

Periodic solution of neutral Lotka–Volterra system with periodic delays[☆]

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Abstract

A nonautonomous n -species Lotka–Volterra system with neutral delays is investigated. A set of verifiable sufficient conditions is derived for the existence of at least one strictly positive periodic solution of this Lotka–Volterra system by applying an existence theorem and some analysis techniques, where the assumptions of the existence theorem are different from that of Gaines and Mawhin's continuation theorem [R.E. Gaines, J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1977] and that of abstract continuation theory for k -set contraction [W. Petryshyn, Z. Yu, *Existence theorem for periodic solutions of higher order nonlinear periodic boundary value problems*, *Nonlinear Anal.* 6 (1982) 943–969]. Moreover, a problem proposed by Freedman and Wu [H.I. Freedman, J. Wu, *Periodic solution of single species models with periodic delay*, *SIAM J. Math. Anal.* 23 (1992) 689–701] is answered. © 2005 Elsevier Inc. All rights reserved.

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1. Introduction

Motivated by the laboratory work of the group led by Halbach [1], Freedman and Wu [2] studied a single-species population growth model with a periodic delay,

$$y'(t) = y(t)[r(t) - a(t)y(t) + b(t)y(t - \tau(t))], \quad (1.1)$$

where the net birth $r(t)$, the self-inhibition rate $a(t)$, the reproduction $b(t)$, and the delay $\tau(t)$ are continuously differentiable ω -periodic functions, and $r(t) > 0$, $a(t) > 0$, $b(t) \geq 0$, $\tau \geq 0$ for $t \in R$. They discussed the existence of a positive ω -periodic solution of (1.1), and obtained the following result.

Theorem A. Assume that the delay functional equation

$$r(t) - a(t)K(t) + b(t)K(t - \tau(t)) = 0 \quad (1.2)$$

has a positive, ω -periodic, continuously differentiable solution $K(t)$. Then Eq. (1.1) has a positive ω -periodic solution.

Besides, Freedman and Wu pointed out that it would be of great interest and difficulty to consider the existence of positive ω -periodic solutions of higher-dimensional systems with periodic delays, such as predator–prey or competitive systems.

In this paper, we will extend system (1.1) to a periodic multiple species Lotka–Volterra system with a neutral delay,

$$y_i'(t) = y_i(t) \left[r_i(t) - \sum_{j=1}^n a_{ij}(t)y_j(t) + \sum_{j=1}^n b_{ij}(t)y_j(t - \tau_{ij}(t)) + \sum_{j=1}^n c_{ij}(t)y_j'(t - \sigma_{ij}(t)) \right], \quad i = 1, \dots, n, \quad (1.3)$$

which is motivated by Gopalsamy's autonomous model in [8],

$$y_i'(t) = y_i(t) \left[r_i - \sum_{j=1}^n a_{ij}y_j(t) + \sum_{j=1}^n b_{ij}y_j(t - \tau_{ij}) + \sum_{j=1}^n c_{ij}y_j'(t - \sigma_{ij}) \right], \quad i = 1, \dots, n, \quad (1.4)$$

where the parameters are constants and $r_i, a_{ij}, b_{ij}, \tau_{ij}, \sigma_{ij} \in (0, \infty)$, $c_{ij} \in R$.

As we know, an important ecological problem in the study of multi-species population interaction in a periodic environment is the global existence of positive periodic solutions. These solutions play the same role as played by the equilibriums of the autonomous models. It is, therefore, reasonable to ask for conditions which guarantee the resulting periodic nonautonomous system to have positive periodic solutions. Note that condition (1.2) in Theorem A is not easy to verify, the purpose of this paper is to obtain “verifiable” sufficient conditions for the existence of positive periodic solutions of system (1.3). To our knowledge, there are few papers published on the existence of positive periodic solutions of system (1.3) so far. Recently, many authors have studied the existence of positive periodic solutions of various neutral delay models [7,9,10,12–15], the basic techniques they used are almost confined to the using of Gaines and Mawhin's continuation theorem (see Li [9], Yang and Cao [12]) or abstract continuation theory

for k -set contraction (see Fang and Li [10], Lu [13,14]). In fact, as pointed out by Fang and Li [10], it is not easy to verify one of important assumptions of Gaines and Mawhin's continuation theorem [11], that is, that the operation $N: \bar{\Omega} \rightarrow \Omega$ is L -compact (for details see [10]). On the other hand, Fang and Li [10] discussed Kuang's open problem 9.2 [3] on single species model with a neutral delay by applying abstract continuation theory for k -set contraction [4], while it may be difficult to deal with system (1.3). In order to obtain sufficient conditions for the existence of positive periodic solutions of system (1.3), we will use a new method, that is, an existence theorem, whose assumptions are different from that of the above two methods.

The organization of this paper is as follows. Notation and several lemmas are introduced in Section 2. In Section 3, with the help of the lemmas stated in Section 2 and some analysis methods we obtain two main results. A brief discussion is given in the last section.

2. Preliminaries

In this section we summarize a few concepts and results from Refs. [5,6] and state the existence theorem mentioned in Section 1.

For a fixed $\tau \geq 0$, let $C =: C([-\tau, 0]; R^n)$. If $x \in C([\sigma - \tau, \sigma + \delta]; R^n)$ for some $\delta > 0$ and $\sigma \in R$, then $x_t \in C$ for $t \in [\sigma, \sigma + \delta]$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau, 0]$. The supremum norm in C is denoted by $\|\cdot\|$, that is, $\|\varphi\| = \max_{\theta \in [-\tau, 0]} |\varphi(\theta)|$ for $\varphi \in C$, where $|\cdot|$ denotes the norm in R^n , and $|u| = \sum_{i=1}^n |u_i|$ for $u = (u_1, \dots, u_n) \in R^n$.

Consider the following neutral functional differential equation:

$$\frac{d}{dt}[x(t) - b(t, x_t)] = f(t, x_t), \quad (2.1)$$

where $f: R \times C \rightarrow R^n$ is completely continuous and $b: R \times C \rightarrow R^n$ is continuous. Moreover, we assume:

- (H1) There exists $\omega > 0$ such that for every $(t, \varphi) \in R \times C$, we have $b(t + \omega, \varphi) = b(t, \varphi)$ and $f(t + \omega, \varphi) = f(t, \varphi)$.
- (H2) There exists a constant $k < 1$ such that $|b(t, \varphi) - b(t, \psi)| \leq k\|\varphi - \psi\|$ for $t \in R$ and $\varphi, \psi \in C$.

