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Norm-linear and norm-additive operators between uniform algebras

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ABSTRACT

Let $A \subset C(X)$ and $B \subset C(Y)$ be uniform algebras with Choquet boundaries δA and δB. A map $T : A \to B$ is called *norm-linear* if $\|\lambda Tf + \mu Tg\| = \|\lambda f + \mu g\|$; *norm-additive*, if $\|Tf + Tg\| = \|f + g\|$, and *norm-additive in modulus*, if $\||Tf| + |Tg|\| = \||f| + |g|\|$ for each $\lambda, \mu \in \mathbb{C}$ and all algebra elements f and g. We show that for any norm-linear surjection $T : A \to B$ there exists a homeomorphism $\psi : \delta A \to \delta B$ such that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$. Sufficient conditions for norm-additive and norm-linear surjections, not assumed *a priori* to be linear, or continuous, to be unital isometric algebra isomorphisms are given. We prove that any unital norm-linear surjection T for which $T(i) = i$, or which preserves the peripheral spectra of $\mathbb{C}$-peaking functions of A, is a unital isometric algebra isomorphism. In particular, we show that if a linear operator between two uniform algebras, which is surjective and norm-preserving, is unital, or preserves the peripheral spectra of $\mathbb{C}$-peaking functions, then it is automatically multiplicative and, in fact, an algebra isomorphism.

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1. Introduction

The problem of characterizing algebra isomorphisms among general maps between Banach algebras has attracted considerable interest. For maps known *a priori* to be linear it has been an active area of research for over a century, particularly for the so-called linear preservers, maps that preserve some specific properties or features of algebra elements (see e.g. [11]). The classical Banach–Stone theorem, for instance, implies that any unital norm-preserving linear surjection between two spaces of type $C(X)$, the algebra of complex-valued, continuous functions on a compact Hausdorff space X, is an isometric algebra isomorphism. One important consequence of the celebrated theorem of Gleason–Kahane–Żelazko [16] states that if a surjective linear map $T : A \to B$ between semisimple commutative Banach algebras preserves the spectra, namely, $\sigma(Tf) = \sigma(f)$ for all $f \in A$, then T is multiplicative and thus an algebra isomorphism. Recall that the *spectrum* of an algebra element $f \in A$ is the compact set $\sigma(f) = \{\lambda \in \mathbb{C}\colon\ (\lambda - f) \notin A^{-1}\}$. A result by Kowalski and Słodkowski [6] implies that if a surjective map $T : A \to B$ between semisimple commutative Banach algebras is weakly additive in the sense that $\sigma(Tf - Tg) = \sigma(f - g)$ for all algebra elements f and g, and $T(0) = 0$, then T is an algebra isomorphism. More on the early stage of this subject can be found in [5,11]. Molnár [11,12] showed that a surjective self-map T of the algebra $C(X)$ with first-countable compact X which is unital and weakly multiplicative in the sense that $\sigma(TfTg) = \sigma(fg)$ for all algebra elements, is an isometric algebra isomorphism. Molnár's result was generalized for arbitrary uniform algebras by Rao and Roy [13], and was extended further in various directions (e.g. [1–3,7–9,14,15]). Recently it was realized that crucial for the isomorphism problem is not the entire spectrum, but merely some of its distinguished parts (e.g. [1,2,7–9,15]).

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Let $A \subset C(X)$ be a uniform algebra on a compact Hausdorff set X. Recall that the *peripheral spectrum* of $f \in A$ is the set $\sigma_\pi(f) = \sigma(f) \cap \{z \in \mathbb{C}\colon |z| = \|f\|\}$ of spectral values of f with maximal modulus. Equivalently, $\sigma_\pi(f)$ is the set of values of f with maximum modulus, i.e. $\sigma_\pi(f) = \{f(x)\colon x \in X \text{ and } |f(x)| = \|f\|\}$. Rao, Tonev and Toneva (e.g. [1]) extended the mentioned Kowalski–Słodkowski's result to so-called *peripherally-additive maps* $T : A \to B$ that are weakly additive in the sense that $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$ for all $f, g \in A$, and have found sufficient conditions for such maps to be unital isometric algebra isomorphisms.

In this paper we show that the peripheral additivity property, considered in [1] and [15], is too restrictive for the isomorphism problem: In fact, it suffices the map to be only norm-additive or norm-linear (see the definitions below). In addition, it is enough the map to be either unital, or to preserve the peripheral spectra of $\mathbb{C}$-peaking functions, rather than of all algebra elements, as required in [15].

Below we describe the main results of the paper. The first proposition generalizes Rao–Tonev–Toneva's additive analogue of Bishop's Lemma [1]. In it $\mathcal{P}_x(A)$ denotes the set of *peaking functions* of A that peak on x, $\mathcal{E}_x(A) = \{f \in A\colon |f(x)| = \|f\|\} = \{f \in A\colon x \in E(f)\}$ is the set of all algebra elements which take their maximum modulus at x, δA is the Choquet boundary of A and $E(f)$ is the *maximum modulus set* of $f \in A$ (see the corresponding definitions in Section 2).

Proposition. *(See Lemma 1, Proposition 4 and Corollary 5.) Let $f \in A$, $f \not\equiv 0$. For any $x_0 \in \delta A$ and arbitrary $r > 1$ (or, $r \geqslant 1$ if $f(x_0) \neq 0$), there exists an $\mathbb{R}$-peaking function $h \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ such that $|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)| = |f(x_0)| + \|h\|$ for every $x \notin E(h)$ and $|f(x)| + |h(x)| = |f(x_0)| + |h(x_0)|$ for all $x \in E(h)$. Consequently, $\||f| + |h|\| = |f(x_0)| + |h(x_0)|$. If $\alpha = \exp\{i \arg f(x_0)\}$, then $f + \alpha h \in \mathbb{C} \cdot \mathcal{P}_{x_0}(A)$, $\sigma_\pi(f + \alpha h) = \{f(x_0) + \alpha h(x_0)\}$ and $E(f + \alpha h) = E(h)$. Given a neighborhood U of x_0, h can be chosen so that $E(h) \subset U$. Moreover, $|f(x_0)| + |h(x_0)| = \||f| + |h|\| = \inf_{\substack{h \in \mathcal{E}_{x_0}(A) \\ \|h\| = r\|f\|}} \||f| + |h|\|$.*

An operator $T : A \to B$ between Banach algebras is called *norm-preserving* if $\|Tf\| = \|f\|$, *norm-linear* if $\|\lambda Tf + \mu Tg\| = \|\lambda f + \mu g\|$, *norm-additive* if $\|Tf + Tg\| = \|f + g\|$, and *norm-additive in modulus* if $\||Tf| + |Tg|\| = \||f| + |g|\|$ (i.e. $\max_{\eta \in \partial B}(|(Tf)(\eta)| + |(Tg)(\eta)|) = \max_{\xi \in \partial A}(|f(\xi)| + |g(\xi)|)$) for all $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$, where $\|\cdot\|$ is the uniform norm on $C(\partial A)$ and $C(\partial B)$ correspondingly. Clearly, every norm-linear operator is norm-additive and every norm-preserving linear (resp. additive) operator is automatically norm-linear (resp. norm-additive).

