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Cohen–Host type idempotent theorems for representations on Banach spaces and applications to Figà-Talamanca–Herz algebras ☆

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Abstract

Let G be a locally compact group, and let $\mathcal{R}(G)$ denote the ring of subsets of G generated by the left cosets of open subsets of G. The Cohen–Host idempotent theorem asserts that a set lies in $\mathcal{R}(G)$ if and only if its indicator function is a coefficient function of a unitary representation of G on some Hilbert space. We prove related results for representations of G on certain Banach spaces. We apply our Cohen–Host type theorems to the study of the Figà-Talamanca–Herz algebras $A_p(G)$ with $p \in (1, \infty)$. For arbitrary G, we characterize those closed ideals of $A_p(G)$ that have an approximate identity bounded by 1 in terms of their hulls. Furthermore, we characterize those G such that $A_p(G)$ is 1-amenable for some—and, equivalently, for all—$p \in (1, \infty)$: these are precisely the abelian groups.

Keywords: Amenability; Bounded approximate identity; Coset ring; Figà-Talamanca–Herz algebra; Locally compact group; Smooth Banach space; Ultrapower; Uniform convexity; Uniformly bounded representation

0. Introduction

Let G be a locally compact abelian group with dual group $\hat{G}$. In [2], P.J. Cohen characterized the idempotent elements of the measure algebra $M(G)$ in terms of their Fourier–Stieltjes trans-

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forms: $\mu \in M(G)$ is idempotent if and only if $\hat{\mu}$ is the indicator function of a set in the coset ring of $\hat{G}$.

In [9], P. Eymard introduced, for a general locally compact group G, the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$. If G is abelian, the Fourier and Fourier–Stieltjes transform, respectively, yield isometric Banach algebra isomorphisms $A(G) \cong L^1(\hat{G})$ and $B(G) \cong M(\hat{G})$. (In the framework of Kac algebras, this extends to a duality between $L^1(G)$ and $A(G)$ for arbitrary G; see [8].) In [22], B. Host extended Cohen's idempotent theorem to Fourier–Stieltjes algebras of arbitrary locally compact groups. Besides being more general than Cohen's theorem, Host's result also has a much simpler proof that only requires elementary operator theory on Hilbert spaces.

Host's result—to which we shall refer as to the *Cohen–Host idempotent theorem* or simply the *Cohen–Host theorem*—has turned out to be a tool of crucial importance in the investigation of $A(G)$ and $B(G)$. We mention only three recent applications:

- *Homomorphisms between Fourier algebras.* Already Cohen used his theorem to describe the algebra homomorphism from $A(G)$ to $B(H)$, where G and H are locally compact abelian groups [3]. Host extended Cohen's result to a setting where only G had to be abelian [22]. This line of research culminated only recently with a complete description of the completely bounded algebra homomorphism from $A(G)$ to $B(H)$ with G amenable [23,24].
- *Ideals of $A(G)$ with a bounded approximate identity.* In [15], the closed ideals of $A(G)$ (for amenable G) that have a bounded approximate identity are completely characterized in terms of their hulls. One direction of this result requires operator space methods (see [7]), but the converse implication relies mainly on the Cohen–Host result.
- *Amenability of $A(G)$.* B.E. Forrest and the author, in [16], characterized those locally compact groups G for which $A(G)$ is amenable in the sense of [25]: they are precisely those with an abelian subgroup of finite index [16, Theorem 2.3]. The proof in [16] relies on the Cohen–Host idempotent theorem only indirectly—through [15]—, but recently, the author gave an alternative proof that invokes the idempotent theorem directly [37].

The Figà-Talamanca–Herz algebra $A_p(G)$ for $p \in (1, \infty)$ were introduced and first studied by C. Herz [20,21]; more recent papers investigating those algebras are, for example, [14,17,29, 33]. They are natural generalizations of $A(G)$ in the sense that $A_2(G) = A(G)$. The algebras $A_p(G)$ share many properties of $A(G)$. For instance, Leptin's theorem [32] extends easily to $A_p(G)$ [21]: G is amenable if and only if $A_p(G)$ has an approximate identity for some—and, equivalently, for all—$p \in (1, \infty)$. Nevertheless, since $A_p(G)$ has no obvious connection with Hilbert space for $p \neq 2$, the powerful methods of operator algebras are not available anymore—or are at least not as easily applicable—for the study of general Figà-Talamanca–Herz algebras. As a consequence, many questions to which the answers are easy—or have at least long been known—for $A(G)$ are still open for general $A_p(G)$.

Since the Cohen–Host theorem is about $B(G)$ its use for the investigation of Figà-Talamanca–Herz algebras is usually limited to the case where $p = 2$. In this paper, we therefore strive for extensions of this result that are applicable to the study of $A_p(G)$ for general $p \in (1, \infty)$. The elements of $B(G)$ can be interpreted as the coefficient functions of the unitary representations of G on Hilbert spaces, so that the Cohen–Host theorem (or rather its difficult direction) can be formulated as follows: if the indicator function of a subset of G is a coefficient function of a unitary representation of G on a Hilbert space, then the set lies in the coset ring of G. We shall prove two Cohen–Host type theorems for representations on Banach spaces. In particular, we shall extend [24, Theorem 2.1] to isometric representations on Banach spaces which are smooth or have a smooth dual.

We apply this Cohen–Host type theorem to the study of general Figà-Talamanca–Herz algebras.

First, we characterize those closed ideals I of $A_p(G)$ that have an approximate identity bounded by 1: we shall see that I has such an approximate identity if and only if I consists precisely of those functions in $A_p(G)$ that vanish outside some left coset of an open, amenable subgroup of G. This result is related to [15, Theorems 2.3 and 4.3], and extends [13, Proposition 3.12] from $p = 2$ to arbitrary $p \in (1, \infty)$.

Secondly, we extend [37, Theorem 3.5] and show that $A_p(G)$ is 1-amenable for some—and, equivalently, for all—$p \in (1, \infty)$ if and only if G is abelian.

1. Cohen–Host type idempotent theorems for representations on Banach spaces

Our notion of a representation of a locally compact group on a Banach space is the usual one:

Definition 1.1. Let G be a locally compact group. Then (π, E) is said to be a *representation* of G on E if E is a Banach space and $\pi : G \to \mathcal{B}(E)$ is a group homomorphism into the invertible operators on E which is continuous with respect to the given topology on G and the strong operator topology on $\mathcal{B}(E)$. We call (π, E) *uniformly bounded* if $\sup_{x\in G} \|\pi(x)\| < \infty$ and *isometric* if $\pi(G)$ only consists of isometries.

