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Positive solution for m-point boundary value problems of fourth order

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Abstract

In this paper, we consider the existence of positive solutions to the fourth order boundary value problem

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t,u), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases}$$

where $\alpha, \beta \in R$, $\xi_i \in (0,1)$, $a_i, b_i \in [0,\infty)$ for $i \in \{1,2,\ldots,m-2\}$ are given constants satisfying some suitable conditions. The proofs are based on the fixed point index theorem in cones.

Keywords: Multipoint boundary value problems; Positive solution; Fixed point index; Cones; Eigenvalue

1. Introduction

Multipoint boundary value problems (BVPs) for ordinary differential equations arise in a variety of areas of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint BVP in [4]; also, many problems in the theory of elastic stability can be handled by multipoint problems in [5].

In [6], Il'in and Moiseev studied the existence of solutions for a linear multipoint BVP. Motivated, Gupta [7] studied certain three-point BVPs for nonlinear ordinary differential equations.

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Since then, more general nonlinear multipoint BVPs have been studied by several authors. We refer the reader to [2,3,7] for some references. When it comes to fourth order BVP, only two-point situation is often seen, here we just refer to [8,9]. Recently, Liu [8] has obtained some existence results for

$$\begin{cases} u'''' = f(t, u, u''), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.1a}$$

under the condition that f is either superlinear or sublinear.

In 2003, Li [1] studied the existence of positive solutions for fourth order BVP without bending term but with two parameters

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t, u), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.1b}$$

under the assumptions:

(H1) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous;
(H2) $\alpha, \beta \in R$ and $\alpha < 2\pi^2$, $\beta \geqslant -\alpha^2/4$, $\alpha/\pi^2 + \beta/\pi^4 < 1$.

He established the following result for (1.1b).

Theorem 1.1. [1] *Assume* (H1) *and* (H2) *hold. Then in each of the following cases*:

(i) $\overline{f}_0 < \pi^4 - \alpha\pi^2 - \beta$, $\underline{f}_\infty > \pi^4 - \alpha\pi^2 - \beta$,
(ii) $\underline{f}_0 > \pi^4 - \alpha\pi^2 - \beta$, $\overline{f}_\infty < \pi^4 - \alpha\pi^2 - \beta$,

the BVP (1.1b) *has at least one positive solution, where*

$$\underline{f}_0 = \liminf_{u \to 0+} \min_{t \in [0,1]} \big(f(t, u)/u\big), \qquad \overline{f}_0 = \limsup_{u \to 0+} \max_{t \in [0,1]} \big(f(t, u)/u\big),$$
$$\underline{f}_\infty = \liminf_{u \to +\infty} \min_{t \in [0,1]} \big(f(t, u)/u\big), \qquad \overline{f}_\infty = \limsup_{u \to +\infty} \max_{t \in [0,1]} \big(f(t, u)/u\big).$$

In this paper, we are interested in the existence of a positive solution for the more general fourth order m-point BVP,

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t, u), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases} \tag{1.1}$$

where $\alpha, \beta \in R, \xi_i \in (0, 1), a_i, b_i \in [0, \infty)$ for $i \in \{1, 2, \ldots, m-2\}$ are given constants. It is clear that when $a_i = b_i = 0$, then (1.1) is reduced to (1.1b). To deal with (1.1), we give an integral equation which is equivalent to (1.1). It is naturally expected that an integral equation equivalent to

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t, u), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} c_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} d_i u''(\xi_i) \end{cases}$$

holds true. It at last fails when we attempt on it. Hence we just consider (1.1).

By a positive solution of (1.1), we understand a function u which is positive on $(0,1)$ and satisfies the differential equation as well as the boundary conditions in (1.1).

The main tools of this paper are the following well-known fixed point index theorems.

Theorem 1.2. [10] *Let E be a Banach space, and let $P \subset E$ be a cone. Assume $\Omega(P)$ is a bounded open set in P. Suppose that $A:\overline{\Omega(P)} \to P$ is a completely continuous operator. If there exists $\psi_0 \in P\backslash\{\theta\}$ such that*

$$\varphi - A\varphi \neq \mu\psi_0, \quad \forall\varphi \in \partial\Omega(P),\ \mu \geqslant 0,$$

then the fixed point index $i(A, \Omega(P), P) = 0$.

Theorem 1.3. [10] *Let E be a Banach space, and let $P \subset E$ be a cone. Assume $\Omega(P)$ is a bounded open set in P with $\theta \in \Omega(P)$. Suppose that $A:\overline{\Omega(P)} \to P$ is a completely continuous operator. If*

$$A\psi \neq \mu\psi, \quad \forall\psi \in \partial\Omega(P),\ \mu \geqslant 1,$$

then the fixed point index $i(A, \Omega(P), P) = 1$.

The paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we obtain an existence result for the BVP (1.1). Finally, in Section 4, we obtain a uniqueness result of positive solution for a kind of special f.

2. Preliminary lemmas

To state and prove the main result of this paper, we need the following lemmas.

Lemma 2.1. *Let* (H2) *holds. Then there exist unique $\varphi_1, \varphi_2, \psi_1, \psi_2$ satisfying*

$$\begin{cases} -\varphi_1'' + \lambda_1\varphi_1 = 0, \\ \varphi_1(0) = 0, \quad \varphi_1(1) = 1; \end{cases} \qquad \begin{cases} -\varphi_2'' + \lambda_1\varphi_2 = 0, \\ \varphi_2(0) = 1, \quad \varphi_2(1) = 0; \end{cases}$$

$$\begin{cases} -\psi_1'' + \lambda_2\psi_1 = 0, \\ \psi_1(0) = 0, \quad \psi_1(1) = 1; \end{cases} \qquad \begin{cases} -\psi_2'' + \lambda_2\psi_2 = 0, \\ \psi_2(0) = 1, \quad \psi_2(1) = 0 \end{cases}$$

respectively. And on $[0,1]$, *$\varphi_1, \varphi_2, \psi_1, \psi_2 \geqslant 0$, where λ_1, λ_2 are the roots for the polynomial equation*

$$\lambda^2 + \alpha\lambda - \beta = 0.$$

That is,

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta}}{2}, \qquad \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 + 4\beta}}{2}.$$

Proof. Let $\omega_i = \sqrt{|\lambda_i|}$ $\{i = 1, 2\}$. We can get by computation that

- if $\lambda_1 > 0$, $\varphi_1(t) = \frac{\sinh\omega_1 t}{\sinh\omega_1}$, $\varphi_2(t) = \frac{\sinh\omega_1(1-t)}{\sinh\omega_1}$;
- if $\lambda_1 = 0$, $\varphi_1(t) = t$, $\varphi_2(t) = 1 - t$;
- if $-\pi^2 < \lambda_1 < 0$, $\varphi_1(t) = \frac{\sin\omega_1 t}{\sin\omega_1}$, $\varphi_2(t) = \frac{\sin\omega_1(1-t)}{\sin\omega_1}$;
- if $\lambda_2 > 0$, $\psi_1(t) = \frac{\sinh\omega_2 t}{\sinh\omega_2}$, $\psi_2(t) = \frac{\sinh\omega_2(1-t)}{\sinh\omega_2}$;

– if $\lambda_2 = 0$, $\psi_1(t) = t$, $\psi_2(t) = 1 - t$;
– if $-\pi^2 < \lambda_2 < 0$, $\psi_1(t) = \frac{\sin \omega_2 t}{\sin \omega_2}$, $\psi_2(t) = \frac{\sin \omega_2(1-t)}{\sin \omega_2}$.

It is obvious that on $[0, 1]$, $\varphi_1, \varphi_2, \psi_1, \psi_2 \geqslant 0$, and $\varphi_1'(0), \psi_1'(0) > 0$. □

In the rest of the paper, we make the following assumptions:

(A1) $\sum_{i=1}^{m-2} a_i \varphi_2(\xi_i) < 1$, $\sum_{i=1}^{m-2} b_i \varphi_1(\xi_i) < 1$;
(A2) $\sum_{i=1}^{m-2} a_i \psi_2(\xi_i) < 1$, $\sum_{i=1}^{m-2} b_i \psi_1(\xi_i) < 1$.

Notation. Set

$$\rho_1 = \varphi_1'(0), \qquad \rho_2 = \psi_1'(0),$$

$$\Delta_1 = \begin{vmatrix} \sum_{i=1}^{m-2} a_i \varphi_1(\xi_i) & \sum_{i=1}^{m-2} a_i \varphi_2(\xi_i) - 1 \\ \sum_{i=1}^{m-2} b_i \varphi_1(\xi_i) - 1 & \sum_{i=1}^{m-2} b_i \varphi_2(\xi_i) \end{vmatrix}, \tag{2.1}$$

$$\Delta_2 = \begin{vmatrix} \sum_{i=1}^{m-2} a_i \psi_1(\xi_i) & \sum_{i=1}^{m-2} a_i \psi_2(\xi_i) - 1 \\ \sum_{i=1}^{m-2} b_i \psi_1(\xi_i) - 1 & \sum_{i=1}^{m-2} b_i \psi_2(\xi_i) \end{vmatrix}. \tag{2.2}$$

Lemma 2.2. *Let* (H2) *holds. Assume that*

(H3) $\Delta_1 \neq 0$.

