular matrix in (3.19). The coefficients matrix (α_{ij}) in Eq. (3.21) has all its elements being smooth real valued functions of their arguments. The lower order terms $I(\omega, \nabla\omega)$ in (3.21) are also all smooth functions of their arguments, the explicit form of which is of no consequence in the following discussion.

For Eq. (3.21) we make the following observations.

Remark 3.1. The transformation (3.1) is a composition of a nonlinear transform of unknown functions (3.2) and a classical partial hodograph transformation (3.3).

An unknown function transformation which does not change the coordinates obviously does not alter the hyperbolicity and the time-like or space-like directions. It has also been shown in [13, Proposition 2.2] that partial hodograph transformation does not change the hyperbolicity and the time-like or space-like directions of a second order partial differential equation. Finally, the introduction of the new unknown function $\omega(y_0, y_1, y_2)$ in (3.12) obviously does not change the hyperbolicity and the time-like or space-like directions.

Consequently, we conclude that the new equation (3.21) is also normally hyperbolic with y_0 being a time-like direction and y_2 a space-like direction.

Next we consider the boundary conditions which should be satisfied by the new unknown function $\omega(y_0, y_1, y_2)$ in the coordinates (y_0, y_1, y_2). These conditions are derived from the conditions in (2.17) and (2.18).

First we look at boundary condition (2.17) on the solid conic surface $y = b(t, \theta)$. Under the transformation (3.1), the solid surface boundary $y = b(t, \theta)$ becomes the new boundary $y_2 = 0$. The boundary condition (2.17) for ψ on the boundary $y = b(t, \theta)$ can be rewritten first as a condition for the function u by (3.9), and then be further written as a condition for the new unknown function ω by (3.12).

On the boundary $y_2 = 0$, we have $u(y_0, y_1, 0) = b(t, \theta)$. Since $b(t, \theta)$ is a known function, the values of u, $\partial_{y_0}u$ and $\partial_{y_1}u$ on the boundary $y_2 = 0$ are therefore all known functions. The boundary condition (2.17) is a relation for the function ψ and its first order derivatives on $y = b(t, \theta)$. Under the transformation (3.1), it becomes a relation for the function $u(y_0, y_1, 0)$ and its first order derivatives on $y_2 = 0$. Since $(u, \partial_{y_0}u, \partial_{y_1}u)$ are all known functions on $y_2 = 0$, consequently the boundary condition derived from (2.17) for

the variable u is nothing but a condition on the normal derivative for $u(y_0, y_1, 0)$ on $y_2 = 0$. We denote this condition for $\partial_{y_2} u$ simply as

$$\partial_{y_2} u = g_1 \quad \text{on } y_2 = 0, \tag{3.23}$$

where g_1 is a smooth known function on $y_2 = 0$. According to (3.12), the value of $\partial_{y_2} u$ on $y_2 = 0$ is simply the value of the new unknown function ω on the boundary $y_2 = 0$. Consequently, the boundary condition (2.17) under the transformation (3.1) becomes the following equivalent new boundary condition for $\omega(y_0, y_1, y_2)$ on $y_2 = 0$:

$$\omega(y_0, y_1, 0) = g_1 \quad \text{on } y_2 = 0. \tag{3.24}$$

Without loss of generality we can further assume $g_1 = 0$, which can be achieved by performing a simple transformation.

The Rankine–Hugoniot condition (2.18) on the free boundary $y = s(t, \theta)$ becomes in the new coordinates (y_0, y_1, y_2) a new nonlinear relation for $\nabla\psi$ on the fixed boundary $y_2 = 1$, on which the value of function ψ is a given constant u_0. By (3.13) the values of $(\psi_t, \psi_\theta, \psi_y)$ can be expressed by $(\xi_t, \xi_\theta, \xi_y)$. Then the condition (2.18) for the function ψ can be expressed as a relation for the unknown function $\nabla\xi$. We denote this relation as

$$G(\psi_t, \psi_\theta, \psi_y, u_0) \equiv \Gamma(\xi_t, \xi_\theta, \xi_y) = 0. \tag{3.25}$$

Therefore, in the new coordinates (y_0, y_1, y_2) the boundary condition (2.18) for ψ becomes an equivalent nonlinear relation for the new unknown function $\omega(y_0, y_1, y_2)$ on the fixed boundary $y_2 = 1$:

$$\Gamma\big(P^{-1}(\omega_{y_0}, \omega_{y_1}, \omega_{y_2})\big) = 0 \quad \text{on } y_2 = 1. \tag{3.26}$$

The transform P^{-1} in (3.26) is the inverse of the transform P in (3.18) which is nonsingular near the background solution.

