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# Optimal birth control for an age-dependent $n$-dimensional food chain model

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**Abstract**

In this paper, we investigate optimal policies for an age-dependent $n$-dimensional food chain model, which is controlled by fertility. By using Dubovitskii–Milyutin's general theory, the maximum principles are obtained for problems with free terminal states, infinite horizon and target sets, respectively.


*Keywords:* Food chain model; Age-dependence; System of partial differential equations; Optimal control; Maximum principle

## 1. Introduction

The study of the control problems of age-structured single species was initiated by Rorres and Fair [1]. Since then, the control problem has received many attentions from several authors [2–9]. For the control problem of multi-species, Albrecht et al. [10], Lenhart et al. [11], Crespo et al. [12] and Ma et al. [16] considered several systems, respectively. However, their results are not concerned with age factor. To the best of our knowledge,

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there are no results on the topic of control problems of age-structured interacting species. In order to bridge this gap, we in the sequel investigate several optimal control problems for an age-dependent $n$-dimensional food chain model.

In paper [2], Chan and Guo studied optimal birth control policies for the following model of Mckendrick type:

$$\begin{cases} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = -\mu(a,t)p(a,t), & 0 < a \leqslant a_+,\ t \geqslant 0, \\ p(a,0) = p_0(a), & 0 < a \leqslant a_+, \\ p(0,t) = \beta(t)\int_{a_1}^{a_2} k(a)h(a)p(a,t)\,da, & t \geqslant 0, \end{cases} \tag{1.1}$$

where $p(a,t)$ stands for the population density of age $a$ at time $t$, $a_+$ is the life expectancy of individuals. Control variable $\beta(t)$ is the average fertility of females at time $t$, $k(a)$ and $h(a)$ denote, respectively, the female ratio and the fertility pattern; $[a_1, a_2]$ is the fertility interval with $\int_{a_1}^{a_2} h(a)\,da = 1$. The functional approach suggested by Dubovitskii and Milyutin was adopted in the investigation of above model (1.1). Maximum principles for problems with free ends, the time optimal control problem, problems with target sects and infinite horizon problems had been derived, respectively.

Motivated by the idea of Chan and Guo [2], the aim of this paper is to establish necessary optimality conditions for the above mentioned optimal control problems by using a powerful functional approach first suggested by Dubovitskii and Milyutin [13] for general extremal problems. In particular, our results extend those of Chan and Guo [2].

The remainder of this paper is organized as follows: In Section 2, we will introduce a basic model and consider its well-posedness. In Sections 3–5, we will establish maximum principles for the control problems with free terminal states, infinite horizon and target sets, respectively.

## 2. The model and its well-posedness

In [15], Webb studied the stability of nontrivial equilibrium solution of the following model:

$$\begin{cases} \frac{\partial l_i}{\partial t} + \frac{\partial l_i}{\partial a} = -[\mu_{i1}(p_{l_1(\cdot,t)}) + \mu_{i2}(p_{l_2(\cdot,t)})]l_i(a,t), & i = 1,2, \\ l_i(0,t) = \int_0^\infty \beta_i(1 - e^{\alpha_i a})l_i(a,t)\,da, & i = 1,2, \\ l_i(a,0) = \varphi_i(a), & i = 1,2, \\ p_{l_i(\cdot,t)} = \int_0^\infty l_i(a,t)\,da, & i = 1,2,\ (a,t) \in (0,\infty)\times(0,\infty), \end{cases}$$

where $l_i(a,t)$ $(i = 1,2)$ are the density with respect to age $a$ of $i$th population at time $t$; $\alpha_1, \alpha_2, \beta_1, \beta_2$ are all positive constants; $\mu_{ij}$ $(i,j = 1,2)$ are all bounded and twice continuously differentiable function from $R$ to $(0,\infty)$. In this article, we consider the effect of age factor for control problems of the interacting species. To do so, motivated by the idea of Webb [15], we introduce the following food chain system composed of $n$ age-dependent species:

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)P_2(t)p_1, \\ \frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial a} = -\mu_i(a,t)p_i + \lambda_{2i-2}(a,t)P_{i-1}(t)p_i - \lambda_{2i-1}(a,t)P_{i+1}(t)p_i, \\ \quad i = 2,3,\ldots,n-1, \\ \frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial a} = -\mu_n(a,t)p_n + \lambda_{2n-2}(a,t)P_{n-1}(t)p_n, \\ p_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\,da, \quad i = 1,2,\ldots,n, \\ p_i(a,0) = p_{i0}(a), \quad i = 1,2,\ldots,n, \\ P_i(t) = \int_0^{a_+} p_i(a,t)\,da, \quad i = 1,2,\ldots,n,\ (a,t) \in Q, \end{cases} \tag{2.1}$$

where $Q = (0,a_+) \times (0,+\infty)$, $[a_1,a_2]$ is the fertility interval, and the other parameters mean as follows (for the sake of convenience, throughout this paper, we suppose that $i = 1,2,\ldots,n$):

| | |
|---|---|
| $p_i(a,t)$: | the density of $i$th population of age $a$ at time $t$; |
| $\mu_i(a,t)$: | the average mortality of $i$th population; |
| $\beta_i(t)$: | the average fertility of $i$th population; |
| $\lambda_k(a,t)$: | the interaction coefficients ($k = 1,2,\ldots,2n-2$); |
| $m_i(a,t)$: | the ratio of females in $i$th population; |
| $p_{i0}(a)$: | the initial age distribution of $i$th population; |
| $a_+$: | the life expectancy, $0 < a_+ < +\infty$. |

Here, without loss of generality, we assume that the $n$ populations have the same life expectancy.

Throughout this paper, we always assume that

(H$_1$) $\mu_i \in L^1_{\mathrm{loc}}(Q)$, $\mu_i(a,t) \geqslant 0$, $\int_0^{a_+} \mu_i(a,t+a)\,da = +\infty$, $(a,t) \in Q$.
(H$_2$) $0 \leqslant \lambda_k(a,t) \leqslant A_k$, $(a,t) \in Q$, $A_k$ are constants ($k = 1,2,\ldots,2n-2$).
(H$_3$) $0 \leqslant m_i(a,t) \leqslant M_i$, $(a,t) \in Q$, $M_i$ are constants, and $m_i(a,t) \equiv 0$, when $a < a_1$ or $a > a_2$.
(H$_4$) $\beta_i \in U_i := \{h_i \in L^\infty(0,\infty)\colon\ 0 \leqslant \beta_0 \leqslant h_i(t) \leqslant \beta^0,\ \forall t > 0\}$, $U = \prod_{i=1}^n U_i$.
(H$_5$) $p_{i0} \in L^\infty(0,a_+)$, $p_{i0}(a) \geqslant 0$, $\forall a \in (0,a_+)$.

For any given $T > 0$ and

$$v = (v_1, v_2, \ldots, v_n) \in L^2(Q_T, R^n), \quad Q_T = (0,a_+) \times (0,T),\ v \geqslant 0,$$

define

$$V_i(t) = \int_0^{a_+} v_i(a,t)\,da, \quad i = 1,2,\ldots,n.$$

Consider the system

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)V_2(t)p_1, \\ \frac{\partial p_k}{\partial t} + \frac{\partial p_k}{\partial a} = -\mu_k(a,t)p_k + \lambda_{2k-2}(a,t)V_{k-1}(t)p_k - \lambda_{2k-1}(a,t)V_{k+1}(t)p_k, \\ \quad k = 2,3,\dots,n-1, \\ \frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial a} = -\mu_n(a,t)p_n + \lambda_{2n-2}(a,t)V_{n-1}(t)p_n, \\ p_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\,da, \\ p_i(a,0) = p_{i0}(a), \\ V_i(t) = \int_0^{a_+} v_i(a,t)\,da, \quad (a,t) \in Q_T. \end{cases} \tag{2.2}$$