Then for any $g \in C[0, 1]$, *the problem*

$$\begin{cases} -u'' + \lambda_1 u = g(t), \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i) \end{cases} \tag{2.3}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t, s) g(s)\, \mathrm{d}s + A(g)\varphi_1(t) + B(g)\varphi_2(t),$$

where

$$G_1(t, s) = \frac{1}{\rho_1} \begin{cases} \varphi_1(t)\varphi_2(s), & t \leqslant s, \\ \varphi_1(s)\varphi_2(t), & s \leqslant t, \end{cases} \tag{2.4}$$

$$A(g) := \frac{1}{\Delta_1} \begin{vmatrix} -\sum_{i=1}^{m-2} a_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s & \sum_{i=1}^{m-2} a_i \varphi_2(\xi_i) - 1 \\ -\sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s & \sum_{i=1}^{m-2} b_i \varphi_2(\xi_i) \end{vmatrix} \tag{2.5}$$

and

$$B(g) := \frac{1}{\Delta_1} \begin{vmatrix} \sum_{i=1}^{m-2} a_i \varphi_1(\xi_i) & -\sum_{i=1}^{m-2} a_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s \\ \sum_{i=1}^{m-2} b_i \varphi_1(\xi_i) - 1 & -\sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s \end{vmatrix}. \tag{2.6}$$

Proof. The proof follows by routine calculations. □

Next we assume

(H4) $\Delta_1 < 0$.

Lemma 2.3. *Let* (A1), (H2) *and* (H4) *hold. Then for any* $g \in C[0,1]$ *and* $g \geqslant 0$, *the unique solution* u *of problem* (2.3) *satisfies*

$$u(t) \geqslant 0, \quad t \in [0,1].$$

Proof. Since $\Delta_1 < 0$ and $G_1 \geqslant 0$ on $[0,1] \times [0,1]$,

$$A(g) \geqslant 0, \qquad B(g) \geqslant 0,$$

the result is obviously true. $\square$

Remark 2.1. In Lemma 2.3, if we change (A1) to either $\sum_{i=1}^{m-2} a_i \varphi_2(\xi_i) > 1$ or $\sum_{i=1}^{m-2} b_i \varphi_1(\xi_i) > 1$, then for any $g \in C[0,1]$ and $g \geqslant 0$, (2.3) has no positive solution.

Consider

$$\begin{cases} -u''(t) + \lambda_1 u(t) = g(t), & 0 < t < 1, \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where $\sum_{i=1}^{m-2} b_i \varphi_1(\xi_i) > 1$. Then

$$\Delta_1 = \sum_{i=1}^{m-2} b_i \varphi_1(\xi_i) - 1, \quad B(g) = 0,$$

$$A(g) = \frac{\sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s}{1 - \sum_{i=1}^{m-2} b_i \varphi_1(\xi_i)} < 0.$$

So we have

$$u(t) = \sum_{i=1}^{m-2} b_i \int_0^1 G_1(t, s) g(s)\, \mathrm{d}s + A(g)\varphi_1(t).$$

Noticing that $\frac{\sum_{i=1}^{m-2} b_i \varphi_1(\xi_i)}{1 - \sum_{i=1}^{m-2} b_i \varphi_1(\xi_i)} < -1$, we have

$$\begin{aligned} u(1) &= \sum_{i=1}^{m-2} b_i u(\xi_i) = \sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s + A(g) \sum_{i=1}^{m-2} b_i \varphi_1(\xi_i) \\ &= \sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s + \frac{\sum_{i=1}^{m-2} b_i \varphi_1(\xi_i)}{1 - \sum_{i=1}^{m-2} b_i \varphi_1(\xi_i)} \sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s \\ &< \sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s - \sum_{i=1}^{m-2} b_i \int_0^1 G_1(\xi_i, s) g(s)\, \mathrm{d}s \\ &= 0. \end{aligned}$$

Similarly, we have

Lemma 2.4. *Let* (H2) *holds. Assume that*

(H5) $\Delta_2 \neq 0$.