By using the continuation theorem for composite coincidence degree, Erbe et al. [5] proved the following existence theorem. See also [6, Theorem 4.7.1].

Lemma 2.1 (Existence theorem). *Suppose that there exists a constant $M > 0$ such that:*

- (i) *for any $\lambda \in (0, 1)$ and any ω -periodic solution x of the system*

$$\frac{d}{dt}[x(t) - \lambda b(t, x_t)] = \lambda f(t, x_t), \quad (2.2)$$

we have $|x(t)| < M$ for $t \in R$;

- (ii) *$g(u) =: \int_0^\omega f(s, \hat{u}) ds \neq 0$ for $u \in \partial B_M(R^n)$, where $B_M(R^n) = \{u \in R^n: |u| < M\}$, and \hat{u} denotes the constant mapping from $[-\tau, 0]$ to R^n with the value $u \in R^n$;*
- (iii) *$\deg(g, B_M(R^n)) \neq 0$.*

Then there exists at least one ω -periodic solution of the system

$$\frac{d}{dt}[x(t) - b(t, x_t)] = f(t, x_t) \quad (2.3)$$

that satisfies $\sup_{t \in R} |x(t)| < M$.

Remark 2.1. [15, Remark 1] In fact, from the proof of Lemma 2.1 [6, Theorem 4.7.1], it is easy to see that Lemma 2.1 still remains valid if assumption (H2) is replaced by

(H2') There exists a constant $k < 1$ such that $|b(t, \varphi) - b(t, \psi)| \leq k\|\varphi - \psi\|$ for $t \in R$ and $\varphi, \psi \in \{\varphi \in C: \|\varphi\| < M\}$ with M given in condition (i) of Lemma 2.1.

The following two lemmas will be used in later sections.

Lemma 2.2. [14, Lemma 2.4] Suppose $\tau(t) \in C^1(R, R)$ is a ω -periodic function and $\tau'(t) < 1$, $\forall t \in [0, \omega]$. Then the function $t - \tau(t)$ has a unique inverse $u(t)$ satisfying $u \in C(R, R)$ with $u(a + \omega) \equiv u(a) + \omega$, $\forall a \in R$.

Lemma 2.3. Let C_ω be a set of continuous ω -periodic functions for $t \in R$. If $g(t) \in C_\omega$ and $\tau(t) \in C^1(R, R)$, $\forall t \in [0, \omega]$. Then $g(u(t)) \in C_\omega$, where $u(t)$ is the inverse function of $t - \tau(t)$.

Proof. By Lemma 2.2, we can see that if $\tau(t) \in C(R, R)$ is continuous ω -periodic function and $\tau'(t) < 1$, $\forall t \in [0, \omega]$, then

$$u(a + \omega) = u(a) + \omega, \quad \forall a \in R, \quad (2.4)$$

if $g \in C_\omega$, then

$$g(u(t + \omega)) = g(u(t) + \omega) = g(u(t)), \quad (2.5)$$

where $u(t)$ is the inverse function of $t - \tau(t)$, which together with $u(t) \in C(R, R)$ implies that $g(u(t)) \in C_\omega$. \square

For convenience, we denote

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t) dt, \quad f_m = \min_{t \in [0, \omega]} f(t), \quad |f|_0 = \max_{t \in [0, \omega]} |f(t)|,$$

where $f(t)$ is a continuous ω -periodic function.

3. Main results

Corresponding to autonomous system (1.4), we will discuss existing results about positive ω -periodic solutions of system (1.3) with periodic coefficients. Throughout this paper, we assume that $a_{ij}(t), b_{ij}(t) \in C(R, [0, \infty))$, $c_{ij}(t) \in C^1(R, (-\infty, 0])$, $\tau_{ij}(t) \in C^1(R, R)$, $\sigma_{ij}(t) \in C^2(R, R)$ are ω -periodic functions. $r_i(t) \in C(R, R)$ are ω -periodic functions with $\int_0^\omega r_i(t) dt > 0$. The growth functions $r_i(t)$ are not necessarily positive, since the environment fluctuates randomly, in bad conditions r_i may be negative. For the ecology sense of system (1.3) we refer to [8] and references cited therein.

We are now in a position to state the first result on the existence of a positive ω -periodic solution of system (1.3).

Theorem 3.1. Assume the following:

- (1) $\tau'_{ij}(t) < 1$, $\sigma'_{ij}(t) < 1$;
- (2) $k_0 =: ce^{M_0} < 1$;

- (3) $\bar{a}_{ii} > \bar{b}_{ii}$, $\bar{r}_i > \sum_{j=1, j \neq i}^n e^{K_j}(\bar{a}_{ij} - \bar{b}_{ij})$, and $\Gamma_{ij}(t) > 0$;
 (4) the system of algebraic equation

$$\sum_{j=1}^n (\bar{a}_{ij} - \bar{b}_{ij}) e^{u_j} = \bar{r}_i, \quad i = 1, \dots, n,$$

has a unique solution $u^* = (u_1^*, \dots, u_n^*) \in R^n$.

Then system (1.3) has at least one positive ω -periodic solution, where

$$c = \max \left\{ \sum_{i=1}^n \sum_{j=1}^n |d_{ij}|_0, \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 \right\}, \quad M_0 = \max \left\{ \sum_{i=1}^n |u_i^*|, K, \sum_{i=1}^n P_i + \omega H_* \right\},$$

$$K = \max_{1 \leq i \leq n} \{K_i\}, \quad K_i = \ln \frac{\bar{r}_i}{\vartheta_{ii}} + \sum_{j=1}^n \frac{\bar{r}_i}{\vartheta_{ij}} + (\bar{R}_i + \Gamma \bar{r}_i) \omega,$$

$$\Gamma_{ij}(t) = a_{ij}(t) - \frac{b_{ij}(u_{ij}(t))}{1 - \tau'_{ij}(u_{ij}(t))} + \frac{d'_{ij}(v_{ij}(t))}{1 - \sigma'_{ij}(v_{ij}(t))}, \quad \bar{R}_i = \frac{1}{\omega} \int_0^\omega |r_i(t)| dt,$$

$$\Gamma_{ij}^1(t) = a_{ij}(t) + \frac{b_{ij}(u_{ij}(t))}{1 - \tau'_{ij}(u_{ij}(t))} + \frac{|d'_{ij}(v_{ij}(t))|}{1 - \sigma'_{ij}(v_{ij}(t))}, \quad d_{ij}(t) = \frac{c_{ij}(t)}{1 - \sigma'_{ij}(t)},$$

