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Dirichlet and *VMOA* domains via Schwarzian derivative ☆

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ABSTRACT

Short proofs of the following results concerning a bounded conformal map g of the unit disc $\mathbb{D}$ are presented: (1) $\log g'$ belongs to the Dirichlet space if and only if the Schwarzian derivative S_g of g satisfies $S_g(z)(1-|z|^2) \in L^2(\mathbb{D})$; (2) $\log g' \in VMOA$ if and only if $|S_g(z)|^2(1-|z|^2)^3$ is a vanishing Carleson measure on $\mathbb{D}$. Analogous results for Besov and $Q_{p,0}$ spaces are also given.

1. Introduction and results

Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in the unit disc $\mathbb{D}$. The *Besov space* B_p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{B_p}^p := \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} dA(z) < \infty, \quad 1 < p < \infty,$$

in particular, B_2 is the *Dirichlet space* $\mathcal{D}$. Moreover, $f \in \mathcal{H}(\mathbb{D})$ belongs to the $Q_{s,0}$ *space* if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) = 0, \quad 0 < s < \infty,$$

in particular, $Q_{1,0}$ is *VMOA*, the space of analytic functions in the Hardy space H^1 whose boundary values have *vanishing mean oscillation* on the unit circle $\mathbb{T}$. Here $\varphi_a(z) := (a-z)/(1-\bar{a}z)$ is the automorphism of $\mathbb{D}$ which satisfies $\varphi_a(\varphi_a(z)) = z$ for all $a, z \in \mathbb{D}$. Both B_p and *VMOA* are subspaces of the *little Bloch space*

$$\mathcal{B}_0 := \left\{f \in \mathcal{H}(\mathbb{D}):\ \lim_{|z|\to 1^-} f'(z)\left(1-|z|^2\right) = 0\right\},$$

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which is in turn a subspace of the *Bloch space*

$$\mathcal{B} := \Big\{ f \in \mathcal{H}(\mathbb{D}):\ \|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} \big|f'(z)\big|\big(1-|z|^2\big) < \infty \Big\}.$$

A positive Borel measure on $\mathbb{D}$ is a *vanishing* (or *compact*) *s-Carleson measure* if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0, \quad 0 < s < \infty,$$

where I is an interval on $\mathbb{T}$ and $S(I) := \{z \in \mathbb{D}:\ 1-|I| \leqslant |z|,\ z/|z| \in I\}$. It is well known that μ is a vanishing s-Carleson measure if and only if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} \big|\varphi_a'(z)\big|^s \, d\mu(z) = 0, \quad 0 < s < \infty, \tag{1}$$

see [3]. The *Schwarzian derivative* of a locally univalent function $g : \mathbb{D} \to \Omega$ is defined as

$$S_g(z) := \left(\frac{g''(z)}{g'(z)}\right)' - \frac{1}{2}\left(\frac{g''(z)}{g'(z)}\right)^2 = f''(z) - \frac{1}{2}\big(f'(z)\big)^2, \quad f := \log g'. \tag{2}$$

Adopting the terminology from [7] for a subspace X of $\mathcal{H}(\mathbb{D})$, we say that Ω is an X *domain* if $\log g' \in X$. Many such domains have been characterized in terms of the Schwarzian derivative of g in the case when g is a conformal map of $\mathbb{D}$. Namely, Becker and Pommerenke [4] characterized bounded $\mathcal{B}_0$ domains in 1978, and in 1991 Astala and Zinsmeister [1] gave a description of *BMOA* domains. Moreover, Q_p domains were characterized by Pau and Peláez [9] in 2009 by using a method developed by Bishop and Jones [5] in 1994. For geometric characterizations of bounded *VMOA* and $\mathcal{B}_0$ domains we refer to a work by Pommerenke [10,11], and in the case of *BMOA* the reader is invited to see the monograph [7] or the original paper by Bishop and Jones [5].

The purpose of this note is to present short proofs of the following two results which complete in part [7, Theorem 8.1(e)] and [9, Theorems 1 and 2].

Theorem 1. *Let* $1 < p < \infty$, *and let* $g : \mathbb{D} \to \Omega$ *be a conformal map such that* $g(\mathbb{T})$ *is a closed Jordan curve. Then* Ω *is a* B_p *domain if and only if*

$$I(g) := \int_{\mathbb{D}} \big|S_g(z)\big|^p \big(1-|z|^2\big)^{2p-2} \, dA(z) < \infty. \tag{3}$$

In particular, Ω *is a Dirichlet domain if and only if* $S_g(z)(1-|z|^2) \in L^2(\mathbb{D})$.

