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On the generalized vector equilibrium problems ☆

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Abstract

In this paper, generalized vector equilibrium problems are studied and some existence theorems of solutions for these problems in the setting of topological vector spaces are proved. Sufficient conditions for the set of solutions to be compact and convex are given. Our results improve some recent results in this field.
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1. Introduction and preliminaries

Throughout the paper, unless otherwise specified, we assume that X and Y are real Hausdorff topological vector spaces, K is a nonempty convex subset of X, $\{C(x)\colon\ x \in K\}$ is a family of convex cone subsets of Y such that $C(x) \neq Y$ and $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$, and $F : K \times K \to 2^Y \setminus \{\emptyset\}$ is a set-valued mapping, where $\operatorname{int} C(x)$ denotes the interior of $C(x)$ and 2^Y the family of all subsets of Y.

Let $f : K \times K \to Y$ be a vector-valued mapping and C be a convex cone, with nonempty interior, in Y. By a vector equilibrium problem, we understand the problem of finding a $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -\operatorname{int} C, \quad \forall y \in K. \tag{1.1}$$

Vector equilibrium problems (for short, VEP) are among the most interesting and intensively studied classes of problems. They include several fundamental mathematical problems, like, vector optimization problems, vector variational inequality problems (see, for example, [7,12,14–16]), Nash equilibrium problem for vector-valued maps and fixed point problems. Vector equilibrium problem has been the focus of attention of many researchers in the recent years (see, for example, [2,3,11,13,17] and references therein).

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There are several possible ways to generalize VEP for a given multi-valued mapping $F : K \times K \to 2^Y \setminus \{\emptyset\}$, for instance,

$$\text{find } \bar{x} \in K \quad \text{such that } F(\bar{x}, y) \subseteq Y \setminus \big(-\operatorname{int} C(\bar{x})\big),\ \forall y \in K, \tag{1.2}$$

$$\text{find } \bar{x} \in K \quad \text{such that } F(\bar{x}, y) \not\subseteq \big(-\operatorname{int} C(\bar{x})\big),\ \forall y \in K, \tag{1.3}$$

$$\text{find } \bar{x} \in K \quad \text{such that } F(\bar{x}, y) \cap Y \setminus (-\operatorname{int} C(\bar{x})) \neq \emptyset,\ \forall y \in K. \tag{1.4}$$

These problems are called generalized vector equilibrium problems. They contain, in particular, generalized vector variational inequality problems and have been studied in [1,3–6,8,17–19]. Besides (1.2)–(1.4), the following problem can be of interest:

$$\text{find } \bar{x} \in K \quad \text{such that } F(\bar{x}, y) \subseteq C(\bar{x}),\ \forall y \in K. \tag{1.5}$$

Clearly, any solution of (1.5) is a solution of (1.2) while any solution of (1.2), in its turn, is a solution of (1.3) and (1.4).

In this paper we study problems (1.2) and (1.5) and establish some existence theorems in Hausdorff topological vector spaces as well as sufficient conditions for the set of solutions to be compact and convex. Our results improve some recent results in this field.

In the rest of this section we recall some known results and definitions which will be used in the sequel.

Definition 1.1. A mapping $\Gamma : K \to 2^X$ is said to be a KKM map if

$$\operatorname{co} A \subseteq \bigcup_{x \in A} \Gamma(x), \quad \forall A \in \mathcal{F}(K),$$

where $\operatorname{co} A$ and $\mathcal{F}(K)$ denote the convex hull of the set A and all of the nonempty finite subsets of K, respectively.

KKM maps were first considered by Knaster, Kuratowski, and Mazurkiewicz in 1920.

Definition 1.2. Let E_1 and E_2 be two topological spaces. A set-valued mapping $T : E_1 \to 2^{E_2}$ is said to be

(i) lower semi-continuous (l.s.c.) at $x \in E_1$ if for any open set V with $T(x) \cap V \neq \emptyset$, there exists an open set U containing x such that for each $t \in U$, $T(t) \cap V \neq \emptyset$; T is said to be l.s.c. on E_1 if it is l.s.c. at all $x \in E_1$.
(ii) Closed if its graph $G_r(T) = \{(x, y)\colon\ x \in E_1,\ y \in T(x)\}$ is a closed subset of $E_1 \times E_2$.