$$\Gamma = \max \left\{ \left(\frac{\Gamma_{ij}^1}{\Gamma_{ij}} \right)_0, 1 \leq i, j \leq n \right\},$$

$$H_* = \frac{\sum_{i=1}^n |r_i|_0 + \sum_{i=1}^n \sum_{j=1}^n (|a_{ij}|_0 + |b_{ij}|_0) e^{K_j}}{1 - \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 e^{K_j}},$$

$$P_i = \max \left\{ \left| \ln \frac{\bar{r}_i}{\bar{a}_{ii} - \bar{b}_{ii}} \right|, \left| \ln \frac{\bar{r}_i - \sum_{j=1, j \neq i}^n e^{K_j}(\bar{a}_{ij} - \bar{b}_{ij})}{\bar{a}_{ii} - \bar{b}_{ii}} \right| \right\},$$

$$\vartheta_{ij} = \frac{(\Gamma_{ij})_m (1 - \sigma'_{ij})_m}{(1 - \sigma'_{ij})_m + |d_{ij}|_0}$$

and $u_{ij}(t)$, $v_{ij}(t)$ represent the inverses of functions $t - \tau_{ij}(t)$ and $t - \sigma_{ij}(t)$, respectively.

To establish the existence of positive periodic solutions of neutral delay system (1.3) we first make the change of variable

$$y_i(t) = e^{x_i(t)}, \quad i = 1, \dots, n. \quad (3.1)$$

Then system (1.3) can be rewritten as

$$\begin{aligned} x'_i(t) &= r_i(t) - \sum_{j=1}^n a_{ij}(t) e^{x_j(t)} + \sum_{j=1}^n b_{ij}(t) e^{x_j(t - \tau_{ij}(t))} \\ &\quad + \sum_{j=1}^n c_{ij}(t) x'_j(t - \sigma_{ij}(t)) e^{x_j(t - \sigma_{ij}(t))}, \quad i = 1, \dots, n. \end{aligned} \quad (3.2)$$

Let Θ denote the linear space of real-valued continuous ω -periodic functions on R . The linear space Θ is a Banach space with the usual norm for $x(t) = (x_1(t), \dots, x_n(t)) \in \Theta$ given by

$$\|x\|_0 = \max_{t \in R} |x(t)| = \max_{t \in R} \sum_{i=1}^n |x_i(t)|.$$

We define the following maps:

$$\begin{aligned} b: R \times C &\rightarrow R^n, \quad b(t, \varphi) = (b_1(t, \varphi), \dots, b_n(t, \varphi)), \\ b_i(t, \varphi) &= \sum_{j=1}^n \frac{c_{ij}(t)}{1 - \sigma'_{ij}(t)} e^{\varphi_j(-\sigma_{ij}(t))}, \\ f: R \times C &\rightarrow R^n, \quad f(t, \varphi) = (f_1(t, \varphi), \dots, f_n(t, \varphi)), \\ f_i(t, \varphi) &= r_i(t) - \sum_{j=1}^n a_{ij}(t) e^{\varphi_j(0)} + \sum_{j=1}^n b_{ij}(t) e^{\varphi_j(-\tau_{ij}(t))} - \sum_{j=1}^n \left(\frac{c_{ij}(t)}{1 - \sigma'_{ij}(t)} \right)' e^{\varphi_j(-\sigma_{ij}(t))}, \\ i &= 1, \dots, n; \quad \varphi = (\varphi_1, \dots, \varphi_n) \in C, \quad t \in R. \end{aligned}$$

Now, system (3.2) becomes

$$\frac{d}{dt} [x(t) - b(t, x_t)] = f(t, x_t). \quad (3.3)$$

To prove Theorem 3.1, we firstly obtain the following two important lemmas.

Lemma 3.1. *If the assumptions of Theorem 3.1 are satisfied and $\Omega = \{\varphi \in C: \|\varphi\| < M\}$, where $M > M_0$ is such that $k =: ce^M < 1$, then*

$$|b(t, \varphi) - b(t, \psi)| \leq k \|\varphi - \psi\|, \quad \text{for } t \in R \text{ and } \varphi, \psi \in \Omega.$$

Proof. For $t \in R$ and $\varphi, \psi \in \Omega$, denoting $d_{ij}(t) = c_{ij}(t)/(1 - \sigma'_{ij}(t))$ we get

$$\begin{aligned} &|b_i(t, \varphi) - b_i(t, \psi)| \\ &\leq \sum_{j=1}^n |d_{ij}(t)| |e^{\varphi_j(-\sigma_{ij}(t))} - e^{\psi_j(-\sigma_{ij}(t))}| \\ &\leq \sum_{j=1}^n |d_{ij}(t)| e^{\eta_{ij}\varphi_j(-\sigma_{ij}(t)) + (1-\eta_{ij})\psi_j(-\sigma_{ij}(t))} |\varphi_j(-\sigma_{ij}(t)) - \psi_j(-\sigma_{ij}(t))|, \end{aligned}$$

for some $\eta_{ij} \in (0, 1)$. Then we have

$$|b_i(t, \varphi) - b_i(t, \psi)| \leq \sum_{j=1}^n |d_{ij}|_0 e^M \|\varphi - \psi\|.$$

Hence,

$$|b(t, \varphi) - b(t, \psi)| \leq \sum_{i=1}^n \sum_{j=1}^n |d_{ij}|_0 e^M \|\varphi - \psi\| \leq ce^M \|\varphi - \psi\| = k \|\varphi - \psi\|.$$

The proof is thus complete. \square

Lemma 3.2. Assume that the assumptions of Theorem 3.1 hold. Then every solution $x(t) \in \Theta$ of the system

$$\frac{d}{dt}[x(t) - \lambda b(t, x_t)] = \lambda f(t, x_t), \quad \lambda \in (0, 1),$$

satisfies $\|x\|_0 \leq M_0$.