In view of [9,14], we know that the above system has a unique nonnegative solution

$$p^v = \left(p_1^v, p_2^v, \dots, p_n^v\right) \in C\left(0,T; L^2(0,a_+;R^n)\right) \cap L^\infty(Q_T;R^n),$$

and

$$p_i^v(a_+,t) = 0, \quad \forall t \in [0,T],\ i = 1,2,\dots,n.$$

Note that, from the comparison principle of linear system [9], it follows that $p_1^v(a,t) \leqslant \bar{p}_1(a,t)$, $(a,t) \in Q_T$, where $\bar{p}_1$ is the solution of the system

$$\begin{cases} \frac{\partial y_1}{\partial t} + \frac{\partial y_1}{\partial a} = -\mu_1(a,t)y_1, \\ y_1(0,t) = \beta_1(t)\int_{a_1}^{a_2} m_1(a,t)y_1(a,t)\,da, \\ y_1(a,0) = p_{10}(a), \quad (a,t) \in Q_T. \end{cases}$$

Similarly, if $v_{k-1}(a,t) \leqslant \bar{p}_{k-1}(a,t)$, $\forall (a,t) \in Q_T$, then $p_k^v(a,t) \leqslant \bar{p}_k(a,t)$, $(a,t) \in Q_T$, in which $\bar{p}_k$ $(k = 2,3,\dots,n)$ is the solution to the system

$$\begin{cases} \frac{\partial y_k}{\partial t} + \frac{\partial y_k}{\partial a} = -\mu_k(a,t)y_k + \lambda_{2k-2}(a,t)y_k\int_0^{a_+} \bar{p}_{k-1}(a,t)\,da, \\ y_k(0,t) = \beta_k(t)\int_{a_1}^{a_2} m_k(a,t)y_k(a,t)\,da, \\ y_k(a,0) = p_{k0}(a), \quad k = 2,3,\dots,n,\ (a,t) \in Q_T. \end{cases}$$

For any $v^k = (v_1^k, v_2^k, \dots, v_n^k) \in L^2(Q_T;R^n)$, $0 \leqslant v_i^k \leqslant \bar{p}_i$, let the corresponding state be $p^k = (p_1^k, p_2^k, \dots, p_n^k)$ $(k = 1,2)$, $x = (x_1, x_2, \dots, x_n) := p^1 - p^2$. It follows from (2.2) that

$$\begin{cases} \frac{\partial x_1}{\partial t} + \frac{\partial x_1}{\partial a} = -\mu_1 x_1 - \lambda_1 V_2^1(t)x_1 - (V_2^1(t) - V_2^2(t))\lambda_1 p_1^2, \\ \frac{\partial x_k}{\partial t} + \frac{\partial x_k}{\partial a} = -\mu_k x_k + \lambda_{2k-2}V_{k-1}^1(t)x_k - \lambda_{2k-1}V_{k+1}^1(t)x_k \\ \qquad\qquad + (V_{k-1}^1(t) - V_{k-1}^2(t))\lambda_{2k-2}p_k^2 - (V_{k+1}^1(t) - V_{k+1}^2(t))\lambda_{2k-1}p_k^2, \\ \quad k = 2,3,\dots,n-1, \\ \frac{\partial x_n}{\partial t} + \frac{\partial x_n}{\partial a} = -\mu_n x_n + \lambda_{2n-2}V_{n-1}^1(t)x_n + (V_{n-1}^1(t) - V_{n-1}^2(t))\lambda_{2n-2}p_n^2, \\ x_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)x_i(a,t)\,da, \\ x_i(a,0) = 0, \\ V_i^k(t) = \int_0^{a_+} v_i^k(a,t)\,da, \quad k = 1,2,\ (a,t) \in Q_T. \end{cases} \tag{2.3}$$

Multiplying $(2.3)_i$ by $x_i$, $i = 1, 2, \dots, n$, and integrating on $(0, a_+) \times (0, t)$ yields

$$\left\|x_1(\cdot\,, t)\right\|^2 \leqslant c \int_0^t \left\|v_2^1(\cdot\,, s) - v_2^2(\cdot\,, s)\right\|^2 ds, \tag{2.4}$$

$$\begin{cases} \|x_k(\cdot\,, t)\|^2 \leqslant c \int_0^t (\|v_{k-1}^1(\cdot\,, s) - v_{k-1}^2(\cdot\,, s)\|^2 \\ \qquad\qquad + \|v_{k+1}^1(\cdot\,, s) - v_{k+1}^2(\cdot\,, s)\|^2)\, ds, \\ k = 2, 3, \dots, n-1, \end{cases} \tag{2.5}$$

and

$$\left\|x_n(\cdot\,, t)\right\|^2 \leqslant c \int_0^t \left\|v_{n-1}^1(\cdot\,, s) - v_{n-1}^2(\cdot\,, s)\right\|^2 ds, \tag{2.6}$$

where $c$ is a constant independent of $v^k$, $k = 1, 2$, $\|\cdot\|$ is the ordinary norm in $L^2(0, a_+)$.

Set

$$I = \left\{v = (v_1, v_2, \dots, v_n) \in L^2(Q_T; R^n)\colon\ 0 \leqslant v_i(a, t) \leqslant \bar{p}_i(a, t),\ \forall (a, t) \in Q_T\right\}.$$

Define the mapping $G : I \to I$,

$$(Gv)(a, t) = p^v(a, t), \quad \forall (a, t) \in Q_T,$$

and an equivalent norm $\|v\|_* = \sum_{i=1}^n \|v_i\|_*$,

$$\|v_i\|_* = \int_0^T \left\|v_i(\cdot\,, t)\right\|^2 e^{-4ct}\, dt, \quad i = 1, 2, \dots, n.$$

Using (2.5) and (2.6), we get that

$$\begin{aligned} \|Gv^1 - Gv^2\|_* &= \|p^1 - p^2\|_* = \int_0^T \left(\sum_{i=1}^n \left\|x_i(\cdot\,, t)\right\|^2\right) e^{-4ct}\, dt \\ &\leqslant \int_0^T \int_0^t c\Bigg(\left\|v_1^1(\cdot\,, s) - v_1^2(\cdot\,, s)\right\|^2 + 2\sum_{i=2}^{n-1} \left\|v_i^1(\cdot\,, s) - v_i^2(\cdot\,, s)\right\|^2 \\ &\quad + \left\|v_n^1(\cdot\,, s) - v_n^2(\cdot\,, s)\right\|^2\Bigg) e^{-4ct}\, ds\, dt \\ &\leqslant \int_0^T \left(\sum_{i=1}^n \left\|v_i^1(\cdot\,, s) - v_i^2(\cdot\,, s)\right\|^2\right) \int_s^T 2ce^{-4ct}\, dt\, ds \\ &\leqslant \frac{1}{2} \int_0^T \left(\sum_{i=1}^n \left\|v_i^1(\cdot\,, s) - v_i^2(\cdot\,, s)\right\|^2\right) e^{-4ct}\, ds = \frac{1}{2}\|v^1 - v^2\|_*. \end{aligned}$$

So, $G$ is a contraction on $(I, \|\cdot\|_*)$ and there is a unique fixed point $v^*$, which is the solution of the system (2.2). Thus, the following result is true.

**Theorem 1.** *For any given $\beta \in U$, there is a unique solution $p^\beta$ of system* (2.1) *such that*

(i) $p^\beta \in C(0, \infty; L^2(0, a_+))$.
(ii) $0 \leqslant p_i^\beta(a, t) \leqslant \bar{p}_i(a, t)$, $\forall (a, t) \in Q$, $i = 1, 2, \ldots, n$.
(iii) *It can be shown in a similar manner that $p^\beta$ depends continuously on $\beta$.*

## 3. Free terminal problem

In this section, we consider the following control problem: determine $(\beta^*, p^*)$, $\beta^* \in U$, such that

$$\begin{cases} J(\beta^*, p^*) = \min\{J(\beta, p),\ \beta \in U, (\beta, p) \text{ is subject to (2.1)}\}, \\ J(\beta, p) = \int_0^T \int_0^{a_+} L(\beta_1(t), \ldots, \beta_n(t), p_1(a, t), \ldots, p_n(a, t), a, t)\, da\, dt \\ \qquad\qquad + (1/2) \sum_{i=1}^n \int_0^{a_+} [p_i(a, t) - \bar{p}_i(a)]^2\, da, \end{cases} \tag{3.1}$$

where $T > 0$ and $\bar{p}_i(a) \geqslant 0$ $(i = 1, 2, \ldots, n)$ are fixed. The functional $L$ defined on

$$\prod_{i=1}^n B_i \times \prod_{i=1}^n L_i \times [0, a_+] \times [0, \infty), \qquad B_i = [\beta_0, \beta^0],\ L_i = L^2(0, a_+),$$

satisfies the following conditions:

(C1) $\partial L/\partial \beta_i$ and $\partial L/\partial p_i$ $(i = 1, 2, \ldots, n)$ are continuous in the first $2n$ arguments, and $L$ is continuous in its all variables.