The primary results of the paper, which follow, reveal the structure of norm-additive and norm-linear operators between uniform algebras and provide sufficient conditions for such operators to be unital isomorphic algebra isomorphisms.

Theorem. *(See Theorem 13.) Any norm-linear surjection $T : A \to B$ between uniform algebras induces an associated homeomorphism $\psi : \delta A \to \delta B$ so that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$.*

The following theorem provides sufficient conditions for a norm-additive operator to be an algebra isomorphism.

Theorem *(Norm-Additive Operators). (See Theorem 16.) A norm-additive surjection $T : A \to B$ between uniform algebras which is norm-additive in modulus induces an associated homeomorphism $\psi : \delta B \to \delta A$ such that $|(Tf)(y)| = |f(\psi(y))|$ for each $f \in A$ and all $y \in \delta B$. If, in addition, $T(1) = 1$ and $T(i) = i$, or if T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, then T is an isometric unital algebra isomorphism.*

As a corollary we obtain the following criteria for a norm-additive operator to be norm-linear.

Corollary. *(See Corollary 17.) Let $T : A \to B$ be a norm-additive surjection for which $T(1) = 1$ and $T(i) = i$, or which preserves the peripheral spectra of $\mathbb{C}$-peaking functions of A. Then T is norm-linear if and only if it is norm-additive in modulus.*

The next theorem gives sufficient conditions for a norm-linear operator to be an algebra isomorphism. Namely,

Theorem *(Norm-Linear Operators). (See Theorem 20.) A norm-linear surjection $T : A \to B$ between two uniform algebras for which $T(1) = 1$ and $T(i) = i$, or which preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, induces an associated homeomorphism $\psi : \delta B \to \delta A$ such that T is a ψ-composition operator on B, and therefore, is an isometric unital algebra isomorphism.*

This theorem yields the following corollary, which extends in a certain sense the corollaries of Banach–Stone's theorem and Gleason–Kahane–Żelazko's theorem mentioned above.

Corollary *(Linear Operators). (See Corollary 21.) If a linear operator between two uniform algebras, which is surjective and norm-preserving, is unital, or preserves the peripheral spectra of $\mathbb{C}$-peaking functions, then it is automatically multiplicative and, in fact, an algebra isomorphism.*

2. Preliminaries

In this section $A \subset C(X)$ will be a uniform algebra on a compact Hausdorff space X. For an $f \in A$ the set $E(f)$ of all x in X at which f attains its maximum modulus is called the *maximum modulus set* of f, i.e. $E(f) = \{x \in X: |f(x)| = \|f\|\} = \{x \in X: f(x) \in \sigma_\pi(f)\} = f^{-1}(\sigma_\pi(f))$. An element $h \in A$ is called a *peaking function of A* if $\sigma_\pi(h) = \{1\}$, i.e., if $\|h\| = 1$ and $|h(x)| < 1$ whenever $h(x) \neq 1$. In this case, the maximum modulus set $E(h) = \{x \in X: h(x) = 1\} = h^{-1}\{1\}$ is called the *peak set* of h. If E is a subset of X such that $E \subset E(h)$ for some peaking function h, we say that h *peaks on* E. The set of all peaking functions in A we denote by $\mathcal{P}(A)$. Clearly, $\mathbb{C} \cdot \mathcal{P}(A)$ is the set of all $f \in A$ with singleton peripheral spectra. The elements of $\mathbb{C} \cdot \mathcal{P}(A)$ (resp. $\mathbb{R} \cdot \mathcal{P}(A)$) we call $\mathbb{C}$*-peaking functions* (resp. $\mathbb{R}$*-peaking functions*) of A. A point $x \in X$ is called a *generalized peak point*, or *p-point*, of A if for every neighborhood V of x there is a peaking function h with $x \in E(h) \subset V$. Recall that the set δA of all generalized peak points of A is the *Choquet* (or the *strong*) *boundary* of A, and $\overline{\delta A} = \partial A$, the Shilov boundary of A. Given an $x \in X$, we denote by $\mathcal{P}_x(A)$ the set of all peaking functions of A which peak on x and by $\mathbb{C} \cdot \mathcal{P}_x(A)$ the set of $\mathbb{C}$-peaking functions of A that peak on x.

The following lemma, which we use on several occasions further, strengthens and generalizes the additive version of Bishop's Lemma from [15].

Lemma 1 *(Strong version of the additive Bishop's Lemma). For any nonzero $f \in A$, $x_0 \in \delta A$ and arbitrary $r > 1$ (or, $r \geqslant 1$ if $f(x_0) \neq 0$) there exists an $\mathbb{R}$-peaking function $h \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ such that*

$$\left|f(x)\right| + \left|h(x)\right| < \left|f(x_0)\right| + \left|h(x_0)\right| \tag{1}$$

for every $x \notin E(h)$ and $|f(x)| + |h(x)| = |f(x_0)| + |h(x_0)|$ for all $x \in E(h)$. Consequently, $\max_{x\in X}(|f(x)| + |h(x)|) = |f(x_0)| + |h(x_0)|$. *Given a neighborhood U of x_0, h can be chosen such that $E(h) \subset U$.*

Proof. Consider first the case when $f(x_0) \neq 0$. For every $n \in \mathbb{N}$ we define the open set $U_n = \{x \in X: |f(x) - f(x_0)| < \frac{|f(x_0)|}{2^{n+1}}\}$. Clearly, $U_n \subset U_{n-1}$ and $x_0 \in U_n$ for every $n \in \mathbb{N}$. Let $r \geqslant 1$. For each n, choose a peaking function $k_n \in \mathcal{P}_{x_0}(A)$ such that $E(k_n) \subset U_n$, and let $h_n \in \mathcal{P}_{x_0}(A)$ be a large enough power of k_n such that $|h_n(x)| < \frac{|f(x_0)|}{2^n r\|f\|}$ on $X \setminus U_n$. One can see that $\bigcap_{n=1}^\infty U_n = f^{-1}(f(x_0))$. Indeed, it is clear that every $x \in f^{-1}(f(x_0))$ belongs to $\bigcap_{n=1}^\infty U_n$; conversely, if $x \in \bigcap_{n=1}^\infty U_n$ then $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2^{n+1}}$ for every $n \in \mathbb{N}$, thus $f(x) = f(x_0)$, i.e. $x \in f^{-1}(f(x_0))$. We claim that the $\mathbb{R}$-peaking function $h = r\|f\| \cdot \sum_1^\infty \frac{h_n}{2^n}$ satisfies inequality (1). Clearly, $h \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ and hence $\|h\| = r\|f\| = |h(x_0)|$. In addition, $E(h) \subset \bigcap_{n=1}^\infty E(h_n) \subset \bigcap_{n=1}^\infty U_n = f^{-1}(f(x_0))$.

For any $x \in E(h)$ we have $|f(x)| + |h(x)| = |f(x_0)| + \|h\|$, while $|f(x)| + |h(x)| = |f(x_0)| + |h(x)| < |f(x_0)| + \|h\|$ holds for any $x \in f^{-1}(f(x_0)) \setminus E(h)$. If $x \notin f^{-1}(f(x_0)) = \bigcap_{n=1}^\infty U_n$, there are two possibilities.

