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Bilinear space–time estimates for linearised KP-type equations on the three-dimensional torus with applications

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ABSTRACT

A bilinear estimate in terms of Bourgain spaces associated with a linearised Kadomtsev–Petviashvili-type equation on the three-dimensional torus is shown. As a consequence, time localized linear and bilinear space–time estimates for this equation are obtained. Applications to the local and global well-posedness of dispersion generalised KP-II equations are discussed. Especially it is proved that the periodic boundary value problem for the original KP-II equation is locally well-posed for data in the anisotropic Sobolev spaces $H^s_x H^\varepsilon_y(\mathbb{T}^3)$, if $s \geqslant \frac{1}{2}$ and $\varepsilon > 0$.

1. Introduction and main results

In a recent paper [7] joint with M. Panthee and J. Silva we investigated local and global well-posedness issues of the Cauchy problem for the dispersion generalised Kadomtsev–Petviashvili-II (KP-II) equation

$$\begin{cases} \partial_t u - |D_x|^\alpha \partial_x u + \partial_x^{-1} \Delta_y u + u \partial_x u = 0, \\ u(0,x,y) = u_0(x,y) \end{cases} \tag{1}$$

on the cylinders $\mathbb{T} \times \mathbb{R}$ and $\mathbb{T} \times \mathbb{R}^2$, respectively. We considered data u_0 satisfying the mean zero condition

$$\int_0^{2\pi} u_0(x,y)\,dx = 0 \tag{2}$$

and belonging to the anisotropic Sobolev spaces $H^s_x(\mathbb{T})H^\varepsilon_y(\mathbb{R}^{n-1})$, $n \in \{2,3\}$. We could prove quite general (with respect to the dispersion parameter α) local well-posedness results, to a large extent optimal—up to the endpoint—(with respect to the Sobolev regularity). In two dimensions and for higher dispersion ($\alpha > 3$) in three dimensions, these local results could be combined with the conservation of the L^2-norm to obtain global well-posedness.

A key tool to obtain these results were certain bilinear space–time estimates for free solutions, similar to Strichartz estimates. A central argument to obtain the space–time estimates was the following simple observation. Consider a linearised version of (1) with a more general phase function

$$\begin{cases} \partial_t u - i\phi(D_x, D_y)u := \partial_t u - i\phi_0(D_x)u + \partial_x^{-1} \Delta_y u = 0, \\ u(0,x,y) = u_0(x,y), \end{cases} \tag{3}$$

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where ϕ_0 is arbitrary at the moment, with solution $u(x,y,t)=e^{it\phi(D_x,D_y)}u_0(x,y)$. Then we can take the partial Fourier transform $\mathcal{F}_x$ with respect to the first spatial variable x only to obtain

$$\mathcal{F}_x e^{it\phi(D_x,D_y)}u_0(k,y)=e^{it\phi_0(k)}e^{i\frac{t}{k}\Delta_y}\mathcal{F}_x u_0(k,y).$$

Fixing k we have a solution of the free Schrödinger equation—with rescaled time variable $s:=\frac{t}{k}$, and multiplied by a phase factor of size one. Now the whole Schrödinger theory—Strichartz estimates, bilinear refinements thereof, local smoothing and maximal function estimates—is applicable to obtain space–time estimates for the linearised KP-type equation (3).

While in two space dimensions this simple argument has to be supplemented by further estimates depending on ϕ_0, we could obtain (almost) sharp estimates in the three-dimensional $\mathbb{T}\times\mathbb{R}^2$-case *only* by using the "Schrödinger trick" described above. In view of Bourgain's L^4_{xt}-estimate for free solutions of the Schrödinger equation with data defined on the two-dimensional torus [3, first part of Prop. 3.6], the question comes up naturally, if our analysis in [7] concerning $\mathbb{T}\times\mathbb{R}^2$ can be extended to KP-type equations on $\mathbb{T}^3$, and that is precisely the aim of the present paper.

To state our main results we have to introduce some more notation: We will consider functions $u,v,\dots$ of $(x,y,t)\in\mathbb{T}\times\mathbb{T}^2\times\mathbb{R}$ with Fourier transform $\widehat{u},\widehat{v},\dots$, sometimes written as $\mathcal{F}u,\mathcal{F}v,\dots$, depending on the dual variables $(\xi,\tau):=(k,\eta,\tau)\in\mathbb{Z}\times\mathbb{Z}^2\times\mathbb{R}$. Throughout the paper we assume $u,v,\dots$ to fulfill the mean zero condition $\widehat{u}(0,\eta,\tau)=0$. For these functions we define the norms

$$\|u\|_{X_{s,\varepsilon,b}}:=\big\||k|^s\langle\eta\rangle^{\varepsilon}\langle\sigma\rangle^b\widehat{u}\big\|_{L^2_{\xi,\tau}},$$

where $\langle x\rangle^2=1+|x|^2$ and $\sigma=\tau-\phi(\xi)=\tau-\phi_0(k)+\frac{|\eta|^2}{k}$. Although some of our arguments do not rely on that, we will always assume ϕ_0 to be odd, in order to have $\|u\|_{X_{s,\varepsilon,b}}=\|\overline{u}\|_{X_{s,\varepsilon,b}}$. For $\varepsilon=0$ we abbreviate $\|u\|_{X_{s,\varepsilon,b}}=\|u\|_{X_{s,b}}$. In these terms our central bilinear space–time estimate reads as follows.

Theorem 1. *Let $b>\frac{1}{2}$, $s_{1,2}\geqslant 0$ with $s_1+s_2>1$ and $\varepsilon_{0,1,2}\geqslant 0$ with $\varepsilon_0+\varepsilon_1+\varepsilon_2>0$. Then the estimate*

$$\big\|D_y^{-\varepsilon_0}(uv)\big\|_{L^2_{xyt}}\lesssim\|u\|_{X_{s_1,\varepsilon_1,b}}\|v\|_{X_{s_2,\varepsilon_2,b}}\tag{4}$$

and its dualized version

$$\|uv\|_{X_{-s_1,-\varepsilon_1,-b}}\lesssim\big\|D_y^{\varepsilon_0}u\big\|_{L^2_{xyt}}\|v\|_{X_{s_2,\varepsilon_2,b}}\tag{5}$$

hold true.

Taking $\varepsilon_0=0$ and $u=v$ we obtain the linear estimate

$$\|u\|_{L^4_{xyt}}\lesssim\|u\|_{X_{s,\varepsilon,b}},\tag{6}$$

whenever $s,b>\frac{1}{2}$ and $\varepsilon>0$. The estimate (4) can be applied to time localised solutions $e^{it\phi(D_x,D_y)}u_0$ and $e^{it\phi(D_x,D_y)}v_0$ of (3) to obtain

$$\big\|D_y^{-\varepsilon_0}\big(e^{it\phi(D_x,D_y)}u_0e^{it\phi(D_x,D_y)}v_0\big)\big\|_{L^2_t([0,1],L^2_{xy})}\lesssim\|u_0\|_{H^{s_1}_xH^{\varepsilon_1}_y}\|v_0\|_{H^{s_2}_xH^{\varepsilon_2}_y},\tag{7}$$

provided $s_{1,2}$ and $\varepsilon_{0,1,2}$ fulfill the assumptions in Theorem 1. Especially for $s>\frac{1}{2}$ and $\varepsilon>0$ we have the linear estimate

$$\big\|e^{it\phi(D_x,D_y)}u_0\big\|_{L^4_t([0,1],L^4_{xy})}\lesssim\|u_0\|_{H^s_xH^\varepsilon_y}.\tag{8}$$

The corresponding estimate for data u_0 defined on $\mathbb{R}^3$ holds global in time and has a $\|u_0\|_{H^{\frac{1}{2}}_xL^2_y}$ on the right-hand side. It goes back to Ben-Artzi and Saut [1]. Dimensional analysis shows that the Sobolev exponent $s=\frac{1}{2}$ is necessary. So we have not lost more than an ε derivative in the x—as well as in the y-variable.