(C2)
$$\int_0^{a_+} \left|\partial L\big(\beta_1, \ldots, \beta_n, p_1(a), \ldots, p_n(a), a, t\big)/\partial \beta_i\right| da$$

and

$$\int_0^{a_+} \left|\partial L\big(\beta_1, \ldots, \beta_n, p_1(a), \ldots, p_n(a), a, t\big)/\partial p_i\right| da \quad (i = 1, 2, \ldots, n)$$

are bounded for any $t \in [0, T]$ and any bounded subset of

$$\prod_{i=1}^n B_i \times \prod_{i=1}^n L_i \times [0, a_+] \times [0, T].$$

In the sequel, we denote $(\beta, p, a, t)$ by

$$\big(\beta_1(t), \ldots, \beta_n(t), p_1(a, t), \ldots, p_n(a, t), a, t\big).$$

**Theorem 2.** *Any solution* $(\beta^*, p^*)$ *of problem* (3.1) *with*

$$\beta_i^*(t)S_i(t) = \max\{\beta_i S_i(t)\colon\ \beta_0 \leqslant \beta_i \leqslant \beta^0\}, \quad \forall t \in [0,T] \text{ a.e.},\ i = 1,2,\ldots,n,$$

*where*

$$S_i(t) = \int_0^{a_+} \left[q_i(0,t)(m_i p_i^*)(a,t) - \partial L(\beta^*, p^*, a, t)/\partial \beta_i\right] da,$$

$q_i$ $(i = 1,2,\ldots,n)$ *is the solution of the adjoint system*

$$\begin{cases} \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} = \mu_1 q_1 - m_1\beta_1^* q_1(0,t) + \lambda_1 q_1 P_2^*(t) \\ \qquad\qquad + \frac{\partial L}{\partial p_1}(\beta^*, p^*, a, t) - \int_0^{a_+}(\lambda_2 p_2^* q_2)(a,t)\,da, \\ \frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial a} = \mu_k q_k - m_k\beta_k^* q_k(0,t) - \lambda_{2k-2} q_k P_{k-1}^*(t) \\ \qquad\qquad + \lambda_{2k-1} q_k P_{k+1}^*(t) + \frac{\partial L}{\partial p_k}(\beta^*, p^*, a, t) \\ \qquad\qquad + \int_0^{a_+}(\lambda_{2k-3} p_{k-1}^* q_{k-1} - \lambda_{2k} p_{k+1}^* q_{k+1})(a,t)\,da, \\ \quad k = 2,3,\ldots,n-1, \\ \frac{\partial q_n}{\partial t} + \frac{\partial q_n}{\partial a} = \mu_n q_n - m_n\beta_n^* q_n(0,t) - \lambda_{2n-2} q_n P_{n-1}^*(t) \\ \qquad\qquad + \frac{\partial L}{\partial p_n}(\beta^*, p^*, a, t) + \int_0^{a_+}(\lambda_{2n-3} p_{n-1}^* q_{n-1})(a,t)\,da, \\ q_i(a,T) = \bar{p}_i(a) - p_i^*(a,T), \\ q_i(a_+,t) = 0, \quad P_i^*(t) = \int_0^{a_+} p_i^*(a,t)\,da, \quad (a,t) \in Q_T. \end{cases} \tag{3.2}$$

**Proof.** For any given $h = (h_1, h_2, \ldots, h_n) \in T_U(\beta^*)$ (the tangent cone to $U$ at $\beta^*$) and $\varepsilon > 0$ small enough, we have $\beta^\varepsilon := \beta^* + \varepsilon h \in U$.

Denoting by $p^\varepsilon$ the state corresponding to $\beta^\varepsilon$, we can write

$$J(\beta^\varepsilon, p^\varepsilon) \geqslant J(\beta^*, p^*),$$

i.e.,

$$\begin{aligned} &\int_0^T\int_0^{a_+} L(\beta^\varepsilon, p^\varepsilon, a, t)\,da\,dt + \frac{1}{2}\sum_{i=1}^n \int_0^{a_+} \left[p_i^\varepsilon(a,T) - \bar{p}_i(a)\right]^2 da \\ &\quad \geqslant \int_0^T\int_0^{a_+} L(\beta^*, p^*, a, t)\,da\,dt + \frac{1}{2}\sum_{i=1}^n \int_0^{a_+} \left[p_i^*(a,T) - \bar{p}_i(a)\right]^2 da. \end{aligned} \tag{3.3}$$

Dividing (3.3) by $\varepsilon$ and passing to the limit as $\varepsilon \to 0^+$, we obtain that

$$\sum_{i=1}^n \left\{ \int_0^T\int_0^{a_+} \left[h_i(t)\frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) + z_i(a,t)\frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\right] da\,dt \right.$$

$$+ \int_0^{a_+} z_i(a,T)\big[p_i^*(a,T) - \bar{p}_i(a)\big]\,da \Bigg\} \geqslant 0, \tag{3.4}$$

where $z_i(a,t) := \lim_{\varepsilon\to 0^+} \varepsilon^{-1}[p_i^\varepsilon(a,t) - p_i^*(a,t)]$ satisfies

$$\begin{cases} \frac{\partial z_1}{\partial t} + \frac{\partial z_1}{\partial a} = -\mu_1 z_1 - \lambda_1 p_1^* Z_2(t) - \lambda_1 P_2^*(t) z_1, \\ \frac{\partial z_k}{\partial t} + \frac{\partial z_k}{\partial a} = -\mu_k z_k + \lambda_{2k-2} p_k^* Z_{k-1}(t) - \lambda_{2k-1} p_k^* Z_{k+1}(t) + \lambda_{2k-2} P_{k-1}^*(t) z_k \\ \qquad\qquad - \lambda_{2k-1} P_{k+1}^*(t) z_k, \quad k = 2,3,\ldots,n-1, \\ \frac{\partial z_n}{\partial t} + \frac{\partial z_n}{\partial a} = -\mu_n z_n + \lambda_{2n-2} p_n^* Z_{n-1}(t) + \lambda_{2n-2} P_{n-1}^*(t) z_n, \\ z_i(0,t) = \beta_i^*(t) \int_{a_1}^{a_2} (m_i z_i)(a,t)\,da + h_i(t) \int_{a_1}^{a_2} (m_i p_i^*)(a,t)\,da, \\ z_i(a,0) = 0, \quad Z_i(t) = \int_0^{a_+} z_i(a,t)\,da, \quad (a,t) \in Q_T. \end{cases} \tag{3.5}$$

Multiplying $(3.5)_i$ by $q_i(a,t)$, $i = 1,2,\ldots,n$, integrating on $Q_T$ and using system (3.2), we derive out that

$$\sum_{i=1}^n \Bigg\{ \int_0^T \int_0^{a_+} z_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\,da\,dt + \int_0^{a_+} z_i(a,T)\big[p_i^*(a,T) - \bar{p}_i(a)\big]\,da \Bigg\}$$
$$= -\sum_{i=1}^n \int_0^T q_i(0,t) \int_0^{a_+} (m_i p_i^*)(a,t)\,da \cdot h_i(t)\,dt. \tag{3.6}$$

Combining (3.4) and (3.6), we are led to that

$$\sum_{i=1}^n \Bigg\{ \int_0^T \int_0^{a_+} \bigg[ q_i(0,t) m_i(a,t) p_i^*(a,t) - \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) \bigg] da \cdot h_i(t)\,dt \Bigg\} \leqslant 0$$

holds for any $h \in T_U(\beta^*)$, so [7] $S_i \in N_{U_i}(\beta_i^*)$ (the normal cone to $U_i$ at $\beta_i^*$). The proof is complete. □

## 4. Infinite horizon problem

In this section, we consider further the optimal control problem. Find $(\beta^*, p^*)$, $\beta^* \in U$, such that

$$\begin{cases} J(\beta^*, p^*) = \min\{J(\beta, p),\ \beta \in U,\ (\beta, p) \text{ is subject to (2.1)}\}, \\ J(\beta, p) = \int_0^\infty \int_0^{a_+} L(\beta_1(t), \ldots, \beta_n(t), p_1(a,t), \ldots, p_n(a,t), a, t)\,da\,dt, \end{cases} \tag{4.1}$$

with other conditions similar to that in problem (3.1). Moreover we suppose that for each admissible pair $(\beta, p)$, the integral in (4.1) is convergent.