Case 1: $x \notin U_1$. In this case $x \notin U_n$ for every $n \in \mathbb{N}$, and hence $|h_n(x)| < \frac{|f(x_0)|}{2^n r\|f\|}$ for every $n \in \mathbb{N}$. Therefore, $|h(x)| < r\|f\| \cdot \sum_1^\infty \frac{|f(x_0)|}{4^n r\|f\|} < |f(x_0)|$, and consequently, $|f(x)| + |h(x)| < r\|f\| + |f(x_0)| = |f(x_0)| + \|h\|$.

Case 2: $x \in U_{n-1} \setminus U_n$ for some $n > 1$. In this case $x \in U_i$ for $1 \leqslant i \leqslant n-1$ and $x \notin U_i$ for every $i \geqslant n$. Therefore, $|h_i(x)| < \frac{|f(x_0)|}{2^i r\|f\|}$ for every $i \geqslant n$. Since $x \in U_{n-1}$, we have $|f(x) - f(x_0)| < \frac{|f(x_0)|}{2^n}$ and hence

$$\left|f(x)\right| + \left|h(x)\right| \leqslant \left|f(x_0)\right| + \left|f(x) - f(x_0)\right| + \left|h(x)\right| < \left|f(x_0)\right| + \frac{|f(x_0)|}{2^n} + r\|f\| \cdot \sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} + r\|f\| \cdot \sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i}.$$

Since each h_n is a peaking function of A, it follows that $|h_n(x)| \leqslant 1$ for any $x \in X$, and therefore, $\sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} \leqslant \sum_{i=1}^{n-1} \frac{1}{2^i} = 1 - \frac{1}{2^{n-1}}$. Moreover, $\sum_{i=n}^\infty \frac{|h_i(x)|}{2^i} < \sum_{i=n}^\infty \frac{|f(x_0)|}{4^i r\|f\|} \leqslant \sum_{i=n}^\infty \frac{1}{4^i} = \frac{1}{3 \cdot 4^{n-1}}$. Hence,

$$\begin{aligned}
\left|f(x)\right| + \left|h(x)\right| &\leqslant \left|f(x_0)\right| + \frac{|f(x_0)|}{2^n} + \left(1 - \frac{1}{2^{n-1}}\right) r\|f\| + \frac{1}{3 \cdot 4^{n-1}} r\|f\| \\
&< \left|f(x_0)\right| + \left(1 - \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{3 \cdot 4^{n-1}}\right) r\|f\| \\
&= \left|f(x_0)\right| + \left(1 - \frac{1}{2^{n-1}}\left(1 - \frac{1}{2} - \frac{1}{3 \cdot 2^{n-1}}\right)\right)\|h\| < \left|f(x_0)\right| + \|h\|.
\end{aligned}$$

We have obtained that $|f(x)| + |h(x)| < |f(x_0)| + \|h\|$ for every $x \notin f^{-1}(f(x_0))$.

If $f(x_0) = 0$, we must show that $|f(x)| + |h(x)| < |h(x_0)| = \|h\|$ for all $x \notin E(h)$. Let $r > 1$. For any $n \in \mathbb{N}$, define the open set $V_n = \{x \in X: |f(x)| < \frac{(r-1)\|f\|}{2^{n+1}}\}$. Clearly, $V_n \subset V_{n-1}$ and $x_0 \in V_n$ for every $n \in \mathbb{N}$. As before, for each n we choose a peaking function $k_n \in \mathcal{P}_{x_0}(A)$ such that $E(k_n) \subset V_n$, and let $h_n \in \mathcal{P}_{x_0}(A)$ be a large enough power of k_n such that $|h_n(x)| < \frac{r-1}{2^n r}$ on $X \setminus V_n$. We claim that in this case the $\mathbb{R}$-peaking function $h = r\|f\| \cdot \sum_{n=1}^\infty \frac{h_n}{2^n}$ satisfies inequality (1). As before, one can see that $E(h) \subset f^{-1}(0) = \bigcap_{n=1}^{\infty} V_n$. Note that $\|h\| = r\|f\|$ since $h \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$. It is clear that if $x \in E(h)$ then $|f(x)| + |h(x)| = \|h\|$, while $|f(x)| + |h(x)| = |h(x)| < \|h\|$ for any $x \in f^{-1}(0) \setminus E(h)$.

Suppose now that $x \notin f^{-1}(0)$. If in addition $x \notin V_1$, then we obtain, as before, that $|h(x)| < r\|f\| \cdot \sum_1^{\infty} \frac{r-1}{4^n r} < (r-1)\|f\|$, and therefore, $|f(x)| + |h(x)| < \|f\| + (r-1)\|f\| = r\|f\| = \|h\|$. If $x \in V_{n-1} \setminus V_n$ for some $n > 1$, then $x \in V_i$ for $1 \leqslant i \leqslant n-1$ and $x \notin V_i$ for every $i \geqslant n$. Therefore, $|h_i(x)| < \frac{r-1}{2^i r}$ for every $i \geqslant n$. Because of $x \in V_{n-1}$ we see that $|f(x)| < \frac{(r-1)\|f\|}{2^n} < \frac{r\|f\|}{2^n}$ and hence $|f(x)| + |h(x)| < \frac{r\|f\|}{2^n} + r\|f\| \sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} + r\|f\| \sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i}$. Since each h_n is a peaking function of A, it follows that $|h_n(x)| \leqslant 1$ for every $x \in X$ and therefore $\sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} \leqslant \sum_{i=1}^{n-1} \frac{1}{2^i} = 1 - \frac{1}{2^{n-1}}$. In addition, $\sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i} < \sum_{i=n}^{\infty} \frac{(r-1)}{4^i r} < \sum_{i=n}^{\infty} \frac{1}{4^i} = \frac{1}{3 \cdot 4^{n-1}}$. Therefore, we have

$$\bigl|f(x)\bigr| + \bigl|h(x)\bigr| \leqslant \frac{r\|f\|}{2^n} + \left(1 - \frac{1}{2^{n-1}}\right) r\|f\| + \frac{1}{3 \cdot 4^{n-1}} r\|f\| \leqslant \left(1 - \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{3 \cdot 4^{n-1}}\right) r\|f\| < r\|f\| = \|h\|.$$

Consequently, $|f(x)| + |h(x)| < \|h\|$ for every $x \notin f^{-1}(f(0))$.

Let U be a neighborhood of x_0. If $h_* \in \mathcal{P}_{x_0}(A)$ is a peaking function of A with $E(h_*) \subset U$, then $|h_*(x)| < 1$ on $X \setminus U$, the function hh_* satisfies inequality (1) and, in addition, $E(hh_*) \subset U$. □

As noted above, Lemma 1 implies the additive version of Bishop's Lemma [15] stated below, which neither specifies the points where $\max_{\xi \in E} |f(\xi)| + \|h\|$ is attained nor treats the case when $f \equiv 0$ on E:

Corollary 2 *(Additive Bishop's Lemma). (See [15].) Let $f \in A$ and E be a peak set for A such that $f \not\equiv 0$ on E. For any $r \geqslant 1$ there exists an $\mathbb{R}$-peaking function $h \in r\|f\| \cdot \mathcal{P}(A)$ with $E(h) \subset E$ such that $|f(x)| + |h(x)| < \max_{\xi \in E} |f(\xi)| + \|h\|$ for all $x \notin E$.*

The next corollary of Lemma 1 strengthens Corollary 2.