Then for any $g \in C[0,1]$, *the problem*

$$\begin{cases} -u'' + \lambda_2 u = g(t), \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i) \end{cases} \tag{2.7}$$

has a unique solution

$$u(t) = \int_0^1 G_2(t,s)g(s)\,\mathrm{d}s + C(g)\psi_1(t) + D(g)\psi_2(t),$$

where

$$G_2(t,s) = \frac{1}{\rho_2}\begin{cases} \psi_1(t)\psi_2(s), & t \leqslant s, \\ \psi_1(s)\psi_2(t), & s \leqslant t, \end{cases} \tag{2.8}$$

$$C(g) := \frac{1}{\Delta_2}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i \int_0^1 G_2(\xi_i,s)g(s)\,\mathrm{d}s & \sum_{i=1}^{m-2} a_i\psi_2(\xi_i) - 1 \\ -\sum_{i=1}^{m-2} b_i \int_0^1 G_2(\xi_i,s)g(s)\,\mathrm{d}s & \sum_{i=1}^{m-2} b_i\psi_2(\xi_i) \end{vmatrix}, \tag{2.9}$$

$$D(g) := \frac{1}{\Delta_2}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\psi_1(\xi_i) & -\sum_{i=1}^{m-2} a_i \int_0^1 G_2(\xi_i,s)g(s)\,\mathrm{d}s \\ \sum_{i=1}^{m-2} b_i\psi_1(\xi_i) - 1 & -\sum_{i=1}^{m-2} b_i \int_0^1 G_2(\xi_i,s)g(s)\,\mathrm{d}s \end{vmatrix}. \tag{2.10}$$

In the following, we need the assumption:

(H6) $\Delta_2 < 0$.

Lemma 2.5. *Let* (A2), (H2) *and* (H6) *hold. Then for any* $g \in C[0,1]$ *and* $g \geqslant 0$, *the unique solution* u *of problem* (2.7) *satisfies*

$$u(t) \geqslant 0, \quad t \in [0,1].$$

Remark 2.2. In Lemma 2.5, if we change (A2) to either $\sum_{i=1}^{m-2} a_i\psi_2(\xi_i) > 1$ or $\sum_{i=1}^{m-2} b_i\psi_1(\xi_i) > 1$, then for any $g \in C[0,1]$ and $g \geqslant 0$, (2.7) has no positive solution.

Now notice that

$$u'''' + \alpha u'' - \beta u = \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \lambda_1\right)\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \lambda_2\right)u,$$

so we can easily get

Lemma 2.6. *Let* (H2), (H4) *and* (H6) *hold. Then for any* $g \in C[0,1]$, *the problem*

$$\begin{cases} u'''' + \alpha u'' - \beta u = g(t), \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases} \tag{2.11}$$

has a unique solution

$$u(t)=\int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)g(s)\,\mathrm{d}s\,\mathrm{d}\tau+\int_0^1 G_2(t,\tau)A(g)\varphi_1(\tau)\,\mathrm{d}\tau$$
$$+\int_0^1 G_2(t,\tau)B(g)\varphi_2(\tau)\,\mathrm{d}\tau+C(h)\psi_1(t)+D(h)\psi_2(t), \tag{2.12}$$

where $G_1, G_2, A(g), B(g), C(g), D(g)$ *are defined as in* (2.4), (2.8), (2.5), (2.6), (2.9), (2.10) *and*

$$h(t)=\int_0^1 G_1(t,s)g(s)\,\mathrm{d}s+A(g)\varphi_1(t)+B(g)\varphi_2(t).$$

In addition, if (A1), (A2) *hold and* $g\geqslant 0$*, then*

$$u(t)\geqslant 0,\quad t\in[0,1].$$

Remark 2.3. In Lemma 2.6, if neither (A1) nor (A2) holds true, then (2.11) has no positive solution.

Lemma 2.7. *For any* $g\in C[0,1]$ *and* $g\geqslant 0$*,* $A(g)$*,* $B(g)$*,* $C(g)$*,* $D(g)$ *are all linear functionals and nondecreasing in* g*.*

Let $E=C[0,1]$ and $P=\{u\in E\colon\ u\geqslant 0\}$. It is obvious that P is a cone in E. Define $T:E\to E$,

$$T:Tu(t)=\int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)f\big(s,u(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau+\int_0^1 G_2(t,\tau)A(f)\varphi_1(\tau)\,\mathrm{d}\tau$$
$$+\int_0^1 G_2(t,\tau)B(f)\varphi_2(\tau)\,\mathrm{d}\tau+C(e)\psi_1(t)+D(e)\psi_2(t), \tag{2.13}$$

where

$$e(t)=\int_0^1 G_1(t,s)f\big(s,u(s)\big)\,\mathrm{d}s+A(f)\varphi_1(t)+B(f)\varphi_2(t).$$