The following two results will be needed in the sequel. The first one is standard and can be found in [9]. The second result is Fan's lemma (see [10]).

Lemma 1.3. *Let E_1 and E_2 be topological spaces and $T : E_1 \to 2^{E_2}$ be a set-valued mapping.*

(i) *T is closed if and only if for any net $\{x_\alpha\}$ converging to x and any net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y$, we have $y \in T(x)$.*
(ii) *T is l.s.c. at $x \in E_1$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$ such that $x_\alpha \to x$, there exists a net $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 1.4. *Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ be a KKM mapping with closed values in K. Assume that there exist a nonempty compact convex subset B of K such that $\bigcap_{x \in B} F(x)$ is compact. Then*

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

2. Main results

In this section we prove some existence results for the problems (1.2) and (1.5) and deduce some consequences. Now we prove an existence result which generalizes Theorem 3.1 in [1].

Theorem 2.1. *Let $\{W(x)\colon x \in K\}$ be a family of nonempty subsets of Y. Suppose that the following assumptions hold*:

(i) *for all $x \in K$, $F(x,x) \subseteq W(x)$;*
(ii) *for all $y \in K$, the set $\{x \in K\colon F(y,x) \subseteq -W(x)\}$ is closed in K;*
(iii) *for all $x \in K$, the set $\{\xi \in K\colon F(x,\xi) \nsubseteq W(x)\}$ is convex;*
(iv) *for all $x, y \in K$, $F(x,y) \subseteq W(x)$ implies $F(y,x) \subseteq -W(x)$;*
(v) *there is a nonempty compact subset D and a nonempty convex compact subset B of K such that*:

$$\forall x \in K\backslash D,\ \exists y \in B; \quad F(y,x) \nsubseteq -W(x).$$

Then, the set $A = \{x \in K\colon F(y,x) \subseteq -W(x),\ \forall y \in K\}$ is nonempty and compact.

Proof. Define $\Gamma : K \to 2^K$ by

$$\Gamma(y) = \big\{x \in K\colon F(y,x) \subseteq -W(x)\big\}, \quad \forall y \in K.$$

We claim that Γ is a KKM mapping. Otherwise, there exist a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq K$ and $z \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$ such that $z \notin \bigcup_{i=1}^n \Gamma(x_i)$. This shows that $F(x_i, z) \nsubseteq W(z)$, for $i = 1, 2, \ldots, n$ and so (iv) implies that $F(z, x_i) \nsubseteq W(z)$. Then by (iii), we get $F(z,z) \nsubseteq W(z)$. This contradicts (i). Hence Γ is a KKM mapping. By (ii) the values of Γ are closed in K and by (v) we get that $\bigcap_{x\in B} \Gamma(x) \subseteq D$. Then $\bigcap_{x\in B} \Gamma(x)$ is a compact set in K. Thus all hypotheses of Lemma 1.4 are satisfied. Hence by Lemma 1.4 there exists $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y\in K} \Gamma(y)$ which implies that there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \subseteq -W(\bar{x})$, for all $y \in K$. This means that $\bar{x} \in A$ and hence $A \neq \emptyset$. Since $A = \bigcap_{y\in K} \Gamma(y)$, it follows, from (v), that $A \subseteq D$. Therefore, since A is closed in K and D is compact, A is compact in K. This completes the proof. □

Remark 2.2. If all conditions (i)–(v) of Theorem 2.1 and the following condition

(vi) for all $x, y \in K$, $F(z,x) \subseteq -W(x)$ for all $z \in]x, y]$ implies $F(x,y) \subseteq W(x)$, where $]x, y]$ denotes the line segment joining x and y but not containing x,

are satisfied, then the solution set to the problem

$$\text{find } \bar{x} \in K \quad \text{such that } F(\bar{x}, y) \subseteq W(\bar{x}),\ \forall y \in K,$$

and that of the problem

$$\text{find } \bar{x} \in K \quad \text{such that } F(y, \bar{x}) \subseteq -W(\bar{x}),\ \forall y \in K,$$

are nonempty, compact and both coincide.

Remark 2.3.