Remarks. (1) Suppose that (π, E) is uniformly bounded. Then

$$|||\xi||| := \sup_{x\in G} \left\|\pi(x)\xi\right\| \quad (\xi \in E)$$

defines an equivalent norm on E such that $(\pi, (E, |||\cdot|||))$ is isometric. This, however, may obscure particular geometric features of the original norm.

(2) Since invertible isometries on a Hilbert space are just the unitary operators, the isometric representations of G on a Hilbert space, are just the usual unitary representations.

(3) Every representation (π, E) of G induces a representation of the group algebra $L^1(G)$ on E through integration, which we denote likewise by (π, E).

We are interested in certain functions associated with representations:

Definition 1.2. Let G be a locally compact group, and let (π, E) a representation of G. A function $f : G \to \mathbb{C}$ is called a *coefficient function* of (π, E) if there are $\xi \in E$ and $\phi \in E^*$ such that

$$f(x) = \left\langle \pi(x)\xi, \phi \right\rangle \quad (x \in G). \tag{1}$$

If $\|\xi\| = \|\phi\| = 1$, we call f *normalized*.

The coefficient functions of the unitary representations of a locally compact group G form an algebra (under the pointwise operations), the *Fourier–Stieltjes algebra* $B(G)$ of G (see [9]). Moreover, $B(G)$ can be identified with the dual space of the full group C^*-algebra of G, which turns it into a commutative Banach algebra.

Extending earlier work by Cohen in the abelian case [2], Host identified the idempotents of $B(G)$ [22]. Since $B(G)$ consists of continuous functions, it is clear that an idempotent of $B(G)$

has to be the indicator function χ_C of some clopen subset C of G. Let $\mathcal{R}(G)$ denote the *coset ring* of G, i.e. the ring of sets generated by all left cosets of open subgroups of G. In [22], Host showed that the idempotents of $B(G)$ are precisely of the form χ_C with $C \in \mathcal{R}(G)$.

Given a representation (π, E), where E is not necessarily a Hilbert space, the set of coefficient functions of (π, E) need not be a linear space anymore, let alone an algebra. Nevertheless, it makes sense to attempt to characterize those subsets C of G for which χ_C is a coefficient function of (π, E).

Without any additional hypotheses, we cannot hope to extend the Cohen–Host theorem:

Example. Let G be any locally compact group, and let $\mathcal{C}_{\mathrm{b}}(G)$ denote the bounded, continuous function on G. For any function $f : G \to \mathbb{C}$ and $x \in G$, define

$$r_x f : G \to \mathbb{C}, \quad y \mapsto f(yx),$$

and call $f \in \mathcal{C}_{\mathrm{b}}(G)$ *right uniformly continuous* if the map

$$G \to \mathcal{C}_{\mathrm{b}}(G), \quad x \mapsto r_x f$$

is continuous with respect to the given topology on G and the norm topology on $\mathcal{C}_{\mathrm{b}}(G)$. The set of all right uniformly continuous function on G is a C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}(G)$, which we denote by $RUC(G)$. Define an isometric representation $(\rho, RUC(G))$ by letting $\rho(x) f := r_x f$ for $x \in G$ and $f \in RUC(G)$. It is then immediate that

$$f(x) = \langle \rho(x) f, \delta_e \rangle \quad \big(f \in RUC(G),\ x \in G\big),$$

where δ_e is the point mass at the identity of G, so that every element of $RUC(G)$ is a coefficient function of $(\rho, RUC(G))$. For discrete G, it is clear that $RUC(G) = \ell^\infty(G)$, so that χ_C is a coefficient function of $(\rho, RUC(G))$ for every $C \subset G$.

If we impose restrictions on both the group and the Banach space on which it is represented, an extension of the Cohen–Host theorem is surprisingly easy to obtain.

For any locally compact group G, denote the component of the identity by G_e; it is a closed, normal subgroup of G. Recall that G is said to be *almost connected* if G/G_e is compact.

Theorem 1.3. *Let G be an almost connected locally compact group. Then the following are equivalent for $C \subset G$:*

(i) $C \in \mathcal{R}(G)$;
(ii) $\chi_C \in B(G)$;
(iii) *χ_C is a coefficient function of a uniformly bounded representation (π, E) of G, where E is reflexive.*

Proof. (i) $\Leftrightarrow$ (ii) is the Cohen–Host theorem, and (ii) $\Rightarrow$ (iii) is straightforward.

(iii) $\Rightarrow$ (i). Let $\xi \in E$ and $\phi \in E^*$ such that χ_C is of the form (1). We can suppose without loss of generality that $\{\pi^*(f)\phi\colon f \in L^1(G)\}$ is dense in E^*: otherwise, replace E^* by $\{\pi^*(f)\phi\colon f \in L^1(G)\}^-$ and E by its quotient modulo $\{\pi^*(f)\phi\colon f \in L^1(G)\}^\perp$.

We claim that $\mathbb{I} := \{\pi(x)\xi\colon x \in G\}$ is uniformly discrete in the norm topology. To see this, let $x_1, x_2 \in G$ be such that $\|\pi(x_1)\xi - \pi(x_2)\xi\| < \frac{1}{C\|\phi\|+1}$, where $C := \sup_{x\in G} \|\pi(x)\|$. We thus have

$$\big|\langle \pi(y)\pi(x_1)\xi, \phi\rangle - \langle \pi(y)\pi(x_2)\xi, \phi\rangle\big| < 1 \quad (y \in G). \tag{2}$$

On the other hand, since $\langle \pi(x)\xi, \phi \rangle \in \{0,1\}$ for $x \in G$, it is clear that $|\langle \pi(y)\pi(x_1)\xi, \phi \rangle - \langle \pi(y)\pi(x_2)\xi, \phi \rangle| \geqslant 1$ whenever $y \in G$ is such that $\langle \pi(y)\pi(x_1)\xi, \phi \rangle \neq \langle \pi(y)\pi(x_2), \phi \rangle$. Combining this with (2) yields

$$\bigl\langle \pi(y)\pi(x_1)\xi, \phi \bigr\rangle = \bigl\langle \pi(y)\pi(x_2), \phi \bigr\rangle \quad (y \in G). \tag{3}$$

Integrating (3) with respect to y, we obtain

$$\bigl\langle \pi(x_1)\xi, \pi(f)^*\phi \bigr\rangle = \bigl\langle \pi(f)\pi(x_1)\xi, \phi \bigr\rangle = \bigl\langle \pi(f)\pi(x_2)\xi, \phi \bigr\rangle = \bigl\langle \pi(x_2)\xi, \pi(f)^*\phi \bigr\rangle$$
$$\bigl(f \in L^1(G)\bigr).$$

Since $\{\pi^*(f)\phi\colon\ f \in L^1(G)\}^- = E^*$, the Hahn–Banach theorem yields that $\pi(x_1)\xi = \pi(x_2)\xi$.