Proof. Let $\frac{d}{dt}[x(t) - \lambda b(t, x_t)] = \lambda f(t, x_t)$ for $x(t) \in \Theta$, that is,

$$\begin{aligned} & \left[x_i(t) - \lambda \sum_{j=1}^n \frac{c_{ij}(t)}{1 - \sigma'_{ij}(t)} e^{x_j(t - \sigma_{ij}(t))} \right]' \\ &= \lambda \left[r_i(t) - \sum_{j=1}^n a_{ij}(t) e^{x_j(t)} + \sum_{j=1}^n b_{ij}(t) e^{x_j(t - \tau_{ij}(t))} - \sum_{j=1}^n \left(\frac{c_{ij}(t)}{1 - \sigma'_{ij}(t)} \right)' e^{x_j(t - \sigma_{ij}(t))} \right], \\ & i = 1, \dots, n, \quad \lambda \in (0, 1), \end{aligned} \quad (3.4)$$

which yield, after integrating these identities from 0 to ω :

$$\begin{aligned} & \int_0^\omega \sum_{j=1}^n [a_{ij}(t) e^{x_j(t)} - b_{ij}(t) e^{x_j(t - \tau_{ij}(t))} + d'_{ij}(t) e^{x_j(t - \sigma_{ij}(t))}] dt = \int_0^\omega r_i(t) dt = \bar{r}_i \omega, \\ & i = 1, \dots, n, \end{aligned} \quad (3.5)$$

where $d_{ij}(t) = c_{ij}(t)/(1 - \sigma'_{ij}(t))$. Since $\tau'_{ij}(t) < 1$, we can let $t - \tau_{ij}(t) = s$, i.e., $t = u_{ij}(s)$, then

$$\int_0^\omega b_{ij}(t) e^{x_j(t - \tau_{ij}(t))} dt = \int_{-\tau_{ij}(0)}^{\omega - \tau_{ij}(\omega)} \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} e^{x_j(s)} ds. \quad (3.6)$$

According to Lemma 2.3, we know

$$\frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} e^{x_j(s)} \in C_\omega.$$

Thus,

$$\int_{-\tau_{ij}(0)}^{\omega - \tau_{ij}(\omega)} \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} e^{x_j(s)} ds = \int_0^\omega \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} e^{x_j(s)} ds,$$

that is,

$$\int_0^\omega b_{ij}(t) e^{x_j(t - \tau_{ij}(t))} dt = \int_0^\omega \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} e^{x_j(s)} ds. \quad (3.7)$$

Similarly,

$$\int_0^\omega d'_{ij}(t) e^{x_j(t - \sigma_{ij}(t))} dt = \int_0^\omega \frac{d'_{ij}(v_{ij}(s))}{1 - \sigma'_{ij}(v_{ij}(s))} e^{x_j(s)} ds. \quad (3.8)$$

So from (3.5), (3.7), (3.8), we can get

$$\int_0^{\omega} \sum_{j=1}^n \Gamma_{ij}(s) e^{x_j(s)} ds = \bar{r}_i \omega, \quad i = 1, \dots, n, \quad (3.9)$$

where

$$\Gamma_{ij}(s) = a_{ij}(s) - \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} + \frac{d'_{ij}(v_{ij}(s))}{1 - \sigma'_{ij}(v_{ij}(s))}.$$

From (3.4) we have:

$$\begin{aligned} & \int_0^{\omega} \left| \left[x_i(t) - \lambda \sum_{j=1}^n d_{ij}(t) e^{x_j(t-\sigma_{ij}(t))} \right]' \right| dt \\ &= \lambda \int_0^{\omega} \left| r_i(t) - \left(\sum_{j=1}^n a_{ij}(t) e^{x_j(t)} - \sum_{j=1}^n b_{ij}(t) e^{x_j(t-\tau_{ij}(t))} + \sum_{j=1}^n d'_{ij}(t) e^{x_j(t-\sigma_{ij}(t))} \right) \right| dt \\ &\leq \lambda \int_0^{\omega} |r_i(t)| dt + \lambda \int_0^{\omega} \left| \sum_{j=1}^n [a_{ij}(t) e^{x_j(t)} - b_{ij}(t) e^{x_j(t-\tau_{ij}(t))} + d'_{ij}(t) e^{x_j(t-\sigma_{ij}(t))}] \right| dt. \end{aligned} \quad (3.10)$$

Note that

$$\begin{aligned} & \int_0^{\omega} \left| \sum_{j=1}^n [a_{ij}(t) e^{x_j(t)} - b_{ij}(t) e^{x_j(t-\tau_{ij}(t))} + d'_{ij}(t) e^{x_j(t-\sigma_{ij}(t))}] \right| dt \\ &\leq \int_0^{\omega} \sum_{j=1}^n [a_{ij}(t) e^{x_j(t)} + b_{ij}(t) e^{x_j(t-\tau_{ij}(t))} + |d'_{ij}(t)| e^{x_j(t-\sigma_{ij}(t))}] dt. \end{aligned}$$

In view of (3.5)–(3.9) and by a similar analysis, we have

$$\begin{aligned} & \int_0^{\omega} \sum_{j=1}^n [a_{ij}(t) e^{x_j(t)} + b_{ij}(t) e^{x_j(t-\tau_{ij}(t))} + |d'_{ij}(t)| e^{x_j(t-\sigma_{ij}(t))}] dt \\ &= \int_0^{\omega} \sum_{j=1}^n \left[a_{ij}(s) + \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} + \frac{|d'_{ij}(v_{ij}(s))|}{1 - \sigma'_{ij}(v_{ij}(s))} \right] e^{x_j(s)} ds \\ &= \int_0^{\omega} \sum_{j=1}^n \Gamma_{ij}^1(s) e^{x_j(s)} ds = \int_0^{\omega} \sum_{j=1}^n \left(\frac{\Gamma_{ij}^1(s)}{\Gamma_{ij}(s)} \right) \Gamma_{ij}(s) e^{x_j(s)} ds \\ &\leq \int_0^{\omega} \sum_{j=1}^n \left(\frac{\Gamma_{ij}^1}{\Gamma_{ij}} \right)_0 \Gamma_{ij}(s) e^{x_j(s)} ds \leq \Gamma \int_0^{\omega} \sum_{j=1}^n \Gamma_{ij}(s) e^{x_j(s)} ds \leq \Gamma \bar{r}_i \omega, \end{aligned} \quad (3.11)$$

where

$$\Gamma = \max \left\{ \left(\frac{\Gamma_{ij}^1}{\Gamma_{ij}} \right)_0, 1 \leq i, j \leq n \right\},$$

$$\Gamma_{ij}^1(s) = a_{ij}(s) + \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} + \frac{|d'_{ij}(v_{ij}(s))|}{1 - \sigma'_{ij}(v_{ij}(s))}.$$