Because of the equivalence of the free boundary value problem (2.16)–(2.18) and the new fixed boundary value problem (3.21), (3.24) and (3.26), the main Theorem 2.1 can be restated for the new boundary value problem (3.21), (3.24) and (3.26) as

Theorem 3.1. *Consider the boundary value problem* (3.21), (3.24) *and* (3.26) *for the unknown function* $\omega(y_0, y_1, y_2)$ *in the domain with fixed boundaries*: $(y_0, y_1, y_2) \in (-\infty, -T) \times [0, 2\pi] \times (0, 1)$.

We assume, similarly as in Theorem 2.1, *that*

- *the solid symmetrical regular cone* $y = b_1$ *satisfy* $b_1 < m_c^{-1}$ *as in* [5, VI.B] *such that a background weak conic shock wave exists*;
- *The non-symmetrical solid conic surface* $y = b(z, \theta)$ *is a sufficiently smooth function and* (2.2) *is satisfied for sufficiently large* N, *i.e., after the transformations in this section, the transformed function* Ω *obtained from the background conic shock wave solution satisfies the equations in the problem* (3.21), (3.24) *and* (3.26) *at* $y_0 = -\infty$ *up to the order* $\mathrm{O}(e^{\nu y_0})$ *with* ν *sufficiently large.*

Then for the fixed boundary value problem of second order hyperbolic equation (3.21), (3.24) *and* (3.26), *there exists a positive large constant* $T \gg 1$ *such that in* $(-\infty, -T)$,

(3.21), (3.24) *and* (3.26) *has a unique classical solution* $\omega(y_0, y_1, y_2)$ *which corresponds to a distorted non-symmetrical conic shock wave.*

Since Theorem 3.1 is equivalent to Theorem 2.1 in Section 2, the next section of the paper is devoted to the proof of Theorem 3.1.

4. Linear iteration and existence

In this section we first establish the energy estimate for the linearization of the boundary value problem (3.21), (3.24) and (3.26) under the assumption of weak shock. Then we use the method of linear iteration to derive the existence of the distorted conic shock wave solution in the Theorem 3.1.

From the assumption that the distorted non-symmetrical solid cone is a small perturbation of the regular symmetrical cone in the sense that it is tangential to the symmetrical regular cone up to sufficiently high order, the transformed background conic shock wave solution Ω can serve as an approximate solution. The linearization and iteration will be carried out in the neighborhood of this background solution. In particular, the transformed background solution Ω is independent of y_1 since the background solution Φ is independent of the variable θ.

This background solution Ω will be used as the first approximation $\omega_0 = \Omega$ in the linear iteration to establish the existence of a classical solution for (3.21), (3.24) and (3.26) by a limiting process.

First we introduce the Sobolev spaces which will be employed in this section to establish the energy estimate and the convergence of the iteration sequence. We will use η-weighted Sobolev norms which were widely adopted in the study of hyperbolic boundary value problems. For smooth functions ω which vanishes near $y_0 = -\infty$, we define the following s-order norms for nonnegative integer s:

$$\|\omega\|^2_{s,\eta,T} = \sum_{k_0+k=s} \int_0^1 \int_0^{2\pi} \int_{-\infty}^{T} e^{-2\eta y_0} \eta^{2k_0} \left|\nabla^k \omega(y_0, y_1, y_2)\right|^2 dy_0\, dy_1\, dy_2. \tag{4.1}$$

On the boundary $y_2 = 1$ we define the boundary norms

$$\langle\omega\rangle^2_{s,\eta,T} = \int_{-\infty}^{-T} \int_0^{2\pi} \sum_{k_0+k=s} e^{-2\eta y_0} \eta^{2k_0} \left|\nabla^k_{y_0,y_1} \omega(y_0, y_1, 1)\right|^2 dy_1\, dy_0.$$

We will use the following norm to indicate that the normal derivatives are also included:

$$\langle\!\langle\omega\rangle\!\rangle^2_{s,\eta,T} = \int_{-\infty}^{-T} \int_0^{2\pi} \sum_{k_0+k=s} e^{-2\eta y_0} \eta^{2k_0} \left|\nabla^k_{y_0,y_1,y_2} \omega(y_0, y_1, y_2)\right|^2_{y_2=1} dy_1\, dy_0. \tag{4.2}$$

Finally we denote

$$|\!|\!|\omega|\!|\!|^2_{s,\eta,T} = \|\omega\|^2_{s,\eta,T} + \langle\!\langle\omega\rangle\!\rangle^2_{s,\eta,T}. \tag{4.3}$$

The completion of smooth functions vanishing near $y_0 = -\infty$ with respect to the norm (4.3) is a Sobolev space and will be denoted as H_s. By the Sobolev's embedding theorem [1], the H_s functions have n-order continuous classical derivatives for $s > 1 + n$.