In order to prove Theorem 1, we will work in Fourier space, where the product uv is turned into the convolution

$$\widehat{u}*\widehat{v}(\xi,\tau)=\int d\tau_1\sum_{\xi_1\in\mathbb{Z}^3}\widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi_2,\tau_2).$$

Here always $(\xi,\tau)=(k,\eta,\tau)=(k_1+k_2,\eta_1+\eta_2,\tau_1+\tau_2)=(\xi_1+\xi_2,\tau_1+\tau_2)$. Observe that there is no contribution to the above sum, whenever $k_1=0$ or $k_2=0$. In the estimation of such convolutions the σ-weights in the $X_{s,\varepsilon,b}$-norms become $\sigma_1=\tau_1-\phi(\xi_1)$ and $\sigma_2=\tau_2-\phi(\xi_2)$. With this notation we introduce the bilinear Fourier multiplier $M^{-\varepsilon}$, which we define by

$$\mathcal{F}M^{-\varepsilon}(u,v)(\xi,\tau):=\chi_{\{k\neq 0\}}\int d\tau_1\sum_{\xi_1\in\mathbb{Z}^3}\langle k_1\eta-k\eta_1\rangle^{-\varepsilon}\widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi_2,\tau_2).$$

Observe that $|k_1\eta - k\eta_1| = |k_1\eta_2 - k_2\eta_1|$, so that we have symmetry between u and v. The operator $M^{-\varepsilon}$ serves to compensate for the unavoidable loss of the D_y^{ε} in (4). A careful examination of the proof of Theorem 1 will give the following.

Theorem 2. *Let $s, b > \frac{1}{2}$ and $\varepsilon > 0$. Then*

$$\left\| M^{-\varepsilon}(u, v) \right\|_{L^2_{xyt}} \lesssim \|u\|_{X_{s,b}} \|v\|_{X_{s,b}}. \tag{9}$$

The proof of the above theorems will be done in Section 2, while Section 3 is devoted to the applications. Here we specialize to the dispersion generalised KP-II equation (1), that is to $\phi_0(k) = |k|^{\alpha}k$. For $\alpha = 2$, which is the original KP-II equation we will use Theorem 1 to show the following local result.

Theorem 3. *Let $s \geqslant \frac{1}{2}$ and $\varepsilon > 0$. Then, for $\alpha = 2$, the Cauchy problem* (1) *is locally well-posed for data $u_0 \in H_x^s H_y^{\varepsilon}(\mathbb{T}^3)$ satisfying the mean zero condition* (2)*.*

For high dispersion, i.e. $\alpha > 3$, one can allow $s < 0$ and $\varepsilon = 0$. In fact, by the aid of Theorem 2 we can prove:

Theorem 4. *Let $3 < \alpha \leqslant 4$ and $s > \frac{3-\alpha}{2}$. Then the Cauchy problem* (1) *is locally well-posed for data $u_0 \in H_x^s L_y^2(\mathbb{T}^3)$ satisfying* (2)*. If $s \geqslant 0$ the corresponding solutions extend globally in time by the conservation of the L^2_{xy}-norm.*

More precise statements of the last two theorems will be given in Section 3. We conclude this introduction with several remarks commenting on our well-posedness results and their context.

1. Concerning the Cauchy problem for the KP-II equation and its dispersion generalisations on $\mathbb{R}^2$ and $\mathbb{R}^3$ there is a rich literature, see e.g. [8,9,12,13,16,17,20,22], this list is by no means exhaustive. For $\alpha = 2$ the theory has even been pushed to the critical space in a recent work of Hadac, Herr, and Koch [10]. On the other hand, for the periodic or semiperiodic problem the theory is much less developed. Besides Bourgain's seminal paper [2] our only references here are the papers [18,19] of Saut and Tzvetkov and our own contribution [7] joint with M. Panthee and J. Silva.

2. The results obtained here for the fully periodic case are as good as those in [7] for the $\mathbb{T} \times \mathbb{R}^2$ case and even as those obtained by Hadac [8] for $\mathbb{R}^3$, which are optimal by scaling considerations. We believe this is remarkable since apart from nonlinear wave and Klein–Gordon equations there are only very few examples in the literature, where the periodic problem is as well behaved as the corresponding continuous case. (One example is of course Bourgain's $L_x^2(\mathbb{T})$ result for the cubic Schrödinger equation [3], but this is half a derivative away from the scaling limit.) On the other hand there are many examples, such as KdV and mKdV, where at least the methods applied here lead to (by $\frac{1}{4}$ derivative) weaker results for the periodic problem. Another example is the KP-II equation itself in two space dimensions, where in [7] we lost $\frac{1}{4}$ derivative when stepping from $\mathbb{R}^2$ to $\mathbb{T} \times \mathbb{R}$. Another loss of $\frac{1}{4}$ derivative in the step from $\mathbb{T} \times \mathbb{R}$ to $\mathbb{T}^2$ is probable.

3. For the semilinear Schrödinger equation

$$iu_t + \Delta u = |u|^p u$$

on the torus, with $2 < p < 4$ in one, $1 < p < 2$ in two dimensions, one barely misses the conserved L_x^2 norm and thus cannot infer global well-posedness. The reason behind that is the loss of an ε derivative in the Strichartz type estimates in the periodic case. A corresponding derivative loss is apparent in Theorem 1 but the usually ignored mixed part of the rather comfortable resonance relation of the dispersion generalised KP-II equation allows (via $M^{-\varepsilon}$) to compensate for this loss, so that for high dispersion ($3 < \alpha \leqslant 4$) we can obtain something global. The author did not expect that, when starting this investigation.

4. We restrict ourselves to the most important (as we believe) values of α. Our arguments work as well for $\alpha \in (2, 3]$ with optimal lower bound for s but possibly with an ε loss in the y variable. For $\alpha > 4$ we probably loose optimality.

5. In [21] Takaoka and Tzvetkov proved a time localised $L^4 - L^2$ Strichartz type estimate *without* derivative loss for free solutions of the Schrödinger equation with data defined on $\mathbb{R} \times \mathbb{T}$. Inserting their arguments in our proof of Theorem 1 we can show a variant thereof with $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$, if the data live on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$. Consequently our well-posedness results are valid in this case, too.