It follows from (3.10) and (3.11), that

$$\int_0^\omega \left| \left[x_i(t) - \lambda \sum_{j=1}^n d_{ij}(t) e^{x_j(t-\sigma_{ij}(t))} \right]' \right| dt < (\bar{R}_i + \Gamma \bar{r}_i) \omega, \quad i = 1, \dots, n. \quad (3.12)$$

In addition, from (3.9) we have

$$\begin{aligned} \bar{r}_i \omega &= \sum_{j=1}^n \int_0^\omega [\Gamma_{ij}(t) e^{x_j(t)} - \vartheta_{ij} e^{x_j(t)} + \vartheta_{ij} d_{ij}(t) e^{x_j(t-\sigma_{ij}(t))}] dt \\ &\quad + \sum_{j=1}^n \int_0^\omega [\vartheta_{ij} e^{x_j(t)} - \vartheta_{ij} d_{ij}(t) e^{x_j(t-\sigma_{ij}(t))}] dt. \end{aligned} \quad (3.13)$$

Similar to (3.5)–(3.9), we can get

$$\begin{aligned} &\sum_{j=1}^n \int_0^\omega [\Gamma_{ij}(t) e^{x_j(t)} - \vartheta_{ij} e^{x_j(t)} + \vartheta_{ij} d_{ij}(t) e^{x_j(t-\sigma_{ij}(t))}] dt \\ &= \sum_{j=1}^n \int_0^\omega \left[\Gamma_{ij}(s) - \vartheta_{ij} + \frac{\vartheta_{ij} d_{ij}(v_{ij}(s))}{1 - \sigma'_{ij}(v_{ij}(s))} \right] e^{x_j(s)} ds. \end{aligned} \quad (3.14)$$

Since $\Gamma_{ij}(t) > 0$, $\sigma'_{ij}(t) < 1$ and $d_{ij}(t) \leq 0$ we can choose positive constants

$$\vartheta_{ij} = \frac{(\Gamma_{ij})_m (1 - \sigma'_{ij})_m}{(1 - \sigma'_{ij})_m + |d_{ij}|_0},$$

such that

$$\Gamma_{ij}(s) - \vartheta_{ij} + \frac{\vartheta_{ij} d_{ij}(v_{ij}(s))}{1 - \sigma'_{ij}(v_{ij}(s))} \geq 0. \quad (3.15)$$

Thus, it follows from (3.13)–(3.15) that

$$\bar{r}_i \omega \geq \sum_{j=1}^n \int_0^\omega [\vartheta_{ij} e^{x_j(t)} - \vartheta_{ij} d_{ij}(t) e^{x_j(t-\sigma_{ij}(t))}] dt,$$

that is,

$$\bar{r}_i \omega \geq \int_0^\omega \sum_{j=1}^n [\vartheta_{ij} e^{x_j(t)} - \vartheta_{ij} d_{ij}(t) e^{x_j(t-\sigma_{ij}(t))}] dt. \quad (3.16)$$

By $\vartheta_{ij} > 0$ and $d_{ij}(t) \leq 0$, we have that there exist points $\xi_i \in [0, \omega]$ ($i = 1, \dots, n$) such that

$$\bar{r}_i \geq \sum_{j=1}^n \vartheta_{ij} e^{x_j(\xi_i)} - \sum_{j=1}^n \vartheta_{ij} d_{ij}(\xi_i) e^{x_j(\xi_i - \sigma_{ij}(\xi_i))}, \quad (3.17)$$

then

$$x_i(\xi_i) < \ln \frac{\bar{r}_i}{\vartheta_{ii}}, \quad -d_{ij}(\xi_i)e^{x_j(\xi_i - \sigma_{ij}(\xi_i))} < \frac{\bar{r}_i}{\vartheta_{ij}}. \quad (3.18)$$

By (3.12) and (3.18), we get

$$\begin{aligned} & x_i(t) - \lambda \sum_{j=1}^n d_{ij}(t)e^{x_j(t - \sigma_{ij}(t))} \\ & \leq x_i(\xi_i) - \lambda \sum_{j=1}^n d_{ij}(\xi_i)e^{x_j(\xi_i - \sigma_{ij}(\xi_i))} + \int_0^\omega \left| \left[x_i(t) - \lambda \sum_{j=1}^n d_{ij}(t)e^{x_j(t - \sigma_{ij}(t))} \right]' \right| dt \\ & < \ln \frac{\bar{r}_i}{\vartheta_{ii}} + \sum_{j=1}^n \frac{\bar{r}_i}{\vartheta_{ij}} + (\bar{R}_i + \Gamma \bar{r}_i)\omega := K_i. \end{aligned}$$

As $\lambda \sum_{j=1}^n d_{ij}(t)e^{x_j(t - \sigma_{ij}(t))} \leq 0$, one can find that

$$x_i(t) < K_i, \quad i = 1, \dots, n. \quad (3.19)$$

Besides, from (3.4) we get

$$\begin{aligned} x'_i(t) = & \lambda \left[r_i(t) - \sum_{j=1}^n a_{ij}(t)e^{x_j(t)} + \sum_{j=1}^n b_{ij}(t)e^{x_j(t - \tau_{ij}(t))} \right. \\ & \left. + \sum_{j=1}^n c_{ij}(t)x'_j(t - \sigma_{ij}(t))e^{x_j(t - \sigma_{ij}(t))} \right], \end{aligned}$$

by (3.19) we have

$$\begin{aligned} |x'_i(t)| & \leq \lambda \left[r_i(t) + \sum_{j=1}^n a_{ij}(t)e^{x_j(t)} + \sum_{j=1}^n b_{ij}(t)e^{x_j(t - \tau_{ij}(t))} \right. \\ & \quad \left. + \sum_{j=1}^n |c_{ij}(t)| |x'_j(t - \sigma_{ij}(t))| e^{x_j(t - \sigma_{ij}(t))} \right] \\ & < |r_i|_0 + \sum_{j=1}^n |a_{ij}|_0 e^{K_j} + \sum_{j=1}^n |b_{ij}|_0 e^{K_j} + \sum_{j=1}^n |c_{ij}|_0 |x'_j|_0 e^{K_j}. \end{aligned} \quad (3.20)$$