We are looking for a solution ω close to ω_0 for the boundary value problem (3.21), (3.24) and (3.26). For this purpose we write the solution ω in the form

$$\omega = \omega_0 + \dot{\omega}, \tag{4.4}$$

where $\dot{\omega}$ is the small perturbation at ω_0. Because the function ω_0 is known, the boundary value problem (3.21), (3.24) and (3.26) for ω can be rewritten into an equivalent boundary value problem for $\dot{\omega}$.

First of all, we rewrite (3.21) into an equivalent equation for $\dot{\omega}$ by applying the Taylor expansion to the lower order terms $I(\omega, \nabla\omega)$ in Eq. (3.21) at the background solution $\omega_0 = \Omega$:

$$\sum_{i,j=0}^{2} \alpha_{ij}\big(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega})\big)\partial_{y_i}\partial_{y_j}\dot{\omega} + \sum_{j=0}^{2} \alpha_j\big(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega})\big)\partial_{y_j}\dot{\omega} + \alpha_3(\omega_0 + \dot{\omega})\dot{\omega} = \dot{f}, \quad \text{in } -\infty < y_0 < -T,\ 0 \leqslant y_1 < 2\pi,\ 0 < y_2 < 1. \tag{4.5}$$

Because ω_0 is the background solution, Eq. (4.5) is satisfied up to the order of $O(e^{\nu y_0})$ near $y_0 = -\infty$ if we take $\dot{\omega} = 0$.

Applying the Taylor expansion similarly to the boundary condition (3.26) at the background solution ω_0, we can rewrite (3.26) into an equivalent form

$$\sum_{j=0}^{2} \beta_{1j}\big(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega})\big)\partial_{y_j}\dot{\omega} + \beta_0(\omega_0 + \dot{\omega})\dot{\omega} = \dot{g}, \quad \text{on } y_2 = 1. \tag{4.6}$$

And again because ω_0 is the background solution, Eq. (4.6) is satisfied up to the order of $O(e^{\nu y_0})$ near $y_0 = -\infty$ for $\dot{\omega} = 0$.

To simplify the notation let's denote

$$\begin{aligned}
&A_{ij}(\dot{\omega}) \equiv \alpha_{ij}\big(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega})\big), \quad i, j = 0, 1, 2;\\
&A_j(\dot{\omega}) \equiv \alpha_j\big(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega})\big), \quad j = 0, 1, 2;\\
&B_{1j}(\dot{\omega}) \equiv \beta_{1j}\big(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega})\big), \quad j = 0, 1, 2;\\
&A_3(\dot{\omega}) \equiv \alpha_3(\omega_0 + \dot{\omega}); \qquad B_0(\dot{\omega}) \equiv \beta_0(\omega_0 + \dot{\omega}).
\end{aligned}$$

Using the expanded equations (4.5) and (4.6) to replace (3.21) and (3.26), we can rewrite the boundary value problem (3.21), (3.24) and (3.26) for ω into an equivalent boundary value problem for $\dot{\omega}$:

$$\begin{aligned}
L(\dot{\omega})\dot{\omega} &\equiv \sum_{i,j=0,1,2} A_{ij}(\dot{\omega})\partial_{y_i}\partial_{y_j}\dot{\omega} + \sum_{j=0}^{2} A_j(\dot{\omega})\partial_{y_j}\dot{\omega} + A_3(\dot{\omega})\dot{\omega}\\
&= \dot{f}, \quad \text{in } 0 < y_2 < 1, \qquad (4.7)\\
\dot{\omega} &= 0, \quad \text{on } y_2 = 0, \qquad (4.8)
\end{aligned}$$

$$M(\dot{\omega})\dot{\omega} \equiv \sum_{j=0}^{2} B_{1j}(\dot{\omega})\partial_{y_j}\dot{\omega} + B_0(\dot{\omega})\dot{\omega} = \dot{g}, \quad \text{on } y_2 = 1. \tag{4.9}$$