Corollary 3. *Let $f \in A$, $f \not\equiv 0$. If E is a peak set for A and $r \geqslant 1$ is arbitrary, then for any $x_0 \in E \cap \delta A$ with $f(x_0) \neq 0$ there exists an $\mathbb{R}$-peaking function $h \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ with $E(h) \subset E$ such that $|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)| = \max_{\xi \in E} |f(\xi)| + \|h\|$ for every $x \notin E$.*

Lemma 1 implies also the next proposition, which we use on several occasions further.

Proposition 4. *Let $f \in A$, $f \not\equiv 0$. If $x_0 \in \delta A$, $\alpha = \exp\{i \arg(f(x_0))\}$ and $r > 1$ (or, $r \geqslant 1$ if $f(x_0) \neq 0$), then there exists an $\mathbb{R}$-peaking function $h \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ such that $E(f + \alpha h) = E(h)$, $|f(x_0) + \alpha h(x_0)| = \|f + \alpha h\|$ and*

$$\bigl|f(x) + \alpha h(x)\bigr| < \|f + \alpha h\| \tag{2}$$

whenever $f(x) + \alpha h(x) \neq f(x_0) + \alpha h(x_0)$. Consequently, $f + \alpha h \in \mathbb{C} \cdot \mathcal{P}_{x_0}(A)$ and $\sigma_\pi(f + \alpha h) = \{f(x_0) + \alpha h(x_0)\}$. Given a neighborhood U of x_0, h can be chosen to be such that $E(f + \alpha h) \subset U$.

Proof. Let the function h be as in Lemma 1. If $\alpha = \exp\{i \arg(f(x_0))\}$, then $|f(x_0) + \alpha h(x_0)| = |f(x_0)| + |h(x_0)|$ and therefore, $\|f + \alpha h\| = \max_{\xi \in X} |f(\xi) + \alpha h(\xi)| \leqslant \max_{\xi \in X} (|f(\xi)| + |h(\xi)|) = |f(x_0)| + |h(x_0)| = |f(x_0) + \alpha h(x_0)| \leqslant \|f + \alpha h\|$. Hence $\|f + \alpha h\| = |f(x_0)| + |h(x_0)| = |f(x_0) + \alpha h(x_0)|$ and therefore, $f(x_0) + \alpha h(x_0) \in \sigma_\pi(f + \alpha h)$. Inequality (1) implies that for any $x \notin E(h)$, we have $|f(x) + \alpha h(x)| \leqslant |f(x)| + |h(x)| < |f(x_0)| + |h(x_0)| = \|f + \alpha h\|$, thus $f(x) + \alpha h(x) \notin \sigma_\pi(f + \alpha h)$ and hence $E(f + \alpha h) \subset E(h)$. Since $E(h) \subset f^{-1}(f(x_0))$, for any $x \in E(h)$ we have $f(x) + \alpha h(x) = f(x_0) + \alpha h(x_0) \in \sigma_\pi(f + \alpha h)$, and therefore, $E(h) \subset E(f + \alpha h)$. Consequently, $E(h) = E(f + \alpha h)$ and $\sigma_\pi(f + \alpha h) = \{f(x_0) + \alpha h(x_0)\}$, as claimed. If U is a neighborhood of x_0, then any function h from Lemma 1 with $E(h) \subset U$ satisfies the inequality (2). □

Given an $x \in X$, we denote by $\mathcal{E}_x(A) = \{f \in A\colon\ |f(x)| = \|f\|\} = \{f \in A\colon\ x \in E(f)\}$ the set of all algebra elements which take their maximum modulus at x. If $h \in \mathcal{P}_x(A)$, then, clearly, x is in the maximum modulus set $E(h)$ of h, so $h \in \mathcal{E}_x(A)$, and therefore, $\mathcal{P}_x(A) \subset \mathcal{E}_x(A)$. One can see that $\mathbb{C} \cdot \mathcal{P}_x(A) = \mathbb{C} \cdot \mathcal{P}(A) \cap \mathcal{E}_x(A) \subset \mathcal{E}_x(A)$. Note that in the case of algebra $C(X)$, families of sets similar to $\mathcal{E}_x(A)$ have been considered by Holsztyński [4] in his proof of Banach–Stone's theorem.

The next result, which we will use on several occasions further in this paper, is a consequence of Lemma 1.

Corollary 5. *Let $f \in A$, $f \not\equiv 0$. If $x_0 \in \delta A$, $r > 1$ (or, $r \geqslant 1$ if $f(x_0) \neq 0$), and $h_0 \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ is as in Lemma 1, then*

$$\bigl|f(x_0)\bigr| + r\|f\| = \bigl|f(x_0)\bigr| + \bigl|h_0(x_0)\bigr| = \bigl\||f| + |h_0|\bigr\| = \inf_{\substack{h \in \mathcal{E}_{x_0}(A) \\ \|h\| = r\|f\|}} \bigl\||f| + |h|\bigr\|. \tag{3}$$

Proof. Let $h_0 \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ be a function as in Lemma 1. For any $h \in \mathcal{E}_{x_0}(A)$ with $\|h\| = r\|f\|$, we have that $\||f| + |h|\| = \max_{\xi \in X}(|f(\xi)| + |h(\xi)|) \geqslant |f(x_0)| + |h(x_0)| = |f(x_0)| + |h_0(x_0)| = \max_{\xi \in X}(|f(\xi)| + |h_0(\xi)|) = \||f| + |h_0|\|$. Consequently, $\inf_{\substack{h \in \mathcal{E}_{x_0}(A) \\ \|h\| = r\|f\|}} \||f| + |h|\| = \||f| + |h_0|\| = |f(x_0)| + |h_0(x_0)| = |f(x_0)| + \|h\|$, according to Lemma 1. □

3. The associated homeomorphism

In this section $A \subset C(X)$ and $B \subset C(Y)$ will be uniform algebras on compact sets X and Y, respectively. We show that under certain conditions any surjective operator $T : A \to B$ between uniform algebras induces in a natural way an associated homeomorphism between δA and δB.

Recall that an operator $T : A \to B$ is *$\mathbb{R}_+$-homogeneous* if $T(rf) = rTf$ for every $f \in A$ and $r \geqslant 0$. For instance, if T is norm-additive, or more generally, peripherally-additive, then T is $\mathbb{R}$-linear (see e.g. Lemma 12 in the next section) and therefore, $\mathbb{R}_+$-homogeneous. The operator T is *monotone increasing in modulus* (see [1]) if the inequality $|f(x)| \leqslant |g(x)|$ on ∂A implies $|(Tf)(y)| \leqslant |(Tg)(y)|$ on ∂B for all $f, g \in A$. For example, surjections $T : A \to B$ that are *norm-additive in modulus* in the sense that $\||Tf| + |Tg|\| = \||f| + |g|\|$ for all $f, g \in A$, are necessarily monotone increasing in modulus (cf. [1]) and also norm-preserving.