So, we have

$$\begin{aligned} \|x'\|_0 & \leq \sum_{i=1}^n |x'_i|_0 \\ & \leq \sum_{i=1}^n |r_i|_0 + \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|_0 e^{K_j} + \sum_{i=1}^n \sum_{j=1}^n |b_{ij}|_0 e^{K_j} + \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 |x'_j|_0 e^{K_j} \\ & \leq \sum_{i=1}^n |r_i|_0 + \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|_0 e^{K_j} + \sum_{i=1}^n \sum_{j=1}^n |b_{ij}|_0 e^{K_j} + \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 \|x'\|_0 e^{K_j}. \end{aligned}$$

By assumption (2) of Theorem 3.1, we see

$$\sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 e^{K_j} \leq \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 e^K \leq \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 e^{M_0} < 1. \quad (3.21)$$

Therefore,

$$\|x'\|_0 < \frac{\sum_{i=1}^n |r_i|_0 + \sum_{i=1}^n \sum_{j=1}^n (|a_{ij}|_0 + |b_{ij}|_0) e^{K_j}}{1 - \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 e^{K_j}} := H_*. \quad (3.22)$$

Next, recalling (3.9) we can see that

$$\bar{r}_i \omega = \sum_{j=1}^n \int_0^\omega \Gamma_{ij}(s) e^{x_j(s)} ds > \int_0^\omega \Gamma_{ii}(s) e^{x_i(s)} ds. \quad (3.23)$$

On the other hand, by (3.19) we have

$$\begin{aligned} \int_0^\omega \Gamma_{ii}(s) e^{x_i(s)} ds &= \bar{r}_i \omega - \sum_{j=1, j \neq i}^n \int_0^\omega \Gamma_{ij}(s) e^{x_j(s)} ds \\ &> \bar{r}_i \omega - \sum_{j=1, j \neq i}^n e^{K_j} \int_0^\omega \Gamma_{ij}(s) ds. \end{aligned} \quad (3.24)$$

We can obtain from (3.23) and (3.24) that

$$\bar{r}_i \omega - \sum_{j=1, j \neq i}^n e^{K_j} \int_0^\omega \Gamma_{ij}(s) ds < \int_0^\omega \Gamma_{ii}(s) e^{x_i(s)} ds < \bar{r}_i \omega, \quad (3.25)$$

then there are points $\eta_i \in [0, \omega]$ ($i = 1, \dots, n$) such that

$$\bar{r}_i \omega - \sum_{j=1, j \neq i}^n e^{K_j} \int_0^\omega \Gamma_{ij}(s) ds < e^{x_i(\eta_i)} \int_0^\omega \Gamma_{ii}(s) ds < \bar{r}_i \omega. \quad (3.26)$$

In view of Lemma 2.2, we can see $u_{ij}(\omega) = u_{ij}(0) + \omega$, so

$$\int_0^\omega \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} ds = \int_{u_{ij}(0)}^{u_{ij}(\omega)} \frac{b_{ij}(t)(1 - \tau'_{ij}(t))}{1 - \tau'_{ij}(t)} dt = \int_{u_{ij}(0)}^{u_{ij}(0)+\omega} b_{ij}(t) dt = \bar{b}_{ij} \omega. \quad (3.27)$$

Similarly,

$$\int_0^\omega \frac{d'_{ij}(v_{ij}(s))}{1 - \sigma'_{ij}(v_{ij}(s))} ds = \int_{v_{ij}(0)}^{v_{ij}(\omega)} \frac{d'_{ij}(t)(1 - \sigma'_{ij}(t))}{1 - \sigma'_{ij}(t)} dt = \int_{v_{ij}(0)}^{v_{ij}(0)+\omega} d'_{ij}(t) dt = 0. \quad (3.28)$$

Thus,

$$\begin{aligned}\bar{r}_{ij} &= \frac{1}{\omega} \int_0^{\omega} \Gamma_{ij}(s) ds = \frac{1}{\omega} \int_0^{\omega} \left[a_{ij}(s) - \frac{b_{ij}(u_{ij}(s))}{1 - \tau'_{ij}(u_{ij}(s))} + \frac{d'_{ij}(v_{ij}(s))}{1 - \sigma'_{ij}(v_{ij}(s))} \right] ds \\ &= (\bar{a}_{ij} - \bar{b}_{ij}),\end{aligned}\quad (3.29)$$

which together with (3.26) yield

$$\bar{r}_i - \sum_{j=1, j \neq i}^n e^{K_j} (\bar{a}_{ij} - \bar{b}_{ij}) < e^{x_i(\eta_i)} (\bar{a}_{ii} - \bar{b}_{ii}) < \bar{r}_i, \quad (3.30)$$

which imply

$$|x_i(\eta_i)| < \max \left\{ \left| \ln \frac{\bar{r}_i}{\bar{a}_{ii} - \bar{b}_{ii}} \right|, \left| \ln \frac{\bar{r}_i - \sum_{j=1, j \neq i}^n e^{K_j} (\bar{a}_{ij} - \bar{b}_{ij})}{\bar{a}_{ii} - \bar{b}_{ii}} \right| \right\} := P_i. \quad (3.31)$$

From (3.22) and (3.31), we see

$$|x_i| \leq |x_i(\eta_i)| + \int_0^{\omega} |x'_i| dt \leq P_i + \int_0^{\omega} |x'_i| dt, \quad i = 1, \dots, n. \quad (3.32)$$

Hence,

$$\|x\|_0 < \sum_{i=1}^n P_i + \int_0^{\omega} \|x'\|_0 dt < \sum_{i=1}^n P_i + H_* \omega \leq M_0. \quad (3.33)$$

Clearly, M_0 is independent of λ . This completes the proof. \square

Finally, we apply Lemma 2.1 and Remark 2.1 to (3.2) and give the proof of Theorem 3.1.

Proof of Theorem 3.1. Obviously, for M given in Lemma 3.1, condition (i) in Lemma 2.1 is satisfied. Let $g(u) = (g_1(u), \dots, g_n(u))$, $B_M(R^n) = \{u \in R^n, \|u\| < M\}$. When $u \in \partial B_M(R^n)$, u is a constant vector in R^n with $\|u\| = M$.