The linearized boundary value problem for (4.7)–(4.9) at $\dot{\omega} = \chi \sim 0$, i.e., near the background solution ω_0, is the following linear problem:

$$L(\chi)\dot{\omega} \equiv \sum_{i,j=0,1,2} A_{ij}(\chi)\partial_{y_i}\partial_{y_j}\dot{\omega} + \sum_{j=0}^{2} A_j(\chi)\partial_{y_j}\dot{\omega} + A_3(\chi)\dot{\omega}$$
$$= \dot{f}, \quad \text{in } 0 < y_2 < 1, \tag{4.10}$$
$$\dot{\omega} = 0, \quad \text{on } y_2 = 0. \tag{4.11}$$
$$M(\chi)\dot{\omega} \equiv \sum_{j=0}^{2} B_{1j}(\chi)\partial_{y_j}\dot{\omega} + B_0(\chi)\dot{\omega} = \dot{g}, \quad \text{on } y_2 = 1. \tag{4.12}$$

For the linear boundary value problem (4.10)–(4.12) and integer $s > 1$, we have the energy estimate for its solution $\dot{\omega}$:

Theorem 4.1. *Let ω_0 correspond to the background conic shock wave solution in Theorems* 2.1 *and* 3.1. *Assume the positive integer $s > 2$ and $\eta \geqslant \eta_0$ be sufficiently large. Then there exists a sufficiently small number $\delta_0 > 0$ such that for all functions $\chi \in H_{s+1}$ and $|||\chi|||_{s+1,\eta,T} \leqslant \delta_0$, the linear boundary value problem* (4.10)–(4.12) *has a unique solution $\dot{\omega}$ satisfying the energy estimate*

$$|||\dot{\omega}|||^2_{s+1,\eta,T} \leqslant C_s\big(\|\dot{f}\|^2_{s,\eta,T} + \langle\dot{g}\rangle^2_{s,\eta,T}\big), \tag{4.13}$$

where the constant C_s depends only on δ_0 and is independent of χ.

Proof. Let $\kappa(y_2)$ be a monotone smooth function which is 0 near $y_2 = 0$ and 1 near $y_2 = 1$. Obviously, Eq. (4.10) multiplied by $1 \equiv (1-\kappa)+\kappa$ will separate the boundaries $y_2 = 0$ and $y_2 = 1$ and we can consider the boundary value problem near $y_2 = 0$ and $y_2 = 1$ separately. Therefore the proof of Theorem 4.1 can be reduced to the study of two boundary value problems: one near the boundary $y_2 = 0$ and the other near the boundary $y_2 = 1$.

At the boundary $y_2 = 0$ which corresponds to the solid non-symmetrical conic surface, the boundary value problem (4.10), (4.11) is the classical Dirichlet problem for a second order hyperbolic equation [7]. Therefore we obviously have the energy estimate (4.13) for the solution $\dot{\omega}$ with $\dot{g} = 0$.

At the boundary $y_2 = 1$ which corresponds to the non-symmetrical conic shock front, the boundary value problem (4.10) and (4.12) is the linearization of problem (3.21), (3.26) which in turn is obtained from the original quasi-linear second order wave equation (1.8) and the Rankine–Hugoniot condition (1.11) by two transformations: the first transformation is the introduction of conic coordinates (2.1) and the second is the generalized nonlinear partial hodograph transformation (3.1).

The conic coordinates introduce the singularity at the tip of the distorted cone $z = 0$ which is formally removed by the singular transform (2.15). The conic coordinates do not change the well-posedness of the boundary value problem at any point $z > 0$ or $y_0 > -\infty$.

The shock wave for second order nonlinear hyperbolic equations was studied in [13] with the method of the partial hodograph transformation. It was shown in [13, Sections 2.5–2.6] that under the partial hodograph transformation, the multi-D linear stability conditions are satisfied for the linearized boundary value problem as long as the Lax' shock inequality is satisfied. The multi-dimensional linear stability conditions are equivalent to the Gårding's conditions for slant derivative boundary value problem of second order hyperbolic equations.