It is easy to prove that the following result is true.

**Lemma 1.** *If* $(\beta^*, p^*)$ *is a solution to problem* (4.1)*, then for any given* $T > 0$*,* $(\beta^*, p^*)$ *is a solution to the problem*

$$\begin{cases} J_T(\beta^*, p^*) = \min\{J_T(\beta, p),\ \beta \in U\}, \\ J_T(\beta, p) = \int_0^T \int_0^{a_+} L(\beta_1(t), \ldots, \beta_n(t), p_1(a,t), \ldots, p_n(a,t), a, t)\, da\, dt, \end{cases} \tag{4.2}$$

*where* $(\beta, p)$ *is subject to the system*

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)P_2(t)p_1, \\ \frac{\partial p_k}{\partial t} + \frac{\partial p_k}{\partial a} = -\mu_k(a,t)p_k + \lambda_{2k-2}(a,t)P_{k-1}(t)p_k - \lambda_{2k-1}(a,t)P_{k+1}(t)p_k, \\ \quad k = 2, 3, \ldots, n-1, \\ \frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial a} = -\mu_n(a,t)p_n + \lambda_{2n-2}(a,t)P_{n-1}(t)p_n, \\ p_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\, da, \\ p_i(a,0) = p_{i0}(a), \quad p_i(a,T) = p_i^*(a,T), \quad (a,t) \in Q_T. \end{cases} \tag{4.3}$$

*Here and in the sequel,* $Q_T := (0, a_+) \times (0, T)$.

Let $X = L^\infty(0,T; R^n) \times C(0,T; L^2(0,a_+; R^n))$. We first investigate the necessary conditions which must be satisfied for the solution of problem (4.2) and (4.3).

Define

$$\Omega_1 = \big\{(\beta, p) \in X\colon\ \beta_0 \leqslant \beta_i(t) \leqslant \beta^0,\ t \in [0,T] \text{ a.e.},\ i = 1, 2, \ldots, n\big\},$$
$$\Omega_2 = \big\{(\beta, p) \in X\colon\ (\beta, p) \text{ solves system (4.3)}\big\}.$$

Then problem (4.2) and (4.3) is equivalent to the following problem: Find $(\beta^*, p^*) \in \Omega_1 \cap \Omega_2$, such that

$$J_T(\beta^*, p^*) = \min\big\{J_T(\beta, p),\ (\beta, p) \in \Omega_1 \cap \Omega_2\big\}. \tag{4.4}$$

In what follows, we will use the general theory of Dubovitskii and Milyutin for extremal problems to deal with problem (4.4), which deeds to determine the corresponding cones.

Under the assumptions for $J(\beta, p)$, the functional $J_T$ is differentiable at any point $(\tilde\beta, \tilde p)$ and

$$J_T'(\tilde\beta, \tilde p)(\beta, p) = \sum_{i=1}^n \int_0^T \int_0^{a_+} \left[\beta_i(t)\frac{\partial L}{\partial \beta_i}(\tilde\beta, \tilde p, a, t) + p_i(a,t)\frac{\partial L}{\partial p_i}(\tilde\beta, \tilde p, a, t)\right] da\, dt.$$

Since $J_T(\beta, p)$ is regularly decreasing at $(\beta^*, p^*)$, its directions of decrease cone is

$$K_0 = \big\{(\beta, p) \in X\colon\ J_T'(\beta^*, p^*)(\beta, p) < 0\big\}.$$

If $K_0 \neq \emptyset$, then for any $f_0 \in K_0^*$ (the dual cone of $K_0$), there exists $\lambda_0 \geqslant 0$ such that

$$f_0(\beta, p) = -\lambda_0 \sum_{i=1}^{n} \int_0^T \int_0^{a_+} \biggl[ \beta_i(t) \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) + p_i(a, t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t) \biggr] \, da \, dt. \tag{4.5}$$

Note that $\Omega_1 = \hat{\Omega}_1 \times C(0, T; L^2(0, a_+; R^n))$ (where $\hat{\Omega}_1 = \{\beta \in L^\infty(0, T; R^n)\colon\ \beta_0 \leqslant \beta_i(t) \leqslant \beta^0\}$) is a closed convex subset of $X$. Thus

$$\operatorname{int}(\Omega_1) = \operatorname{int}(\hat{\Omega}_1) \times C\bigl(0, T; L^2(0, a_+; R^n)\bigr) \neq \emptyset,$$

where $\operatorname{int}(\Omega_1)$ denotes the interior of $\Omega_1$. Hence the feasible directions cone for $\Omega_1$ at $(\beta^*, p^*)$ is $K_1 = \{\lambda(\operatorname{int}(\Omega_1) - (\beta^*, p^*))\colon\ \lambda > 0\} := \{\lambda((\beta, p) - (\beta^*, p^*))\colon\ (\beta, p) \in \operatorname{int}(\Omega_1),\ \lambda > 0\}$. For any functional $f_1 \in K_1^*$, if there exists $a_i(t) \in L^1(0, T)$ $(i = 1, 2, \ldots, n)$ such that

$$f_1(\beta, p) = \sum_{i=1}^{n} \int_0^T a_i(t) \beta_i(t) \, dt, \tag{4.6}$$

then [13, p. 76]

$$\sum_{i=1}^{n} a_i(t) \bigl[\beta_i - \beta_i^*(t)\bigr] \geqslant 0, \quad \forall \beta_i \in [\beta_0, \beta^0],\ t \in [0, T] \text{ a.e.} \tag{4.7}$$

Next we determine the tangent directions cone for $\Omega_2$ at $(\beta^*, p^*)$. As far as the mild solutions are concerned, system (4.3) is equivalent to the system

$$\begin{cases} u_1(a, t) := \int_0^a [p_1(\tau, t) - p_{10}(\tau)] \, d\tau + \int_0^t p_1(a, s) \, ds \\ \qquad - \int_0^t \int_{a_1}^{a_2} \beta_1(s) m_1(a, s) p_1(a, s) \, da \, ds \\ \qquad + \int_0^t \int_0^a p_1(\tau, s) [\mu_1(\tau, s) + \lambda_1(\tau, s) P_2(s)] \, d\tau \, ds = 0, \\ u_k(a, t) := \int_0^a [p_k(\tau, t) - p_{k0}(\tau)] \, d\tau + \int_0^t p_k(a, s) \, ds \\ \qquad - \int_0^t \int_{a_1}^{a_2} \beta_k(s) m_k(a, s) p_k(a, s) \, da \, ds \\ \qquad + \int_0^t \int_0^a p_k(\tau, s) [\mu_k(\tau, s) - \lambda_{2k-2}(\tau, s) P_{k-1}(s) \\ \qquad + \lambda_{2k-1}(\tau, s) P_{k+1}(s)] \, d\tau \, ds = 0, \quad k = 2, 3, \ldots, n-1, \\ u_n(a, t) := \int_0^a [p_n(\tau, t) - p_{n0}(\tau)] \, d\tau + \int_0^t p_n(a, s) \, ds \\ \qquad - \int_0^t \int_{a_1}^{a_2} \beta_n(s) m_n(a, s) p_n(a, s) \, da \, ds \\ \qquad + \int_0^t \int_0^a p_n(\tau, s) [\mu_n(\tau, s) - \lambda_{2n-2}(\tau, s) P_{n-1}(s)] \, d\tau \, ds = 0, \\ p_i(a, T) = p_i^*(a, T), \quad i = 1, 2, \ldots, n. \end{cases} \tag{4.8}$$

Define the operator $G : X \to C(0, T; L^2(0, a_+; R^n)) \times L^2(0, T; R^n)$,

$$\big[G(\beta, p)\big](a,t) = \big(u_1(a,t), \dots, u_n(a,t),\\ p_1(a,T) - p_1^*(a,T), \dots, p_n(a,T) - p_n^*(a,T)\big).$$