(a) If X is a real reflexive Banach space and K is a nonempty, bounded and closed convex subset of X (see [1, Theorem 3.1]), then condition (v) from Theorem 2.1, can be eliminated.
(b) The result in Theorem 2.1 is still valid if we replace assumptions (i)–(iii) in it by the following conditions:
(i′) $F(x,x) \subseteq -W(x)$, $\forall x \in K$;
(ii′) for all $x \in K$, the set $\{\xi \in K\colon F(\xi, x) \nsubseteq -W(x)\}$ is convex.

The next theorem establish a sufficient conditions for the nonemptiness and compactness of the solution set to the problem (1.2). Moreover it improves Theorem 2.1 in [19].

Theorem 2.4. *Assume that the following conditions hold*:

(i) $F(x,x) \subseteq Y \setminus -\operatorname{int} C(x)$, $\forall x \in K$;
(ii) *for all* $x \in K$, *the set* $\{\xi \in K\colon\ F(\xi,x) \subseteq Y\setminus(-\operatorname{int} C(\xi))\}$ *is closed in* K;
(iii) *for all* $x \in K$, *the set* $\{\xi \in K\colon\ F(x,\xi) \nsubseteq Y\setminus(-\operatorname{int} C(x))\}$ *is convex*;
(iv) *there is a nonempty compact subset* D *and a nonempty convex compact subset* B *of* K *such that*:

$$\forall x \in K\setminus D,\ \exists y \in B;\quad F(x,y) \nsubseteq Y\setminus\big(-\operatorname{int} C(x)\big).$$

Then, the solution set of the problem (1.2)*, that is, the set*

$$A = \big\{x \in K\colon\ F(x,y) \subseteq Y\setminus\big(-\operatorname{int} C(x)\big),\ \forall y \in K\big\},$$

is nonempty and compact.

Proof. Define $\Gamma : K \times K \to 2^Y$ by

$$\Gamma(y) = \big\{x \in K\colon\ F(x,y) \subseteq Y\setminus\big(-\operatorname{int} C(x)\big)\big\}$$

for each $y \in K$. We shall first show that Γ is a KKM mapping. Indeed, suppose on the contrary there exist a finite subset $\{y_1, y_2, \ldots, y_n\}$ of K and $z \in \operatorname{co}\{y_1, y_2, \ldots, y_n\}$ such that $F(z,y_i) \nsubseteq Y\setminus(-\operatorname{int} C(z))$, for all $i = 1,2,\ldots,n$. Thus from (iii) we have $F(z,z) \nsubseteq Y\setminus(-\operatorname{int} C(z))$ which is a contradiction. Hence Γ is a KKM mapping. By (ii), $\Gamma(y)$ is closed in K, for all $y \in K$ and by (iv), $\bigcap_{x\in K}\Gamma(x) \subseteq D$. Therefore all the conditions of Lemma 1.4 are satisfied. By Lemma 1.4, $\bigcap_{y\in K}\Gamma(y) \neq \emptyset$. Take any $\bar{x} \in \bigcap_{y\in K}\Gamma(y)$, then $F(\bar{x},y) \subseteq Y\setminus(-\operatorname{int} C(\bar{x}))$, for all $y \in K$. Hence $\bar{x} \in A$. Consequently A is nonempty. By (ii), A is a closed set in K and by (iv), it is a subset of the compact set D. Thus A is a compact subset of K. This completes the proof. □

Remark 2.5. (a) It is clear that if $F(x,.)$ is $C(x)$-convex, i.e., for all $y_1, y_2 \in K$ and $t \in [0,1]$,

$$tF(x,y_1) + (1-t)F(x,y_2) \subseteq F\big(x, ty_1 + (1-t)y_2\big) + C(x),$$

for all $x \in K$, then condition (iii) of Theorem 2.4 holds. In fact, if it were false, then there exist $x, \xi_1, \xi_2 \in K$ and $t \in\,]0,1[$ such that

$$F\big(x, t\xi_1 + (1-t)\xi_2\big) \subseteq Y\setminus\big(-\operatorname{int} C(x)\big), \tag{2.1}$$

and

$$F(x,\xi_i) \nsubseteq Y\setminus\big(-\operatorname{int} C(x)\big),\quad \forall i = 1,2. \tag{2.2}$$

By (2.2), there exists $w_i \in F(x,\xi_i)$ such that $w_i \in -\operatorname{int} C(x)$ for $i = 1,2$. Since $F(x,.)$ is $C(x)$-convex, there exist $\alpha \in F(x, t\xi_1 + (1-t)\xi_2)$ and $\beta \in C(x)$ such that $tw_1 + (1-t)w_2 = \alpha + \beta$. Consequently,

$$\alpha = tw_1 + (1-t)w_2 - \beta \in -\operatorname{int} C(x) - C(x) \subseteq -\operatorname{int} C(x),$$

which is a contradiction (with (2.1)).