According to the assumption of weak shock (2.10), the normal directional velocities of the fluid satisfy the Lax' shock inequality that the velocity is supersonic in front of the shock and subsonic behind of the shock. We notice that the generalized nonlinear partial hodograph transformation (3.1) is a composition of the usual partial hodograph transformation and a nonlinear transformation of unknown functions. A nonlinear transformation of unknown function introduces only a factor into the principal parts of the differential equation and the boundary conditions and this obviously does not change the nature of the boundary value problem.

Therefore, except for a non-zero factor the linearized boundary value problem (4.10) and (4.12) has the same principal parts as the boundary value problem obtained from the second order hyperbolic equation by the usual partial hodograph transformation. Hence the linearized boundary value problem (4.10) and (4.12) satisfies the Gårding conditions for the second order hyperbolic equation and the solution satisfies an *a priori* energy estimate of the form of (4.13) as in [6]. Consequently, the solution of (4.10) and (4.12) satisfies the energy estimate (4.13) near the boundary $y_2 = 1$.

Combining the energy estimates for the boundary value problems at $y_2 = 0$ and $y_2 = 1$, we obtain the energy estimate (4.13) for the solution $\dot{\omega}$ in the domain $0 < y_2 < 1$.

The dependency of the constant C_s is obtained from two considerations. First of all, it comes from the fact that Sobolev spaces H_s with $s > [\frac{n}{2}]$ are Banach algebras and for which we have the Gagliardo–Nirenberg Inequality [1]. Secondly, because both the Dirichlet boundary value problem and the Gårding boundary conditions are stable under small perturbation of the coefficients in the differential equation as well as in the boundary conditions, so once Theorem 4.1 is proved for $\chi = 0$, it remains to be true for all small χ. And this concludes the proof of Theorem 4.1. □

Using the energy estimate (4.13) obtained in Theorem 4.1, we will employ the linear iteration to establish the existence of the solution for the problem (3.21), (3.24) and (3.26) and hence prove Theorem 3.1.

Proof of Theorem 3.1. We need only to prove the existence of the solution $\dot{\omega}$ for the boundary value problem (4.7)–(4.9) which is equivalent to the problem (3.21), (3.24) and (3.26).

First we note that the problem (4.7)–(4.9) is simply the first-order Taylor expansion of (3.21), (3.24) and (3.26) at ω_0. The coefficients $A_{ij}(\dot{\omega}) \equiv \alpha_{ij}(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega}))$ and $B_{1j}(\dot{\omega}) \equiv \beta_{1j}(\omega_0 + \dot{\omega}, \nabla(\omega_0 + \dot{\omega}))$ are perturbations of the corresponding coefficients α_{ij} and β_{1j} in (4.5) and (4.6) at ω_0 and we have

$$\dot{f} = -\sum_{i,j=0,1,2} \alpha_{ij}(\omega_0, \nabla\omega_0)(\omega_0)_{y_i y_j} - I(\omega_0, \nabla\omega_0), \tag{4.14}$$

$$\dot{g} = -G\big(P^{-1}\big((\omega_0)_{y_0}, (\omega_0)_{y_1}, (\omega_0)_{y_2}\big), u_0\big). \tag{4.15}$$

By the assumption that the distorted non-symmetrical solid cone is tangential to the symmetrical regular cone up to sufficiently high order and the corresponding property of the first approximate solution ω_0, the right-side terms $\dot{f}$ and $\dot{g}$ in (4.10) and (4.12) vanish of sufficiently high order at $y_0 = -\infty$. Therefore for fixed s and η, and for any given $\delta > 0$, we can choose $-T \gg 1$ such that

$$\|\dot{f}\|^2_{s,\eta,T} + \langle\dot{g}\rangle^2_{s,\eta,T} < \delta. \tag{4.16}$$

For the problem (4.7)–(4.9) we perform the linear iteration as follows. We take $\dot{\omega}_0 = 0$ the first term in the iteration, and define inductively $\dot{\omega}_{k+1}$ as the solution of the linear boundary value problem

$$\sum_{i,j=0,1,2} A_{ij}(\dot{\omega}_k)\partial_{y_i}\partial_{y_j}\dot{\omega}_{k+1} + \sum_{j=0}^{2} A_j(\dot{\omega}_k)\partial_{y_j}\dot{\omega}_{k+1} + A_3(\dot{\omega}_k)\dot{\omega}_{k+1} = \dot{f}, \quad \text{in } 0 < y_2 < 1, \tag{4.17}$$

$$\dot{\omega}_{k+1} = 0, \quad \text{on } y_2 = 0. \tag{4.18}$$

$$\sum_{j=0}^{2} B_{1j}(\dot{\omega}_k)\partial_{y_j}\dot{\omega}_{k+1} + B_0(\dot{\omega}_k)\dot{\omega}_{k+1} = \dot{g}, \quad \text{on } y_2 = 1. \tag{4.19}$$