2. Proof of Theorem 1

The main ingredient in the proof of Bourgain's Schrödinger estimate

$$\left\| e^{it\Delta} u_0 \right\|_{L^4_{xt}(\mathbb{T}^3)} \lesssim \|u_0\|_{H_x^{\varepsilon}(\mathbb{T}^2)} \quad (\varepsilon > 0)$$

is the well-known estimate on the number of representations of an integer $r > 0$ as a sum of two squares: For any $\varepsilon > 0$ there exists c_ε such that

$$\#\{\eta \in \mathbb{Z}^2:\ |\eta|^2 = r\} \leqslant c_\varepsilon r^\varepsilon. \tag{10}$$

For (10), see [11, Theorem 338]. Our proof of Theorem 1 relies on the following variant thereof.

Lemma 1. *Let $r \in \mathbb{N}$, $\delta \in \mathbb{R}^2$. Then for any $\varepsilon > 0$ there exists c_ε, independent of r and δ, such that*

$$\#\{\eta \in \mathbb{Z}^2:\ r \leqslant |\eta - \delta|^2 < r+1\} \leqslant c_\varepsilon r^\varepsilon. \tag{11}$$

Proof. In the case where $\delta \in \mathbb{Z}^2$, this follows by translation from (10). So we may assume $\delta \in [0,1]^2$, and we start by considering the special case $\delta = (\frac{1}{2}, \frac{1}{2})$. Here

$$\#\{\eta \in \mathbb{Z}^2:\ r \leqslant |\eta - \delta|^2 < r+1\} = \#\{\eta \in \mathbb{Z}^2:\ 4r \leqslant |2\eta - 2\delta|^2 < 4(r+1)\} = \sum_{l=4r}^{4r+3} \#\{\eta \in \mathbb{Z}^2:\ |2\eta - 2\delta|^2 = l\}.$$

But $|2\eta - 2\delta|^2 = 4|\eta|^2 - 4\langle \eta, 2\delta\rangle + 2 \equiv 2 \pmod 4$, so the only contribution to the above sum comes from $l = 4r+2$. Thus, by (10), for any $\varepsilon' > 0$ there exists $c_{\varepsilon'}$ such that

$$\#\{\eta \in \mathbb{Z}^2:\ r \leqslant |\eta - \delta|^2 < r+1\} \leqslant c_{\varepsilon'}(4r+2)^{\varepsilon'} \leqslant c_{\varepsilon'}(6r)^{\varepsilon'}. \tag{12}$$

Next we observe that for $\delta \in \{(0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, 1), (1, \frac{1}{2})\}$ we have $|2\eta - 2\delta|^2 \equiv 1 \pmod 4$, so that the estimate (12) is valid in these cases, too. Iterating the argument, we obtain for $\delta = (\frac{m_1}{2^m}, \frac{m_2}{2^m})$ with $m \in \mathbb{N}$ and $0 \leqslant m_{1,2} \leqslant 2^m$ the estimate

$$\#\{\eta \in \mathbb{Z}^2:\ r \leqslant |\eta - \delta|^2 < r+1\} \leqslant c_{\varepsilon'}\big(6^m r\big)^{\varepsilon'}. \tag{13}$$

Now for an arbitrary $\delta \in [0,1]^2$ we choose $\delta' = (\frac{m_1}{2^{m'}}, \frac{m_2}{2^{m'}})$ with $|\delta - \delta'| \sim r^{-\frac{1}{2}}$, so that

$$\{\eta \in \mathbb{Z}^2:\ r \leqslant |\eta - \delta|^2 < r+1\} \subset \{\eta \in \mathbb{Z}^2:\ r-1 \leqslant |\eta - \delta|^2 < r+2\}$$

and hence, by (13),

$$\#\{\eta \in \mathbb{Z}^2:\ r \leqslant |\eta - \delta|^2 < r+1\} \leqslant 3c_{\varepsilon'}\big(6^{m'} r\big)^{\varepsilon'}.$$

Such a δ exists for $2^{m'} \sim r^{\frac{1}{2}}$, estimating roughly, for $6^{m'} \leqslant r^{\frac{3}{2}}$. So we have the bound

$$\#\{\eta \in \mathbb{Z}^2:\ r \leqslant |\eta - \delta|^2 < r+1\} \leqslant 3c_{\varepsilon'} r^{\frac{5\varepsilon'}{2}}.$$

Choosing $\varepsilon' = \frac{2\varepsilon}{5}$, $c_\varepsilon = 3c_{\varepsilon'}$, we obtain (11). □

Corollary 1. *If B is a disc (or square) of arbitrary position and of radius (sidelength) R, then for any $\varepsilon > 0$ there exists c_ε such that*

$$\sum_{\substack{\eta_1 \in \mathbb{Z}^2 \\ r \leqslant |\eta_1 - \delta|^2 < r+1}} \chi_B(\eta_1) \leqslant c_\varepsilon R^\varepsilon. \tag{14}$$

Proof. If $R \gtrsim r^{\frac{1}{6}}$, the estimate (14) follows from Lemma 1. If $R \ll r^{\frac{1}{6}}$, there are at most two lattice points on the intersection of B with the circle of radius $\simeq r^{\frac{1}{2}}$ around δ, by Lemma 4.4 of [4]. □

In the sequel we will use the following projections: For a subset $M \subset \mathbb{Z}^2$ we define P_M by $\mathcal{F}P_M u(k,\eta,\tau) = \chi_M(\eta)\mathcal{F}u(k,\eta,\tau)$. Especially, if M is a ball of radius 2^l centered at the origin, we will write P_l instead of P_M. Furthermore we have $P_{\Delta l} = P_l - P_{l-1}$, and the P-notation will also be used in connection with a sequence $\{Q_\alpha^l\}_{\alpha \in \mathbb{Z}^2}$ of squares of sidelength 2^l, centered at $2^l\alpha$. Double sized squares with the same centers will be denoted by $\widetilde{Q}_\alpha^l$.

Theorem 5. *Let $s > 1$, $b > \frac{1}{2}$ and $\varepsilon > 0$. Then for a disc (or square) B of arbitrary position with radius (sidelength) R we have*

$$\big\|(P_B u)v\big\|_{L^2_{xyt}} \lesssim R^\varepsilon \|u\|_{X_{0,b}} \|v\|_{X_{s,b}}. \tag{15}$$

Proof. Choose f, g with $\|f\|_{L^2_{\xi\tau}} = \|u\|_{X_{0,b}}$ and $\|g\|_{L^2_{\xi\tau}} = \|v\|_{X_{s,b}}$. Then the left-hand side of (15) becomes

$$\left\| \int d\tau_1 \sum_{k_1\in\mathbb{Z}} \sum_{\eta_1\in\mathbb{Z}^2} \chi_B(\eta_1) f(\xi_1,\tau_1)\langle\sigma_1\rangle^{-b} g(\xi_2,\tau_2)|k_2|^{-s}\langle\sigma_2\rangle^{-b} \right\|_{L^2_{\xi\tau}}. \tag{16}$$