Since $\{\pi(x)\xi\colon\ x \in G_e\}$ is connected in the norm topology of E, we conclude that $\pi(x)\xi = \xi$ for all $x \in G_e$. As a consequence, $\pi(x)\xi$ with $x \in G$ only depends on the coset of x in G/G_e. Hence, the map

$$G/G_e \to E, \quad xG_e \mapsto \pi(x)\xi$$

is well defined, is continuous with respect to the norm topology on E, and clearly has $\mathbb{I}$ as its range. Since G/G_e is compact, it follows that $\mathbb{I}$ is compact and thus finite.

Let G_d denote the group G equipped with the discrete topology. Define a unitary representation $\tilde{\pi}$ of G_d on $\ell^2(\mathbb{I})$ by letting

$$\tilde{\pi}(x)\delta_\eta := \delta_{\pi(x)\eta} \quad (x \in G,\ \eta \in \mathbb{I}).$$

Since $\mathbb{I}$ is finite, the restriction of ϕ to $\mathbb{I}$ can be identified with an element of $\ell^2(\mathbb{I})^*$, which we denote by $\tilde{\phi}$. By construction, we have

$$\bigl\langle \tilde{\pi}(x)\delta_\xi, \tilde{\phi} \bigr\rangle = \bigl\langle \pi(x)\xi, \phi \bigr\rangle = \chi_C(x) \quad (x \in G),$$

so that $\chi_C \in B(G_d)$. Since C is clopen, χ_C is continuous, so that actually $\chi_C \in B(G)$ by [9, (2.24), Corollaire 1]. From [22], we conclude that $C \in \mathcal{R}(G)$. $\square$

In [24], M. Ilie and N. Spronk proved a variant of the Cohen–Host theorem for normalized coefficient functions in the Fourier–Stieltjes algebra: they showed that these are precisely the indicator functions of left cosets of open subgroups [24, Theorem 2.1]. As Spronk pointed out to the author, the argument used in [24] can be adapted to certain Banach spaces.

The following definition is crucial (see [28], for instance):

Definition 1.4. A Banach space E is said to be *smooth* if, for each $\xi \in E \setminus \{0\}$, there is a unique $\phi \in E^*$ such that $\|\phi\| = 1$ and $\langle \xi, \phi \rangle = \|\xi\|$.

Extending [24, Theorem 2.1], we obtain:

Theorem 1.5. *Let G be a locally compact group. Then the following are equivalent for $C \subset G$:*

(i) *C is a left coset of an open subgroup of G;*
(ii) *$\chi_C \in B(G)$ with $\|\chi_C\| = 1$;*
(iii) *$\chi_C \not\equiv 0$ is a normalized coefficient function of an isometric representation (π, E) of G, where E or E^* is smooth.*

Proof. (i) $\Leftrightarrow$ (ii) is [24, Theorem 2.1(i)], and (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Suppose that $\chi_C \not\equiv 0$ is of the form (1) with $\xi \in E$ and $\phi \in E^*$ such that $\|x\| = \|\phi\| = 1$.

We first treat the case where E^* is smooth. Fix $x \in C$, and set

$$H := \{y \in G\colon\ xy \in C\}.$$

By definition, we have for $y \in G$ that

$$y \in H \quad\Longleftrightarrow\quad \langle \pi(xy)\xi, \phi\rangle = \langle \pi(y)\xi, \pi(x)^*\phi\rangle = \langle \xi, \pi(x)^*\phi\rangle = 1.$$

Since E^* is smooth, there is a *unique* $\Psi \in E^{**}$ such that $\langle \pi(x)^*\phi, \Psi\rangle = 1$. From this uniqueness assertion, it follows that $\Psi = \xi = \pi(y)\xi$ for all $y \in H$ and that

$$H = \{y \in G\colon\ \pi(y)\xi = \xi\}.$$

Consequently, H is a subgroup of G, and it is immediate that $C = xH$. Since χ_C is continuous, C—and thus H—is clopen. This proves (i).

If E is smooth, an analogous argument yields that there are $x \in G$ and an open subgroup H of G such that $C = Hx$. Since $Hx = x(x^{-1}Hx)$, this also proves (i). $\square$

We conclude with a look at those spaces to which we shall apply Theorem 1.5 in the next section:

Example. The *modulus of convexity* of a Banach space E is defined, for $\epsilon \in (0,2]$ as

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{\|\xi + \eta\|}{2}\colon\ \xi, \eta \in E,\ \|\xi\| \leqslant 1,\ \|\eta\| \leqslant 1,\ \|\xi - \eta\| \geqslant \epsilon\right\} > 0;$$

if $\delta_E(\epsilon) > 0$ for each $\epsilon \in (0,2]$, then E is called *uniformly convex* [11, Definition 9.1]. All uniformly convex Banach spaces are reflexive [11, Theorem 9.12]. If X is any measure space and $p \in (1,\infty)$, then $L^p(X)$ is uniformly convex [11, Theorem 9.3]. More generally, whenever E is a uniformly convex Banach space, X is any measure space, and $p \in (1,\infty)$, the vector valued L^p-space $L^p(X,E)$ is again uniformly convex [6]; in particular, for any two measure spaces X and Y and $p,q \in (1,\infty)$, the Banach space $L^p(X, L^q(Y))$ is uniformly convex. If E is uniformly convex, then E^* is smooth by [11, Lemma 8.4(i) and Theorem 9.10]. Hence, if G is a locally compact group and $C \subset G$ is such that χ_C is a normalized coefficient function of an isometric representation on a Banach space, which is uniformly convex or has a uniformly convex dual, then C is a left coset of an open subgroup of G by Theorem 1.5.

The proof of the general Cohen–Host theorem from [22] relies heavily on some (elementary) facts on Hilbert space operators, for which there seem to be no analogs in a more general Banach space setting. Concluding this section, we shall see that there is *no* general Cohen–Host theorem for isometric representations on uniformly convex Banach spaces:

Example. A subset L of a group G is called a *Leinert set* (see [30,31]) if, for any $x_1, \ldots, x_{2n} \in L$ with $x_j \neq x_{j+1}$ for $j = 1, \ldots, 2n-1$, we have $x_1^{-1}x_2x_3^{-1}\cdots x_{2n-1}^{-1}x_{2n} \neq e$. For instance, the subset $\{a^nb^n\colon\ n \in \mathbb{Z}\}$ of the free group $\mathbb{F}_2$ generated by a and b is a Leinert set [30, (1.10)]. By the proof of [31, (12) Korollar], the indicator function of an infinite Leinert subset of $\mathbb{F}_2$ does not lie in $B(\mathbb{F}_2)$, so that the set does not belong to $\mathcal{R}(\mathbb{F}_2)$. On the other hand, the indicator function of every Leinert subset of a group G is a coefficient function of a uniformly bounded representation of G on some Hilbert space [12, 1.1 Theorem]. By [1, Lemma 2], this Hilbert space can be equipped with an equivalent, still uniformly convex norm such that the given representation of G becomes isometric. Hence, there are subsets of $\mathbb{F}_2$ that do not lie in $\mathcal{R}(\mathbb{F}_2)$, but are nevertheless coefficient functions of isometric representations of $\mathbb{F}_2$ on uniformly convex Banach spaces.