By Lemma 2.6, we know that u is the fixed point of T in P is equivalent to u is the positive solution of (1.1). Define $L:E\to E$,

$$L:Lu(t)=\int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,\mathrm{d}s\,\mathrm{d}\tau+\int_0^1 G_2(t,\tau)A(u)\varphi_1(\tau)\,\mathrm{d}\tau$$
$$+\int_0^1 G_2(t,\tau)B(u)\varphi_2(\tau)\,\mathrm{d}\tau+C(h)\psi_1(t)+D(h)\psi_2(t), \tag{2.14}$$

where

$$h(t) = \int_0^1 G_1(t,s)u(s)\,\mathrm{d}s + A(u)\varphi_1(t) + B(u)\varphi_2(t).$$

Lemma 2.8. *Let* (A1), (A2), (H1), (H2), (H4) *and* (H6) *hold. Then* $T : P \to P$ *is completely continuous.*

Lemma 2.9. *Let* (A1), (A2), (H2), (H4) *and* (H6) *hold. Then* $L : P \to P$ *is completely continuous.*

Notation.

$$\underline{f}_0 = \liminf_{u\to 0+} \min_{t\in[0,1]} \big(f(t,u)/u\big), \qquad \bar{f}_0 = \limsup_{u\to 0+} \max_{t\in[0,1]} \big(f(t,u)/u\big),$$
$$\underline{f}_\infty = \liminf_{u\to +\infty} \min_{t\in[0,1]} \big(f(t,u)/u\big), \qquad \bar{f}_\infty = \limsup_{u\to +\infty} \max_{t\in[0,1]} \big(f(t,u)/u\big).$$

Lemma 2.10 *(Krein–Rutmann).* [11] *Let* $\mathcal{L} : E \to E$ *be a continuous linear operator and* $\mathcal{L}(P) \subset P$. *If there exist* $\psi \in E \setminus (-P)$ *and a positive constant c such that* $c\mathcal{L}\psi \geqslant \psi$, *then the spectral radius* $r(\mathcal{L}) \neq 0$ *and T have a positive eigenfunction corresponding to its first eigenvalue* $\lambda_* = r(\mathcal{L})^{-1}$.

Lemma 2.11. *If* (A1), (A2), (H2), (H4) *and* (H6) *are satisfied, then for the operator T defined by* (2.13), *the spectral radius* $r(L) \neq 0$ *and L have a positive eigenfunction corresponding to its first eigenvalue* $\lambda_* = r(L)^{-1}$.

Proof. It is easy to see that there is $t_1 \in (0,1)$ such that $G_1(t_1,t_1)G_2(t_1,t_1) > 0$. Thus there exists $[\alpha,\beta] \subset (0,1)$ such that $t_1 \in (\alpha,\beta)$ and

$$G_2(t,\tau)G_1(\tau,s) > 0, \quad t,\tau,s \in [\alpha,\beta].$$

Take $u \in E$ such that $u(t) \geqslant 0$, $\forall x \in [0,1]$, $u(t_1) > 0$ and $u(t) = 0$, $\forall t \notin [\alpha,\beta]$. Then for $t \in [\alpha,\beta]$,

$$\begin{aligned}
Lu(t) &= \int_0^1 \int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau)A(u)\varphi_1(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^1 G_2(t,\tau)B(u)\varphi_2(\tau)\,\mathrm{d}\tau + C(h)\psi_1(t) + D(h)\psi_2(t) \\
&\geqslant \int_\alpha^\beta \int_\alpha^\beta G_2(t,\tau)G_1(\tau,s)u(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_\alpha^\beta G_2(t,\tau)A(u)\varphi_1(\tau)\,\mathrm{d}\tau \\
&\quad + \int_\alpha^\beta G_2(t,\tau)B(u)\varphi_2(\tau)\,\mathrm{d}\tau + C(h)\psi_1(t) + D(h)\psi_2(t) \\
&> 0.
\end{aligned}$$

So there exists a constant $c > 0$ such that $t \in [0,1]$, $c(Lu)(t) \geqslant u(t)$. From Lemma 2.10, we know that the spectral radius $r(L) \neq 0$ and L have a positive eigenfunction corresponding to its first eigenvalue $\lambda_* = r(T)^{-1}$. □

3. Existence result for the BVP (1.1)

In this section, we obtain the following existence result for a positive solution of BVP (1.1). It is our main result.