Since $\|uv\|_{L^2_{xyt}} = \|u\bar{v}\|_{L^2_{xyt}}$, which corresponds to $\|\widehat{u} * \widehat{v}\|_{L^2_{\xi\tau}} = \|\widehat{u} * \widehat{w}\|_{L^2_{\xi\tau}}$ on Fourier side, where $\widehat{w}(\xi,\tau) = \overline{\widehat{v}}(-\xi,-\tau)$, and since the phase function ϕ is assumed to be odd, so that $\|u\|_{X_{s,b}} = \|\bar{u}\|_{X_{s,b}}$, we may assume in the estimation on (16), that k_1 and k_2 have the same sign, cf. Remark 4.7 in [2]. So it is sufficient to consider $0 < |k_2| \leqslant |k_1| < |k|$. Now, using Minkowski's inequality we estimate (16) by

$$\begin{aligned} &\left\| \sum_{k_1\in\mathbb{Z}} |k_2|^{-s} \left\| \int d\tau_1 \sum_{\eta_1\in\mathbb{Z}^2} \chi_B(\eta_1) f(\xi_1,\tau_1)\langle\sigma_1\rangle^{-b} g(\xi_2,\tau_2)\langle\sigma_2\rangle^{-b} \right\|_{L^2_{\eta\tau}} \right\|_{L^2_k} \\ &\quad\leqslant \left\| |k_2|^{-\frac12} \left\| \int d\tau_1 \sum_{\eta_1\in\mathbb{Z}^2} \chi_B(\eta_1) f(\xi_1,\tau_1)\langle\sigma_1\rangle^{-b} g(\xi_2,\tau_2)\langle\sigma_2\rangle^{-b} \right\|_{L^2_{\eta\tau}} \right\|_{L^2_{kk_1}}, \end{aligned}$$

where Cauchy–Schwarz was applied to $\sum_{k_1\in\mathbb{Z}}$. Thus it is sufficient to show that

$$\left\| \int d\tau_1 \sum_{\eta_1\in\mathbb{Z}^2} \chi_B(\eta_1) f(\xi_1,\tau_1)\langle\sigma_1\rangle^{-b} g(\xi_2,\tau_2)\langle\sigma_2\rangle^{-b} \right\|_{L^2_{\eta\tau}} \lesssim R^\varepsilon |k_2|^{\frac12} \|f(k_1,\cdot,\cdot)\|_{L^2_{\eta_1\tau_1}} \|g(k_2,\cdot,\cdot)\|_{L^2_{\eta\tau}}. \tag{17}$$

By the "Schwarz-method" developed in [14,15] and by [5, Lemma 4.2], (17) follows from

$$\sum_{\eta_1\in\mathbb{Z}^2} \chi_B(\eta_1) \left\langle \tau - \phi_0(k_1) - \phi_0(k_2) + \frac{|\eta_1|^2}{k_1} + \frac{|\eta_2|^2}{k_2} \right\rangle^{-2b} \lesssim R^{2\varepsilon}|k_2|. \tag{18}$$

For $\omega := \eta_1 - \frac{k_1}{k}\eta$ we have $\frac{|\eta_1|^2}{k_1} + \frac{|\eta_2|^2}{k_2} = \frac{|\eta|^2}{k} + \frac{k}{k_1k_2}|\omega|^2$, so that with $a := \tau - \phi_0(k_1) - \phi_0(k_2) + \frac{|\eta|^2}{k}$ the left-hand side of (18) becomes

$$\sum_{\eta_1\in\mathbb{Z}^2} \chi_B(\eta_1) \left\langle a + \frac{k}{k_1k_2}|\omega|^2 \right\rangle^{-2b} = \sum_{r\geqslant 0} \left\langle a + \frac{k}{k_1k_2} r \right\rangle^{-2b} \sum_{r\leqslant |\eta_1 - \frac{k_1}{k}\eta|^2 < r+1} \chi_B(\eta_1).$$

By Corollary 1 the inner sum is controlled by $c_\varepsilon R^\varepsilon$, while

$$\sum_{r\geqslant 0} \left\langle a + \frac{k}{k_1k_2} r \right\rangle^{-2b} \lesssim \frac{|k_1k_2|}{|k|} \lesssim |k_2|,$$

which proves (18). □

Remark. The quantity, which we precisely loose in the application of Lemma 1, is

$$r^\varepsilon \simeq \left| \eta_1 - \frac{k_1}{k}\eta \right|^{2\varepsilon} \leqslant \langle k\eta_1 - k_1\eta\rangle^{2\varepsilon},$$

which is the symbol of the Fourier multiplier $M^{2\varepsilon}$. Rereading carefully the calculation in the previous proof, we see that—instead of (17)—the following estimate holds true as well

$$\left\| \int d\tau_1 \sum_{\eta_1\in\mathbb{Z}^2} \langle k\eta_1 - k_1\eta\rangle^{-\varepsilon} f(\xi_1,\tau_1)\langle\sigma_1\rangle^{-b} g(\xi_2,\tau_2)\langle\sigma_2\rangle^{-b} \right\|_{L^2_{\eta\tau}} \lesssim \frac{|k_1k_2|^{\frac12}}{|k|^{\frac12}} \|f(k_1,\cdot,\cdot)\|_{L^2_{\eta_1\tau_1}} \|g(k_2,\cdot,\cdot)\|_{L^2_{\eta\tau}}. \tag{19}$$

(Introducing the $M^{-\varepsilon}$ we cannot justify the sign assumption on $k, k_{1,2}$ any more.) Multiplying by $|k|^{\frac12}$ and summing up over k_1 using Cauchy–Schwarz we obtain

$$\left\| \mathcal{F}_x D_x^{\frac12} M^{-\varepsilon}(u,v) \right\|_{L^\infty_k L^2_{yt}} \lesssim \|u\|_{X_{\frac12,b}} \|v\|_{X_{\frac12,b}}, \tag{20}$$

from which (9) follows by a further application of the Cauchy–Schwarz inequality. So Theorem 2 is proved.

Proof of Theorem 1. Since in Corollary 1 the position of the disc is arbitrary, we may replace $\chi_B(\eta_1)$ by $\chi_B(\eta_2)$ in the proof of Theorem 5, which gives

$$\left\| u(P_B v) \right\|_{L^2_{xyt}} \lesssim R^\varepsilon \|u\|_{X_{0,b}} \|v\|_{X_{s,b}}. \tag{21}$$

Now we have symmetry between u and v, so that we may interpolate bilinearly to obtain

$$\big\|(P_B u)v\big\|_{L^2_{xyt}} \lesssim R^{\varepsilon}\|u\|_{X_{s_1,b}}\|v\|_{X_{s_2,b}} \tag{22}$$

for $s_{1,2} \geqslant 0$ with $s_1 + s_2 > 1$. Decomposing dyadically we obtain with $0 < \varepsilon' < \varepsilon$,

$$\|uv\|_{L^2_{xyt}} \leqslant \sum_{l\geqslant 0}\big\|(P_{\Delta l}u)v\big\|_{L^2_{xyt}} \lesssim \sum_{l\geqslant 0} 2^{l\varepsilon'}\|P_{\Delta l}u\|_{X_{s_1,b}}\|v\|_{X_{s_2,b}} \lesssim \sum_{l\geqslant 0} 2^{l(\varepsilon'-\varepsilon)}\|u\|_{X_{s_1,\varepsilon,b}}\|v\|_{X_{s_2,b}} \lesssim \|u\|_{X_{s_1,\varepsilon,b}}\|v\|_{X_{s_2,b}}.$$