Remark. The foregoing example leaves it open whether the following are equivalent for $C \subset G$:

(i) $C \in \mathcal{R}(G)$;
(ii) χ_C is a coefficient function of a uniformly bounded representation (π, E) of G, where is a Banach space such that both E and E^* are uniformly convex.

As pointed out in the last remark of this paper below, the equivalence of these two assertions is crucial to obtain a characterization of those locally compact groups with amenable Figà-Talamanca–Herz algebras.

2. Applications to $A_p(G)$

We shall now turn to applications of Theorem 1.5 to Figà-Talamanca–Herz algebras on locally compact groups.

Let G be a locally compact group. For any function $f : G \to \mathbb{C}$, we define $\check{f} : G \to \mathbb{C}$ by letting $\check{f}(x) := f(x^{-1})$ for $x \in G$. Let $p \in (1, \infty)$, and let $p' \in (1, \infty)$ be dual to p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The *Figà-Talamanca–Herz algebra* $A_p(G)$ consists of those functions $f : G \to \mathbb{C}$ such that there are sequences $(\xi_n)_{n=1}^\infty$ in $L^p(G)$ and $(\eta_n)_{n=1}^\infty$ in $L^{p'}(G)$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|_{L^p(G)} \|\eta_n\|_{L^{p'}(G)} < \infty \tag{4}$$

and

$$f = \sum_{n=1}^{\infty} \xi_n * \check{\eta}_n. \tag{5}$$

The norm on $A_p(G)$ is defined as the infimum over all sums (4) such that (5) holds. It is clear that $A_p(G)$ is a Banach space that embeds contractively into $\mathcal{C}_0(G)$, the algebra of all continuous functions on G vanishing at infinity. It was shown by C. Herz [20] (see also [10] or [34]) that $A_p(G)$ is closed under pointwise multiplication and thus a Banach algebra. More specifically [21, Proposition 3 and Theorem 3], $A_p(G)$ is a regular, Tauberian, commutative Banach algebra whose character space can be canonically identified with G. If $p = 2$, the algebra $A_2(G)$ is Eymard's *Fourier algebra* $A(G)$ [9]. (With our notation, we follow [10]—as does [34]—rather than [20,21] like most authors do: $A_p(G)$ in our sense is $A_{p'}(G)$ in [20,21].)

The algebras $A_p(G)$ are related to certain isometric representations of G. Let $\lambda_{p'} : G \to \mathcal{B}(L^{p'}(G))$ be the *regular left representation* of G on $L^{p'}(G)$, i.e.

$$\big(\lambda_{p'}(x)\xi\big)(y) = \xi\big(x^{-1}y\big) \quad \big(x, y \in G,\ \xi \in L^{p'}(G)\big).$$

The algebra of *p'-pseudomeasures* $\mathrm{PM}_{p'}(G)$ is defined as the w^*-closure of $\lambda_{p'}(L^1(G))$ in the dual Banach space $\mathcal{B}(L^{p'}(G))$. There is a canonical duality $\mathrm{PM}_{p'}(G) \cong A_p(G)^*$ via

$$\langle \xi * \check{\eta}, T\rangle := \langle T\eta, \xi\rangle \quad \big(\xi \in L^{p'}(G),\ \eta \in L^p(G),\ T \in \mathrm{PM}_{p'}(G)\big).$$

In particular, we have

$$(\xi * \check{\eta})(x) = \langle \lambda_{p'}(x)\eta, \xi \rangle \quad \big(\xi \in L^{p'}(G),\ \eta \in L^p(G),\ x \in G\big).$$

Hence, even though it seems to be still unknown (see [10, 9.2]) if $A_p(G)$ consists of coefficient functions of $\lambda_{p'}$—except if $p = 2$, of course—, the elements of $A_p(G)$ are nevertheless not far from being coefficient functions of $\lambda_{p'}$ and are, in fact, coefficient functions of a representation closely related to $\lambda_{p'}$:

Proposition 2.1. *Let G be a locally compact group, let $p \in (1,\infty)$, and let $\lambda_{p'}^\infty : G \to \mathcal{B}(\ell^{p'}(L^{p'}(G)))$ be defined by letting*

$$\lambda_{p'}^\infty(x) := \mathrm{id}_{\ell^p} \otimes \lambda_{p'}(x) \quad (x \in G).$$

Then $(\lambda_{p'}^\infty, \ell^{p'}(L^{p'}(G)))$ is an isometric representation of G and every $f \in A_p(G)$ is a coefficient function of $(\lambda_{p'}^\infty, \ell^{p'}(L^{p'}(G)))$. More precisely, for every $\epsilon > 0$, there are $\eta \in \ell^{p'}(L^{p'}(G))$ and $\xi \in \ell^p(L^p(G))$ such that $\|\eta\|\|\xi\| < \|f\| + \epsilon$ and

$$f(x) = \langle \lambda_{p'}^\infty(x)\eta, \xi \rangle \quad (x \in G).$$

Proof. To check that $(\lambda_{p'}^\infty, \ell^{p'}(L^{p'}(G)))$ is an isometric representation of G is straightforward.

Let $f \in A_p(G)$ and let $\epsilon > 0$. By the definition of $A_p(G)$, there are sequences $(\tilde{\xi}_n)_{n=1}^\infty$ in $L^p(G)$ and $(\tilde{\eta}_n)_{n=1}^\infty$ in $L^{p'}(G)$ such that

$$f(x) = \sum_{n=1}^\infty \langle \lambda_{p'}(x)\tilde{\eta}_n, \tilde{\xi}_n \rangle \quad (x \in G),$$

and

$$\sum_{n=1}^\infty \|\tilde{\xi}_n\|\|\tilde{\eta}_n\| < \|f\| + \epsilon.$$

For $n \in \mathbb{N}$, set

$$\xi_n := \begin{cases} \|\tilde{\xi}_n\|^{-1+\frac{1}{p}} \|\tilde{\eta}_n\|^{\frac{1}{p}} \tilde{\xi}_n, & \text{if } \tilde{\xi}_n \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\eta_n := \begin{cases} \|\tilde{\eta}_n\|^{-1+\frac{1}{p'}} \|\tilde{\xi}_n\|^{\frac{1}{p'}} \tilde{\eta}_n, & \text{if } \tilde{\eta}_n \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that,

$$\left(\sum_{n=1}^\infty \|\xi_n\|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^\infty \|\tilde{\xi}_n\|\|\tilde{\eta}_n\|\right)^{\frac{1}{p}} < \big(\|f\| + \epsilon\big)^{\frac{1}{p}}$$

and, similarly,

$$\left(\sum_{n=1}^\infty \|\eta_n\|^{p'}\right)^{\frac{1}{p'}} < \big(\|f\| + \epsilon\big)^{\frac{1}{p'}}.$$