Since

$$\begin{aligned}g_i(u) &= \int_0^{\omega} f_i(s, \hat{u}) ds = \int_0^{\omega} r_i(t) dt - \sum_{j=1}^n \int_0^{\omega} a_{ij}(t) dt e^{u_j} + \sum_{j=1}^n \int_0^{\omega} b_{ij}(t) dt e^{u_j} \\ &= \left[\bar{r}_i - \sum_{j=1}^n (\bar{a}_{ij} - \bar{b}_{ij}) e^{u_j} \right] \omega\end{aligned}\quad (3.34)$$

and $M > \sum_{i=1}^n |u_i^*|$, we have $g(u) \neq 0$ for any $u \in \partial B_M(R^n)$. That is, condition (ii) in Lemma 2.1 holds. At last, we verify that condition (iii) of Lemma 2.1 also holds. By assumption (4) of Theorem 3.1 and the formula for Brouwer degree (see Theorem 2.2.3 in Ref. [6]), a straightforward calculation shows that

$$\begin{aligned}\deg(g, B_M(R^n)) &= \sum_{u \in g^{-1}(0) \cap B_M(R^n)} \text{sign det } Dg(u) = \text{sign} \left\{ (-1)^n [\text{det}(\bar{a}_{ij} - \bar{b}_{ij})] e^{\sum_{j=1}^n u_j^*} \right\} \\ &\neq 0.\end{aligned}\quad (3.35)$$

By now all the assumptions required in Lemma 2.1 hold. It follows by Lemma 2.1 and Remark 2.1 that system (3.2) has a ω -periodic solution. By the change of $y_i(t) = e^{x_i(t)}$, we obtain that (1.3) has at least one positive ω -periodic solution. The proof of Theorem 3.1 is complete. \square

Next, we state and prove the second main result.

Theorem 3.2. *Assume the following:*

- (1) $\tau_{ij}(t) = m_{ij}\omega$, $\sigma_{ij}(t) = n_{ij}\omega$;
- (2) $k_1 =: ce^{M_1} < 1$;
- (3) $\bar{a}_{ii} > \bar{b}_{ii}$, $\bar{r}_i > \sum_{j=1, j \neq i}^n e^{Q_j}(\bar{a}_{ij} - \bar{b}_{ij})$ and $a_{ij}(t) - b_{ij}(t) + c'_{ij}(t) > 0$;
- (4) the system of algebraic equations

$$\sum_{j=1}^n (\bar{a}_{ij} - \bar{b}_{ij})e^{u_j} = \bar{r}_i, \quad i = 1, \dots, n,$$

has a unique solution $u^* = (u_1^*, \dots, u_n^*) \in R^n$ and m_{ij} , n_{ij} are nonnegative integers.

Then system (1.3) has at least one positive ω -periodic solution, where

$$c = \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0, \quad M_1 = \max \left\{ \sum_{i=1}^n |u_i^*|, Q, \sum_{i=1}^n H_i + \omega P_* \right\},$$

$$Q = \max_{1 \leq i \leq n} \{Q_i\}, \quad Q_i = \ln \frac{\bar{r}_i}{(a_{ii} - b_{ii} + c'_{ii})_m} + \sum_{j=1}^n \frac{\bar{r}_i |c_{ij}|_0}{(a_{ij} - b_{ij} + c'_{ij})_m} + (\bar{R}_i + \bar{r}_i)\omega,$$

$$P_* = \frac{\sum_{i=1}^n |r_i|_0 + \sum_{i=1}^n \sum_{j=1}^n (|a_{ij}|_0 + |b_{ij}|_0)e^{Q_j}}{1 - \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 e^{Q_j}}, \quad \bar{R}_i = \frac{1}{\omega} \int_0^\omega |r_i(t)| dt,$$

$$H_i = \max \left\{ \left| \ln \frac{\bar{r}_i}{\bar{a}_{ii} - \bar{b}_{ii}} \right|, \left| \ln \frac{\bar{r}_i - \sum_{j=1, j \neq i}^n e^{Q_j}(\bar{a}_{ij} - \bar{b}_{ij})}{\bar{a}_{ii} - \bar{b}_{ii}} \right| \right\}.$$

Proof. When $\tau_{ij}(t) = m_{ij}\omega$, $\sigma_{ij}(t) = n_{ij}\omega$, m_{ij} and n_{ij} are nonnegative integers, it is not difficult to see that if $y^*(t) = (y_1^*(t), \dots, y_n^*(t))$ is a positive ω -periodic solution of the following equation:

$$y'_i(t) = y_i(t) \left[r_i(t) - \sum_{j=1}^n a_{ij}(t)y_j(t) + \sum_{j=1}^n b_{ij}(t)y_j(t) + \sum_{j=1}^n c_{ij}(t)y'_j(t) \right],$$

$$i = 1, \dots, n, \quad (3.36)$$

then $y^*(t)$ is also a positive ω -periodic solution of (1.3).

In order to establish the existence of a positive ω -periodic solution of (3.36), we consider the equation

$$x'_i(t) = r_i(t) - \sum_{j=1}^n (a_{ij}(t) - b_{ij}(t))e^{x_j(t)} + \sum_{j=1}^n c_{ij}(t)x'_j(t)e^{x_j(t)}, \quad i = 1, \dots, n. \quad (3.37)$$

Similarly to Theorem 3.1, we can define the following maps:

$$\begin{aligned} b: R \times C &\rightarrow R^n, \quad b(t, \varphi) = (b_1(t, \varphi), \dots, b_n(t, \varphi)), \\ b_i(t, \varphi) &= \sum_{j=1}^n c_{ij}(t) e^{\varphi_j(0)}; \\ f: R \times C &\rightarrow R^n, \quad f(t, \varphi) = (f_1(t, \varphi), \dots, f_n(t, \varphi)), \\ f_i(t, \varphi) &= r_i(t) - \sum_{j=1}^n a_{ij}(t) e^{\varphi_j(0)} + \sum_{j=1}^n b_{ij}(t) e^{\varphi_j(0)} - \sum_{j=1}^n c'_{ij}(t) e^{\varphi_j(0)}, \\ i &= 1, \dots, n; \quad \varphi = (\varphi_1, \dots, \varphi_n) \in C, \quad t \in R. \end{aligned}$$

Since Lemma 3.3 is similar to Lemma 3.1, we omit its proof.