Exchanging u and v again we have shown for $s_{1,2} \geqslant 0$ with $s_1 + s_2 > 1$ and $\varepsilon_{1,2} \geqslant 0$ with $\varepsilon_1 + \varepsilon_2 > 0$ that

$$\|uv\|_{L^2_{xyt}} \lesssim \|u\|_{X_{s_1,\varepsilon_1,b}}\|v\|_{X_{s_2,\varepsilon_2,b}}, \tag{23}$$

which is the $\varepsilon_0 = 0$ part of (4) in Theorem 1. To see the $\varepsilon_0 > 0$ part, we decompose

$$\big\|D_y^{-\varepsilon_0}(uv)\big\|_{L^2_{xyt}} \leqslant \sum_{l\geqslant 0} 2^{-l\varepsilon_0}\big\|P_{\Delta l}(uv)\big\|_{L^2_{xyt}},$$

where for fixed l,

$$\big\|P_{\Delta l}(uv)\big\|^2_{L^2_{xyt}} = \sum_{\alpha,\beta\in\mathbb{Z}^2}\big\langle P_{\Delta l}\big((P_{Q^l_\alpha}u)v\big), P_{\Delta l}\big((P_{Q^l_\beta}u)v\big)\big\rangle_{L^2_{xyt}}.$$

Now for $\eta_1 \in Q^l_\alpha$, $|\eta| \leqslant 2^l$ we have $\eta_2 = \eta - \eta_1 \in \widetilde{Q}^l_{-\alpha}$, so that the latter can be estimated by

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{Z}^2}\big\langle (P_{Q^l_\alpha}u)(P_{\widetilde{Q}^l_{-\alpha}}v), (P_{Q^l_\beta}u)(P_{\widetilde{Q}^l_{-\beta}}v)\big\rangle_{L^2_{xyt}} &\leqslant \sum_{\alpha,\beta\in\mathbb{Z}^2}\big\langle (P_{\widetilde{Q}^l_\alpha}u)(P_{\widetilde{Q}^l_\beta}\bar{v}), (P_{\widetilde{Q}^l_\beta}u)(P_{\widetilde{Q}^l_\alpha}\bar{v})\big\rangle_{L^2_{xyt}}\\ &\leqslant \sum_{\alpha,\beta\in\mathbb{Z}^2}\big\|(P_{\widetilde{Q}^l_\alpha}u)(P_{\widetilde{Q}^l_{-\beta}}v)\big\|_{L^2_{xyt}}\big\|(P_{\widetilde{Q}^l_\beta}u)(P_{\widetilde{Q}^l_{-\alpha}}v)\big\|_{L^2_{xyt}}\\ &\leqslant \sum_{\alpha,\beta\in\mathbb{Z}^2}\big\|(P_{\widetilde{Q}^l_\alpha}u)(P_{\widetilde{Q}^l_{-\beta}}v)\big\|^2_{L^2_{xyt}}.\end{aligned}$$

Using (22) and the almost orthogonality of the sequence $\{P_{\widetilde{Q}^l_\alpha}v\}_{\alpha\in\mathbb{Z}^2}$ we estimate the latter by

$$2^{2l\varepsilon}\sum_{\alpha,\beta\in\mathbb{Z}^2}\|P_{\widetilde{Q}^l_\alpha}u\|^2_{X_{s_1,b}}\|P_{\widetilde{Q}^l_{-\beta}}v\|^2_{X_{s_2,b}} \lesssim 2^{2l\varepsilon}\|u\|^2_{X_{s_1,b}}\|v\|^2_{X_{s_2,b}}.$$

Choosing $\varepsilon < \varepsilon_0$ the sum over l remains finite and we arrive at

$$\big\|D_y^{-\varepsilon_0}(uv)\big\|_{L^2_{xyt}} \lesssim \|u\|_{X_{s_1,b}}\|v\|_{X_{s_2,b}}.$$

Finally we remark that (4) and (5) are equivalent by duality. □

3. Applications to KP-II type equations

Here the phase function is specified as $\phi_0(k) = |k|^\alpha k$, $\alpha \geqslant 2$, so that the mixed weight becomes $\sigma = \tau - |k|^\alpha k + \frac{|\eta|^2}{k}$. To prove the well-posedness results in Theorem 3 and 4, we need some more norms and function spaces, respectively. In both cases we use the spaces $X_{s,\varepsilon,b;\beta}$ with additional weights, introduced in [2] and defined by

$$\|f\|_{X_{s,\varepsilon,b;\beta}} := \left\|\langle k\rangle^s\langle\eta\rangle^\varepsilon\langle\sigma\rangle^b\left(1 + \frac{\langle\sigma\rangle}{\langle k\rangle^{\alpha+1}}\right)^\beta\hat{f}\right\|_{L^2_{\tau\xi}}. \tag{24}$$

We will always have $\beta \geqslant 0$, so that

$$\|f\|_{X_{s,b}} \leqslant \|f\|_{X_{s,b;\beta}}. \tag{25}$$

Observe that

$$\|f\|_{X_{s,b}} \sim \|f\|_{X_{s,b;\beta}}, \tag{26}$$

if $\langle\sigma\rangle \leqslant \langle k\rangle^{\alpha+1}$.

The case $\alpha = 2$ corresponding to the original KP-II equation becomes a limiting case in our considerations, where we have to choose the parameter $b = \frac{1}{2}$. Thus we also need the auxiliary norms

$$\|f\|_{Y_{s,\varepsilon;\beta}} := \left\| \langle k\rangle^s \langle\eta\rangle^\varepsilon \langle\sigma\rangle^{-1} \left(1+\frac{\langle\sigma\rangle}{\langle k\rangle^{\alpha+1}}\right)^\beta \hat{f} \right\|_{L^2_\xi(L^1_\tau)}, \tag{27}$$

cf. [5], and

$$\|f\|_{Z_{s,\varepsilon;\beta}} := \|f\|_{Y_{s,\varepsilon;\beta}} + \|f\|_{X_{s,\varepsilon,-\frac{1}{2};\beta}}. \tag{28}$$

As before, for $\varepsilon = 0$ we will write $X_{s,b;\beta}$ instead of $X_{s,\varepsilon,b;\beta}$, and if the exponent β of the additional weight is zero, we use $X_{s,\varepsilon,b}$ as abbreviation for $X_{s,\varepsilon,b;\beta}$. Similar for the Y- and Z-norms. In these terms the crucial bilinear estimate leading to Theorem 3 is the following.

Lemma 2. *Let $\alpha = 2$, $s \geqslant \frac{1}{2}$ and $\varepsilon > 0$. Then there exists $\gamma > 0$, such that for all u, v supported in $[-T, T] \times \mathbb{T}^3$ the estimate*

$$\left\|\partial_x(uv)\right\|_{Z_{s,\varepsilon;\frac{1}{2}}} \lesssim T^\gamma \|u\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}} \|v\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}} \tag{29}$$

holds true.