Consequently, $\xi := (\xi_n)_{n=1}^{\infty} \in \ell^p(L^p(G))$ and $\eta := (\eta_n)_{n=1}^{\infty} \in \ell^{p'}(L^{p'}(G))$ satisfy

$$\|\xi\|\|\eta\| < \|f\| + \epsilon$$

as well as

$$f(x) = \sum_{n=1}^{\infty} \langle \lambda_{p'}(x)\tilde{\eta}_n, \tilde{\xi}_n \rangle = \sum_{n=1}^{\infty} \langle \lambda_{p'}(x)\eta_n, \xi_n \rangle = \langle \lambda_{p'}^{\infty}(x)\eta, \xi \rangle \quad (x \in G).$$

This completes the proof. □

Remark. The proof of Proposition 2.1 is patterned after that of [5, Proposition 5].

2.1. Ideals with a bounded approximate identity

In this section, we shall characterize—for arbitrary G and $p \in (1, \infty)$—those closed ideals of $A_p(G)$ that have an approximate identity bounded by 1.

Given a locally compact group G, $p \in (1, \infty)$, and a closed subset F of G, we let

$$I(F) := \{ f \in A_p(G) \colon f|_F \equiv 0 \}.$$

Theorem 2.2. *Let G be a locally compact group, and let $p \in (1, \infty)$. Then the following are equivalent for a closed ideal I of $A_p(G)$:*

(i) *I has an approximate identity bounded by* 1;
(ii) *there are $x \in G$ and an open, amenable subgroup H of G such that $I = I(G \setminus xH)$.*

Our key to proving Theorem 2.2 is the following proposition:

Proposition 2.3. *Let G be a locally compact group, let $p \in (1, \infty)$, and let $(f_\alpha)_{\alpha \in \mathbb{A}}$ be a bounded net in $A_p(G)$ that converges pointwise on G to a function $f : G \to \mathbb{C}$. Then there is a measure space X and an isometric representation $(\pi, L^{p'}(X))$ of G_d such that f is a coefficient function of $(\pi, L^{p'}(X))$. More precisely, if $C \geqslant 0$ is such that $\sup_\alpha \|f_\alpha\| \leqslant C$, then there are $\eta \in L^{p'}(X)$ and $\xi \in L^p(X)$ with $\|\eta\|\|\xi\| \leqslant C$ and*

$$f(x) = \langle \pi(x)\eta, \xi \rangle \quad (x \in G).$$

Before we prove Proposition 2.3, we recall a few facts about ultrapowers of Banach spaces (see [19,38]).

Let E be a Banach space, and let $\mathbb{I}$ be any index set. We denote the Banach space of all bounded families $(\xi_i)_{i \in \mathbb{I}}$ in E, equipped with the supremum norm, by $\ell^\infty(\mathbb{I}, E)$. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{I}$, and define

$$\mathcal{N}_{\mathfrak{U}} := \Big\{ (\xi_i)_{i \in \mathbb{I}} \in \ell^\infty(\mathbb{I}, E) \colon \lim_{\mathfrak{U}} \|\xi_i\| = 0 \Big\}.$$

Then $\mathcal{N}_{\mathfrak{U}}$ is a closed subspace of $\ell^\infty(\mathbb{I}, E)$. The quotient space $\ell^\infty(\mathbb{I}, E)/\mathcal{N}_{\mathfrak{U}}$ is called an *ultrapower* of E and denoted by $(E)_{\mathfrak{U}}$. For any $(\xi_i)_{i \in \mathbb{I}} \in \ell^\infty(\mathbb{I}, E)$, we denote its equivalence class in $(E)_{\mathfrak{U}}$ by $(\xi_i)_{\mathfrak{U}}$; it is easy to see that

$$\big\| (\xi_i)_{\mathfrak{U}} \big\|_{(E)_{\mathfrak{U}}} = \lim_{\mathfrak{U}} \|\xi_i\|_E. \tag{6}$$

We require the following facts about ultrapowers:

- If $E = L^p(X)$ for $p \in (1, \infty)$ and some measure space X, then there is a measure space Y such that $(E)_{\mathfrak{U}} \cong L^p(Y)$ [19, Theorem 3.3(ii)].
- There is a canonical isometric embedding of $(E^*)_{\mathfrak{U}}$ into $(E)_{\mathfrak{U}}^*$, via the duality

$$\big\langle (\xi_i)_{\mathfrak{U}}, (\phi_i)_{\mathfrak{U}} \big\rangle := \lim_{\mathfrak{U}} \langle \xi_i, \phi_i \rangle \quad \big((\xi_i)_{\mathfrak{U}} \in (E)_{\mathfrak{U}},\ (\phi_i)_{\mathfrak{U}} \in \big(E^*\big)_{\mathfrak{U}} \big),$$

  which, in general, need not be surjective [19, p. 87].
- If E is uniformly convex, then so is $(E)_{\mathfrak{U}}$ [38, §10, Proposition 6].

Proof of Proposition 2.3. Let $C \geqslant 0$ such that $\sup_\alpha \|f_\alpha\| \leqslant C$.

For each $\alpha \in \mathbb{A}$ and $\epsilon > 0$, Proposition 2.1 provides $\eta_{\alpha,\epsilon} \in \ell^{p'}(L^{p'}(G))$ and $\xi_{\alpha,\epsilon} \in \ell^p(L^p(G))$ such that

$$\|\eta_{\alpha,\epsilon}\| \|\xi_{\alpha,\epsilon}\| \leqslant C + \epsilon \tag{7}$$

and

$$f_\alpha(x) = \big\langle \lambda_{p'}^\infty(x)\eta_{\alpha,\epsilon}, \xi_{\alpha,\epsilon} \big\rangle \quad (x \in G). \tag{8}$$