Theorem 3.1. *Suppose that the conditions* (A1), (A2), (H1), (H2), (H4) *and* (H6) *are satisfied, and*

$$\underline{f}_0 > \lambda_*, \qquad \overline{f}_\infty < \lambda_*,$$

where λ_* is the first eigenvalue of L defined by (2.14). *Then the m-point boundary value problem* (1.1) *has at least one positive solution.*

Proof. From $\underline{f}_0 > \lambda_*$ we know that there exists $r_1 > 0$ such that

$$f(t,u) \geqslant \lambda_* u, \quad \forall t \in [0,1],\ u \in [0, r_1]. \tag{3.1}$$

Let u^* be a positive eigenfunction of L corresponding to λ_*, thus $u^* = \lambda_* L u^*$.

For every $u \in \partial B_{r_1} \cap P$, it follows from (3.1) that

$$\begin{aligned}
Tu(t) &= \int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s) f\big(s,u(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau)A(f)\varphi_1(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^1 G_2(t,\tau)B(f)\varphi_2(\tau)\,\mathrm{d}\tau + C(e)\psi_1(t) + D(e)\psi_2(t) \\
&\geqslant \lambda_* \int_0^1\int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau)A(u)\varphi_1(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^1 G_2(t,\tau)B(u)\varphi_2(\tau)\,\mathrm{d}\tau + C(h)\psi_1(t) + D(h)\psi_2(t) \\
&= \lambda_*(Lu)(t), \quad t \in [0,1].
\end{aligned} \tag{3.2}$$

We may suppose that T has no fixed point on $\partial B_{r_1} \cap P$ (otherwise, the proof is complete). Now we show that

$$u - Tu \neq \mu u^*, \quad \forall u \in \partial B_{r_1} \cap P,\ \mu \geqslant 0. \tag{3.3}$$

If otherwise, there exist $u_1 \in \partial B_{r_1} \cap P$ and $\tau_0 \geqslant 0$ such that

$$u_1 - Tu_1 = \tau_0 u^*,$$

hence $\tau_0 > 0$ and

$$u_1 = Tu_1 + \tau_0 u^* \geqslant \tau_0 u^*.$$

Put

$$\tau^* = \sup\{\tau\colon\ u_1 \geqslant \tau u^*\}.$$

It is easy to show that $\tau^* \geqslant \tau_0 > 0$ and $u_1 \geqslant \tau^* u^*$. We find from $L(P) \subset P$ that

$$\lambda_* L u_1 \geqslant \tau^* \lambda_* L u^* = \tau^* \varphi^*.$$

Therefore by (3.2),

$$u_1 = T u_1 + \tau_0 u^* \geqslant \lambda_* L u_1 + \tau_0 u^* \geqslant (\tau^* + \tau_0) u^*,$$

which contradicts the definition of τ^*. Hence (3.3) is true and we have from Theorem 1.2 that

$$i(T, B_{r_1} \cap P, P) = 0. \tag{3.4}$$

From $\overline{f}_\infty < \lambda_*$ we know that there exist $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$f(t, u) \leqslant \sigma \lambda_* u, \quad \forall t \in [0, 1],\ u \in [r_2, +\infty).$$

Let $L_1 u = \sigma \lambda_* L u$, $u \in E$. Then $L_1 : E \to E$ is a bounded linear operator and $L_1(P) \subset P$. Let

$$\begin{aligned} M^* = \max_{u \in \overline{B}_{r_2} \cap P,\ t \in [0,1]} & \int_0^1 \int_0^1 G_2(t, \tau) G_1(\tau, s) f\bigl(s, u(s)\bigr)\, \mathrm{d}s\, \mathrm{d}\tau \\ & + \int_0^1 G_2(t, \tau) A(f) \varphi_1(\tau)\, \mathrm{d}\tau + \int_0^1 G_2(t, \tau) B(f) \varphi_2(\tau)\, \mathrm{d}\tau \\ & + C(e)\psi_1(t) + D(e)\psi_2(t). \end{aligned}$$

It is clear that $M^* < +\infty$. Let

$$W = \{u \in P\colon\ u = \mu T u,\ 0 \leqslant \mu \leqslant 1\}.$$

In the following, we prove that W is bounded.