From Theorem 4.1, we have the energy estimate (4.13) for the solution $\dot{\omega}_{k+1}$ of (4.17)–(4.19) as long as $\dot{\omega}_k$ in the coefficients of (4.17)–(4.19) satisfies

$$|||\dot{\omega}_k|||_{s+1,\eta,T} \leqslant \delta_0 \tag{4.20}$$

for $s > 2$.

Under the assumption (4.20) we obtain by (4.13) the estimate for $\dot{\omega}_{k+1}$:

$$|||\dot{\omega}_{k+1}|||^2_{s+1,\eta,T} \leqslant C_s\big(\|\dot{f}\|^2_{s,\eta,T} + \langle\dot{g}\rangle^2_{s,\eta,T}\big), \tag{4.21}$$

and the constant C_s in (4.21) depends only on δ_0.

For this fixed δ_0, because $\dot{f}$ and $\dot{g}$ vanish up to the order of $O(e^{\nu y_0})$ at $y_0 = -\infty$ for large ν, then by (4.16) we can choose $-T_0 \gg 1$ and $T < T_0$ such that

$$C_s\big(\|\dot{f}\|^2_{s,\eta,T} + \langle\dot{g}\rangle^2_{s,\eta,T}\big) < \delta_0. \tag{4.22}$$

(4.20) is obviously satisfied for $k = 0$. (4.21) and (4.22) show that if (4.20) is true for k, then it is also true for $k + 1$. Hence (4.20) is satisfied for all k with such choice of T_0. This proves the uniform boundedness of the sequence $\dot{\omega}_k$ in the $s + 1$ order norm $|||\cdot|||_{s+1,\eta,T}$.

The proof of the convergence of the sequence $\{\dot{\omega}_k\}$ is standard. Let

$$\widetilde{\omega}_k = \dot{\omega}_{k+1} - \dot{\omega}_k.$$

Then $\widetilde{\omega}_k$ satisfies the boundary value problem:

$$L(\dot{\omega}_k)\widetilde{\omega}_k = -\big[L(\dot{\omega}_k) - L(\dot{\omega}_{k-1})\big]\dot{\omega}_k \quad \text{in } 0 < y_2 < 1, \tag{4.23}$$

$$\widetilde{\omega}_k = 0 \quad \text{on } y_2 = 0, \tag{4.24}$$

$$M(\dot{\omega}_k)\widetilde{\omega}_k = -\big[M(\dot{\omega}_k) - M(\dot{\omega}_{k-1})\big]\dot{\omega}_k \quad \text{on } y_2 = 1. \tag{4.25}$$

By the Gagliardo–Nirenberg Inequality for the norms of product and composition [1], we have

$$\left\|\left[L(\dot{\omega}_k) - L(\dot{\omega}_{k-1})\right]\dot{\omega}_k\right\|_{s-2,\eta,T} \leqslant C|||\widetilde{\omega}_{k-1}|||_{s-1,\eta,T}|||\dot{\omega}_k|||_{s,\eta,T},$$
$$\left\langle\left[M(\dot{\omega}_k) - M(\dot{\omega}_{k-1})\right]\dot{\omega}_k\right\rangle_{s-2,\eta,T} \leqslant C|||\widetilde{\omega}_{k-1}|||_{s-1,\eta,T}|||\dot{\omega}_k|||_{s,\eta,T}.$$

Then from the energy estimate in Theorem 4.1, $\widetilde{\omega}_k$ satisfies

$$|||\widetilde{\omega}_k|||^2_{s-1,\eta,T} \leqslant C_{s-1}|||\widetilde{\omega}_{k-1}|||^2_{s-1,\eta,T}|||\dot{\omega}_k|||^2_{s,\eta,T}. \tag{4.26}$$

When we choose $\delta_0^2 < C_{s-1}^{-1}$ so that

$$C_{s-1}|||\dot{\omega}_k|||^2_{s,\eta,T} < 1,$$

then sequence $\{\dot{\omega}_k\}$ converges in the H_{s-1}-norm. This completes the proof of Theorem 3.1. □

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