Turn $\mathbb{I} := \mathbb{A} \times (0, \infty)$ into a directed set via

$$(\alpha_1, \epsilon_1) \preccurlyeq (\alpha_2, \epsilon_2) \quad :\Longleftrightarrow \quad \alpha_1 \preccurlyeq \alpha_2 \quad \text{and} \quad \epsilon_1 \geqslant \epsilon_2,$$

and let $\mathfrak{U}$ be an ultrafilter on $\mathbb{I}$ that dominates the order filter. Since $\ell^{p'}(L^{p'}(G)) \cong L^{p'}(\mathbb{N} \times G)$ is an $L^{p'}$-space, there is a measure space X such that $(\ell^{p'}(L^{p'}(G)))_{\mathfrak{U}} \cong L^{p'}(X)$. Define $\pi : G \to \mathcal{B}(L^{p'}(X))$ by letting

$$\pi(x)(\zeta_i)_{\mathfrak{U}} := \big(\lambda_{p'}^\infty(x)\zeta_i\big)_{\mathfrak{U}} \quad \big(x \in G,\ (\zeta_i)_{\mathfrak{U}} \in L^{p'}(X)\big).$$

It is then clear that $(\pi, L^{p'}(X))$ is an isometric representation, if not of G, but at least of G_d. Set $\eta := (\eta_{\alpha,\epsilon})_{\mathfrak{U}}$ and $\xi := (\xi_{\alpha,\epsilon})_{\mathfrak{U}}$, so that $\eta \in L^{p'}(X)$ and $\xi \in L^{p'}(X)^* \cong L^p(X)$. From (7), it is immediate that $\|\eta\| \|\xi\| \leqslant C$, and from (8), we obtain

$$f(x) = \lim_{\mathfrak{U}} f_\alpha(x) = \lim_{\mathfrak{U}} \big\langle \lambda_{p'}^\infty(x)\eta_{\alpha,\epsilon}, \xi_{\alpha,\epsilon} \big\rangle = \big\langle \pi(x)\eta, \xi \big\rangle \quad (x \in G).$$

This completes the proof. □

Remark. The idea to use ultrapowers to "glue together" representations of groups or algebras seems to appear for the first time in [4] and also—less explicitly and, as it seems, independently of [4]—in [5].

Another ingredient of the proof of Theorem 2.2 is:

Lemma 2.4. *Let G be a locally compact group, let $p \in (1, \infty)$, and let H be an open subgroup of G. Then we have a canonical isometric isomorphism of $A_p(H)$ and $\{f \in A_p(G)\colon \operatorname{supp}(f) \subset H\}$.*

Proof. By [21, Theorem 1], restriction to H is a quotient map from $A_p(G)$ onto $A_p(H)$. Consequently, we have a contractive inclusion

$$\big\{ f \in A_p(G)\colon \operatorname{supp}(f) \subset H \big\} \subset A_p(H).$$

(This does not require H to be open.)

Since H is open, we may view $L^p(H)$ and $L^{p'}(H)$, respectively, as closed subspaces of $L^p(G)$ and $L^{p'}(G)$, respectively. From the definition of $A_p(G)$ and $A_p(H)$ it is then immediate that $A_p(H)$ contractively embeds into $A_p(G)$. $\square$

Proof of Theorem 2.2. (i) $\Rightarrow$ (ii). Let $F \subset G$ be the *hull* of I, i.e.

$$F := \{x \in G\colon\ f(x) = 0 \text{ for all } f \in I\}.$$

Then F is obviously closed, and $I \subset I(F)$ holds.

Let $(e_\alpha)_\alpha$ be an approximate identity for I bounded by 1. Let $x \in G \setminus F$. Then there is $f \in I$ such that $f(x) \neq 0$. Since $\lim_\alpha e_\alpha(x) f(x) = f(x)$, it follows that $\lim_\alpha e_\alpha(x) = 1$. We conclude that $e_\alpha \to \chi_{G\setminus F}$ pointwise on G. By Proposition 2.3, there is thus a measure space X and an isometric representation $(\pi, L^{p'}(G))$ of G_d such that $\chi_{G\setminus F}$ is a normalized coefficient function of $(\pi, L^{p'}(G))$. Since $L^{p'}(X)$ is smooth, Theorem 1.5 yields that $G \setminus F = xH$ for some $x \in G$ and a subgroup H of G. Since F is closed, xH—and thus H—must be open.

What remains to be shown is the amenability of H. Without loss of generality, suppose that $F = G \setminus H$, so that

$$I \subset I(F) = \{f \in A_p(G)\colon\ \operatorname{supp}(f) \subset H\} \cong A_p(H)$$

by Lemma 2.4. Since the Banach algebra $A_p(H)$ is Tauberian, and since the hull of I in H is empty, it follows that $I = A_p(H)$, so that $A_p(H)$ has a bounded approximate identity. By [21, Theorem 6], this means that H is amenable.

(ii) $\Rightarrow$ (i). Without loss of generality, suppose that $I = I(G \setminus H)$ for some open subgroup of G, so that—again by Lemma 2.4—

$$I = \{f \in A_p(G)\colon\ \operatorname{supp}(f) \subset H\} \cong A_p(H).$$

Since H is amenable, $A_p(H)$ has an approximate identity bounded by 1 [21, Theorem 6], which proves the claim. $\square$

Remarks. (1) In the $p = 2$ case, Theorem 2.2 is [13, Proposition 3.12].

(2) For a locally compact group G, let

$$\mathcal{R}_c(G) := \{F \in \mathcal{R}(G_d)\colon\ F \text{ is closed}\}.$$

If G is amenable, then a closed ideal I of $A(G)$ has a bounded approximate identity if and only if $I = I(F)$ for some $F \in \mathcal{R}_c(G)$ [15, Theorem 2.3]. The "if" part of this result remains true with $A(G)$ replaced by $A_p(G)$ for arbitrary $p \in (1, \infty)$ [15, Theorem 4.3]. If the Cohen–Host idempotent theorem could be extended to isometric representations on L^p-spaces for general $p \in (1, \infty)$, then the proof of Theorem 2.2 can easily be adapted to extend both directions of [15, Theorem 2.3] to arbitrary Figà-Talamanca–Herz algebras.

2.2. *Amenability*

The theory of amenable Banach algebras begins with B.E. Johnson's memoir [25]. The choice of terminology is motivated by [25, Theorem 2.5]: a locally compact group is amenable (in the usual sense; see [34], for example), if and only if its group algebra $L^1(G)$ is an amenable Banach algebra.

Johnson's original definition of an amenable Banach algebra was in terms of cohomology groups [25]. We prefer to give another approach, which is based on a characterization of amenable Banach algebras from [26].

Let $\mathfrak{A}$ be a Banach algebra, and let $\hat{\otimes}$ stand for the (completed) projective tensor product of Banach spaces. The space $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in a canonical manner via

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya \quad (a, x, y \in \mathfrak{A}),$$

and the *diagonal operator*

$$\Delta_{\mathfrak{A}} : \mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A}, \quad a \otimes b \mapsto ab$$

is a homomorphism of Banach $\mathfrak{A}$-bimodules.