So, $\Omega_2 = \{(\beta, p) \in X\colon\ G(\beta, p) = 0\}$, and

$$G'(\beta^*, p^*)(\beta, p) = \big(v_1(a,t), \dots, v_n(a,t), p_1(a,T), \dots, p_n(a,T)\big),$$

where

$$\begin{aligned} v_1(a,t) = {} & \int_0^a p_1(\tau,t)\,d\tau + \int_0^t p_1(a,s)\,ds + \int_0^t\int_0^a (\mu_1 p_1)(\tau,s)\,d\tau\,ds \\ & - \int_0^t\int_{a_1}^{a_2} m_1(a,s)\big[\beta_1^*(s)p_1(a,s) + \beta_1(s)p_1^*(a,s)\big]\,da\,ds \\ & + \int_0^t\int_0^a \lambda_1(\tau,s)\big[p_1^*(\tau,s)P_2(s) + p_1(\tau,s)P_2^*(s)\big]\,d\tau\,ds, \end{aligned} \tag{4.9}$$

$$\begin{aligned} v_k(a,t) = {} & \int_0^a p_k(\tau,t)\,d\tau + \int_0^t p_k(a,s)\,ds + \int_0^t\int_0^a (\mu_k p_k)(\tau,s)\,d\tau\,ds \\ & - \int_0^t\int_{a_1}^{a_2} m_k(a,s)\big[\beta_k^*(s)p_k(a,s) + \beta_k(s)p_k^*(a,s)\big]\,da\,ds \\ & - \int_0^t\int_0^a \lambda_{2k-2}(\tau,s)\big[p_k^*(\tau,s)P_{k-1}(s) + p_k(\tau,s)P_{k-1}^*(s)\big] \\ & + \int_0^t\int_0^a \lambda_{2k-1}(\tau,s)\big[p_k^*(\tau,s)P_{k+1}(s) + p_k(\tau,s)P_{k+1}^*(s)\big]\,d\tau\,ds, \\ & k = 2,3,\dots,n-1, \end{aligned} \tag{4.10}$$

$$\begin{aligned} v_n(a,t) = {} & \int_0^a p_n(\tau,t)\,d\tau + \int_0^t p_n(a,s)\,ds + \int_0^t\int_0^a (\mu_n p_n)(\tau,s)\,d\tau\,ds \\ & - \int_0^t\int_{a_1}^{a_2} m_n(a,s)\big[\beta_n^*(s)p_n(a,s) + \beta_n(s)p_n^*(a,s)\big]\,da\,ds \\ & - \int_0^t\int_0^a \lambda_{2n-2}(\tau,s)\big[p_n^*(\tau,s)P_{n-1}(s) + p_n(\tau,s)P_{n-1}^*(s)\big]\,d\tau\,ds. \end{aligned} \tag{4.11}$$

To show that $G'(\beta^*, p^*)$ is an onto mapping, we solve equation $G'(\beta^*, p^*)(\beta, p) = (w_1, w_2, \ldots, w_{2n})$, i.e.,

$$\begin{cases} \int_0^a p_1(\tau,t)\,d\tau + \int_0^t p_1(a,s)\,ds + \int_0^t \int_0^a (\mu_1 p_1)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t \int_{a_1}^{a_2} m_1(a,s)[\beta_1^*(s)p_1(a,s) + \beta_1(s)p_1^*(a,s)]\,da\,ds \\ \quad + \int_0^t \int_0^a \lambda_1(\tau,s)[p_1^*(\tau,s)P_2(s) + p_1(\tau,s)P_2^*(s)]\,d\tau\,ds = w_1(a,t), \\ \int_0^a p_k(\tau,t)\,d\tau + \int_0^t p_k(a,s)\,ds + \int_0^t \int_0^a (\mu_k p_k)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t \int_{a_1}^{a_2} m_k(a,s)[\beta_k^*(s)p_k(a,s) + \beta_k(s)p_k^*(a,s)]\,da\,ds \\ \quad - \int_0^t \int_0^a \lambda_{2k-2}(\tau,s)[p_k^*(\tau,s)P_{k-1}(s) + p_k(\tau,s)P_{k-1}^*(s)]\,d\tau\,ds \\ \quad + \int_0^t \int_0^a \lambda_{2k-1}(\tau,s)[p_k^*(\tau,s)P_{k+1}(s) + p_k(\tau,s)P_{k+1}^*(s)]\,d\tau\,ds = w_k(a,t), \\ \quad k = 2, 3, \ldots, n-1, \\ \int_0^a p_n(\tau,t)\,d\tau + \int_0^t p_n(a,s)\,ds + \int_0^t \int_0^a (\mu_n p_n)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t \int_{a_1}^{a_2} m_n(a,s)[\beta_n^*(s)p_n(a,s) + \beta_n(s)p_n^*(a,s)]\,da\,ds \\ \quad - \int_0^t \int_0^a \lambda_{2n-2}(\tau,s)[p_n^*(\tau,s)P_{n-1}(s) + p_n(\tau,s)P_{n-1}^*(s)]\,d\tau\,ds = w_n(a,t), \\ p_i(a,T) = w_{i+3}(a), \quad i = 1, 2, \ldots, n, \\ \quad \to \quad p_i(a,T) = w_{i+n}(a), \quad 1 = 1, 2, \ldots, n, \end{cases} \tag{4.12}$$

where $(w_1, w_2, \ldots, w_{2n})$ is prescribed.

Note that the linearized system of (2.1) at $(\beta^*, p^*)$ is

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)[P_2^*(t)p_1 + P_2(t)p_1^*], \\ \frac{\partial p_k}{\partial t} + \frac{\partial p_k}{\partial a} = -\mu_k(a,t)p_k + \lambda_{2k-2}(a,t)[P_{k-1}^*(t)p_k + P_{k-1}(t)p_k^*] \\ \qquad\qquad - \lambda_{2k-1}(a,t)[P_{k+1}^*(t)p_k + P_{k+1}(t)p_k^*], \\ \quad k = 2, 3, \ldots, n-1, \\ \frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial a} = -\mu_n(a,t)p_n + \lambda_{2n-2}(a,t)[P_{n-1}^*(t)p_n + P_{n-1}(t)p_n^*], \\ p_i(0,t) = \int_{a_1}^{a_2} m_i(a,t)[\beta_i^*(t)p_i(a,t) + \beta_i(t)p_i^*(a,t)]\,da, \\ p_i(a,0) = 0, \quad (a,t) \in Q_T. \end{cases} \tag{4.13}$$

It is easily seen that each mild solution of (4.13) satisfies $v_i(a,t) = 0$ $(i = 1, 2, \ldots, n)$. So there is at least one solution to (4.12) if system (4.13) is exactly controllable. In fact, there exists $(\hat\beta_1, \hat\beta_2)$ such that the corresponding solution of system (4.13) satisfies

$$\hat p_i(a,T) = w_{i+3}(a) - \gamma_i(a,T), \quad i = 1, 2, \ldots, n,$$

$$\to \quad \hat p_i(a,T) = w_{i+n}(a) - \gamma_i(a,T), \quad i = 1, 2, \ldots, n,$$

where $\gamma_i(a,t)$ is the solution to the system

$$\begin{cases} \int_0^a \gamma_1(\tau,t)\,d\tau + \int_0^t \gamma_1(a,s)\,ds + \int_0^t\int_0^a (\mu_1\gamma_1)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t\int_{a_1}^{a_2} m_1(a,s)\beta_1^*(s)\gamma_1(a,s)\,da\,ds \\ \quad + \int_0^t\int_0^a \lambda_1(\tau,s)[p_1^*(\tau,s)\Gamma_2(s) + \gamma_1(\tau,s)P_2^*(s)]\,d\tau\,ds = w_1(a,t), \\ \int_0^a \gamma_k(\tau,t)\,d\tau + \int_0^t \gamma_k(a,s)\,ds + \int_0^t\int_0^a (\mu_k\gamma_k)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t\int_{a_1}^{a_2} m_k(a,s)\beta_k^*(s)\gamma_k(a,s)\,da\,ds \\ \quad - \int_0^t\int_0^a \lambda_{2k-2}(\tau,s)[p_k^*(\tau,s)\Gamma_{k-1}(s) + \gamma_k(\tau,s)P_{k-1}^*(s)]\,d\tau\,ds \\ \quad + \int_0^t\int_0^a \lambda_{2k-1}(\tau,s)[p_k^*(\tau,s)\Gamma_{k+1}(s) + \gamma_k(\tau,s)P_{k+1}^*(s)]\,d\tau\,ds = w_k(a,t), \\ \quad k = 2,3,\ldots,n-1, \\ \int_0^a \gamma_n(\tau,t)\,d\tau + \int_0^t \gamma_n(a,s)\,ds + \int_0^t\int_0^a (\mu_n\gamma_n)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t\int_{a_1}^{a_2} m_n(a,s)\beta_n^*(s)\gamma_n(a,s)\,da\,ds \\ \quad - \int_0^t\int_0^a \lambda_{2n-2}(\tau,s)[p_n^*(\tau,s)\Gamma_{n-1}(s) + \gamma_n(\tau,s)P_{n-1}^*(s)]\,d\tau\,ds = w_n(a,t), \\ \Gamma_i(s) = \int_0^{a_+} \gamma_i(a,s)\,da. \end{cases}$$

Then it is not difficult to show that $(\hat\beta_1,\ldots,\hat\beta_n,\hat p_1+\gamma_1,\ldots,\hat p_n+\gamma_n)$ solves system (4.12). Now the tangent directions cone $K_2$ consists of the kernel of $G'(\beta^*,p^*)$.