Lemma 6. *If a norm-preserving operator $T : A \to B$ is $\mathbb{R}_+$-homogeneous and monotone increasing in modulus, then for any generalized peak point $x \in \delta A$, the set*

$$E_x = \bigcap_{f \in \mathcal{E}_x(A)} E(Tf) \tag{4}$$

is nonempty and $E_x \cap \delta B \neq \emptyset$.

Proof. Let $x \in \delta A$. We will show that the family $\{E(Tf)\colon\ f \in \mathcal{E}_x(A)\}$ has the finite intersection property. Let $f_1, \dots, f_n \in \mathcal{E}_x(A)$ and define $f = f_1 \cdots f_n$. Then

$$\|f_1 \cdots f_n\| = \|f\| \geqslant \big|f(x)\big| = \big|(f_1 \cdots f_n)(x)\big| = \big|f_1(x) \cdots f_n(x)\big| = \big|f_1(x)\big| \cdots \big|f_n(x)\big| = \|f_1\| \cdots \|f_n\| \geqslant \|f_1 \cdots f_n\|,$$

so $|f(x)| = \|f\| = \prod_{j=1}^n \|f_j\|$ and hence $f \in \mathcal{E}_x(A)$. For any $\xi \in \partial A$ and every fixed $k = 1, \dots, n$, we have

$$\big|f(\xi)\big| = \big|f_1(\xi)\big| \cdots \big|f_n(\xi)\big| \leqslant \left(\prod_{j \neq k} \|f_j\|\right) \cdot \big|f_k(\xi)\big| = \left|\left(\prod_{j \neq k} \|f_j\|\right) \cdot f_k(\xi)\right|.$$

Since T is monotone increasing in modulus and $\mathbb{R}_+$-homogeneous, $|(Tf)(\eta)| \leqslant |T((\prod_{j \neq k} \|f_j\|) \cdot f_k)(\eta)| = (\prod_{j \neq k} \|f_j\|) \cdot |(Tf_k)(\eta)| \leqslant \prod_{j=1}^n \|f_j\| = \|f\| = \|Tf\|$ for all $\eta \in \partial B$. If $y \in E(Tf) \cap \partial B$ then $\|Tf\| = |(Tf)(y)| \leqslant (\prod_{j \neq k} \|f_j\|) \cdot |(Tf_k)(\eta)| \leqslant \|Tf\|$, thus $(\prod_{j \neq k} \|f_j\|) \cdot |(Tf_k)(y)| = \|Tf\| = \prod_{j=1}^n \|f_j\|$ and hence $|(Tf_k)(y)| = \|f_k\| = \|Tf_k\|$. Hence $y \in E(Tf_k)$ and therefore, $E(Tf) \cap \partial B \subset E(Tf_k)$. Since this holds for every $k = 1, \dots, n$ we conclude that $E(Tf) \cap \partial B \subset \bigcap_{j=1}^n E(Tf_j)$. Consequently, the family $\{E(Tf)\colon\ f \in \mathcal{E}_x(A)\}$ has the finite intersection property, as claimed. Since each $E(Tf)$ is a closed subset of Y, a compact set, the above family must have nonempty intersection.

Observe that the set $E(Tf) = (Tf)^{-1}(\sigma_\pi(Tf))$ is a union of peak sets of B since $(Tf)^{-1}(u)$ is a peak set for any $u \in \sigma_\pi(Tf)$. Hence, every $y \in E_x$ belongs to an intersection $F_y = \bigcap_{f \in \mathcal{E}_x(A)} F_{y,f} \subset E_x$ of peak sets $F_{y,f} \subset E(Tf)$ of B. Therefore, F_y meets δB (cf. [10, p. 165]), and so does E_x. $\square$

If, in addition, T preserves the peripheral spectra of algebra elements, sets similar to (4), involving only peaking functions, are considered in [15].

Lemma 7. *Let $T : A \to B$ be $\mathbb{R}_+$-homogeneous and norm-additive in modulus surjection. If $x \in \delta A$ and $y \in E_x \cap \delta B$, then $T^{-1}(\mathcal{E}_y(B)) \subset \mathcal{E}_x(A)$.*

Proof. Since T is norm-additive in modulus, then it is norm-preserving and, as noted above, monotone increasing in modulus. Therefore, $E_x \neq \emptyset$ by Lemma 6. Fix a $k \in \mathcal{E}_y(B)$ and let $h \in T^{-1}(k)$. To prove that $h \in \mathcal{E}_x(A)$, it suffices to show that $|h(x)| = \|h\|$. Take an open neighborhood V of x and a $\mathbb{C}$-peaking function $p \in \|h\| \cdot \mathcal{P}_x(A)$ such that $E(p) \subset V$. Since $y \in E_x = \bigcap_{f \in \mathcal{E}_x(A)} E(Tf) \subset E(Tp)$ it follows that $|(Tp)(y)| = \|Tp\|$. Hence, $Tp \in \mathcal{E}_y(B)$. Since T is norm-additive in modulus, it preserves the norms and therefore, $|k(y)| = \|k\| = \|h\| = \|p\| = \|Tp\|$. Hence,

$$\|h\| + \|p\| \geqslant \big\||h| + |p|\big\| = \big\||k| + |Tp|\big\| \geqslant \big|k(y)\big| + \big|(Tp)(y)\big| = \|k\| + \|Tp\| = \|h\| + \|p\|.$$

Consequently, $\||h| + |p|\| = \|h\| + \|p\|$ and there must be an $x_V \in \partial A$ such that $|h(x_V)| = \|h\|$ and $|p(x_V)| = \|p\|$. Therefore, $x_V \in E(p) \subset V$ and any neighborhood V of x must contain a point x_V with $|h(x_V)| = \|h\|$. The continuity of h implies that $|h(x)| = \|h\|$, so $h \in \mathcal{E}_x(A)$. Hence, $T^{-1}(\mathcal{E}_y(B)) \subset \mathcal{E}_x(A)$, as claimed. $\square$

Corollary 8. *If $T : A \to B$ is as in Lemma 7, then the set E_x is a singleton and belongs to δB for any generalized peak point $x \in \delta A$.*

Proof. Let $y \in E_x$. If there exists a $z \in E_x \setminus \{y\}$ then there is a function $k \in \mathcal{E}_y(B)$ such that $|k(z)| < \|k\|$. For any $h \in T^{-1}(k) \subset \mathcal{E}_x(A)$, we have $E(k) = E(Th) \supset E_x$. Hence, the function $|k| = |Th|$ is identically equal to $\|k\|$ on E_x, in contradiction with $|k(z)| < \|k\|$. Therefore, the set E_x contains no points other than y. $\square$

Let $T : A \to B$ be as in Lemma 7 and $x \in \delta A$. Define $\tau(x)$ to be the single element of the set E_x, i.e.,

$$\{\tau(x)\} = E_x = \bigcap_{h \in \mathcal{E}_x(A)} E(Th). \tag{5}$$

Hence T induces an associated mapping $\tau : x \mapsto \tau(x)$ from δA to δB. Lemma 7 shows that $\mathcal{E}_{\tau(x)}(B) = \mathcal{E}_y(B) \subset T(\mathcal{E}_x(A))$. If $h \in \mathcal{E}_x(A)$, then (4) implies $E(Th) \supset E_x = \{\tau(x)\}$. Consequently,

$$\bigl|(Th)\bigl(\tau(x)\bigr)\bigr| = \|Th\| = \|h\| = \bigl|h(x)\bigr| \tag{6}$$

for any $h \in \mathcal{E}_x(A)$.