Theorem 2. *Let* $0 < s \leqslant 1$, *and let* $g : \mathbb{D} \to \Omega$ *be a conformal map such that* $g(\mathbb{T})$ *is a closed Jordan curve. Then* Ω *is a* $Q_{s,0}$ *domain if and only if* $|S_g(z)|^2(1-|z|^2)^{2+s}\, dA(z)$ *is a vanishing s-Carleson measure. In particular,* Ω *is a VMOA domain if and only if* $|S_g(z)|^2(1-|z|^2)^3\, dA(z)$ *is a vanishing Carleson measure.*

The rest of this note is devoted to proofs. But before presenting them, few words about the notation. We write $a \lesssim b$, if $a \leqslant Cb$ for some positive constant C, independent of a and b, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \simeq b$. We will repeatedly use the following well-known result on the norm of the *weighted Bergman space* A^p_α:

$$\|h\|^p_{A^p_\alpha} := \int_{\mathbb{D}} \big|h(z)\big|^p \big(1-|z|^2\big)^{\alpha} \, dA(z) \simeq \int_{\mathbb{D}} \big|h'(z)\big|^p \big(1-|z|^2\big)^{p+\alpha} \, dA(z) + \big|h(0)\big|^p, \tag{4}$$

valid for all $h \in \mathcal{H}(\mathbb{D})$, $0 < p < \infty$ and $-1 < \alpha < \infty$.

2. Proof of Theorem 1

If Ω is a B_p domain, i.e., $f = \log g' \in B_p$, then (2) and (4) yield

$$I(g) \lesssim \|f''\|^p_{A^p_{2p-2}} + \|f'\|_{A^{2p}_{2p-2}} \lesssim \|f\|^p_{B_p} + \|f\|^p_{\mathcal{B}}\|f\|^p_{B_p}$$

from which (3) follows since $B_p \subset \mathcal{B}$ with $\|f\|_{\mathcal{B}} \lesssim \|f\|^p_{B_p}$.

If (3) is satisfied, then the subharmonicity of $|S_g|^p$ yields $S_g(z)(1-|z|^2)^2 \to 0$, as $|z| \to 1^-$, and hence $f \in \mathcal{B}_0$ by [10, Theorem 11.1]. Therefore for a given $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that $|f'(z)|(1-|z|^2) \leqslant \varepsilon$ whenever $|z| \geqslant \delta/(2-\delta)$.

Let $r \in (\frac{1}{2-\delta}, 1)$, and denote $f_r(z) := f(rz)$. Since all polynomials belong to B_p, we may assume $f'(0) = 0$. Then the identities (2) and (4) imply

$$\|f_r\|_{B_p}^p \simeq \|f_r''\|_{A_{2p-2}^p}^p \lesssim \|S_{g_r}\|_{A_{2p-2}^p}^p + \|f_r'\|_{A_{2p-2}^{2p}}^{2p}$$
$$\lesssim \|S_{g_r}\|_{A_{2p-2}^p}^p + \varepsilon^p \|f_r\|_{B_p}^p + \int_{D(0,\delta)} \left|f_r'(z)\right|^{2p} \left(1-|z|^2\right)^{2p-2} dA(z),$$

where $D(0,\delta) := \{z\colon\ |z| < \delta\}$. This and Hardy's convexity theorem [6, Theorem 1.5] show that

$$\|f_r\|_{B_p}^p \leqslant C\left(\|S_{g_r}\|_{A_{2p-2}^p}^p + \varepsilon^p \|f_r\|_{B_p}^p + \int_{D(0,\delta)} \left|f'(z)\right|^{2p}\left(1-|z|^2\right)^{2p-2} dA(z)\right)$$

for some positive constant C. Choosing $\varepsilon^p = 1/(2C)$ and fixing δ accordingly, re-arranging terms, applying Fatou's lemma and [8, Proposition 1.3], we obtain

$$\|f\|_{B_p}^p \leqslant \liminf_{r\to 1^-} \|f_r\|_{B_p}^p \leqslant 2C\left(\|S_g\|_{A_{2p-2}^p}^p + \int_{D(0,\delta)} \left|f'(z)\right|^{2p}\left(1-|z|^2\right)^{2p-2} dA(z)\right).$$

Therefore $f \in B_p$ and thus Ω is a B_p domain as desired.