Define the linear subspaces of $X$ by

$$K_{11} = \bigl\{(\beta,p)\in X:\ v_i(a,t)\equiv 0,\ i=1,2,\ldots,n\bigr\},$$

$$K_{12} = \bigl\{(\beta,p)\in X:\ p_i(a,T)\equiv 0,\ i=1,2,\ldots,n\bigr\},$$

where $v_i$ $(i=1,2,\ldots,n)$ are given by (4.9)–(4.11). Then $K_2 = K_{11}\cap K_{12}$, $K_2^* = K_{11}^* + K_{12}^*$. For any $f_2\in K_2^*$, $f_2 = f_{11}+f_{12}$, $f_{1k}\in K_{1k}^*$ $(k=1,2)$, there exists $\alpha_i(a)\in L^2(0,a_+)$, such that

$$f_{12}(\beta,p) = \sum_{i=1}^n \int_0^{a_+} \alpha_i(a)p_i(a,T)\,da. \tag{4.14}$$

According to Dubovitskii–Milyutin's theorem [13, Theorem 6.1], there exists functionals $f_0\in K_0^*$, $f_1\in K_1^*$, $f_{1k}\in K_{1k}^*$ $(k=1,2)$, not all zero, such that

$$f_0+f_1+f_{11}+f_{12}=0. \tag{4.15}$$

For any $\beta\in L^\infty(0,T)$, select $p$ such that the first three equations in (4.12) holds. Then $(\beta,p)\in K_{11}$ and $f_{11}(\beta,p)=0$ [13, Theorem 10.1], from which

$$f_1(\beta,p) = -f_0(\beta,p) - f_{12}(\beta,p)$$

$$= \sum_{i=1}^{n} \left\{ \lambda_0 \int_0^T \int_0^{a_+} \left[ \beta_i(t) \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) + p_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t) \right] da\, dt \right.$$
$$\left. - \int_0^{a_+} \alpha_i(a) p_i(a,T)\, da \right\}. \tag{4.16}$$

Define the adjoint system

$$\begin{cases} \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} = \mu_1 q_1 - m_1 \beta_1^* q_1(0,t) + \lambda_1 q_1 P_2^*(t) \\ \qquad + \lambda_0 \frac{\partial L}{\partial p_1}(\beta^*, p^*, a, t) - \int_0^{a_+} (\lambda_2 p_2^* q_2)(a,t)\, da, \\ \frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial a} = \mu_k q_k - m_k \beta_k^* q_k(0,t) - \lambda_{2k-2} q_k P_{k-1}^*(t) \\ \qquad + \lambda_{2k-1} q_k P_{k+1}^*(t) + \lambda_0 \frac{\partial L}{\partial p_k}(\beta^*, p^*, a, t) \\ \qquad + \int_0^{a_+} (\lambda_{2k-3} p_{k-1}^* q_{k-1} - \lambda_{2k} p_{k+1}^* q_{k+1})(a,t)\, da, \\ \frac{\partial q_n}{\partial t} + \frac{\partial q_n}{\partial a} = \mu_n q_n - m_n \beta_n^* q_n(0,t) - \lambda_{2n-2} q_n P_{n-1}^*(t) \\ \qquad + \lambda_0 \frac{\partial L}{\partial p_n}(\beta^*, p^*, a, t) + \int_0^{a_+} (\lambda_{2n-3} p_{n-1}^* q_{n-1})(a,t)\, da, \\ \quad k = 2, 3, \ldots, n-1, \\ q_i(a,T) = \alpha_i(a), \\ q_i(a_+, t) = 0, \quad (a,t) \in Q_T. \end{cases} \tag{4.17}$$

Then we can prove that

$$\sum_{i=1}^{n} \left[ \lambda_0 \int_0^T \int_0^{a_+} p_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\, da\, dt - \int_0^{a_+} \alpha_i(a) p_i(a,T)\, da \right]$$
$$= -\sum_{i=1}^{n} \int_0^T q_i(0,t) \int_0^{a_+} m_i(a,t) p_i^*(a,t)\, da \cdot \beta_i(t)\, dt. \tag{4.18}$$

From (4.16) and (4.18),

$$f_1(\beta, p) = \sum_{i=1}^{n} \int_0^T \int_0^{a_+} \left[ \lambda_0 \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) \right.$$
$$\left. - q_i(0,t) m_i(a,t) p_i^*(a,t) \right] da \cdot \beta_i(t)\, dt. \tag{4.19}$$

Consequently (4.7) leads us to

$$\begin{cases} \sum_{i=1}^{n} \int_0^{a_+} [\lambda_0 \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) - q_i(0,t) m_i(a,t) p_i^*(a,t)]\, da \\ \quad \times [\beta_i - \beta_i^*(t)] \geqslant 0, \\ \quad \forall \beta_i \in [\beta_0, \beta^0],\ t \in [0,T] \text{ a.e.} \end{cases} \tag{4.20}$$

We claim that there is no possibility for both $\lambda_0$ and $\alpha(a) = (\alpha_1(a), \alpha_2(a), \alpha_3(a)) \to \alpha(a) = (\alpha_1(a), \dots, \alpha_n(a))$ being zero. Otherwise $f_0 = 0$, $f_{12} = 0$, $q_i(a,t) = 0$, $f_1 = 0$ [8, p. 297]. Then from (4.15), $f_{11} = 0$. This contradicts the fact that $f_0, f_1, f_{11}, f_{12}$ are not all identically zero.

On the other hand, if $K_0 = \emptyset$, then for any $(\beta, p) \in X$,

$$\sum_{i=1}^{n} \int_0^T \int_0^{a_+} \left[ \beta_i(t) \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) + p_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t) \right] da\, dt = 0. \tag{4.21}$$

Choosing $\lambda_0 = 1$ and $\alpha(a) = 0$ in (4.18) yields

$$\begin{cases} \sum_{i=1}^n \int_0^T \int_0^{a_+} p_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\, da\, dt \\ \quad = -\sum_{i=1}^n \int_0^T q_i(0,t) \int_0^{a_+} m_i(a,t) p_i^*(a,t)\, da \cdot \beta_i(t)\, dt. \end{cases} \tag{4.22}$$

Combining (4.21) and (4.22), we still get relation (4.20).

Finally, if the adjoint system (4.17) has a nonzero solution such that

$$\int_0^{a_+} q_i(0,t) m_i(a,t) p_i^*(a,t)\, da = 0, \quad \forall t \in [0,T] \text{ a.e., } i = 1, 2, \dots, n, \tag{4.23}$$

then let $\lambda_0 = 0$, inequality (4.20) is also satisfied. If for any nonzero solution of (4.17), we always have

$$\left( \int_0^{a_+} q_1(0,t) m_1(a,t) p_1^*(a,t)\, da, \dots, \int_0^{a_+} q_n(0,t) m_n(a,t) p_n^*(a,t)\, da \right) \neq 0, \tag{4.24}$$

then system (4.13) must be controllable; otherwise there exists $\alpha(a) \in L^2(0, a_+; R^n)$ such that

$$\sum_{i=1}^{n} \int_0^{a_+} \alpha_i p_i(a,t)\, da = 0, \quad \alpha(a) \neq 0.$$

Choosing $\lambda_0 = 0$ in (4.18), we obtain that

$$\sum_{i=1}^{n} \int_0^T q_i(0,t) \int_0^{a_+} m_i(a,t) p_i^*(a,t)\, da \cdot \beta_i(t)\, dt = 0$$

holds for arbitrary $\beta_i(t) \in [\beta_0, \beta^0]$, which yields (4.23). This contradicts (4.24). Therefore system (4.13) is controllable.