(b) Conditions (iii) and (iv) of Theorem 2.1 in [19] (that is, $F(x,y)$ is l.s.c. in x, for all $y \in K$, and the map $W : K \to 2^Y$ defined by $W(x) = Y\setminus(-\operatorname{int} C(x))$, for each $x \in K$, is closed) guarantee condition (ii) in Theorem 2.4. To see this, let $x \in K$, $\xi_\alpha \to \xi$ and $F(\xi_\alpha,x) \subseteq Y\setminus(-\operatorname{int} C(\xi_\alpha))$. We prove that $F(\xi,x) \subseteq Y\setminus(-\operatorname{int} C(\xi))$. For this let $w \in F(\xi,x)$, by Lemma 1.4(ii), there exists a net $w_\alpha \in F(\xi_\alpha,x)$ such that $w_\alpha \to w$. Now from $(\xi_\alpha, w_\alpha) \to (\xi, w)$, $w_\alpha \in Y\setminus(-\operatorname{int} C(w_\alpha))$ and the closeness of the graph of $W(x) = Y\setminus(-\operatorname{int} C(x))$ through Lemma 1.3(i) we get $w \in Y\setminus(-\operatorname{int} C(\xi))$. This completes the proof.

The following is an example which shows that the converse of Remark 2.5(b) does not hold in general. Furthermore, it is an example for which Theorem 2.4 is applicable while Theorem 2.1 in [19] is not.

Example 2.6. Let $X = K = Y = \mathbb{R}$ and $C(x) = [0,+\infty)$ for all $x \in K$. Define $F : K \times K \to 2^Y$ by

$$F(x,y) = \begin{cases} [0,1] & \text{if } x = 0,\\ \{0\} & \text{if } x \neq 0, \end{cases}$$

for each $(x, y) \in K \times K$. It is easy to check that F satisfies all assumptions of Theorem 2.4. However, for each $y \in K$, the map $x \to F(x, y)$ is not lower semi-continuous, for instance at $x = 0$, on K and hence Theorem 2.1 in [19] is not applicable.

The next theorem extends the result in Theorem 3 of [19].

Theorem 2.7. *If all conditions* (i)–(iv) *of Theorem* 2.4 *and the following condition*

(v) *the set* $\{x \in K\colon\ F(x, y) \subseteq Y \backslash (-\operatorname{int} C(x))\}$ *is convex,* $\forall y \in K$,

are satisfied, then the solution set of problem (1.2) *is nonempty, compact and convex.*

Proof. If we denote the solution set of problem (2.1) by A then by Theorem 2.4 A is nonempty and compact in K. It is obvious that

$$A = \bigcap_{y \in K} \left\{x \in K\colon\ F(x, y) \subseteq Y \backslash \left(-\operatorname{int} C(x)\right)\right\}$$

and hence by (v), A is convex. This completes the proof. □

Remark 2.8. If $C(.)$ is a constant mapping, that is, $C(x) = C$ for all $x \in K$, and for each $y \in K$, the map $x \to F(x, y)$ is properly quasi-concave with respect to C, that is, for all $x_1, x_2 \in K$, $t \in [0, 1]$, $x = tx_1 + (1 - t)x_2$, $w \in F(x, y)$, there is $z_1 \in F(x_1, y)$ or $z_2 \in F(x_2, y)$ such that $w \in z_1 + C$ or $w \in z_2 + C$ (see [9, p. 22]), then condition (v) in Corollary 2.7 holds.

The following example shows that Theorem 2.7 is really a generalization of Theorem 3 in [19]. Incidently, it shows that the converse of Remark 2.8 does not hold in general.