For any $u \in W$, set $\tilde{u}(t) = \min\{u(t), r_2\}$ and denote $s(u) = \{t \in [0, 1]\colon\ u(t) > r_2\}$, $\tilde{f}(t) = f(t, \tilde{u}(t))$. Then

$$\begin{aligned} u(t) &= \mu (T u)(t) \leqslant (T u)(t) \\ &= \int_0^1 \int_{s(u)} G_2(t, \tau) G_1(\tau, s) f\bigl(s, u(s)\bigr)\, \mathrm{d}s\, \mathrm{d}\tau + \int_0^1 G_2(t, \tau) A_{s(u)}(f) \varphi_1(\tau)\, \mathrm{d}\tau \\ &\quad + \int_0^1 G_2(t, \tau) B_{s(u)}(f) \varphi_2(\tau)\, \mathrm{d}\tau + C(e_{s(u)})\psi_1(t) + D(e_{s(u)})\psi_2(t) \\ &\quad + \int_0^1 \int_{[0,1]\setminus s(u)} G_2(t, \tau) G_1(\tau, s) f\bigl(s, u(s)\bigr)\, \mathrm{d}s\, \mathrm{d}\tau \\ &\quad + \int_0^1 G_2(t, \tau) A_{[0,1]\setminus s(u)}(f) \varphi_1(\tau)\, \mathrm{d}\tau + \int_0^1 G_2(t, \tau) B_{[0,1]\setminus s(u)}(f) \varphi_2(\tau)\, \mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&+ C(e_{[0,1]\setminus s(u)})\psi_1(t) + D(e_{[0,1]\setminus s(u)})\psi_2(t) \\
&\leqslant \sigma\lambda_* \Bigg[\int_0^1 \int_0^1 G_2(t,\tau)G_1(\tau,s)u(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau)A(u)\varphi_1(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^1 G_2(t,\tau)B(u)\varphi_2(\tau)\,\mathrm{d}\tau + C(h)\psi_1(t) + D(h)\psi_2(t) \Bigg] \\
&\quad + \Bigg[\int_0^1 \int_0^1 G_2(t,\tau)G_1(\tau,s)f\big(s,\tilde{u}(s)\big)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau)A(\tilde{f})\varphi_1(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^1 G_2(t,\tau)B(\tilde{f})\varphi_2(\tau)\,\mathrm{d}\tau + C(\tilde{f})\psi_1(t) + D(\tilde{f})\psi_2(t) \Bigg] \\
&\leqslant (L_1u)(t) + M^*, \quad t \in [0,1],
\end{aligned}
$$

where

$$
A_{s(u)}(f) := \frac{1}{\Delta_1} \begin{vmatrix} -\sum_{i=1}^{m-2} a_i \int_{s(u)} G_1(\xi_i,s)f(s,u(s))\,\mathrm{d}s & \sum_{i=1}^{m-2} a_i\varphi_2(\xi_i) - 1 \\ -\sum_{i=1}^{m-2} b_i \int_{s(u)} G_1(\xi_i,s)f(s,u(s))\,\mathrm{d}s & \sum_{i=1}^{m-2} b_i\varphi_2(\xi_i) \end{vmatrix},
$$

and $B_{s(u)}$, $A_{[0,1]\setminus s(u)}$, $B_{[0,1]\setminus s(u)}$ have the similar meaning and

$$
e_{s(u)}(t) = \int_{s(u)} G_1(t,s)f\big(s,u(s)\big)\,\mathrm{d}s + A_{s(u)}(f)\varphi_1(t) + B_{s(u)}(f)\varphi_2(t).
$$

Thus $((I-L_1)u)(t) \leqslant M^*$, $t \in [0,1]$. Since λ_* is the first eigenvalue of L and $0 < \sigma < 1$, the first eigenvalue of L_1, $(r(L_1))^{-1} > 1$. Therefore, the inverse operator $(I-L_1)^{-1}$ exists and

$$
(I-L_1)^{-1} = I + L_1 + L_1^2 + \cdots + L_1^n + \cdots.
$$

It follows from $L_1(P) \subset P$ that $(I-L_1)^{-1}P \subset P$. So we know that $u(t) \leqslant (I-L_1)^{-1}M^*$, $t \in [0,1]$ and W is bounded. We denote by $\sup W$ the bound of W.

Select $r_3 > \max\{r_2, \sup W\}$. Let $h = I - \mu T$ be a homotopy. Then from the homotopy invariance property of the fixed point index combining Theorem 1.3, we have

$$
i(T, B_{r_3} \cap P, P) = i(\theta, B_{r_3} \cap P, P) = 1. \tag{3.5}
$$

By (3.4) and (3.5), we have that

$$
i\big(T, (B_{r_3} \cap P) \setminus (\overline{B}_{r_1} \cap P), P\big) = i(T, B_{r_3} \cap P, P) - i(T, B_{r_1} \cap P, P) = 1.
$$

Then T has at least one fixed point on $(B_{r_3} \cap P) \setminus (\overline{B}_{r_1} \cap P)$, which means that m-point boundary value problem (1.1) has at least one positive solution. □

Remark 3.1. $\lambda_* = \pi^4 - \alpha\pi^2 - \beta$ when $a_i = b_i = 0$ $(i = 1, 2, \ldots)$. In such a case, Theorem 2.1 is reduced to the second part of Theorem 1.1 in [1].