Correspondingly for Theorem 4, we have

Lemma 3. *Let $3 < \alpha \leqslant 4$. Then, for $s > \frac{3-\alpha}{2}$ there exist $b' > -\frac{1}{2}$ and $\beta \in [0, -b']$, such that for all $b > \frac{1}{2}$,*

$$\left\|D_x^{s+1+\varepsilon} M^{-\varepsilon}(u,v)\right\|_{X_{0,b';\beta}} \lesssim \|u\|_{X_{s,b;\beta}} \|v\|_{X_{s,b;\beta}}, \tag{30}$$

whenever $\varepsilon > 0$ is sufficiently small, and

$$\left\|\partial_x(uv)\right\|_{X_{s,b';\beta}} \lesssim \|u\|_{X_{s,b;\beta}} \|v\|_{X_{s,b;\beta}}. \tag{31}$$

In the proof of both Lemmas above the resonance relation for the KP-II-type equation with quadratic nonlinearity plays an important role. We have

$$\sigma_1 + \sigma_2 - \sigma = r(k, k_1) + \frac{|k\eta_1 - k_1\eta|^2}{kk_1k_2}, \tag{32}$$

where

$$\left|r(k,k_1)\right| = \left||k|^\alpha k - |k_1|^\alpha k_1 - |k_2|^\alpha k_2\right| \sim |k_{\max}|^\alpha |k_{\min}|,$$

see [9]. Both terms on the right of (32) have the same sign, so that

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |k_{\min}||k_{\max}|^\alpha + \frac{|k\eta_1 - k_1\eta|^2}{|kk_1k_2|}. \tag{33}$$

The proof of Lemma 2 is almost the same as that of Lemma 4 in [7], it is repeated here—with minor modifications—for the sake of completeness. We need a variant of Theorem 1 with $b < \frac{1}{2}$. To obtain this, we first observe that, if $s_{1,2} \geqslant 0$ with $s_1 + s_2 > \frac{1}{2}$, $\varepsilon_{0,1,2} \geqslant 0$ with $\varepsilon_0 + \varepsilon_1 + \varepsilon_2 > 1$, $1 \leqslant p \leqslant 2$, and $b > \frac{1}{2p}$, then

$$\left\|\mathcal{F} D_y^{-\varepsilon_0}(uv)\right\|_{L^2_\xi L^p_\tau} \lesssim \|u\|_{X_{s_1,\varepsilon_1,b}} \|v\|_{X_{s_2,\varepsilon_2,b}}. \tag{34}$$

This follows from Sobolev type embeddings and applications of Young's inequality. Dualizing the $p = 2$ part of (34) we obtain

$$\|uv\|_{X_{-s_1,-\varepsilon_1,-b}} \lesssim \left\|D_y^{\varepsilon_0} u\right\|_{L^2_{xyt}} \|v\|_{X_{s_2,\varepsilon_2,b}}. \tag{35}$$

Now bilinear interpolation with Theorem 1 gives the following.

Corollary 2. *Let $s_{1,2} \geqslant 0$ with $s_1 + s_2 = 1$ and $\varepsilon_{1,2} \geqslant 0$ with $\varepsilon_1 + \varepsilon_2 > 0$, then there exist $b < \frac{1}{2}$ and $p < 2$ such that*

$$\left\|\mathcal{F}(uv)\right\|_{L^2_\xi L^p_\tau} + \|uv\|_{L^2_{xyt}} \lesssim \|u\|_{X_{s_1,\varepsilon_1,b}} \|v\|_{X_{s_2,\varepsilon_2,b}} \tag{36}$$

and

$$\|uv\|_{X_{-s_1,-b}} \lesssim \left\|D_y^{\varepsilon_1} u\right\|_{L^2_{xyt}} \|v\|_{X_{s_2,\varepsilon_2,b}} \tag{37}$$

hold true.

The purpose of the $p<2$ part in the above corollary is to deal with the Y-contribution to the Z-norm in Lemma 2. Its application will usually follow on an embedding

$$\left\|\langle\sigma\rangle^{-\frac{1}{2}}\hat{f}\right\|_{L^2_\xi L^1_\tau} \lesssim \|\hat{f}\|_{L^2_\xi L^p_\tau},$$

where $p<2$ but arbitrarily closed to 2. Now we are prepared to establish Lemma 2.

Proof of Lemma 2. Without loss of generality we may assume that $s=\frac{1}{2}$. The proof consists of the following case by case discussion.

Case a: $\langle k\rangle^3 \leqslant \langle\sigma\rangle$. First we observe that

$$\left\|\partial_x(uv)\right\|_{Z_{s,\varepsilon;\frac{1}{2}}} \lesssim \left\|D_x^{s+1}\left(D_y^\varepsilon u\cdot v\right)\right\|_{Z_{0,0;\frac{1}{2}}} + \left\|D_x^{s+1}\left(u\cdot D_y^\varepsilon v\right)\right\|_{Z_{0,0;\frac{1}{2}}}. \tag{38}$$

The first contribution to (38) equals

$$\left\|\mathcal{F}\left(D_y^\varepsilon u\cdot v\right)\right\|_{L^2_{\xi,\tau}} + \left\|\langle\sigma\rangle^{-\frac{1}{2}}\mathcal{F}\left(D_y^\varepsilon u\cdot v\right)\right\|_{L^2_\xi L^1_\tau} \lesssim \left\|\mathcal{F}\left(D_y^\varepsilon u\cdot v\right)\right\|_{L^2_{\xi,\tau}\cap L^2_\xi L^p_\tau} \lesssim \|u\|_{X_{s,\varepsilon,b}}\|v\|_{X_{s,\varepsilon,b}}$$

by (36), for some $b<\frac{1}{2}$. Using the fact[2] that under the support assumption on u the inequality

$$\|u\|_{X_{s,\varepsilon,b}} \lesssim T^{\tilde{b}-b}\|u\|_{X_{s,\varepsilon,\tilde{b}}} \tag{39}$$

holds, whenever $-\frac{1}{2}<b<\tilde{b}<\frac{1}{2}$, this can, for some $\gamma>0$, be further estimated by $T^\gamma\|u\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}}\|v\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}}$ as desired. The second contribution to (38) can be treated in precisely the same manner.

Case b: $\langle k\rangle^3 \geqslant \langle\sigma\rangle$. Here the additional weight on the left is of size one, so that we have to show

$$\left\|\partial_x(uv)\right\|_{Z_{s,\varepsilon}} \lesssim T^\gamma\|u\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}}\|v\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}}.$$

Subcase b.a: σ maximal. Exploiting the resonance relation (33), we see that the contribution from this subcase is bounded by

$$\left\|\mathcal{F}D_xD_y^\varepsilon\left(D_x^{-\frac{1}{2}}u\cdot D_x^{-\frac{1}{2}}v\right)\right\|_{L^2_{\xi,\tau}\cap L^2_\xi L^p_\tau} \lesssim \left\|\mathcal{F}\left(D_x^{\frac{1}{2}}D_y^\varepsilon u\cdot D_x^{-\frac{1}{2}}v\right)\right\|_{L^2_{\xi,\tau}\cap L^2_\xi L^p_\tau} + \cdots,$$

where $p<2$. The dots stand for the other possible distributions of derivatives on the two factors, in the same norms, which—by (36) of Corollary 2—can all be estimated by $c\|u\|_{X_{s,\varepsilon,b}}\|v\|_{X_{s,\varepsilon,b}}$ for some $b<\frac{1}{2}$. The latter is then further treated as in case a.