Example 2.9. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = [1, \infty]$, $C(x) = C = \{(x, y) \in \mathbb{R}^2\colon\ x \geqslant 0,\ 0 \leqslant y \leqslant 4x\}$ and define a mapping $F : K \times K \to 2^Y$ by

$$F(x, y) = \left[(0, 0), \left(-x, \frac{1}{x}\right)\right],$$

for each $(x, y) \in K \times K$, where $[(0, 0), (-x, \frac{1}{x})]$ denotes the line segment joining points $(0, 0)$ and $(-x, \frac{1}{x})$. We claim that, for each $y \in K$, the map $x \to F(x, y)$ is not properly quasi-concave. Indeed, if we take $x_1 = 1$, $x_2 = 2$ and $w = (\frac{3}{2}, \frac{-2}{3}) \in F(tx_1 + (1 - t)x_2, y)$ for $t = \frac{1}{2}$, then there is no $z_1 \in F(1, y) = [(0, 0), (-1, 1)]$ or $z_2 \in F(2, y) = [(0, 0), (-2, \frac{1}{2})]$ such that $w - z_1 \in C$ or $w - z_2 \in C$. Hence Theorem 3 in [19] is not applicable. However, condition (v) of Theorem 2.7 trivially holds. In fact, we have $\{x \in K\colon\ F(x, y) \subseteq Y \backslash (-\operatorname{int} C(x))\} = K$ which is convex, for all $y \in K$. Now if K is restricted to each nonempty compact convex subset of $[1, \infty]$, then, it is not hard to verify that F satisfy all the other assumptions of Theorem 2.7. Therefore Theorem 2.7 is applicable and further the solution set of problem (1.2) equals to the set K.

The following theorem improves Theorem 2 in [19].

Theorem 2.10. *Assume that* $F, G : K \times K \to 2^Y$ *are set-valued mappings such that*

(i) $\forall x \in K$, $G(x, x) \subseteq Y \backslash (-\operatorname{int} C(x))$;
(ii) *for all* $x \in K$, *the set* $\{\xi \in K\colon\ F(\xi, x) \subseteq Y \backslash (-\operatorname{int} C(\xi))\}$ *is closed in* K;
(iii) *for all* $x \in K$, *the set* $\{\xi \in K\colon\ G(x, \xi) \nsubseteq Y \backslash (-\operatorname{int} C(x))\}$ *is convex*;
(iv) $\forall x, y \in K$, $\forall w \in F(x, y)$, $\exists z \in G(x, y)$ *such that* $z - w \in -\operatorname{int} C(x)$;
(v) *there is a nonempty compact subset* D *and a nonempty convex compact subset* B *of* K *such that*: $\forall x \in K \backslash D$, $\exists y \in B$, $F(x, y) \nsubseteq Y \backslash (-\operatorname{int} C(x))$.

Then the solution set of problem (1.2) *is nonempty and compact.*

Proof. We define two set-valued maps Γ and $\hat{\Gamma}$ from K to 2^K by the rule that for $y \in K$,

$$\hat{\Gamma}(y) = \{x \in K\colon\ G(x, y) \subseteq Y \backslash -\operatorname{int} C(x)\},$$

and

$$\Gamma(y) = \{x \in K\colon\ F(x, y) \subseteq Y \backslash -\operatorname{int} C(x)\}.$$

It is clear that $\hat{\Gamma}(y) \subseteq \Gamma(y)$, for all $y \in K$. In fact, if we let $x \in \hat{\Gamma}(y)$, then we have $G(x, y) \subseteq Y \backslash -\operatorname{int} C(x)$, and so for each $w \in F(x, y)$, from (iv) and $(Y \backslash -\operatorname{int} C(x)) + \operatorname{int} C(x) \subseteq Y \backslash -\operatorname{int} C(x)$, we get $w \in Y \backslash -\operatorname{int} C(x)$. Thus $F(x, y) \subseteq Y \backslash -\operatorname{int} C(x)$ and hence $x \in \Gamma(y)$. As seen in the proof of Theorem 2.4 one can observe that $\hat{\Gamma}$ is a KKM mapping and hence so is Γ. By (ii), the values of Γ are closed in K, and by (iv) the inclusion $\bigcap_{y \in B} \Gamma(y) \subseteq D$ is valid. Now F satisfies all conditions of Lemma 1.4 and so $\bigcap_{y \in K} \Gamma(y) \neq \emptyset$. This means, since the solution set of problem (1.2) equals to $\bigcap_{y \in K} \Gamma(y)$, that the solution set of problem (1.2) is nonempty. The compactness of the solution set (1.2) follows from (ii) and (iv). This completes the proof of Theorem 2.10. □

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