Definition 2.5. Let $\mathfrak{A}$ be a Banach algebra. An *approximate diagonal* for $\mathfrak{A}$ is a bounded net $(d_\alpha)_\alpha$ in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad (a \in \mathfrak{A}), \tag{9}$$

and

$$a \Delta_{\mathfrak{A}} d_\alpha \to a \quad (a \in \mathfrak{A}). \tag{10}$$

Definition 2.6. A Banach algebra $\mathfrak{A}$ is said to be *C-amenable* with $C \geqslant 1$ if there is an approximate diagonal for $\mathfrak{A}$ bounded by C. If $\mathfrak{A}$ is C-amenable for some $C \geqslant 1$, then $\mathfrak{A}$ is called *amenable*.

Examples. (1) Let G be a locally compact group. As mentioned already, $L^1(G)$ is amenable in the sense of Definition 2.6 if and only if G is amenable, and by [39], $L^1(G)$ is 1-amenable if and only if G is amenable. Hence, for $L^1(G)$, amenability and 1-amenability are equivalent.

(2) A C^*-algebra $\mathfrak{A}$ is amenable if and only if it is nuclear (see [35, Chapter 6] for a relatively self-contained exposition of this deep result). In fact, the nuclearity of $\mathfrak{A}$ implies already that it is 1-amenable [18, Theorem 3.1]. Hence, amenability and 1-amenability are also equivalent for C^*-algebras.

(3) In general, 1-amenability is far more restrictive than mere amenability: $A(G)$ is amenable for any finite group G, but is 1-amenable only if and only if G is abelian [27, Proposition 4.3].

For more examples and a modern account of the theory of amenable Banach algebras, see [35].

It is straightforward from Definitions 2.6 and 2.5 that an amenable Banach algebra must have a bounded approximate identity. It is therefore immediate from Leptin's theorem [32] and its generalization to Figà-Talamanca–Herz algebras by Herz [21, Theorem 6] that, for a locally compact group G, the Fourier algebra $A(G)$—or, more generally, $A_p(G)$ for any $p \in (1, \infty)$—can be amenable only if G is amenable. The tempting conjecture that $A(G)$ is amenable if and only if G is amenable, turned out to be wrong, however: in [27], Johnson exhibited examples of compact groups G such that $A(G)$ is not amenable. Eventually, Forrest and the author [16, Theorem 2.3] gave a characterization of those G for which $A(G)$ is amenable: they are precisely the *almost abelian* group, i.e. those with an abelian subgroup of finite index.

In [37], the author gave a more direct proof of [16, Theorem 2.3] that made direct appeal to the Cohen–Host idempotent theorem. Invoking [24, Theorem 2.1], the author also proved that $A(G)$ is 1-amenable for a locally compact group G if and only if G is abelian [37, Theorem 3.5]. In this section, we shall extend this latter result to general Figà-Talamanca–Herz algebras.

Even though our arguments in this section, parallel those in the last one, we now have to consider representations on spaces more general than L^p-spaces (which, nevertheless, will still turn out to be uniformly convex). Given two locally compact groups G and H, and $p, q \in (1, \infty)$, the left regular representation of $G \times H$ on $L^p(G, L^q(H))$ is defined as

$$\lambda_{p,q} : G \times H \to \mathcal{B}\big(L^p\big(G, L^q(H)\big)\big), \quad (x, y) \mapsto \lambda_p(x) \otimes \lambda_q(x).$$

It is immediate that $(\lambda_{p,q}, L^p(G, L^q(H)))$ is an isometric representation of $G \times H$, as is $(\lambda^\infty_{p,q}, \ell^p(L^p(G, L^q(H))))$, where

$$\lambda^\infty_{p,q}(x, y) := \mathrm{id}_{\ell^p} \otimes \lambda_{p,q}(x, y) \quad (x \in G,\ y \in H).$$

In analogy with Proposition 2.1, we have:

Lemma 2.7. *Let G and H be locally compact groups, let $p, q \in (1, \infty)$, and let $f \in A_p(G) \hat{\otimes} A_q(H)$. Then the Gelfand transform of f on $G \times H$ is a coefficient function of $(\lambda^\infty_{p,q}, \ell^p(L^p(G, L^q(H))))$, and for each $\epsilon > 0$, there are $\eta \in \ell^{p'}(L^{p'}(G,^{q'}(H)))$ and $\xi \in \ell^p(L^p(G, L^q(H)))$ such that*

$$\|\eta\|\|\xi\| < \|f\| + \epsilon \tag{11}$$

and

$$f(x, y) = \big\langle \lambda^\infty_{p',q'}(x, y)\eta, \xi \big\rangle \quad (x \in G,\ y \in H). \tag{12}$$

Proof. Let $\epsilon > 0$. From the definition of $A_p(G)$ and $A_q(H)$ and from the fact the projective tensor product is compatible with quotients, it follows that there are sequences $(\xi_{n,p})_{n=1}^\infty$ in $L^p(G)$, $(\xi_{n,q})_{n=1}^\infty$ in $L^q(H)$, $(\eta_{n,p})_{n=1}^\infty$ in $L^{p'}(G)$, and $(\eta_{n,q})_{n=1}^\infty$ in $L^{q'}(H)$ such that

$$f = \sum_{n=1}^\infty (\xi_{n,p} * \check{\eta}_{n,p}) \otimes (\xi_{n,q} * \check{\eta}_{n,q}) \tag{13}$$

and

$$f = \sum_{n=1}^\infty \|\xi_{n,p}\|_{L^p(G)} \|\check{\eta}_{n,p}\|_{L^{p'}(G)} \|\xi_{n,q}\|_{L^q(H)} \|\check{\eta}_{n,q}\|_{L^{q'}(H)} < \|f\| + \epsilon. \tag{14}$$

For $n \in \mathbb{N}$, set

$$\xi_n := \xi_{n,p} \otimes \xi_{n,q} \in L^p\big(G, L^q(H)\big) \quad \text{and} \quad \eta_n := \eta_{n,p} \otimes \eta_{n,q} \in L^{p'}\big(G, L^{q'}(H)\big).$$

From (13) and (14), it is then obvious that

$$f(x, y) = \sum_{n=1}^\infty \big\langle \lambda_{p',q'}(x, y)\eta_n, \xi_n \big\rangle \quad (x \in G,\ y \in H),$$

and

$$\sum_{n=1}^\infty \|\xi_n\|_{L^p(G, L^q(H))} \|\eta_n\|_{L^{p'}(G, L^{q'}(H))} < \|f\| + \epsilon.$$

As in the proof of Proposition 2.1, we eventually obtain $\eta \in \ell^{p'}(L^{p'}(G, L^{q'}(H)))$ and $\xi \in \ell^p(L^p(H, L^q(H)))$ that satisfy (11) and (12). $\square$

Proposition 2.8. *Let G and H be locally compact groups, let $p, q \in (1, \infty)$, and let $(f_\alpha)_{\alpha \in \mathbb{A}}$ be a bounded net in $A_p(G) \hat{\otimes} A_q(H)$ that converges pointwise on $G \times H$ to a function $f : G \times H \to \mathbb{C}$. Then there is an isometric representation (π, E) of $G_d \times H_d$ on a uniformly convex Banach space such that, if $C \geqslant 0$ is such that $\sup_\alpha \|f_\alpha\| \leqslant C$, there are $\eta \in E$ and $\xi \in E^*$ with $\|\eta\| \|\xi\| \leqslant C$ and*

$$f(x, y) = \langle \pi(x, y)\eta, \xi \rangle \quad (x \in G,\ y \in H).$$

Proof. The proof parallels that of Proposition 2.3, so that we can afford being somewhat sketchy.