Subcase b.b: σ_1 maximal. Here we start with the observation that by Cauchy–Schwarz and (39), for every $b'>-\frac{1}{2}$ there is a $\gamma>0$ such that

$$\left\|\partial_x(uv)\right\|_{Z_{s,\varepsilon}} \lesssim T^\gamma\left\|D_x^{s+1}(uv)\right\|_{X_{0,\varepsilon,b'}}.$$

With the notation $\Lambda^b=\mathcal{F}^{-1}\langle\sigma\rangle^b\mathcal{F}$ we obtain from the resonance relation that

$$\begin{aligned}\left\|D_x^{s+1}(uv)\right\|_{X_{0,\varepsilon,b'}} &\lesssim \left\|D_x\left(D_x^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}u\cdot D_x^{-\frac{1}{2}}v\right)\right\|_{X_{0,\varepsilon,b'}} \lesssim \left\|\left(D_x^{\frac{1}{2}}D_y^\varepsilon\Lambda^{\frac{1}{2}}u\right)\left(D_x^{-\frac{1}{2}}v\right)\right\|_{X_{0,b'}} + \left\|\left(D_x^{\frac{1}{2}}\Lambda^{\frac{1}{2}}u\right)\left(D_x^{-\frac{1}{2}}D_y^\varepsilon v\right)\right\|_{X_{0,b'}} \\ &\quad + \left\|\left(D_x^{-\frac{1}{2}}D_y^\varepsilon\Lambda^{\frac{1}{2}}u\right)\left(D_x^{\frac{1}{2}}v\right)\right\|_{X_{0,b'}} + \left\|\left(D_x^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}u\right)\left(D_x^{\frac{1}{2}}D_y^\varepsilon v\right)\right\|_{X_{0,b'}}.\end{aligned}$$

Using (37) the first two contributions can be estimated by $c\|u\|_{X_{s,\varepsilon,\frac{1}{2}}}\|v\|_{X_{s,\varepsilon,b}}$ as desired. The third and fourth term only appear in the frequency range $|k|\ll|k_1|\sim|k_2|$, where the additional weight in the $\|u\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}}$-norm on the right becomes $\frac{|k_2|}{|k_1|}$, thus shifting a whole derivative from the high frequency factor v to the low frequency factor u. So, using (37) again, these contributions can be estimated by

$$c\|u\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}}\|v\|_{X_{s,\varepsilon,b}} \lesssim \|u\|_{X_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}}\|v\|_{X_{s,\varepsilon,b;\frac{1}{2}}}. \qquad \Box$$

Now we turn to the proof of Lemma 3, where the restrictions to the b-parameters can be relaxed slightly, so that the auxiliary Y- and Z-norms are not needed. We use again the Λ-notation, i.e. $\Lambda^b=\mathcal{F}^{-1}\langle\sigma\rangle^b\mathcal{F}$.

[2] For a proof see e.g. Lemma 1.10 in [6].

Proof of Lemma 3. First we show how (30) implies (31). By the resonance relation (33) we have

$$|k_1\eta - k\eta_1|^2 \lesssim |kk_1k_2|\big(\langle\sigma\rangle + \langle\sigma_1\rangle + \langle\sigma_2\rangle\big) \leqslant |kk_1k_2|\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle,$$

so that (31) is reduced to

$$\big\|D_x^{s+1+\frac{\varepsilon}{2}}M^{-\varepsilon}(u,v)\big\|_{X_{0,b'+\frac{\varepsilon}{2};\beta}} \lesssim \|u\|_{X_{s-\frac{\varepsilon}{2},b-\frac{\varepsilon}{2};\beta}}\|v\|_{X_{s-\frac{\varepsilon}{2},b-\frac{\varepsilon}{2};\beta}}.$$

Relabelling appropriately and choosing ε sufficiently small, we see that (31) follows from (30). To prove the latter, we may assume $s \leqslant 0$. Next we choose ε small and b' close to $-\frac{1}{2}$ so that

$$s > 2 + (\alpha+1)b' + 3\varepsilon \tag{40}$$

and $\beta := \frac{s-b'}{\alpha} \in [0, -b']$. Now the proof consists again of a case by case discussion.

Case a: $\langle k\rangle^{\alpha+1} \leqslant \langle\sigma\rangle$. Here it is sufficient to show

$$\big\|D_x^{s+1+\varepsilon-\alpha\beta-\beta}M^{-\varepsilon}(u,v)\big\|_{X_{0,b'+\beta}} \lesssim \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}. \tag{41}$$

Subcase a.a: $|k| \ll |k_1| \sim |k_2|$.

Subsubcase triple a: $\langle\sigma\rangle \geqslant \langle\sigma_{1,2}\rangle$. Here we use the resonance relation (33) to see that the left-hand side of (41) is bounded by

$$\big\|D_x^{s+1+\varepsilon-\alpha\beta+b'}M^{-\varepsilon}\big(D_x^{\frac{\alpha(b'+\beta)}{2}}u, D_x^{\frac{\alpha(b'+\beta)}{2}}v\big)\big\|_{L^2_{xyt}} \lesssim \big\|M^{-\varepsilon}\big(D_x^{\frac{s+1+\varepsilon+(\alpha+1)b'}{2}}u, D_x^{\frac{s+1+\varepsilon+(\alpha+1)b'}{2}}v\big)\big\|_{L^2_{xyt}},$$

where we have used the assumption on the frequency sizes in this subcase. Observe that our choice of β implies $s+1+\varepsilon-\alpha\beta+b' = 1+\varepsilon+2b' \geqslant 0$. Now the bilinear estimate (9) is applied to obtain the upper bound

$$\big\|D_x^{\frac{s+2+3\varepsilon+(\alpha+1)b'}{2}}u\big\|_{X_{0,b}}\big\|D_x^{\frac{s+2+3\varepsilon+(\alpha+1)b'}{2}}v\big\|_{X_{0,b}} \lesssim \|u\|_{X_{s,b}}\|v\|_{X_{s,b}},$$

where in the last step we have used (40).

Subsubcase a.a.b: $\langle\sigma_1\rangle \geqslant \langle\sigma\rangle, \langle\sigma_2\rangle$. Here the resonance relation (33) gives that the left-hand side of (41) is bounded by

$$\big\|D_x^{s+1+\varepsilon-\alpha\beta+b'}M^{-\varepsilon}\big(D_x^s\Lambda^b u, D_x^{\alpha(b'+\beta)-s}v\big)\big\|_{X_{0,-b}} \lesssim \big\|D_x^{-\frac{1}{2}-\varepsilon}M^{-\varepsilon}\big(D_x^s\Lambda^b u, D_x^{\frac{3}{2}+2\varepsilon+(\alpha+1)b'}v\big)\big\|_{X_{0,-b}}.$$

Now the dual version of estimate (9), that is

$$\big\|M^{-\varepsilon}(u,v)\big\|_{X_{-\frac{1}{2}-,-\frac{1}{2}-}} \lesssim \|u\|_{L^2_{xyt}}\|v\|_{X_{\frac{1}{2}+,\frac{1}{2}+}} \tag{42}$$

is applied, which gives, together with the assumption (40), that the latter is bounded by $c\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}$. This completes the discussion of subcase a.a. Concerning subcase a.b, where $|k| \gtrsim |k_{1,2}|$, we solely remark that it can be reduced to the estimation in subsubcase triple a.