Remark 3.2. As for another case: $\overline{f}_0 < \lambda_*$ and $\underline{f}_\infty > \lambda_*$, we still do not know whether a similar result holds.

Remark 3.3. Since λ_* is the first eigenvalue of the linear problem corresponding to (1.1), the positive result cannot be guaranteed when the strict inequality is weakened to nonstrict inequality. So our result is optimal.

4. Uniqueness result

In this section, we establish the uniqueness result for the positive solution of

$$\begin{cases} u'''' + \alpha u'' - \beta u = h(t)u^r, \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases} \tag{4.1}$$

where $r \in (0,1)$, $\alpha, \beta \in R$, $\xi_i \in (0,1)$, $a_i, b_i \in [0,\infty)$ for $i \in \{1, 2, \dots, m-2\}$ are given constants.

Theorem 4.1. *Suppose that* (A1), (A2), (H1), (H2), (H4) *and* (H6) *are satisfied, then* (4.1) *has at most one positive solution.*

Proof. Suppose u_1, u_2 are two positive solutions of (4.1). Set

$$\Lambda = \bigl\{\lambda \in (0,\infty)\colon\ u_1(t) - \lambda u_2(t) \geqslant 0,\ t \in [0,1]\bigr\}. \tag{4.2}$$

It is obvious that $\Lambda \neq \Phi$. Let

$$\lambda^* = \sup \Lambda. \tag{4.3}$$

We claim that

$$\lambda^* \geqslant 1.$$

In fact, if $\lambda^* < 1$, it follows from (4.2) that $u_1(t) - \lambda^* u_2(t) \geqslant 0$, $t \in [0,1]$. So we have

$$\begin{aligned} u_1(t) &= \int_0^1 \int_0^1 G_2(t,\tau) G_1(\tau,s) h(s) u_1^r(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau) A(f_1)\varphi_1(\tau)\,\mathrm{d}\tau \\ &\quad + \int_0^1 G_2(t,\tau) B(f_1)\varphi_2(\tau)\,\mathrm{d}\tau + C(e_1)\psi_1(t) + D(e_1)\psi_2(t) \\ &\geqslant \int_0^1 \int_0^1 G_2(t,\tau) G_1(\tau,s) h(s) (\lambda^* u_2)^r(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau) A(f_*)\varphi_1(\tau)\,\mathrm{d}\tau \\ &\quad + G_2(t,\tau) B(f_*)\varphi_2(\tau)\,\mathrm{d}\tau + C(e_*)\psi_1(t) + D(e_*)\psi_2(t) \\ &= \lambda^{*r}\Biggl[\int_0^1 \int_0^1 G_2(t,\tau) G_1(\tau,s) h(s) u_2^r(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^1 G_2(t,\tau) A(f_2)\varphi_1(\tau)\,\mathrm{d}\tau \\ &\quad + \int_0^1 G_2(t,\tau) A(f_2)\varphi_1(\tau)\,\mathrm{d}\tau + C(e_2) + D(e_2)\Biggr] \\ &= \lambda^{*r} u_2(t), \end{aligned}$$

where $f_i(t,u) = h(t)u_i^r\,(i = 1,2)$, $f_*(t,u) = h(t)(\lambda^* u_2)^r$,

$$e_i(t) = \int_0^1 G_1(t,s) f_i\bigl(t,u(s)\bigr)\,\mathrm{d}s + A(f_i)\varphi_1(t) + B(f_i)\varphi_2(t) \quad (i = 1,2),$$

$$e_*(t) = \int_0^1 G_1(t,s) f_*\bigl(t,u(s)\bigr)\,\mathrm{d}s + A(f_*)\varphi_1(t) + B(f_*)\varphi_2(t).$$

Hence $\lambda^{*r} \in \Lambda$. Noticing that $0 < r < 1$, we know $\lambda^{*r} > \lambda^*$, which contradicts (4.3). Therefore

$$u_1(t) \geqslant \lambda^* u_2(t) \geqslant u_2(t), \quad t \in [0,1].$$

We can have by a similar way that

$$u_2(t) \geqslant u_1(t), \quad t \in [0,1].$$

We conclude at last that

$$u_1(t) \equiv u_2(t), \quad t \in [0,1]. \qquad \square$$

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