In all cases, relation (4.20) remains valid. We have proved

**Theorem 3.** *If* $(\beta^*, p^*)$ *is a solution to problem* (4.2) *and* (4.3)*, then there exists* $\lambda_{0T} \geqslant 0$, $\alpha_T(a) \in L^2(0, a_+; R^n)$, *not all zero, such that*

$$\beta^*(t) \cdot H(\beta^*, p^*) = \max\{\beta \cdot H(\beta^*, p^*) \colon \beta \in [\beta_0, \beta^0]\}, \quad \forall t \in [0,T] \text{ a.e.},$$

*where $\cdot$ denotes the scalar product in $R^n$,*

$$H(\beta^*, p^*) = \left( \int_0^{a_+} \left[ q_1(0,t) m_1(a,t) p_1^*(a,t) - \lambda_0 \frac{\partial L}{\partial \beta_1}(\beta^*, p^*, a, t) \right] da, \right.$$
$$\int_0^{a_+} \left[ q_2(0,t) m_2(a,t) p_2^*(a,t) - \lambda_0 \frac{\partial L}{\partial \beta_2}(\beta^*, p^*, a, t) \right] da, \ldots,$$
$$\left. \int_0^{a_+} \left[ q_n(0,t) m_n(a,t) p_n^*(a,t) - \lambda_0 \frac{\partial L}{\partial \beta_n}(\beta^*, p^*, a, t) \right] da \right),$$

$q_i$ $(i = 1, 2, \ldots, n)$ *is the solution of the adjoint system* (4.17) *corresponding to* $\lambda_0 = \lambda_{0T}$, $\alpha_i = \alpha_{iT}$.

Now return to the infinite time problem (4.1). We suppose

$$\lambda_{0T} + \left\| q_T(a,\cdot) \right\|_{L^2(0,T;R^n)} \leqslant M, \quad \forall a \in [0, a_+] \text{ a.e.,} \tag{4.25}$$

where $M > 0$ is a constant. Choose $T_N \to \infty$ such that $\lambda_{0T_N} \to \lambda_\infty$. For any fixed $t > 0$ and $T_N$ large enough, by the means of characteristic line, we derive out that

$$\begin{cases} q_{1T_N}(0,t) = \int_t^{T_N} \exp\{-\int_t^s [\mu_1(\rho - t, \rho) + \lambda_1(\rho - t, \rho) P_2^*(\rho)]\, d\rho\} \\ \qquad \times \big[ m_1(s-t, s) \beta_1^*(s) q_{1T_N}(0, s) \\ \qquad + \int_0^{a_+} (\lambda_2 p_2^* q_{2T_N})(a, s)\, da - \lambda_{0T_N} \frac{\partial L}{\partial p_1}(\beta^*, p^*, s-t, s) \big]\, ds, \\ q_{kT_N}(0,t) = \int_t^{T_N} \exp\{-\int_t^s [\mu_k(\rho - t, \rho) - \lambda_{2k-2}(\rho - t, \rho) P_{k-1}^*(\rho) \\ \qquad + \lambda_{2k-1}(\rho - t, \rho) P_{k+1}^*(\rho)] d\rho\} \\ \qquad \times \big[ m_k(s-t, s) \beta_k^*(s) q_{kT_N}(0, s) \\ \qquad - \int_0^{a_+} (\lambda_{2k-3} p_{k-1}^* q_{(k-1)T_N}\, da - \lambda_{2k} p_{k+1}^* q_{(k+1)T_N})(a, s)\, da \\ \qquad - \lambda_{0T_N} \frac{\partial L}{\partial p_k}(\beta^*, p^*, s-t, s) \big]\, ds, \quad k = 2, 3, \ldots, n-1, \\ q_{nT_N}(0,t) = \int_t^{T_N} \exp\{-\int_t^s [\mu_n(\rho - t, \rho) - \lambda_{2n-2}(\rho - t, \rho) P_{n-1}^*(\rho)]\, d\rho\} \\ \qquad \times \big[ m_n(s-t, s) \beta_n^*(s) q_{nT_N}(0, s) \\ \qquad - \int_0^{a_+} (\lambda_{2n-3} p_{n-1}^* q_{(n-1)T_N})(a, s)\, da \\ \qquad - \lambda_{0T_N} \frac{\partial L}{\partial p_n}(\beta^*, p^*, s-t, s) \big]\, ds, \end{cases} \tag{4.26}$$

Note that $\int_t^s \mu_i(\rho - t, \rho)\, d\rho = +\infty$ when $s \geqslant t + a_+$. So the integration interval $[t, T_N]$ in (4.26) can be replaced by $[t, t + a_+]$. From (4.25) it follows that $\|q_{T_N}(a,\cdot)\|_{L^2(t,t+a_+;R^n)} \leqslant M$. Thus there is a subsequence of time (also denoted by $\{T_N\}$) such that

$$q_{T_N}(a,\cdot) \to q_\infty(a,\cdot) \quad \text{weakly in } L^2(t, t+a_+; R^n). \tag{4.27}$$

From (4.26) and (4.27), it is not difficult to prove that

$$\begin{cases} q_{1\infty}(0,t) = \int_t^{t+a_+} \exp\{-\int_t^s [\mu_1(\rho-t,\rho)+\lambda_1(\rho-t,\rho)P_2^*(\rho)]\,d\rho\} \\ \qquad \times \big[m_1(s-t,s)\beta_1^*(s)q_{1\infty}(0,s) \\ \qquad + \int_0^{a_+}(\lambda_2 p_2^* q_{2\infty})(a,s)\,da - \lambda_\infty \frac{\partial L}{\partial p_1}(\beta^*,p^*,s-t,s)\big]\,ds, \\ q_{k\infty}(0,t) = \int_t^{t+a_+} \exp\{-\int_t^s [\mu_k(\rho-t,\rho)-\lambda_{2k-2}(\rho-t,\rho)P_{k-1}^*(\rho) \\ \qquad + \lambda_{2k-1}(\rho-t,\rho)P_{k+1}^*(\rho)]\,d\rho\} \\ \qquad \times \big[m_k(s-t,s)\beta_k^*(s)q_{k\infty}(0,s) \\ \qquad - \int_0^{a_+}(\lambda_{2k-3}p_{k-1}^* q_{(k-1)\infty} - \lambda_{2k}p_{k+1}^* q_{(k+1)\infty})(a,s)\,da \\ \qquad - \lambda_\infty \frac{\partial L}{\partial p_k}(\beta^*,p^*,s-t,s)\big]\,ds, \quad k=2,3,\ldots,n-1, \\ q_{n\infty}(0,t) = \int_t^{t+a_+} \exp\{-\int_t^s [\mu_n(\rho-t,\rho)-\lambda_{2n-2}(\rho-t,\rho)P_{n-1}^*(\rho)]\,d\rho\} \\ \qquad \times \big[m_n(s-t,s)\beta_n^*(s)q_{n\infty}(0,s) \\ \qquad - \int_0^{a_+}(\lambda_{2n-3}p_{n-1}^* q_{(n-1)\infty})(a,s)\,da \\ \qquad - \lambda_\infty \frac{\partial L}{\partial p_n}(\beta^*,p^*,s-t,s)\big]\,ds, \end{cases}$$

which enables us to state

**Theorem 4.** *Let* $(\beta^*, p^*)$ *be a solution for problem* (4.1)*, then there exist* $\lambda_\infty \geqslant 0$ *and a function* $q:[0,\infty) \to R^n$*, not simultaneously zero, such that*

$$\beta^*(t)\cdot H(\beta^*,p^*) = \max\big\{\beta\cdot H(\beta^*,p^*):\ \beta\in[\beta_0,\beta^0]\big\},$$
$$\forall t\in[0,\infty]\ \text{a.e.},$$