Lemma 3.3. *If the assumptions of Theorem 3.2 are satisfied and $\Omega_1 = \{\varphi \in C, \|\varphi\| < M_1\}$, where $M > M_1$ is such that $k_1 =: ce^{M_1} < 1$, then*

$$|b(t, \varphi) - b(t, \psi)| \leq k \|\varphi - \psi\|, \quad \text{for } t \in R \text{ and } \varphi, \psi \in \Omega_1.$$

Corresponding to the following identity

$$\frac{d}{dt}[x(t) - \lambda b(t, x_t)] = \lambda f(t, x_t), \quad \lambda \in (0, 1),$$

we have

$$\begin{aligned} &\left[x_i(t) - \lambda \sum_{j=1}^n c_{ij}(t) e^{x_j(t)} \right]' \\ &= \lambda \left[r_i(t) - \sum_{j=1}^n a_{ij}(t) e^{x_j(t)} + \sum_{j=1}^n b_{ij}(t) e^{x_j(t)} - \sum_{j=1}^n c'_{ij}(t) e^{x_j(t)} \right], \\ &i = 1, \dots, n; \quad \lambda \in (0, 1). \end{aligned} \tag{3.38}$$

Integrating the above identities over $[0, \omega]$, we get

$$\int_0^\omega \sum_{j=1}^n (a_{ij}(t) - b_{ij}(t) + c'_{ij}(t)) e^{x_j(t)} dt = \int_0^\omega r_i(t) dt = \bar{r}_i \omega, \quad i = 1, \dots, n. \tag{3.39}$$

From (3.38), (3.39) and assumption (3) of Theorem 3.2 we get

$$\begin{aligned} \int_0^\omega \left| \left[x_i(t) - \lambda \sum_{j=1}^n c_{ij}(t) e^{x_j(t)} \right]' \right| dt &= \lambda \int_0^\omega \left| r_i(t) - \sum_{j=1}^n (a_{ij}(t) - b_{ij}(t) + c'_{ij}(t)) e^{x_j(t)} \right| dt \\ &< \int_0^\omega |r_i(t)| dt + \int_0^\omega \sum_{j=1}^n (a_{ij}(t) - b_{ij}(t) + c'_{ij}(t)) e^{x_j(t)} dt \\ &= (\bar{R}_i + \bar{r}_i) \omega. \end{aligned} \tag{3.40}$$

That is,

$$\int_0^\omega \left| \left[x_i(t) - \lambda \sum_{j=1}^n c_{ij}(t) e^{x_j(t)} \right]' \right| dt < (\bar{R}_i + \bar{r}_i) \omega, \quad i = 1, \dots, n. \quad (3.41)$$

From (3.39) we see that there exist points $\theta_i \in [0, \omega]$ such that

$$\sum_{j=1}^n (a_{ij}(\theta_i) - b_{ij}(\theta_i) + c'_{ij}(\theta_i)) e^{x_j(\theta_i)} = \bar{r}_i, \quad i = 1, \dots, n, \quad (3.42)$$

which imply that

$$e^{x_j(\theta_i)} < \frac{\bar{r}_i}{(a_{ij} - b_{ij} + c'_{ij})_m}, \quad i, j = 1, \dots, n. \quad (3.43)$$

It follows from (3.41), (3.43) and $c_{ij}(t) \leq 0$ that

$$\begin{aligned} x_i(t) - \lambda \sum_{j=1}^n c_{ij}(t) e^{x_j(t)} &\leq x_i(\theta_i) - \lambda \sum_{j=1}^n c_{ij}(\theta_i) e^{x_j(\theta_i)} + \int_0^\omega \left| \left[x_i(t) - \lambda \sum_{j=1}^n c_{ij}(t) e^{x_j(t)} \right]' \right| dt \\ &< \ln \frac{\bar{r}_i}{(a_{ii} - b_{ii} + c'_{ii})_m} + \sum_{j=1}^n \frac{\bar{r}_i |c_{ij}|_0}{(a_{ij} - b_{ij} + c'_{ij})_m} + (\bar{R}_i + \bar{r}_i) \omega := Q_i. \end{aligned}$$

As $\lambda \sum_{j=1}^n c_{ij}(t) e^{x_j(t)} \leq 0$, we have

$$x_i(t) < Q_i, \quad i = 1, \dots, n. \quad (3.44)$$

In view of (3.39) and by a similar method we can obtain

$$\begin{aligned} \bar{r}_i \omega &> \int_0^\omega (a_{ii}(t) - b_{ii}(t) + c'_{ii}(t)) e^{x_i(t)} dt \\ &> \bar{r}_i \omega - \sum_{j=1, j \neq i}^n e^{Q_j} \int_0^\omega (a_{ij}(t) - b_{ij}(t) + c'_{ij}(t)) dt, \end{aligned} \quad (3.45)$$

then there exists $\zeta_i \in [0, \omega]$ such that

$$|x_i(\zeta_i)| < \max \left\{ \left| \ln \frac{\bar{r}_i}{\bar{a}_{ii} - \bar{b}_{ii}} \right|, \left| \ln \frac{\bar{r}_i - \sum_{j=1, j \neq i}^n e^{Q_j} (\bar{a}_{ij} - \bar{b}_{ij})}{\bar{a}_{ii} - \bar{b}_{ii}} \right| \right\} := H_i. \quad (3.46)$$

Again, by (3.37) and (3.44) we have

$$|x'_i| \leq |r_i|_0 + \sum_{j=1}^n (|a_{ij}|_0 + |b_{ij}|_0) e^{Q_j} + \sum_{j=1}^n |c_{ij}|_0 |x'_j|_0 e^{Q_j}, \quad i = 1, \dots, n.$$

Under condition (2) in Theorem 3.2, we have

$$\|x'\|_0 < \frac{\sum_{i=1}^n |r_i|_0 + \sum_{i=1}^n \sum_{j=1}^n (|a_{ij}|_0 + |b_{ij}|_0) e^{Q_j}}{1 - \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|_0 e^{Q_j}} := P_*. \quad (3.47)$$

The rest of the proof is similar to that of Theorem 3.1, and the details are omitted. The proof of Theorem 3.2 is also complete. \square

4. Discussion

In this paper, based on Gopalsamy's autonomous model (1.4) in [8] we consider the existence of a positive periodic solution of a periodic neutral Lotka–Volterra system with delays. We obtain two main results by applying some analysis techniques and an existence theorem whose assumptions are different from that of Gaines and Mawhin's continuation theorem [11] and that of abstract continuation theory for k -set contraction [4]. But we only consider the case when $c_{ij}(t) \leq 0$. We would like to mention here an interesting but challenging problem associated with the existence of a positive periodic solution of system (1.3) when $c_{ij}(t) > 0$. Of course, the uniqueness and global stability of positive ω -periodic solutions of system (1.3) would also be of great interest. We leave the above problems for future work.

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