We mention that if, in addition, $\sigma_\pi(Th) = \sigma_\pi(h)$ for some $h \in \mathbb{C} \cdot \mathcal{P}_x(A)$, then $(Th)(\tau(x)) = h(x)$.

Corollary 9. *If $T : A \to B$ is as in Lemma 7 then $T(\mathcal{E}_x(A)) = \mathcal{E}_{\tau(x)}(B)$.*

Proof. Let $h \in \mathcal{E}_x(A)$ for some $x \in \delta A$ and let $k = Th$. By Eq. (6) we have $|k(\tau(x))| = |(Th)(\tau(x))| = |h(x)| = \|h\| = \|k\|$. Consequently, $k \in \mathcal{E}_{\tau(x)}(B)$ and therefore $T(\mathcal{E}_x(A)) \subset \mathcal{E}_{\tau(x)}(B)$. The opposite inclusion follows from Lemma 7. $\square$

The next proposition shows that Eq. (6) in fact holds for every $f \in A$ and $x \in \delta A$.

Proposition 10. *If $T : A \to B$ is an $\mathbb{R}_+$-homogeneous and norm-additive in modulus surjection between uniform algebras, then the induced associated mapping τ is continuous and the equation*

$$\bigl|(Tf)\bigl(\tau(x)\bigr)\bigr| = \bigl|f(x)\bigr| \tag{7}$$

holds for every $x \in \delta A$ and all $f \in A$. If, in addition, T is bijective, then τ is a homeomorphism from δA onto δB, and then

$$\bigl|(Tf)(y)\bigr| = \bigl|f\bigl(\psi(y)\bigr)\bigr| \tag{8}$$

for every $y \in \delta B$, where $\psi : \delta B \to \delta A$ is the inverse mapping of τ.

Proof. First we will show that $|(Tf)(\tau(x))| = |f(x)|$ for every $x \in \delta A$ and for all $f \in A$. Let $x \in \delta A$, $f \in A$ and $r > 1$. If $h_0 \in r\|f\| \cdot \mathcal{P}_x(A)$ is a function as in Lemma 1, then $\|Th_0\| = \|h_0\| = r\|f\| = r\|Tf\|$ and $Th_0 \in r\|Tf\| \cdot \mathcal{E}_{\tau(x)}(B)$. Since T is norm-additive in modulus, the strong version of Bishop's Lemma 1, (3) and Corollary 9 imply

$$\bigl|f(x)\bigr| + r\|f\| = \inf_{\substack{h \in \mathcal{E}_x(A)\\ \|h\| = r\|f\|}} \bigl\||f| + |h|\bigr\| = \inf_{\substack{h \in \mathcal{E}_x(A)\\ \|h\| = r\|f\|}} \bigl\||Tf| + |Th|\bigr\| = \inf_{\substack{k \in \mathcal{E}_{\tau(x)}(A)\\ \|k\| = r\|f\|}} \bigl\||Tf| + |k|\bigr\| = \bigl|(Tf)\bigl(\tau(x)\bigr)\bigr| + r\|f\|.$$

Consequently, $|(Tf)(\tau(x))| = |f(x)|$, as claimed.

To show the continuity of τ let $x \in \delta A$ and $p \in (0,1)$. Choose an open set V of $\tau(x)$ in δB and a peaking function $k \in \mathcal{P}_{\tau(x)}(B)$ such that $E(k) \subset V$ and $|k(y)| < p$ on $\delta B \setminus V$. If $h \in T^{-1}(k)$, then $h \in \mathcal{E}_x(A)$, and according to Eq. (6), $|h(x)| = |(Th)(\tau(x))| = |k(\tau(x))| = 1 > p$. Therefore, the open set $W = \{\xi \in \delta A\colon\ |h(\xi)| > p\}$ contains x. The first part of the proof shows that for every $\xi \in W$, we have $|k(\tau(\xi))| = |(Th)(\tau(\xi))| = |h(\xi)| > p$, which implies that $\tau(\xi) \in V$ since $|k(\eta)| < p$ on $\delta B \setminus V$. Consequently, $\tau(W) \subset V$, and thus τ is continuous.

Now suppose that T is bijective. It is easy to see that $T^{-1} : B \to A$ is also an $\mathbb{R}_+$-homogeneous operator. Since the equation $\||Tf| + |Tg|\| = \||f| + |g|\|$ is symmetric with respect to f and Tf, the operator $T^{-1} : B \to A$ is also norm-additive in modulus. By the first part of the proof, T^{-1} induces its own associated continuous map $\psi : \delta B \to \delta A$ such that $|(T^{-1}k)(\psi(\eta))| = |k(\eta)|$ for all $\eta \in \delta B$ and for any $k \in \mathcal{E}_{\psi(\eta)}(B)$. Let $x \in \delta A$ and $y = \tau(x) \in \delta B$. If $h \in \mathcal{E}_x(A)$, then $k = Th \in \mathcal{E}_y(B)$ by Corollary 9; Thus $|h(\psi(y))| = |(T^{-1}(k))(\psi(y))| = |k(y)| = |(Th)(y)| = |(Th)(\tau(x))| = |h(x)| = \|h\|$. Hence $\psi(y) \in E(h)$ for any $h \in \mathcal{E}_x(A)$. Since $\bigcap_{h \in \mathcal{E}_x(A)} E(h) = \{x\}$, we deduce that $\psi(\tau(x)) = \psi(y) = x$ for every $x \in \delta A$. Similar arguments show that $\tau(\psi(y)) = y$ for all $y \in \delta B$. Consequently, τ and ψ are both bijective and $\psi = \tau^{-1}$. Therefore, τ is a homeomorphism. Eq. (8) follows immediately from Eq. (7). $\square$

A version of Proposition 10 for norm-multiplicative operators T is given in [9]. Proposition 10 yields one of the basic results in [15] stated below, where, in particular, T is assumed to be peripherally-additive.

Corollary 11. *(See [15, Lemma 14 and Corollary 6].) For any peripherally-additive and norm-additive in modulus surjection $T : A \to B$ there exists a homeomorphism $\tau : \delta A \to \delta B$ such that the equation $|(Tf)(\tau(x))| = |f(x)|$ holds for every $f \in A$ and $x \in \delta A$.*

In general, the moduli in Eqs. (7) and (8) cannot be omitted, since if so, then T would also be multiplicative. For instance, the operator $Tf = if$ satisfies the hypotheses of Proposition 10 without being multiplicative. However, if T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions, then, by the remark prior to Corollary 9, $(Th)(\tau(x)) = h(x)$ for every $x \in \delta A$ and all $h \in \mathbb{C} \cdot \mathcal{P}_x(A)$.