*where*

$$H(\beta^*,p^*) = \Bigg(\int_0^{a_+}\bigg[q_1(0,t)m_1(a,t)p_1^*(a,t) - \lambda_\infty \frac{\partial L}{\partial \beta_1}(\beta^*,p^*,a,t)\bigg]\,da,$$
$$\int_0^{a_+}\bigg[q_2(0,t)m_2(a,t)p_2^*(a,t) - \lambda_\infty \frac{\partial L}{\partial \beta_2}(\beta^*,p^*,a,t)\bigg]\,da,\ldots,$$
$$\int_0^{a_+}\bigg[q_n(0,t)m_n(a,t)p_n^*(a,t) - \lambda_\infty \frac{\partial L}{\partial \beta_n}(\beta^*,p^*,a,t)\bigg]\,da\Bigg),$$

*$q_i(a,t)$ is given by the adjoint system*

$$\begin{cases} \frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial a}=\mu_1 q_1-m_1\beta_1^* q_1(0,t)+\lambda_1 q_1 P_2^*(t) \\ \qquad\qquad +\lambda_\infty \frac{\partial L}{\partial p_1}(\beta^*,p^*,a,t)-\int_0^{a_+}(\lambda_2 p_2^* q_2)(a,t)\,da, \\ \frac{\partial q_k}{\partial t}+\frac{\partial q_k}{\partial a}=\mu_k q_k-m_k\beta_k^* q_k(0,t)-\lambda_{2k-2} q_k P_{k-1}^*(t) \\ \qquad\qquad +\lambda_{2k-1} q_k P_{k+1}^*(t)+\lambda_\infty \frac{\partial L}{\partial p_k}(\beta^*,p^*,a,t) \\ \qquad\qquad +\int_0^{a_+}(\lambda_{2k-3} p_{k-1}^* q_{k-1}-\lambda_{2k} p_{k+1}^* q_{k+1})(a,t)\,da, \\ \quad k=2,3,\dots,n-1, \\ \frac{\partial q_n}{\partial t}+\frac{\partial q_n}{\partial a}=\mu_n q_n-m_n\beta_n^* q_n(0,t)-\lambda_{2n-2} q_n P_{n-1}^*(t) \\ \qquad\qquad +\lambda_\infty \frac{\partial L}{\partial p_n}(\beta^*,p^*,a,t)+\int_0^{a_+}(\lambda_{2n-3} p_{n-1}^* q_{n-1})(a,t)\,da, \\ q_i(a,\infty)=0, \\ q_i(a_+,t)=0, \quad (a,t)\in Q, \quad i=1,2,\dots,n. \end{cases}$$

## 5. Constrained end point problem

Problem (4.2) and (4.3) leads us to the problem

$$\text{Minimize} \quad J(\beta,p)=\int_0^T\int_0^{a_+} L\big(\beta_1(t),\dots,\beta_n(t), \\ p_1(a,t),\dots,p_n(a,t),a,t\big)\,da\,dt, \tag{5.1}$$

where $T>0$ is fixed, $\beta\in U$ and $(\beta,p)$ is subject to system (2.1) and

$$p_i(\cdot,T)\in V_i, \quad V_i=\big\{p\in L^2(0,a_+)\colon\ \|p-p_i^0\|\leqslant\varepsilon\big\}, \quad i=1,2,\dots,n, \tag{5.2}$$

in which $p_i^0$ and $\varepsilon$ are prescribed. The assumptions on $L$ and the definition of $X$ and $\Omega_1$ are as before.

Let

$$\Omega_2=\big\{(\beta,p)\in X\colon\ p_i(\cdot,T)\in V_i,\ i=1,2,\dots,n\big\},$$
$$\Omega_3=\big\{(\beta,p)\in X\colon\ (\beta,p)\text{ satisfies (1)}\big\}.$$

Suppose that $(\beta^*,p^*)$ solves problem (5.1) and (5.2). Clearly the directions of decrease cone and its dual are as in Section 4; so are the feasible directions cone for $\Omega_1$ and its dual. Since $\Omega_2$ is a closed convex set and $\mathrm{int}(\Omega_2)\neq\emptyset$, any functional $f_2$ in the dual of the feasible directions cone for $\Omega_2$ is supporting; that is,

$$f_2(\beta,p)\geqslant f_2(\beta^*,p^*), \quad \forall p(a,T)\in\prod_{i=1}^n V_i.$$

Obviously there exists $\alpha \in L^2(0, a_+; R^n)$ such that

$$f_2(\beta, p) = \int_0^{a_+} \alpha(a) \cdot p(a, T)\, da.$$

Therefore [8, p. 300]

$$\alpha(a) = \tilde{\lambda}_0\big[p^0(a) - p^*(a, T)\big], \quad \tilde{\lambda}_0 \geqslant 0.$$

Then by a reasoning similar to that in Section 4, we arrive at

**Theorem 5.** *If* $(\beta^*, p^*)$ *is a solution to problem* (5.1) *and* (5.2), *then there exist* $\lambda_0 \geqslant 0$ *and* $\tilde{\lambda}_0 \geqslant 0$, *not both zero, such that*

$$\beta^*(t) \cdot H(\beta^*, p^*) = \max\big\{\beta \cdot H(\beta^*, p^*)\colon\ \beta \in [\beta_0, \beta^0]\big\}, \quad \forall t \in [0, T] \text{ a.e.},$$

$$\begin{aligned} H(\beta^*, p^*) = \Bigg( & \int_0^{a_+} \bigg[q_1(0, t) m_1(a, t) p_1^*(a, t) - \lambda_0 \frac{\partial L}{\partial \beta_1}(\beta^*, p^*, a, t)\bigg] da, \\ & \int_0^{a_+} \bigg[q_2(0, t) m_2(a, t) p_2^*(a, t) - \lambda_0 \frac{\partial L}{\partial \beta_2}(\beta^*, p^*, a, t)\bigg] da, \ldots, \\ & \int_0^{a_+} \bigg[q_n(0, t) m_n(a, t) p_n^*(a, t) - \lambda_0 \frac{\partial L}{\partial \beta_n}(\beta^*, p^*, a, t)\bigg] da\Bigg), \end{aligned}$$

*and* $q_i$ *is the solution of the adjoint system*

$$\begin{cases} \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} = \mu_1 q_1 - m_1 \beta_1^* q_1(0, t) + \lambda_1 q_1 P_2^*(t) \\ \qquad\qquad + \lambda_0 \frac{\partial L}{\partial p_1}(\beta^*, p^*, a, t) - \int_0^{a_+} (\lambda_2 p_2^* q_2)(a, t)\, da, \\ \frac{\partial q_k}{\partial t} + \frac{\partial q_k}{\partial a} = \mu_k q_k - m_k \beta_k^* q_k(0, t) - \lambda_{2k-2} q_k P_{k-1}^*(t) \\ \qquad\qquad + \lambda_{2k-1} q_k P_{k+1}^*(t) + \lambda_0 \frac{\partial L}{\partial p_k}(\beta^*, p^*, a, t) \\ \qquad\qquad + \int_0^{a_+} (\lambda_{2k-3} p_{k-1}^* q_{k-1} - \lambda_{2k} p_{k+1}^* q_{k+1})(a, t)\, da, \\ \quad k = 2, 3, \ldots, n-1, \\ \frac{\partial q_n}{\partial t} + \frac{\partial q_n}{\partial a} = \mu_n q_n - m_n \beta_n^* q_n(0, t) - \lambda_{2n-2} q_n P_{n-1}^*(t) \\ \qquad\qquad + \lambda_0 \frac{\partial L}{\partial p_n}(\beta^*, p^*, a, t) + \int_0^{a_+} (\lambda_{2n-3} p_{n-1}^* q_{n-1})(a, t)\, da, \\ q_i(a, T) = \tilde{\lambda}_0[p_i^0(a) - p_i^*(a, T)], \\ q_i(a_+, t) = 0, \quad (a, t) \in Q_T. \end{cases}$$

**Remark 1.** Note that just for the sake of simplicity, the average fertility of female individuals $\beta_i(t)$ in system (2.1) is choosing to be independent of age $a$. Replacing $\beta_i(t)$ with $\beta_i(a, t)$ forms no essential obstacles to the previous treatment.

**Remark 2.** We can easily check that the all results but the time-optimal problem in [2] are contained by this work.

**Remark 3.** If $\lambda_k(a,t) \equiv 0$, $k = 1, 2, \dots, 2n-2$, $\forall (a,t) \in Q$, then our results reduce to these of Chan and Guo [2].

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