Case b: $\langle k\rangle^{\alpha+1} \geqslant \langle\sigma\rangle$. Here the additional weight in the norm on the left of (30) is of size one, so our task is to show

$$\big\|D_x^{s+1+\varepsilon}M^{-\varepsilon}(u,v)\big\|_{X_{0,b'}} \lesssim \|u\|_{X_{s,b;\beta}}\|v\|_{X_{s,b;\beta}}. \tag{43}$$

Subcase b.a: $\langle\sigma\rangle \geqslant \langle\sigma_{1,2}\rangle$. Here we may assume by symmetry that $|k_1| \geqslant |k_2|$. We apply (33) and (9) to see that for $\delta \geqslant 0$ the left-hand side of (43) is controlled by

$$\big\|M^{-\varepsilon}\big(D_x^{s+1+\varepsilon+\alpha b'+\delta}u, D_x^{b'-\delta}v\big)\big\|_{L^2_{xyt}} \lesssim \big\|D_x^{\frac{3}{2}+2\varepsilon+\alpha b'+\delta}u\big\|_{X_{s,b}}\big\|D_x^{b'+\frac{1}{2}+\varepsilon-\delta}v\big\|_{X_{0,b}}.$$

The latter is bounded by $c\|u\|_{X_{s,b}}\|v\|_{X_{s,b}}$, provided $\frac{3}{2}+2\varepsilon+\alpha b'+\delta \leqslant 0$ and $b'+\frac{1}{2}+\varepsilon-\delta \leqslant s$, which can be fulfilled by a proper choice of $\delta \geqslant 0$, since (40) holds.

Subcase b.b: $\langle\sigma_1\rangle \geqslant \langle\sigma\rangle, \langle\sigma_2\rangle$.

Subsubcase b.b.a: $|k_1| \gtrsim |k_2|$. With $\delta \geqslant 0$ as in subcase b.a the contribution here is bounded by

$$\big\|D_x^{-\frac{1}{2}-\varepsilon}M^{-\varepsilon}\big(D_x^{s+\frac{3}{2}+2\varepsilon+\alpha b'+\delta}\Lambda^b u, D_x^{b'-\delta}v\big)\big\|_{X_{0,-b}} \lesssim \big\|D_x^{\frac{3}{2}+2\varepsilon+\alpha b'+\delta}u\big\|_{X_{s,b}}\big\|D_x^{b'+\frac{1}{2}+\varepsilon-\delta}v\big\|_{X_{0,b}} \lesssim \|u\|_{X_{s,b}}\|v\|_{X_{s,b}},$$

where (33) and the dual version (42) of Theorem 2 were used again. Finally we turn to:

Subsubcase triple b, where $|k_1| \ll |k| \sim |k_2|$. Here the additional weight in $\|u\|_{X_{s,b;\beta}}$ on the right of (43) behaves like

$$\left(\frac{|k|}{|k_1|}\right)^{\alpha\beta} \sim \left(\frac{|k_2|}{|k_1|}\right)^{\alpha\beta},$$

so that it is sufficient to show

$$\left\| D_x^{s+1+\varepsilon} M^{-\varepsilon}\left(u, D_x^{-\alpha\beta} v\right)\right\|_{X_{0,b'}} \lesssim \|u\|_{X_{s-\alpha\beta,b}} \|v\|_{X_{s,b}}. \tag{44}$$

Now by (33) the left-hand side of (44) can be controlled by

$$\left\| D_x^{-\frac{1}{2}-\varepsilon} M^{-\varepsilon}\left(D_x^{b'} \Lambda^b u, D_x^{s+\frac{3}{2}+2\varepsilon+\alpha(b'-\beta)} v\right)\right\|_{X_{0,-b}} \lesssim \left\| D_x^{b'} u\right\|_{X_{0,b}} \left\| D_x^{2+3\varepsilon+(\alpha+1)b'} v\right\|_{X_{0,b}}$$

by (42). Since $b' = s - \alpha\beta$ the first factor equals $\|u\|_{X_{s-\alpha\beta,b}}$, while by (40) the second is dominated by $\|v\|_{X_{s,b}}$. This proves (44). $\square$

Finally we recall the definition of the Fourier restriction norm spaces from [2]. For a time slab $I = (-\delta, \delta) \times \mathbb{T}^3$ they are given by

$$X^{\delta}_{s,\varepsilon,b;\beta} := \{u|_I\colon\ u \in X_{s,\varepsilon,b;\beta}\}$$

with norm

$$\|u\|_{X^{\delta}_{s,\varepsilon,b;\beta}} := \inf\left\{\|\widetilde{u}\|_{X_{s,\varepsilon,b;\beta}}\colon\ \widetilde{u} \in X_{s,\varepsilon,b;\beta},\ \widetilde{u}|I = u\right\}.$$

Now our well-posedness results read as follows.

Theorem 6 *(Precise version of Theorem 3). Let $s \geqslant \frac{1}{2}$ and $\varepsilon > 0$. Then for $u_0 \in H^s_x H^\varepsilon_y(\mathbb{T}^3)$ satisfying (2) there exist $\delta = \delta(\|u_0\|_{H^s_x H^\varepsilon_y}) > 0$ and a unique solution $u \in X^{\delta}_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}$ of the Cauchy problem (1) with $\alpha = 2$. This solution is persistent and the mapping $u_0 \mapsto u$, $H^s_x H^\varepsilon_y(\mathbb{T}^3) \to X^{\delta_0}_{s,\varepsilon,\frac{1}{2};\frac{1}{2}}$ is locally Lipschitz for any $\delta_0 \in (0,\delta)$.*

Theorem 7 *(Precise version of Theorem 4). Let $3 < \alpha \leqslant 4$, $s \geqslant s' > \frac{3-\alpha}{2}$ and $\varepsilon \geqslant 0$. Then for $u_0 \in H^s_x H^\varepsilon_y(\mathbb{T}^3)$ satisfying (2) there exist $b > \frac{1}{2}$, $\beta > 0$, $\delta = \delta(\|u_0\|_{H^{s'}_x L^2_y}) > 0$ and a unique solution $u \in X^{\delta}_{s,\varepsilon,b;\beta} \subset C^0((-\delta,\delta), H^s_x H^\varepsilon_y(\mathbb{T}^3))$. This solution depends continuously on the data, and extends globally in time, if $s \geqslant 0$ and $\varepsilon = 0$.*

With the estimates from Lemmas 2 and 3 at our disposal the proof of these theorems is done by the contraction mapping principle, cf. [2,5,14,15]. The reader is also referred to Section 1.3 of [6], where the related arguments are gathered in a general local well-posedness theorem.

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