Note that a mapping similar to τ, but involving only peaking functions, is considered in [1] and [15], where T is assumed in addition to be peripherally-additive in the sense that $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$ for all $f, g \in A$ and consequently, to preserve the peripheral spectra of all $f \in A$.

4. When are norm-linear or norm-additive operators algebra isomorphisms?

Here we provide sufficient conditions for norm-linear and norm-additive operators between uniform algebras to be isometric algebra isomorphisms.

Lemma 12. *If $T : A \to B$ is norm-additive operator and $f, g \in A$, then*

(a) *$T0 = 0$,*
(b) *$T(-f) = -Tf$,*
(c) *T is norm-preserving,*
(d) *T preserves the distances, i.e. $\|Tf - Tg\| = \|f - g\|$,*
(e) *T is injective, and*
(f) *T is continuous.*

If T is also surjective, then T is $\mathbb{R}$-linear, thus additive and $\mathbb{R}_+$-homogeneous.

Proof. Let $f = g = 0$. Then $0 = \|f + g\| = \|Tf + Tg\| = \|2T0\| = 2\|T0\|$, so $T0 = 0$. For property (b) note that $\|Tf + T(-f)\| = \|f + (-f)\| = \|0\| = 0$, so $Tf + T(-f) = 0$, which implies that $T(-f) = -Tf$. If $g = 0$, then $\|Tf + T0\| = \|Tf\| = \|f\|$. Property (d) follows from (b) since $\|f - g\| = \|Tf + T(-g)\| = \|Tf - Tg\|$. The injectivity of T follows from property (d) since if $Tf = Tg$, then $\|f - g\| = \|Tf - Tg\| = 0$ and thus $f = g$. The continuity of T is a direct consequence of (d). Finally, the $\mathbb{R}$-linearity of T follows from the Mazur–Ulam theorem. □

As shown in [15], any norm-linear operator is norm-additive in modulus. Proposition 10 and Lemma 12 imply the following

Theorem 13. *If $T : A \to B$ is a norm-linear surjection between uniform algebras, then the induced associated mapping $\psi : \delta A \to \delta B$ is a homeomorphism and T is a ψ-composition operator in modulus on δB, i.e. $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$.*

Let $T : A \to B$ be a bijection, $\tau : \delta A \to \delta B$ be as in Proposition 10, and $\psi = \tau^{-1}$. We call T a *ψ-composition operator in modulus* on δB if $|(Tf)(y)| = |f(\psi(y))|$, and *ψ-composition operator* on δB if $(Tf)(y) = f(\psi(y))$ for all $f \in A$ and $y \in \delta B$. Eq. (8) shows that T is a ψ-composition operator in modulus on δB.

The following lemma provides sufficient conditions under which the moduli in Eq. (8) can be omitted.

Lemma 14. *Let the ψ-composition operator in modulus $T : A \to B$ on δB from Proposition 10 be additive. If $T(1) = 1$ and $T(i) = i$, or if T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, (i.e. $\sigma_\pi(Tf) = \sigma_\pi(f)$ for all $f \in \mathbb{C} \cdot \mathcal{P}(A)$), then $(Tf)(y) = f(\psi(y))$ for every $f \in A$ and all $y \in \delta B$, thus T is a ψ-composition operator on δB.*

Proof. Suppose first that $T(1) = 1$ and $T(i) = i$. Fix an $f \in A$ and $y_0 \in \delta B$ such that $f(\psi(y_0)) \neq 0$. Since T is ψ-composition operator in modulus, then $|(T(1 + f))(y_0)| = |(1 + f)(\psi(y_0))| = |1 + f(\psi(y_0))|$. On the other hand, the additivity of T implies $|(T(1 + f))(y_0)| = |1 + (Tf)(y_0)|$. Hence, $|1 + f(\psi(y_0))| = |1 + (Tf)(y_0)|$. Consequently, either $(Tf)(y_0) = f(\psi(y_0))$, or, $(Tf)(y_0) = \overline{f(\psi(y_0))}$, which holds for every $f \in A$ and $y_0 \in \delta B$. We claim that $(Tf)(y_0) = f(\psi(y_0))$. Without loss of generality we may assume that $\operatorname{Im}\{(Tf)(y_0)\} \neq 0$. Assume that $(Tf)(y_0) = \overline{f(\psi(y_0))}$ and suppose, in addition, that $(T(i + f))(y_0) = \overline{(i + f)(\psi(y_0))} = -i + \overline{f(\psi(y_0))}$. Then $|i + (Tf)(y_0)| = |(T(i + f))(y_0)| = |-i + \overline{f(\psi(y_0))}| = |-i + (Tf)(y_0)| = |i + \overline{(Tf)(y_0)}|$. Therefore, $(Tf)(y_0) = \overline{(Tf)(y_0)}$, thus $\operatorname{Im}\{(Tf)(y_0)\} = 0$ which contradicts the assumption for f. If $(T(i + f))(y_0) = (i + f)(\psi(y_0)) = i + f(\psi(y_0))$, then $|i + (Tf)(y_0)| = |(T(i + f))(y_0)| = |(i + f)(\psi(y_0))| = |i + f(\psi(y_0))| = |i + \overline{(Tf)(y_0)}|$, which is impossible. Consequently, $(Tf)(y_0) = f(\psi(y_0))$ for every $f \in A$, as claimed.

Suppose now that T preserves the peripheral spectra of $\mathbb{C}$-peaking functions. Then $T(1) = 1$ and $T(i) = i$ since $|T(1)| = |T(i)| = 1$ by (7), and the first part of the proof applies, but we will provide also an alternative proof. Fix an $y_0 \in \delta A$ and let $f \in A$. Since T is a ψ-composition operator in modulus on δB, without loss of generality we can assume that $f(x_0) \neq 0$, where $x_0 = \tau^{-1}(y_0)$. The strong version of the additive Bishop's Lemma, and more precisely its consequence Proposition 4 with $r = 1$, implies that there is an $h \in \|f\| \cdot \mathcal{P}_{x_0}(A)$ such that $\sigma_\pi(f + \alpha h) = f(x_0) + \alpha h(x_0)$, where $\alpha = \exp\{i \arg(f(x_0))\}$. If T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions, then as noted at the end of the previous section, $(Th)(\tau(x_0)) =$

$h(x_0)$, $(T(\alpha h))(\tau(x_0)) = \alpha h(x_0)$, and $(T(f+\alpha h))(\tau(x_0)) = f(x_0) + \alpha h(x_0)$, since $h, \alpha h$ and $f + \alpha h$ belong to $\mathbb{C} \cdot \mathcal{P}_{x_0}(A)$. Since T is additive, $(T(f+\alpha h))(\tau(x_0)) = (Tf)(\tau(x_0)) + (T(\alpha h))(\tau(x_0)) = (Tf)(\tau(x_0)) + \alpha h(x_0)$. Consequently, $f(x_0) + \alpha h(x_0) = (T(f+\alpha h))(\tau(x_0)) = (Tf)(\tau(x_0)) + \alpha h(x_0)$, and thus $(Tf)(\tau(x_0)) = f(x_0)$. Hence $(Tf)(y_0) = f(\psi(y_0))$, where $\psi = \tau^{-1}$. □

Since the elements in a uniform algebra are uniquely determined by their restrictions on the Choquet boundary, uniform algebras are isometrically and algebraically isomorphic to the restriction algebras on their Choquet boundaries. Therefore, any map $T : A \to B$ induces automatically an associated map $T^\dagger : A|_{\delta A} \to B|_{\delta B}$ between the restriction algebras on their corresponding Choquet boundaries.