3. Proof of Theorem 2

If $f \in Q_{s,0}$, i.e., $|f'(z)|^2(1-|z|^2)^s\,dA(z)$ is a vanishing s-Carleson measure by (1), then so is $|f''(z)|^2(1-|z|^2)^{2+s}\,dA(z)$ by [2, Theorem 2]. The identity (2) yields

$$\left|S_g(z)\right|^2\left(1-|z|^2\right)^{2+s} \lesssim \left|f''(z)\right|^2\left(1-|z|^2\right)^{2+s} + \|f\|_{\mathcal{B}}^2 \left|f'(z)\right|^2\left(1-|z|^2\right)^s,$$

and since $Q_{s,0} \subset \mathcal{B}$, it follows that $|S_g(z)|^2(1-|z|^2)^{2+s}\,dA(z)$ is a vanishing s-Carleson measure.

To see the converse, note first that, by (1), $|S_g(z)|^2(1-|z|^2)^{2+s}\,dA(z)$ is a vanishing s-Carleson measure if and only if

$$\lim_{|a|\to 1^-} T(g,a) := \lim_{|a|\to 1^-} \int_{\mathbb{D}} \left|S_g(z)\right|^2\left(1-|z|^2\right)^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) = 0.$$

Again the subharmonicity of $|S_g|^2$ yields $S_g(z)(1-|z|^2)^2 \to 0$, as $|z| \to 1^-$, and therefore $f \in \mathcal{B}_0$ by [10, Theorem 11.1]. Since all polynomials belong to $Q_{s,0}$, we may assume $f'(0) = 0$. Then (4), with $h(z) := f'(z)/(1-\bar{a}z)^s$, and (2) yield

$$I_1(f,a) := \int_{\mathbb{D}} \left|f'(z)\right|^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)$$
$$\lesssim \int_{\mathbb{D}} \left|f''(z)\right|^2\left(1-|z|^2\right)^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) + \int_{\mathbb{D}} \left|\frac{f'(z)}{1-\bar{a}z}\right|^2\left(1-|z|^2\right)^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)$$
$$\lesssim T(g,a) + \int_{\mathbb{D}} \left|f'(z)\right|^4\left(1-|z|^2\right)^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)$$
$$+ \int_{\mathbb{D}} \left|\frac{f'(z)}{1-\bar{a}z}\right|^2\left(1-|z|^2\right)^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z). \tag{5}$$

Since $f \in \mathcal{B}_0$, for $\varepsilon_1 > 0$, there exists $\delta \in (0,1)$ such that $|f'(z)|(1-|z|^2) \leqslant \varepsilon_1$ whenever $|z| \geqslant \delta$. This and the triangular inequality applied to the denominator of the last term in (5) implies

$$I_1(f,a) \lesssim T(g,a) + \int_{D(0,\delta)} \left|f'(z)\right|^4\left(1-|z|^2\right)^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)$$
$$+ \varepsilon_1^2 I_1(f,a) + \int_{D(0,\delta)} \left|f'(z)\right|^2\left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z)$$
$$+ \varepsilon_1^2 \int_{\mathbb{D}} \frac{(1-|a|^2)^s\left(1-|z|^2\right)^s}{|1-\bar{a}z|^{2s+2}}\, dA(z). \tag{6}$$

The last integral in (6) is uniformly bounded for all $a \in \mathbb{D}$ by the Forelli–Rudin estimates [8, Theorem 1.7], and therefore

$$\begin{aligned} I_1(f,a) \lesssim T(g,a) &+ \int\limits_{D(0,\delta)} \left|f'(z)\right|^4 \left(1-|z|^2\right)^2 \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) \\ &+ \varepsilon_1^2 I_1(f,a) + \int\limits_{D(0,\delta)} \left|f'(z)\right|^2 \left(1-\left|\varphi_a(z)\right|^2\right)^s dA(z) + \varepsilon_1^2. \end{aligned} \tag{7}$$

Let now $\varepsilon > 0$ be given. Choosing ε_1 sufficiently small and fixing δ accordingly, and re-arranging terms in (7) we obtain $I_1(f,a) < \varepsilon$ for all $a \in \mathbb{D}$ sufficiently close to the boundary. Thus $f \in Q_{s,0}$ as desired.

Note that there is no need to use dilatations as in the proof of Theorem 1 since $I_1(f,a)$ is uniformly bounded by [9, Theorem 1].

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