Let $C \geqslant 0$ such that $\sup_\alpha \|f_\alpha\| \leqslant C$. For each $\alpha \in \mathbb{A}$ and $\epsilon > 0$, Lemma 2.7 provides $\eta_{\alpha,\epsilon} \in \ell^{p'}(L^{p'}(G, L^{q'}(H)))$ and $\xi_{\alpha,\epsilon} \in \ell^p(L^p(G, L^q(H)))$ such that

$$\|\eta_{\alpha,\epsilon}\| \|\xi_{\alpha,\epsilon}\| \leqslant C + \epsilon$$

and

$$f_\alpha(x, y) = \langle \lambda^\infty_{p',q'}(x, y)\eta_{\alpha,\epsilon}, \xi_{\alpha,\epsilon} \rangle \quad (x \in G,\ y \in H).$$

As in the proof of Proposition 2.3, turn $\mathbb{I} := \mathbb{A} \times (0, \infty)$ into a directed set, and let $\mathfrak{U}$ be an ultrafilter on $\mathbb{I}$ that dominates the order filter. Since $\ell^{p'}(L^{p'}(G, L^{q'}(H)))$ is uniformly convex by [6], so is $E := (\ell^{p'}(L^{p'}(G, L^{q'}(H))))_{\mathfrak{U}}$. Define $\pi : G \times H \to \mathcal{B}(E)$ by letting

$$\pi(x, y)(\zeta_i)_{\mathfrak{U}} := \left(\lambda^\infty_{p',q'}(x, y)\zeta_i\right)_{\mathfrak{U}} \quad \left(x \in G,\ y \in H,\ (\zeta_i)_{\mathfrak{U}} \in E\right),$$

and set $\eta := (\eta_{\alpha,\epsilon})_{\mathfrak{U}}$ and $\xi := (\xi_{\alpha,\epsilon})_{\mathfrak{U}}$.

As in the proof of Proposition 2.3, it is seen that (π, E), η, and ξ have the desired properties. $\square$

We obtain finally:

Theorem 2.9. *Let G be a locally compact group. Then the following are equivalent*:

(i) *G is abelian*;
(ii) *$A_p(G)$ is 1-amenable for each $p \in (1, \infty)$*;
(iii) *$A(G)$ is 1-amenable*;
(iv) *there is $p \in (1, \infty)$ such that $A_p(G)$ is 1-amenable.*

Proof. (i) ⇒ (iii). Suppose that G is abelian. Then $A(G) \cong L^1(\hat{G})$ is 1-amenable [39].

(iii) ⇒ (ii). Suppose that $A(G)$ is 1-amenable, and let $p \in (1, \infty)$. Since G must be amenable, [20, Theorem C] yields that $A(G)$ is contained in $A_p(G)$ such that the inclusion is contractive. A glance at the proof of [35, Proposition 2.3.1] shows that $A_p(G)$ then must be 1-amenable, too.

(ii) ⇒ (iv) is trivial.

(iv) ⇒ (i). Let $p \in (1, \infty)$ be such that $A_p(G)$ is 1-amenable, and let $(d_\alpha)_\alpha$ be an approximate diagonal for $A_p(G)$, bounded by 1. Since

$$^\vee : A_p(G) \to A_{p'}(G), \quad f \mapsto \check{f}$$

is an isometric isomorphism of Banach algebras, the net $((\mathrm{id}_{A_p(G)} \otimes {}^\vee)d_\alpha)_\alpha$, which lies in $A_p(G) \hat{\otimes} A_{p'}(G)$, is also bounded by 1. From (9) and (10), it is immediate that

$$\left((\mathrm{id}_{A_p(G)} \otimes {}^\vee)d_\alpha\right)_\alpha$$

converges pointwise on $G \times G$ to χ_Γ, where

$$\Gamma := \{(x, x^{-1}) \colon x \in G\}.$$

By Proposition 2.8, there is therefore an isometric representation of $G_d \times G_d$ on a uniformly convex Banach space such that χ_Γ is a normalized coefficient function of (π, E). From Theorem 1.5, we conclude that Γ is a left coset of a subgroup of $G \times G$. Since Γ contains the identity of $G \times G$, it follows that Γ is, in fact, a subgroup of $G \times G$. This is possible only if G is abelian. □

Remark. Let G be a locally compact group, and consider the following statements:

(i) G is almost abelian;
(ii) $A_p(G)$ is amenable for each $p \in (1, \infty)$;
(iii) $A(G)$ is amenable;
(iv) there is $p \in (1, \infty)$ such that $A_p(G)$ is amenable.

It is known that (i) ⇒ (ii) [36, Corollary 8.4], and (ii) ⇒ (iii) ⇒ (iv) are trivial. We believe, but have been unable to prove, that (i), (ii), (iii), and (iv) are equivalent. An inspection of the proof of Theorem 2.9 reveals that the main obstacle in the way of proving (iv) ⇒ (i) is the lack of a suitable Cohen–Host type idempotent theorem for isometric representations on uniformly convex Banach spaces with uniformly convex duals.

Note added in proof

After this paper was submitted, M. Daws brought the paper "Property (T) and rigidity for actions on Banach spaces" (arXiv: math.GR/0506361) by U. Bader, A. Furman, T. Gelander, and N. Monod to the author's attention. Proposition 2.3 of that paper asserts that, for every uniformly bounded representation on a superreflexive Banach space E, there is an equivalent norm on E such that the representation becomes isometric and that both E and E^* are uniformly convex with respect to this new norm. In view of the example at the end of Section 1, this means that there can be no Cohen–Host type idempotent theorem for isometric representations on uniformly convex Banach spaces with uniformly convex duals. It remains open, however, whether such a theorem still holds true for the specific spaces occurring in the proof of Proposition 2.8.

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