Because an additive and norm-additive in modulus operator is norm-additive, Lemma 12, Proposition 10 and Lemma 14 imply the following

Corollary 15. *Any additive and norm-additive in modulus surjection $T : A \to B$ is a ψ-composition operator in modulus on δB. If, in addition, $T(1) = 1$ and $T(i) = i$, or if T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, then T is a ψ-composition operator on δB, hence, the induced by T operator $T^\dagger : A|_{\delta A} \to B|_{\delta B}$ is an algebraic isomorphism and the restricted algebras $A|_{\delta A}$ and $B|_{\delta B}$ are algebraically isomorphic.*

Note that, since the operator T in Corollary 15 is $\mathbb{R}$-homogeneous, it preserves the peripheral spectra of all $\mathbb{C}$-peaking functions if it preserves the peripheral spectra of $\mathbb{T}$-peaking functions, where $\mathbb{T}$ is the unit circle. Since, by Lemma 12, every surjective norm-additive operator is additive, Proposition 10 and Corollary 15 yield:

Theorem 16 *(Norm-Additive Operators). Any norm-additive and norm-additive in modulus surjection $T : A \to B$ between uniform algebras is a ψ-composition operator in modulus on δB. If, in addition, $T(1) = 1$ and $T(i) = i$, or if T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, then T is an isometric unital algebra isomorphism.*

Note that the operator T in Theorem 16 is not assumed *a priori* to be linear or continuous. Theorem 16 holds also for surjective norm-additive in modulus isometries T (i.e. such that $\|Tf - Tg\| = \|f - g\|$) with $T(0) = 0$, and therefore extends in a certain sense to the case of uniform algebras the corollary of Banach–Stone's theorem, mentioned in the beginning. Theorem 16 implies the following criteria for a norm-additive operator to be norm-linear.

Corollary 17. *Let $T : A \to B$ be a norm-additive surjection for which $T(1) = 1$ and $T(i) = i$, or which preserves the peripheral spectra of $\mathbb{C}$-peaking functions of A. Then T is norm-linear if and only if it is norm-additive in modulus.*

Indeed, by Theorem 16, T is a ψ-composition operator in modulus, and therefore, $\|\lambda Tf + \mu Tg\| = \sup_{y \in \delta B} |\lambda (Tf)(y) + \mu (Tg)(y)| = \sup_{y \in \delta B} |\lambda f(\psi(y)) + \mu g(\psi(y))| = \sup_{x \in \delta A} |\lambda f(x) + \mu g(x)| = \|\lambda f + \mu g\|$. Consequently, T is norm-linear. Conversely, any norm-linear operator is norm-additive and norm-additive in modulus, as shown in [1] and [15].

The second part of Theorem 16 generalizes the main result in [15] stated below, where, in particular, T is assumed to preserve the peripheral spectra of all algebra elements.

Corollary 18. *(See [1,15].) Any peripherally-additive and norm-additive in modulus surjection $T : A \to B$ is an isometric algebra isomorphism.*

As shown in [15], if T satisfies the equation $\|Tf + \alpha Tg\| = \|f + \alpha g\|$ for all $f, g \in A$ and every $\alpha \in \mathbb{T}$, then T is norm-additive and norm-additive in modulus. Therefore, Theorem 16 implies also the following

Corollary 19. *Any surjection $T : A \to B$ which satisfies the equation $\|Tf + \alpha Tg\| = \|f + \alpha g\|$ for every $f, g \in A$ and all $\alpha \in \mathbb{T}$, is a ψ-composition operator in modulus on δB. If, in addition, $T(1) = 1$ and $T(i) = i$, or if T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, then T is an isometric unital algebra isomorphism.*

Since, according to Corollary 17, every norm-linear operator is norm-additive and norm-additive in modulus, Theorem 16 yields:

Theorem 20 *(Norm-Linear Operators). A norm-linear surjection $T : A \to B$ between uniform algebras for which $T(1) = 1$ and $T(i) = i$, or which preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, is a ψ-composition operator on δB, and therefore, an isometric unital algebra isomorphism.*

Note that the operator T in Theorem 20 is not assumed *a priori* to be linear or continuous. Multiplicative analogues of Corollary 18 are proven in [7] and [9].

Both the norm-linearity and the preservation of peripheral spectra of all $\mathbb{C}$-peaking functions are necessary conditions for T in Theorem 20. Indeed, the operator $Tf = -f$ is norm-linear but does not preserve the peripheral spectra and also is not multiplicative, while the operator $Tf = \frac{f|f|}{\|f\|}$, $f \neq 0$, on $C(X)$ preserves the peripheral spectra of algebra elements but is not norm-linear.

As noted before, a linear operator which preserves the norms of algebra elements is norm-linear. Therefore, Theorem 20 implies the next characterization of algebra isomorphisms, which extends in a certain sense the corollaries of Banach–Stone's theorem and Gleason–Kahane–Żelazko's theorem mentioned at the beginning.

Corollary 21 *(Linear Operators). If a linear operator between two uniform algebras, which is surjective and norm-preserving, is unital, or preserves the peripheral spectra of $\mathbb{C}$-peaking functions, then it is automatically multiplicative and, in fact, an algebra isomorphism.*

Since weakly peripherally-linear operators, in the sense that $\sigma_\pi(\lambda Tf + \mu Tg) \cap \sigma_\pi(\lambda f + \mu g) \neq \emptyset$ for all $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$, are norm-linear, Theorem 20 also implies the following:

Corollary 22. *Any weakly peripherally-linear surjection $T : A \to B$ is a ψ-composition operator in modulus on δB. If, in addition, $T(1) = 1$ and $T(i) = i$, or if T preserves the peripheral spectra of all $\mathbb{C}$-peaking functions of A, then T is an isometric unital algebra isomorphism.*

Corollary 19 implies that the weak peripheral linearity of T in Corollary 22 can be replaced by the more relaxed property $\sigma_\pi(Tf + \alpha Tg) \cap \sigma_\pi(f + \alpha g) \neq \emptyset$ for every $f, g \in A$ and all $\alpha \in \mathbb{T}$. A version of Corollary 22 for weakly peripherally-multiplicative operators in the sense that $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \emptyset$ is proven in [9]. Corollaries 21 and 22 improve previous results from [15], where, in particular, T is assumed to preserve the peripheral spectra of all $f \in A$.

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